Birational Classification of Algebraic Threefolds

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§1. Introduction

Let us begin by explaining the background of the birational classification. We will work over the field $\mathbb{C}$ of complex numbers unless otherwise mentioned.

Let X be a non-singular projective variety of dimension r. The canonical divisor class K_X is the only divisor class (up to multiples) naturally defined on an arbitrary X. Its sheaf $\mathcal{O}_X(K_X)$ is the sheaf of holomorphic r-forms. An alternative description is $K_X = -c_1(X)$, where $c_1(X)$ is the first Chern class of X. Therefore it is natural to expect some role of K_X in the classification of algebraic varieties.

The classification of non-singular projective curves C is classical, and summarized in the following table, where $g(C)$ is the genus (the number of holes) of C, $H = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$ and Γ is a subgroup of $SL_2(\mathbb{R})$:

(1.1)

$g(C)$	0	1	≥ 2
$\deg K_C$	-2	0	$2g(C)-2$
C	$\mathbb{P}^1$	$\mathbb{C}/(\text{lattice})$	H/Γ

Here we see three different situations. For instance, everything is explicit if $g(C) = 0$; the moduli (to parametrize curves) is the main interest if $g(C) \geq 2$.

Our interest is in generalizing this to higher dimensions. The first difficulty which arises in the surface case is that there are too many varieties for genuine classification (*biregular classification*).

(1.2) For a non-singular projective surface X and an arbitrary point $x \in X$, there is a birational morphism $\pi : B_xX \to X$ from a non-singular projective surface B_xX such that $E = \pi^{-1}(x)$ is isomorphic to $\mathbb{P}^1$ (E is called a *(-1)-curve*) and π induces an isomorphism $B_xX - E \simeq X - x$.

In view of (1.2), it is impractical to distinguish X from B_xX, $B_yB_xX, \ldots$ if we want a reasonable classification list. More generally, we say that two algebraic varieties X and Y are *birationally equivalent* and we write $X \sim Y$ if there is a birational mapping $X \cdots \to Y$ or equivalently if their rational function fields $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ are isomorphic function fields over $\mathbb{C}$. We did not face this phenomenon in the curve case, since $X \simeq Y$ iff $X \sim Y$ for curves X and Y.

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In view of the list (1.1) for curves, we need to divide the varieties into several classes to formulate more precise problems. This is why the *Kodaira dimension* $\kappa(X)$ of a non-singular projective variety X was introduced by [Iitaka1] and [Moishezon].

(1.3) Let $H^0(X, \mathcal{O}(\nu K_X))$ be the space of global ν-ple holomorphic r-forms ($\nu \geq 0, r = \dim X$), and $\phi_0, \ldots, \phi_N$ be its basis. If $N \geq 0$, then

$$\Phi_{\nu K_X} : X \cdots \to \mathbb{P}^N \text{ given by } \Phi_{\nu K}(x) = (\phi_0(x) : \cdots : \phi_N(x))$$

is a rational map. We set $P_\nu(X) = N + 1$. It is important that $H^0(X, \mathcal{O}(\nu K_X))$ and $\Phi_{\nu K}$ are birational invariants, that is $X \sim Y$ induces $H^0(X, \mathcal{O}(\nu K_X)) = H^0(Y, \mathcal{O}(\nu K_Y))$ for $\nu > 0$. We set $\kappa(X) = -\infty$ if $P_\nu(X) = 0$ for all $\nu > 0$. If $P_e(X) > 0$ for some $e > 0$, then

$$\kappa(X) := \mathrm{Max}\{\dim \Phi_{\nu K_X}(X) \mid \nu > 0\}.$$

In particular, $P_\nu(X)$ and $\kappa(X)$ are birational invariants of X.

We remark that $\kappa(X) \in \{-\infty, 0, 1, \ldots, \dim X\}$, and that X with $\kappa(X) = \dim X$ is said to be *of general type*. We have the following table for curves.

(1.4)

$g(C)$	0	1	≥ 2
$\kappa(C)$	$-\infty$	0	1

To have some idea on higher dimensions, we can use the easy result $\kappa(X \times Y) = \kappa(X) + \kappa(Y)$. In particular,

(1.5) case ($\kappa(X) = -\infty$) $\quad \kappa(\mathbb{P}^1 \times Y) = -\infty$,

(1.6) case ($0 < \kappa(X) < \dim X$)

$$\kappa(\underbrace{E \times \cdots \times E}_{a \text{ times}} \times \underbrace{C \times \cdots \times C}_{b \text{ times}}) = b \text{ if } g(E) = 1 \text{ and } g(C) \geq 2.$$

The case $0 < \kappa(X) < \dim X$ is studied by the Iitaka fibration.

(1.7) **Iitaka Fibering Theorem** [Iitaka2]. *Let X be a non-singular projective variety with $0 < \kappa(X) < \dim X$. Then there is a morphism $f : X' \to Y'$ of non-singular projective varieties with connected fibers such that $X' \sim X$, $\dim Y' = \kappa(X)$ and $\kappa(f^{-1}(y)) = 0$ for a sufficiently general point $y \in Y'$.*

In (1.7), we cannot expect $\kappa(Y') = \dim Y'$ or even $\kappa(Y') \geq 0$. Therefore X' is not so simple as (1.6). Nevertheless (1.7) reduces the case $0 < \kappa(X) < \dim X$ to the cases $\kappa(X) = -\infty, 0, \dim X$. Thus we can explain the birational classification as in (1.1) for higher dimensions.

§2. Birational Classification

For a non-singular projective variety X, we define a graded ring (called the *canonical ring*)

$$R(X) = \oplus_{\nu \geq 0} H^0(X, \mathcal{O}(\nu K_X)).$$

If $\kappa(X) \geq 0$, the ν-canonical image $\Phi_{\nu K}(X)$ is a birational invariant of X. The existence of stable canonical image is interpreted in terms of $R(X)$ by the following easy proposition.

(2.1) **Proposition.** *Let X be a non-singular projective variety of $\kappa(X) \geq 0$. Then $\Phi_{\nu K}(X)$ for sufficiently divisible $\nu > 0$ are all naturally isomorphic iff $R(X)$ is a finitely generated $\mathbb{C}$-algebra.*

(2.2) For X of general type, constructing moduli spaces is one of our main interests. One standard way is to try to find a uniform ν such that $\Phi_{\nu K} : X \cdots \to \Phi_{\nu K}(X)$ is birational for all X and classify the image. One can expect nice properties of the image (canonical model to be explained later) if there is a stable canonical image. Therefore we would like to ask whether the canonical ring is finitely generated for X of general type (2.1).

(2.3) The case $\kappa(X) = \dim X$ suggests to reduce the birational classification of all varieties to the biregular classification of standard models (like $\Phi_{\nu K}(X)$ for sufficiently divisible ν). However, when $\kappa(X) < \dim X$, there are no obvious candidates for the standard models. For $0 \leq \kappa(X)$, we can ask to find some "standard" models.

We only say the following for $\kappa(X) \leq 0$ at this point.

(2.4) For X with $\kappa(X) = 0$, we would like to find some "standard" model $Y \sim X$ and to classify all such Y.

(2.5) For many X with $\kappa(X) = -\infty$, there exist infinitely many "standard" models $\sim X$. To study the relation among these models is a role of birational geometry. We would like to have a structure theorem of such models. One general problem is to see if all such X are *uniruled*, i.e. there exists a rational curve through an arbitrary point of X, or equivalently there is a dominating rational map $\mathbb{P}^1 \times Y \cdots \to X$ for some Y of dimension $n-1$. (It is easy to see that uniruled varieties have $\kappa = -\infty$ as in (1.5).)

Since we use the formulation by Iitaka and Moishezon, one basic problem will be the deformation invariance of κ.

(2.6) **Conjecture** [Iitaka1, Moishezon]. *Let $f : X \to Y$ be a smooth projective morphism with connected fibers and connected Y. Then $\kappa(f^{-1}(y))$ and $P_\nu(f^{-1}(y))$ $(\nu \geq 1)$ are independent of $y \in Y$.*

§3. Surface Case

We review a few classical results on surfaces which may help the reader to understand the results for 3-folds.

The basic result is the inverse process of (1.2).

(3.1) **Castelnuovo-Enriques.** *Let E be a curve on a non-singular projective surface X'. Then E is a (-1)-curve (i.e. $X' = B_x X$ and E is the inverse image of x for some non-singular projective surface X and $x \in X$) iff $E \simeq \mathbb{P}^1$ and $(E \cdot K_{X'}) = -1$. We write* $\text{cont}_E : X' \to X$ *and call it the contraction of the* (-1)*-curve E.*

Finding a (-1)-curve in every exceptional set, we have the following:

(3.2) **Factorization of Birational Morphisms.** *Let $f : X \to Y$ be a birational morphism of non-singular projective surfaces. Then f is a composition of a finite number of contractions of* (-1)*-curves.*

Starting with a non-singular projective surface X, we can keep contracting (-1)-curves if there are any. After a finite number of contractions, we get a non-singular projective surface $Y(\sim X)$ with no (-1)-curves. Depending on whether K_Y is *nef* ($(K_Y \cdot C) \geq 0$ for all curves C), $\kappa(X)$ takes different values.

(3.3) Case where K_Y is nef. Then Y is the only non-singular projective surface $\sim X$ with no (-1)-curves. To be precise, if Y' is a such surface, then the composite $Y \cdots \to X \cdots \to Y'$ is an isomorphism. This Y is called the *minimal model* of X and denoted by X_{min}. In this case, we have $\kappa(X) \geq 0$.

(3.4) Case where K_Y is not nef. Then an arbitrary Y' (including Y) which is birational to X and has no (-1)-curves is isomorphic to either $\mathbb{P}^2$ or a $\mathbb{P}^1$-bundle over some non-singular curve. In this case, X has no minimal models and we have $\kappa(X) = -\infty$ by (1.5).

The above (3.3) together with (3.4) says that the birational classification of X with $\kappa \geq 0$ is equivalent to the biregular classification of minimal models.

Based on (3.3) and (3.4), the canonical model is defined.

(3.5) Let X be a non-singular projective surface of general type. Then there exists exactly one normal projective surface $Z(\sim X)$ such that Z has only Du Val (rational double) points and K_Z is ample, where Du Val points are defined by one of the following list.

$$A_n : xy + z^{n+1} = 0 \ (n \geq 0),$$
$$D_n : x^2 + y^2 z + z^{n-1} = 0 \ (n \geq 4),$$
$$E_6 : x^2 + y^3 + z^4 = 0,$$
$$E_7 : x^2 + y^3 + yz^3 = 0,$$
$$E_8 : x^2 + y^3 + z^5 = 0.$$

Such Z is called the *canonical model* of X and denoted by X_{can}. The natural map $X_{\min} \cdots \to X_{\text{can}}$ is a morphism which contracts all the rational curves C with $(C \cdot K_{X_{\min}}) = 0$ into Du Val points and is isomorphic elsewhere.

(3.5.1) **Remark.** This X_{can} can also be obtained as $\Phi_{\nu K}(X_{\min}) = \Phi_{\nu K}(X)$ for an arbitrary $\nu \geq 5$ (Bombieri).

(3.6) Let X be a non-singular projective minimal surface with $\kappa = 0$. Thus X has torsion K_X, i.e. some non-zero multiple of it is trivial. There is a precise classification of all such X.

(3.7) The deformation invariance of $\kappa(x)$ and $P_\nu(X)$ was done by [Iitaka3] using the classification of surfaces. [Levine] gave a simple proof without using classification.

§4. The Extremal Ray Theory (The Minimal Model Theory)

The first problem in generalizing the results in §3 to higher dimensions is to find some class of varieties in which there is a reasonable contraction theorem because there is no immediate generalization of (3.1) to 3-folds, since the contraction process inevitably introduces singularities [Mori1]. To define the necessary class of singularities, the first important step was taken by Reid [Reid1,3].

(4.1) **Definition** [Reid3]. Let (X, P) be a normal germ of an algebraic variety (or an analytic space) which is normal. We say that (X, P) has *terminal singularities* (resp. *canonical singularities*) iff

(i) K_X is a $\mathbb{Q}$-*Cartier* divisor, i.e. rK_X is Cartier for some positive integer r (minimal such r is called the *index* of (X, P)), and

(ii) for some (or equivalently, every) resolution $\pi : Y \to (X, P)$, we have $a_i > 0$ (resp. $a_i \geq 0$) for all i in the expression:

$$rK_Y = \pi^*(rK_Y) + \sum a_i E_i,$$

where E_i are all the exceptional divisors and $a_i \in \mathbb{Z}$.

For surfaces, a terminal (resp. canonical) singularity is smooth (resp. a Du Val point). We note that, for projective varieties X with only canonical singularities, the same definitions of $P_\nu(X)$, $\Phi_{\nu K}$ and $\kappa(X)$ work and these are still birational invariants. We can also talk about the ampleness of K_X and the intersection number $(K_X \cdot C) \in \mathbb{Q}$ for such X.

The idea of the cone of curves which is the core of the extremal ray theory was first introduced in Hironaka's thesis [Hironaka].

(4.2) **Definition.** Let X be a projective n-fold. A 1-cycle $\sum a_C C$ is a formal finite sum of irreducible curves C on X with coefficients $a_C \in \mathbb{Z}$. For a 1-cycle Z and a $\mathbb{Q}$-Cartier divisor D, the intersection number $(Z \cdot D) \in \mathbb{Q}$ is defined. Then

$$N_1(X)_{\mathbb{Z}} = \{\text{1-cycles}\}/\{\text{1-cycles } Z \mid (Z \cdot D) = 0 \text{ for all } D\}$$

is a free abelian group of finite rank $\varrho(X) < \infty$. Thus $N_1(X) = N_1(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ is a finite dimensional Euclidean space. The classes $[C]$ of all the irreducible curves C span a convex cone $NE(X)$ in $N_1(X)$. Taking the closure for the metric topology, we have a closed convex cone $\overline{NE}(X)$. Then

(4.3) **Cone Theorem.** *If X has only canonical singularities, then there exist countably many half lines $R_i \subset NE(X)$ such that*

(i) $\overline{NE}(X) = \sum_i R_i + \{z \in \overline{NE}(X) \mid (z \cdot K_X) \geq 0\}$,

(ii) for an arbitrary ample divisor H of X and arbitrary $\varepsilon > 0$, there are only finitely many R_i's contained in

$$\{z \in \overline{NE}(X) \mid (z \cdot K_X) \leq -\varepsilon(z \cdot H)\}.$$

Such an R_i is called an *extremal ray* of X if it cannot be omitted in (i) of (4.3). We note that an extremal ray exists on X iff K_X is not nef. Each extremal ray R_i defines a contraction of X.

(4.4) **Contraction Theorem.** *Let R be an extremal ray of a projective n-fold X with only canonical singularities. Then there exists a morphism $f : X \to Y$ to a projective variety Y (unique up to isomorphism) such that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and an irreducible curve $C \subset X$ is sent to a point by f iff $[C] \in R$. Furthermore* $\mathrm{Pic}\,Y = \mathrm{Ker}[(C\cdot) : \mathrm{Pic}\,X \to \mathbb{Z}]$ *for such a contracted curve C. This f is called the contraction of R and denoted by* cont_R.

The contraction of an extremal ray is not always birational.

(4.5) *Let X be a smooth projective surface with an extremal ray R. Then* cont_R *is one of the following.*

(i) the contraction of a (-1)-curve,

(ii) a $\mathbb{P}^1$-bundle structure $X \to C$ over a non-singular curve,

(iii) a morphism to one point, when $X \simeq \mathbb{P}^2$.

The description of all the possible contractions for a nonsingular projective 3-fold X is given in [Mori1]. Here we only remark that $\mathrm{cont}_R X$ can have a terminal singularity $\mathbb{C}^3/ < \sigma >$ of index 2, where σ is the involution $\sigma(x, y, z) = (-x, -y, -z)$.

(4.6) The category of varieties in which we play the game of the minimal model program is the category $\mathscr{C}$ of projective varieties with only terminal singularities which are $\mathbb{Q}$-*factorial* (i.e. every Weil divisor is $\mathbb{Q}$-Cartier). The goal of the game is to get a *minimal* (resp. *canonical*) model, i.e. a projective n-fold X with only terminal (resp. canonical) singularities such that K_X is nef (resp. ample). Let us first state the minimal model program which involves two conjectures.

(4.7) Let X be an n-fold $\in \mathscr{C}$. If K_X is nef, then X is a minimal model and we are done. Otherwise, X has an extremal ray R. Then $\mathrm{cont}_R : X \to X'$ satisfies one of the following.

(4.7.1) Case where $\dim X' < \dim X$. Then cont_R is a surjective morphism with connected fibers of dimension > 0 and relatively ample $-K_X$ (like $\mathbb{P}^1$-bundle), and X is uniruled ([Miyaoka-Mori]). This is the case where we can never get a minimal model, and we stop the game since we have the global structure of X, $\mathrm{cont}_R : X \to X'$.

(4.7.2) Case where $\mathrm{cont}_R : X \to X'$ is birational and contracts a divisor. This cont_R is called a *divisorial contraction*. In this case $X' \in \mathscr{C}$ and $\varrho(X') < \varrho(X)$. Therefore we can work on X' instead of X.

(4.7.3) Case where $\mathrm{cont}_R : X \to X'$ is birational and contracts no divisors. In this case, $K_{X'}$ is not $\mathbb{Q}$-Cartier and $X' \notin \mathscr{C}$. So we cannot continue the game with X'. This is the new phenomenon in dimension ≥ 3.

To get around the trouble in (4.7.3) and to continue the game, Reid proposed the following.

(4.8) **Conjecture (Existence of Flips).** *In the situation of (4.7.3), there is an n-fold $X^+ \in \mathscr{C}$ with a birational morphism $f^+ : X^+ \to X'$ which contracts no divisors and such that K_{X^+} is f^+-ample. The map $X \cdots \to X^+$ is called a flip.*

Since $\varrho(X^+) = \varrho(X)$ in (4.8), the divisorial contraction will not occur for infinitely many times. Therefore the following will guarantee that the game will be over after finitely many steps.

(4.9) **Conjecture (Termination of Flips).** *There does not exist an infinite sequence of flips $X_1 \cdots \to X_2 \cdots \to \cdots$.*

Therefore the minimal model program is completed only when the conjectures (4.8) and (4.9) are settled affirmatively.

The conjecture (4.9) was settled affirmatively by [Shokurov1] for 3-folds and by Kawamata-Matsuda-Matsuki [KMM] for 4-folds. (4.8) was first done by [Tsunoda], [Shokurov2], [Mori3] and [Kawamata6] in a special but important case. Finally (4.8) was done for 3-folds by [Mori5] using the work of [Kawamata6] mentioned above.

(4.10) Thus for 3-folds, we can operate divisorial contractions and flips for a finite number of times and get either a minimal model $\in \mathscr{C}$ or an $X \in \mathscr{C}$ which has an extremal ray R of type (4.7.1). Thus we can get 3-fold analogues of results in §3.

(4.11) For simplicity of the exposition, we did not state the results in the strongest form and we even omitted various results. Therefore we would like to mention names and give a quick review.

After the prototype of the extremal ray theory was given in [Mori1], the theory has been generalized to the relative setting with a larger class of singularities (toward the conjectures of Reid [Reid3,4]) by Kawamata, Benveniste, Reid, Shokurov and Kollár (in the historical order) and perhaps some others.

First through the works of [Benveniste] and [Kawamata2], Kawamata introduced a technique [Kawamata3] which was an ingenious application of the Kawamata-Viehweg vanishing ([Kawamata1] and [Viehweg2]). Based on the works by [Shokurov1] (Non-vanishing theorem) and [Reid2] (Rationality theorem), [Kawamata4] developped the technique to prove the Base point freeness theorem (and others) in arbitrary dimensions. The discreteness of the extremal rays was later done by [Kollár1]. As for this section, we refer the reader to the talk of Kawamata.

§5. Applications of the Minimal Model Program (MMP) to 3-Folds

Considering MMP in relative setting, one has the factorization generalizing (3.2):

(5.1) **Theorem.** *Let $f : X \to Y$ be a birational morphism of projective 3-folds with only $\mathbb{Q}$-factorial terminal singularities. Then f is a composition of divisorial contractions and flips.*

Since minimal 3-folds have $\kappa \geq 0$ by the hard result of Miyaoka [Miyaoka1–3], one has the following (cf. (3.3) and (3.4)).

(5.2) **Theorem.** *A 3-fold X has a minimal model iff $\kappa(X) \geq 0$.*

Unlike the surface case, the minimal model of a 3-fold X is not unique; it is unique only in codimension 1. If we are given a $\mathbb{Q}$-factorial minimal model $X_{\min}$, every other $\mathbb{Q}$-factorial minimal model of X is obtained from $X_{\min}$ by operating a simple operation called a *flop* for a finite number of times ([Kawamata6], [Kollár4]). Many important invariants computed by minimal models do not depend on the choice of the minimal model. We refer the reader to the talk of Kollár.

(5.3) **Theorem.** *For a 3-fold X, the following are equivalent.*

(i) $\kappa(X) = -\infty$,

(ii) X is uniruled,

(iii) X is birational to a projective 3-fold Y with only $\mathbb{Q}$-factorial terminal sigularities which has an extremal ray of type (4.7.1).

It will be an important but difficult problem to classify all the possible Y in (iii) of (5.3). There are only finitely many families of such Y with $\varrho(Y) = 1$ ([Kawamata7]).

Since a canonical model exists if a minimal model does ([Benveniste] and [Kawamata2]), one has the following (cf. (3.5)).

(5.4) **Theorem.** *If X is a 3-fold of general type, then X has a canonical model and the canonical ring $R(X)$ is a finitely generated $\mathbb{C}$-algebra.*

The argument for (5.4) can be considered as a generalization of the argument for (3.5.1). However the effective part "$\nu \geq 5$" of (3.5.1) has not yet been generalized to dimension ≥ 3.

To study varieties X with $\kappa \geq 0$, [Kawamata4] posed the following.

(5.5) **Conjecture (Abundance Conjecture).** *If X is a minimal variety, then rK_X is base point free for some $r > 0$.*

For 3-folds, there are works by [Kawamata4] and [Miyaoka4] (cf. [KMM]). However the torsionness of K for minimal 3-folds with $\kappa = 0$ is unsolved, and it remains to prove:

(5.6) **Problem.** *Let X be a minimal 3-fold with H an ample divisor such that $(K_X^3) = 0$ and $(K_X^2 \cdot H) > 0$. Then prove that $\kappa(X) = 2$.*

(5.7) **Remark** ($\kappa = 0$). The 3-folds X with $\kappa(X) = 0$ and $H^1(X, \mathcal{O}_X) \neq 0$ were classified by [Viehweg1] and (5.5) holds for these. This was based on Viehweg's solution of the addition conjecture for 3-folds, and we refer the reader to [Iitaka4]. However not much is known about the 3-folds X with $\kappa(X) = 0$ (or even K_X torsion) and $H^1(X, \mathcal{O}_X) = 0$: so far many examples have been constructed and it is not known if there are only finitely many families. There is a conjecture of [Reid6] in this direction.

By studying the flips more closely, [Kollár-Mori] proved the deformation invariance of κ and P_ν (cf. (3.7)):

(5.8) **Theorem.** *Let $f : X \to \Delta$ (unit disk) be a projective morphism whose fibers are connected 3-folds with only $\mathbb{Q}$-factorial terminal singularities. Then*

(i) $\kappa(X_t)$ is independent of $t \in \Delta$, where $X_t = f^{-1}(t)$,

(ii) $P_\nu(X_t)$ is independent of $t \in \Delta$ for all $\nu \geq 0$ if $\kappa(X_0) \neq 0$.

Indeed for such a family X/Δ, the simultaneous minimal model program is proved and the (modified) work of [Levine] is used to prove (5.8). We cannot drop the condition "$\kappa(X_0) \neq 0$" at present since the abundance conjecture is not completely solved for 3-folds.

As for other applications (e.g. addition conjecture, deformation space of quotient surface singularities, birational moduli), we refer the reder to [KMM] and [Kollár6].

§6. Comments on the Proofs for 3-Folds

Many results on 3-folds are proved by using only the formal definitions of terminal singularities. However some results on 3-folds rely on the classification of 3-fold terminal singularities [Reid3], [Danilov], [Morrison-Stevens], [Mori2] and [KSB] (cf. Reid's survey [Reid5] and [Stevens].) The existence of flips and flops heavily rely on it. Thus generalizing their proofs to higher dimension seems hopeless. At present, there is no evidence for the existence of flips in higher dimensions except that they fit in the MMP beautifully. I myself would accept them as working hypotheses. A more practical problem will be to complete the log-version of the minimal model program for 3-folds [KMM]. This is related to the birational classification of open 3-folds and n-folds with $\kappa = 3$. Since log-terminal singularities have no explicit classification, this might be a good place to get some idea on higher dimension. Shokurov made some progress in this direction [Shokurov3].

There are two other results relying on the classification.

(6.1) **Theorem** [Mori4]. *Every 3-dimensional termal singularity deforms to a finite sum of cyclic quotient terminal singularities (i.e. points of the form $\mathbb{C}^3/\mathbb{Z}_r(1,-1,a)$ for some relatively prime positive integers a and r).*

This was used in the Barlow-Fletcher-Reid plurigenus formula for 3-folds [Fletcher] and [Reid5] (cf. also [Kawamata5]). Given a 3-fold X with only terminal singularities, each singularity of X can be deformed to a sum of cyclic quotient singularities $\mathbb{C}^3/\mathbb{Z}_r(1,-1,a)$. Let $S(X)$ be the set of all such (counted with multiplicity). For each $P = \mathbb{C}^3/\mathbb{Z}_r(1,-1,a) \in S(X)$, we let

$$\phi_P(m) = (m - \{m\}_r)\frac{r^2-1}{12r} + \sum_{j=0}^{\{m\}_r-1} \frac{\{aj\}_r(r-\{aj\}_r)}{2r},$$

where $\{m\}_r$ is the integer $s \in [0, r-1]$ such that $s \equiv m(\text{mod } r)$. For a line bundle L on X, let $\chi(L) = \sum_j(-1)^j \dim H^j(X,L)$ and let $c_2(X)$ be the second Chern class of X, which is well-defined since X has only isolated singularities. Then the formula is stated as the following.

(6.2) **The Barlow-Fletcher-Reid Plurigenus Formula.**

$$\chi(\mathcal{O}_X(mK_X)) = \frac{m(m-1)(2m-1)}{12}(K_X^3) + (1-2m)\chi(\mathcal{O}_X) + \sum_{P\in S(X)} \phi_P(m),$$

$$\chi(\mathcal{O}_X) = -\frac{1}{24}(K_X \cdot c_2(X)) + \sum_{P\in S(X)} \frac{r^2-1}{24r}.$$

This is important for effective results on 3-folds (cf. §7).

(6.3) **Theorem** ([KSB]). *A small deformation of a 3-dimensional terminal singularity is terminal.*

This is indispensable in the construction of birational moduli. An open problem in this direction is

(6.4) **Problem.** *Is every small deformation of a 3-dimensional canonical singularity canonical?*

Since this remains unsolved, we cannot put an algebraic structure on

$$\{\text{canonical 3-folds}\}/\text{isomorphisms}.$$

§7. Related Results

I would like to list some of the directions, which I could not mention in the previous sections. This is by no means exhaustive. For instance, I could not mention the birational automorphism groups (cf. [Iskovskih] for the works before 1983) due to the lack of my knowledge.

(7.1) *Effective Classification.* The Kodaira dimension κ is not a simple invariant. For instance, we know that $\kappa(X) = -\infty$ iff $P_\nu(X) = 0 (\forall \nu > 0)$. Therefore $P_{12}(X) = 0$ was an effecitve criterion for a surface X to be ruled, while $\kappa(X) = -\infty$ was not. The 3-dimensional analogue is not known yet.

There are results by Kollár [Kollár2] in the case $\dim H^1(X, \mathcal{O}_X) \geq 3$ (cf. [Mori4]). The Barlow-Fletcher-Reid plurigenus formula (6.2) is applied for instance to get $aK_X \sim 0$ with some effectively given $a > 0$ for 3-folds X with numerically trivial K_X by [Kawamata5] and [Morrison], and to get $P_{12}(X) > 0$ for canonical 3-folds X with $\chi(\mathcal{O}_X) \leq 1$ by [Fletcher].

(7.2) *Differential Geometry.* As shown by [Yau], there are differential geometric results (especially when K is positive) which seem out of reach of algebraic geometry. Therefore we welcome differential geometric approaches. In this direction is Tsuji's construction of Kähler-Einstein metrics on canonical 3-folds [Tsuji].

(7.3) *Characteristic p.* [Kollár5] generalized [Mori1] (extremal rays of smooth projective 3-folds over $\mathbb{C}$) to char p. This suggests the possibility of little use of vanishing theorems in MMP for 3-folds. A goal will be the MMP for 3-folds in char p. However even the classification of terminal singularities is open.

(7.4) *Mixed Characteristic Case.* One can ask about the extremal rays (and so on) for arithmetic 3-folds X/S. The methods of [Shokurov2] and [Tsunoda] might work, if X/S is semistable. In the general case, I do not know any results in this direction.

(7.5) *Analytic or Non-projective 3-Folds.* Studying analytic or non-projective 3-folds will require a substitute for the cone of curves modulo numerical equivalence. However analytic or non-projective minimal 3-folds can be handled by the flop [Kollár4]. There is a work of [Kollár6].

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