Harmonic Analysis and Nonlinear Partial Differential Equations

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1 Introduction

The aim of this report is to describe some recent research in the area of nonlinear evolution equations. The choice of the topics is largely influenced by the author's own interests and it is in no way a complete survey of this field, which would be nearly impossible to achieve in a single exposition. Some very outstanding achievements in recent years such as, for instance, the work of Christodoulou and Klainerman [C-K] on global solutions of the Einstein equations will not be discussed here.

We will be mainly concerned with nonlinear Hamiltonian equations on bounded domains (Dirichlet or periodic boundary conditions say) and the following issues:

(i) The Cauchy problem; i.e., local and global wellposedness results for individual data

(ii) Behavior of solutions for time $\to \infty$

(iii) Behavior of the flow in phase space.

These issues are rather well understood in the integrable case, because of the presence of a large set of invariants of motion. The integrable Hamiltonian evolution equations form a small and distinguished class, including, for instance, the

1D cubic nonlinear Schrödinger (NLS) equation $iu_t + u_{xx} + u|u|^2 = 0$

Korteweg de Vries (KdV) equation $u_t + \partial_x^3 u + uu_x = 0$

Modified KdV equation $u_t + \partial_x^3 u + u^2 u_x = 0$.

These invariants of motion allow us to control for a given data $u(0) = \phi$ the solution $u(t)$ for all time. In the general Hamiltonian case on the other hand, one only disposes of a few conserved quantities, namely the Hamiltonian itself, sometimes the L^2-norm $\|u(t)\|_{L^2}$. Hence, to establish global existence of solutions even for smooth data, one needs to study the local wellposedness problem for data of low regularity, because the existence time should only depend on the conserved quantities. This procedure leads to estimates on higher smoothness norms that

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are exponential in time, thus $\|u(t)\|_{H^s} < C^{|t|}$, for some $C > 1$ depending on the initial data. A natural problem is whether this is the "true" behavior or only a crude estimate, and the lack of a rigorous mathematical approach here is in sharp contrast to the situation in integrable models. On the other hand, certain information on the global behavior of the flow of the equation in an appropriate phase space may be obtained from methods of statistical mechanics and symplectic geometry. The Gibbs measure construction from statistical mechanics gives a normalization procedure of the formally invariant Liouville measure on an infinite-dimensional phase space and permits us to obtain Poincaré recurrence properties for the flow. Other symplectic invariants, called symplectic capacities, originating from Gromov's pioneering work [Grom], allow us to study "squeezing properties" and energy transitions in the symplectic normalization of phase space. These normalizations of the phase space are however such that the resulting theories deal with low-regularity solutions and consequently, as a first step, require us again to establish the existence of the flow for such data. Our investigations on the Cauchy problem have been mainly pursued for periodic boundary conditions (i.e. the space variable ranges in a d-dimensional torus $\mathbb{T}^d$), which is also the context for the discussion above. In fact, the literature on the initial value problems (IVPs) in the periodic case is far less extensive than that on the line and the theory is less developed. It turned out that the analysis in the periodic situation is significantly different (due for instance to the absence of dispersion) and requires new ideas, some of which also eventually lead to an advance on the corresponding problem for the line. Several results in this direction will be discussed in the next section.

A third important method borrowed from classical mechanics is the KAM method to establish persistency of time periodic or quasi-periodic solutions of small Hamiltonian perturbations of linear or integrable equations. The main contributor in adapting the KAM technology to the PDE setting is Kuksin [Kuk_1]. His work gives satisfactory results for 1D problems with Dirichlet boundary conditions. A different approach, avoiding some of the limitations of the KAM technique, has been elaborated by Craig and Wayne [$\text{C-W}_{1,2}$] and the author [B_1] and permits us to deal with 1D periodic boundary conditions.

2 Initial value problems for KdV type equations

There are numerous investigations of the Cauchy problem for the standard KdV equation on $\mathbb{R}$

$$\begin{cases} u_t + \partial_x^3 u + u u_x = 0 \\ \\ u(0) = \phi(x) \end{cases} \tag{2.1}$$

using either fixpoint techniques or inverse scattering methods. The advantage of the fixpoint approach is its large range of applicability, and it is the only method we consider here. The setup is given by Duhamel's integral formula

$$u(t) = S(t)\phi - \int_0^t S(t-\tau)\, w(\tau)\, d\tau \quad , \quad w = u u_x \tag{2.2}$$

where $S(t)\psi$ solves the linear problem

$$\begin{cases} u_t + \partial_x^3 u = 0 \\ u(0) = \psi \end{cases} \tag{2.3}$$

and is explicitly given by the oscillatory integral ($\widehat{\psi}$ denotes the Fourier transform of ψ)

$$S(t)\psi(x) = \int \widehat{\psi}(\lambda)\ e^{i(\lambda x + \lambda^3 t)}\ d\lambda \tag{2.4}$$

on the line ($\mathbb{R}$-case) and the exponential sum

$$S(t)\psi(x) = \sum_{n\in\mathbb{Z}} \widehat{\psi}(n)\ e^{i(nx+n^3 t)} \tag{2.5}$$

in the periodic case ($\mathbb{T}$-case).

Solving (2.2) by a fixpoint argument in the $\mathbb{R}$-case is mainly based on the regularizing properties of the linear group $S(t)$, such as Stricharti's inequality and Kato's smoothing. In the periodic case, no smoothing properties may be expected. We introduced in [B_2] new space-time norms defined in terms of the Fourier transform of u. These norms exploit some arithmetical features, which form a substitute for the smoothing properties of (2.4). In fact, the regularity gains here are due both to the linear part of the equation and the specific structure of the nonlinear part. This method and its application in conjunction with earlier techniques lead to the "best" known results on the IVP for the KdV equation.

THEOREM 2.6. ($\mathbb{T}$) *There is local wellposedness for data $\phi \in H^s(\mathbb{T})$, $s \geq -\frac{1}{2}$; global wellposedness for data $\phi \in L^2(\mathbb{T})$; the solutions resulting from L^2-data are almost periodic in time (see* [B_3]*).*

($\mathbb{R}$) *There is local wellposedness for data $\phi \in H^s(\mathbb{R})$, $s > -\frac{3}{4}$; global wellposedness for data $\phi \in L^2(\mathbb{R})$ (see* [K-P-V_1], [K]*).*

The solution depends real analytically on the data.

REMARKS.

(i) By wellposedness, we mean the construction of a unique solution for a certain class of data (coinciding with the classical solution in the smooth case) and depending continuously on the data.

(ii) The almost periodicity of KdV solutions is a subject with a long history that we will not recall here. Important steps are due to Gardner-Kruskal-Miura [G-K-M], Lax [Lax], Novikov [Nov], and McKean-Trubowitz [M-T]. The statement for L^2-data is a consequence of the work of [M-T] and the existence of regular L^2-flow.

A direct generalization of KdV are equations of the form

$$u_t + \partial_x^3 u + f(u)u_x = 0 \tag{2.7}$$

where f is a smooth function of u. The Hamiltonian is given by

$$H(\phi) = \frac{1}{2} \int (\phi_x)^2 - \int F_2(\phi) \tag{2.8}$$

where F_2 is the second primitive of f $(F_2(0) = F_2'(0) = 0)$ and equation (2.7) is equivalent to

$$u_t = D_x \frac{\partial H}{\partial u} \,. \tag{2.9}$$

The mean $\int \phi$, the L^2-norm $\int \phi^2$, and $H(\phi)$ are preserved under the flow. In the general case ($f(u) = u, u^2$ are special), these are the only invariants of motion at our disposal. Construction of global solutions based on these a priori bounds requires a local theory for H^1-data. In the $\mathbb{R}$-case, results along these lines appear in the works of Kenig, Ponce, and Vega (see [K-P-V_2] for instance). We state the theorem in the periodic case (see [B_2]).

THEOREM 2.10. *The* IVP

$$u_t + \partial_x^3 u + f(u)u_x = 0 \quad ; \quad u(0) = \phi$$

is globally well posed for sufficiently smooth data as long as the H^1-norm remains bounded. This is in particular the case for small data.

The proof uses many of the techniques developped for the periodic KdV case. We just want to mention one additional point, which is a certain "renormalization" of the nonlinearity, in the spirit of "Wick-ordering" discussed below. Rewrite (2.7) in the form

$$u_t + \partial_x^3 u + v(t)u = [v(t) - f(u)]\, u_x \quad \text{with} \quad v(t) = \int_{\mathbb{T}} f(u)\, dx \tag{2.11}$$

redefining the linear and nonlinear parts of the equation. Observe that for $f(u) = u$ or u^2, $v(t)$ is time independent. The different setup (2.11) seems necessary for a regularizing interaction between linear and nonlinear terms in a fixpoint argument.

REMARK. There has been recent work by Klainerman and Machedon [K-M_1] on nonlinear wave equations in the same spirit as Theorem 2.10. In particular, a local H^1-theory is developed considering appropriate space-time norms, and a careful analysis of the nonlinear term is needed. See also [K-M_2] for IVP results related to Yang-Mills equations.

A 2-dimensional generalization of the KdV equation is given by the Kadomstev-Petviashvili equation (KP). The KP-II equation

$$u_t + \partial_x^3 u + uu_x + D_x^{-1} u_{yy} = 0 \tag{2.12}$$

is integrable. However, the conserved integrals

$$\int u \, dx \, dy$$

$$\int u^2 \, dx \, dy$$

$$\int \left[u_x^2 - \tfrac{1}{3} u^3 - (D_x^{-1} u_y)^2\right] dx \, dy$$

do not imply immediately a priori bounds on the solution, except for the L^2-norm (because of sign features). Techniques closely related to those used in proving the periodic results for the KdV equation (Theorem 2.6) yield the following.

THEOREM 2.13. [B4] *The* KP-II *equation is globally well posed for data* $\phi \in H^s(\mathbb{T}^2)$ *or* $\phi \in H^s(\mathbb{R}^2)$, $s \geq 0$.

There is a rich algebraic theory around the KP-II equation and explicit solutions may be expressd in terms of logarithmic derivatives of θ-functions associated to certain (possibly infinite genus) Riemann surfaces. This theory has been developed by Its, Novikov, and Krichever among others (see [Kr]). Very recently, Knörrer and Trubowitz proved the following analogue of the [M-T] result for the KdV equation.

THEOREM 2.14. [K-Tr] *Solutions of the periodic* KP-II *equations for smooth periodic data are almost periodic in time.*

3 Nonlinear Schrödinger equations and invariant Gibbs measures

We consider the nonlinear Schrödinger (NLS) equation

$$iu_t + \Delta u \pm u|u|^{p-2} = 0 \tag{3.1}$$

with periodic boundary conditions. Thus, u is a complex function on $\mathbb{T}^d \times I$ (local) or $\mathbb{T}^d \times \mathbb{R}$ (global). The equation may be rewritten in Hamiltonian format as

$$u_t = i \, \frac{\partial H}{\partial \, \overline{u}} \tag{3.2}$$

where $H(\phi) = \frac{1}{2} \int_{\mathbb{T}^d} |\nabla \phi|^2 \mp \frac{1}{p} \int_{\mathbb{T}^d} |\phi|^p$. Both the Hamiltonian $H(\phi)$ and the L^2-norm $\int |\phi|^2$ are preserved under the flow. The $1D$ case $p = 4$ is special (1D cubic NLS) because it is integrable and there are many invariants of motion. This aspect will however play no role in the present discussion. The possible sign choice $\pm$ in (3.1) corresponds to the focusing (resp. defocusing) case. In the focusing case, the Hamiltonian may be unbounded from below and blowup phenomena may occur (for $p \geq 2 + \frac{4}{d}$). The canonical coordinates are $(\operatorname{Re} \phi, \operatorname{Im} \phi)$ or alternatively $(\operatorname{Re} \widehat{\phi}, \operatorname{Im} \widehat{\phi})$. The formal Gibbs measure on this infinite-dimensional phase is given by

$$d\gamma_\beta = e^{-\beta H(\phi)} \prod_x d\phi(x) = e^{\pm \frac{\beta}{p} \int |\phi|^p} \cdot e^{-\frac{\beta}{2} \int |\nabla \phi|^2} \prod_x d\phi(x). \tag{3.3}$$

($\beta > 0$ is the reciprocal temperature and we may take $\beta = 1$ in this discussion.)

From Liouville's theorem, (3.3) defines an invariant measure for the flow of (3.1). Making this statement precise requires us to clarify the following two issues:

(i) The rigorous construction (normalization) of the measure (3.3)

(ii) The existence problem for the flow of (3.1) on the support of the measure.

The first issue is well understood in the defocusing case. The case $D = 1$ is trivial, the case $D = 2$, p even integer is based on the Wick-ordering procedure (see [G-J]), and the normalization for $D = 3$, $p = 4$ is due to Jaffe [Ja]. In the focusing case, only the case $D = 1$ is understood [L-R-S] and normalization of the measure is possible for $p \leq 6$, restricting ϕ to an appropriate ball in $L^2(\mathbb{T})$.

The construction of a flow is clearly a PDE issue. The author succeeded in this in the $D = 1$ and $D = 2$, $p = 4$ cases ([B$_5$], [B$_6$]). For $D = 2$, $p = 4$ there is a natural PDE counterpart of the Wick-ordering procedure and equation (3.1) has to be suitably modified (this modification seems physically inessential however). We may summarize the results as follows.

THEOREM 3.4. *($D = 1$)* (i) *In the defocusing case, the measure* (3.3) *appears as a weighted Wiener measure, the density being given by the first factor. The same statement is true in the focusing case for $p \leq 6$, provided one restricts the measure to an L^2-ball $[\|\phi\|_2 \leq B]$. The choice of B is arbitrary for $p < 6$ and B has to be sufficiently small if $p = 6$.*

(ii) *Assuming the measure exists, the corresponding* 1D *equation* (3.1) *is globally well posed on a K_σ set A of data, $A \subset \bigcap_{s<\frac{1}{2}} H^s(\mathbb{T})$, carrying the Gibbs measure γ_β. The set A and the Gibbs measure γ_β are invariant under the flow.*

REMARKS.

(i) In dimension 1, the L^2-restriction is acceptable, because L^2 is a conserved quantity and a typical ϕ in the support of the Wiener measure is a function in $H^s(\mathbb{T})$, for all $s < \frac{1}{2}$. Instead of restricting to an L^2-ball, one may alternativelymultiply with a weight function with a suitable exponential decay in $\|\phi\|_2$.

(ii) Let for each $N = 1, 2, \ldots$

$$P_N\phi = \phi_N = \sum_{|n|\leq N} \widehat{\phi}(n)\ e^{i\langle n,x\rangle} \tag{3.5}$$

be the restriction operator to the N first Fourier modes. Finite dimensional versions of the PDE model are obtained considering "truncated" equations

$$\begin{cases} iu_t^N + u_{xx}^N \pm P_N\left(u^N|u^N|^{p-2}\right) = 0 \\ u^N(0) = P_N\phi\ . \end{cases} \tag{3.6}$$

It is proved that for typical ϕ, the solutions u^N of (3.6) converge in the space $C_{H^s(\mathbb{T})}[0,T]$ for all time T and $s < \frac{1}{2}$ to a (strong) solution of

$$\begin{cases} iu_t + u_{xx} \pm P\left(u|u|^{p-2}\right) = 0 \\ u(0) = \phi\ . \end{cases} \tag{3.7}$$

THEOREM 3.8. *($D = 2$, $p = 4$) (i) Denote $\widetilde{H}_N$ the Wick-ordered Hamiltonians, obtained replacing*

$$|\phi_N|^4 \quad by \quad |\phi_N|^4 - 4a_N|\phi_N|^2 + 2|a_N|^2 \qquad \left(a_N = \sum_{|n|\le N} \frac{1}{|n|^2+\rho} \sim \log N\right).$$

The corresponding measures $e^{-\beta \widetilde{H}_N(\phi)} \prod d\phi$ converge for $N \to \infty$ to a weighted 2-dimensional Wiener measure whose density belongs to all L^p-spaces. Denote by $\widetilde{\gamma}_\beta$ this "Wick-ordered" Gibbs measure.

(ii) *The measure $\widetilde{\gamma}_\beta$ is invariant under the flow of the "Wick-ordered" equation*

$$iu_t + \Delta u - \left(u|u|^2 - 2u \int |u|^2\right) = 0 \tag{3.9}$$

which is well defined. More precisely, denoting by u^N the solutions of

$$\begin{cases} iu_t^N + \Delta u^N - P_N\left(u^N|u^N|^2 - 2u^N \int |u^N|^2\right) = 0 \\ u^N(0) = P_N\phi \end{cases} \tag{3.10}$$

the sequence

$$u^N(t) - \sum_{|n|\le N} \widehat{\phi}(n)\ e^{i(\langle n,x\rangle + |n|^2 t)} \tag{3.11}$$

converges for typical ϕ in $C_{H^s(\mathbb{T}^2)}[0,T]$ for some $s > 0$, all time T, to

$$u(t) - \sum \widehat{\phi}(n)\ e^{i(\langle n,x\rangle + |n|^2 t)}. \tag{3.12}$$

REMARKS.

(i) We repeat that the novelty of Theorem 3.8 lies in the second statement on the existence of a flow. The first statement is a classical result.

(ii) The second terms in (3.11), (3.12) are the solutions to the linear problem

$$\begin{cases} iu_t + \Delta u = 0 \\ u(0) = \phi\ . \end{cases}$$

Here a typical ϕ is a distribution, not a function. However the difference (3.12) between solutions of the linear and nonlinear equation is an H^s-function for some $s > 0$, which is a rather remarkable fact.

(iii) The failure in $D = 2$ of typical ϕ to be an L^2-function makes the [L-R-S] construction for $D = 1$ inadequate to deal with the $D = 2$ focusing case. Some recent work on this issue is due to Jaffe, but for cubic nonlinearities in the Hamiltonian only. The problem for $D = 2$, $p = 4$ in the focusing case is open and intimately related to blowup phenomena ($p = 4$ is critical in 2D).

The $1D$ cubic NLS equation appears as the limit of the $1D$ Zakharov model (ZE)

$$\begin{cases} iu_t = -u_{xx} + nu \\ n_{tt} - c^2 n_{xx} = c^2 \left(|u|^2\right)_{xx} \end{cases} \tag{3.13}$$

when $c \to \infty$. The physical meaning of u, n, c are resp. the electrostatic envelope field, the ion density fluctuation field, and the ion sound speed. This model is discussed in [L-R-S]. Defining an auxiliary field $V(x,t)$ by

$$\begin{cases} n_t = -c^2\ V_x \\ V_t = -n_x - |u|_x^2 \end{cases} \tag{3.14}$$

we may write (3.13) in a Hamiltonian way, where

$$H = \tfrac{1}{2} \int \left[|u_x|^2 + \tfrac{1}{2}\ (n^2 + c^2\ V^2) + n|u|^2\right]\ dx \tag{3.15}$$

and $(\operatorname{Re} u, \operatorname{Im} u)$, $\left(\widetilde{n}, \widetilde{V}\right)$ with $\widetilde{n} = 2^{-1/2} n$, $\widetilde{V} = 2^{-1/2} \int^x V$ as pairs of conjugate variables. Considering the associated Gibbs measure

$$e^{-\beta H} \cdot \chi_{\left\{\int |u|^2 dx \leq B\right\}} \prod_x d^2 u(x)\ d\widetilde{n}(x)\ d\widetilde{V}(x) \tag{3.16}$$

one gets the 1D cubic NLS Gibbs measure as marginal distribution of the u-field.

THEOREM 3.17. [B_7] *The* $1D$ *(ZE) is globally well posed for almost all data* $\left(u_0, \widetilde{u}_0,\ \widetilde{V}_0\right)$ *in the support of the Gibbs measure, which is invariant under the resulting flow.*

REMARKS.

(i) In the study of invariant Gibbs measures, it suffices to establish local wellposedness of the IVP for typical data in the support of the measure. One may then exploit the invariance of the measure as a conservation law and generate a global flow. For instance, for the 1D NLS $iu_t + u_{xx} \pm u|u|^{p-2} = 0$, there is for $p = 4$ a global wellposedness result for L^2-data (L^2 is conserved).

However, for $p > 4$, we only dispose presently of a local result (in the periodic case) for data ϕ satisfying

$$\begin{cases} \phi \in H^s \ , \ s > 0 \qquad (p \leq 6) \\ \phi \in H^s \ , \ s > s_* \ , \ p = 2 + \frac{4}{1-2s_*} \qquad (p > 6) \end{cases} \tag{3.18}$$

and a global flow is established from the invariant measure considerations.

(ii) There have been other investigations in 1D on invariant measures, mostly by more probabilistic arguments. In this respect, we mention the works of McKean-Vaninski and in particular Mckean [McK] on the 1D cubic NLS. These methods are more general but give less information on the flow.

4 Symplectic capacities, squeezing and growth of higher derivatives

The works of Gromov and Ekeland, Hofer, Zehnder, and Viterbo lead to new finite-dimensional symplectic invariants, different from Liouville measure on the phase space. Let us recall the following construction of a symplectic capacity for open domains O in $\mathbb{R}^n \times \mathbb{R}^n$, $dp \wedge dq$. Call a smooth function f m-admissible ($m > 0$) if $f = 1$ on a neighborhood of O and $f = 0$ on a nonempty subdomain of O. Denote V_f the associated Hamiltonian vector field $\left(\frac{\partial f}{\partial p}\ , \ -\frac{\partial f}{\partial q}\right)$. Define the symplectic invariant

$$\begin{aligned} c_{2n}(O) = \ & \inf \{m > 0 \mid V_f \text{ has nontrivial periodic orbit of period } \leq 1\ , \\ & \text{whenever } f \text{ is } m\text{-admissible for } O\}. \end{aligned} \tag{4.1}$$

Then $c_{2n}(\cdot)$ is monotonic and translation invariant and scales as $c_{2n}(\tau O) = \tau^2 c_{2n}(O)$. The main property is that

$$c_{2n}(B_\rho) = \pi\rho^2 = c_{2n}(\Pi_\rho) \tag{4.2}$$

where B_ρ is the ball $B_\rho = \left\{|p|^2 + |q|^2 < \rho^2\right\}$ and $\prod_\rho$ a cylinder, say $\prod_\rho = \{p_1^2 + q_1^2 < \rho^2\}$. As a corollary, there is no symplectic squeezing of a ρ-ball in a cylinder of width ρ', $\rho' < \rho$.

Exploiting such an invariant in Hamiltonian PDE requires an infinite-dimensional setting. Notice that although the theory described above is finite-dimensional, a conclusion such as (4.2) is dimension free. An appropriate "finite-dimensional approximation" appears to be possible if the flow S_t of the considered equation is of the form

$$\text{linear operator} + \text{"smooth compact operator"} \tag{4.3}$$

or, more generally, if the evolution of individual Fourier modes on a finite time interval is approximately the same as in a truncated model $\dot{v} = J\nabla H(v, x, t)$, $v = P_N v$. Here the cutoff N should only depend on the required approximation, the time interval $[0, T]$, and the size of the initial data in phase space. Here and also in (4.3), the phase space has to be defined in a specific way, corresponding to the finite-dimensional normalizations. Hence, the flow properties derived this way relate to a specific "symplectic Hilbert space", for instance

L^2 for nonlinear Schrödinger equations (in any dimension)

$H^{1/2} \times H^{1/2}$ for nonlinear wave equations (in any dimension)

$H^{-1/2}$ for KdV type equations,

and "nonsqueezing" refers to that particular space.

THEOREM 4.4. ([B_8], [Kuk_2]) *There is nonsqueezing of balls in cylinders of smaller width*

(i) *For nonlinear wave equations $u_{tt} = \nabla u + p(u; t, x)$ with smooth nonlinearity of arbitrary polynomial growth in u in dimension 1 and polynomial in u of degree ≤ 4 (resp. ≤ 2) in dimension 2 (resp. 3, 4).*

(ii) For certain 1D nonlinear Schrödinger equations.

The interest of the squeezing or nonsqueezing properties lies in its relevance to the energy transition to higher modes, more precisely whether, for instance, part of the energy may leave a given Fourier mode, which would correspond to squeezing in a small cylinder. The nonsqueezing implies also the lack of uniform asymptotic stability of bounded solutions; i.e., $\operatorname{diam} S_t(B_\rho)$ does not tend to 0 for $t \to \infty$ if $\rho > 0$.

The drawback of those results is that they do not relate to properties of the flow in a classical sense, because of the phase space topology. On the other hand, Kuksin showed recently that in fact certain squeezing of balls in cylinders may occur in spaces of higher smoothness, if one considers for instance a nonlinear wave equation $u_{tt} = \rho\Delta u + p(u)$ where ρ is a small parameter (small dispersion). The squeezing phenomena appear in some finite time and are stronger when $\rho \to 0$.

As far as the behavior of individual smooth solutions concerns, some examples are obtained in [B_2] and [B_8] of Hamiltonian PDE (in NLS or KdV form) defined as a smooth perturbation of a linear equation, showing in particular that higher derivatives of solutions $u(t)$ for smooth data $u(0) = \phi$ need not be bounded in time. For instance

PROPOSITION 4.5. *There is a Hamiltonian* NLS *equation with smooth and local nonlinearity such that $S_t(B^s(\delta))$, $t > 0$, is not a bounded subset of H^{s_0}, for any $s < \infty$, $\delta > 0$. Here $B^s(\delta)$ denote $\{\varphi \in H^s \mid \|\varphi\|_s < \delta\}$ and s_0 is numerical.*

Another example, closely related to the discussion in the next section, is the following. Considering a linear Schrödinger equation

$$-iu_t = -u_{xx} + V(x)u \tag{4.6}$$

where $V(x)$ is a real smooth periodic potential and the periodic spectrum $\{\lambda_k\}$ of $-\frac{d^2}{dx^2} + V$ satisfies a "near resonance" property

$$\operatorname{dist}(\lambda_{n_j}, \mathbb{Z}\lambda_{n_0}) \to 0 \quad \text{rapidly for} \quad j \to \infty \tag{4.7}$$

for some subsequence $\{n_j\}$. We construct a Hamiltonian perturbation $\Gamma(u) = \frac{\partial}{\partial \bar{u}} G$ such that the solution $u_{\varepsilon,q}$ of the IVP

$$\begin{cases} -iu_t = -u_{xx} + V(x)u + \varepsilon\, \Gamma(u) \\ u(0) = q \end{cases} \tag{4.8}$$

satisfies

$$\inf_{q\in O} \sup_t \|u_{\varepsilon,q}(t)\|_{H^{s_0}} \to \infty \quad \text{for} \quad \varepsilon \to 0. \tag{4.9}$$

Here s_0 is again some positive integer and O is some nonempty open subset of $H^{s_0}(\mathbb{T})$.

5 Persistency of periodic and quasi-periodic solutions under perturbation

One of the most exciting recent developments in nonlinear PDE is the use of the classical KAM-type techniques to construct time quasi-periodic solutions of Hamiltonian equations obtained by perturbation of a linear or integrable PDE. This subject is rapidly developing. Results so far are only obtained in 1D and in this brief discussion, we only consider perturbations of linear equations. We work in the real analytic category. Important contributions are due to Kuksin [Kuk$_1$], using the standard KAM scheme and more precisely infinite-dimensional versions of Melnikov's theorem on the persistency of n-dimensional tori in systems with $N > n$ degrees of freedom. His work yields a rather general theory and we mention only some typical examples of applications to 1D nonlinear wave or Schrödinger equations

$$w_{tt} = \left(\frac{\partial^2}{\partial_{x^2}} - V(x;a)\right) w - \varepsilon\, \frac{\partial\varphi}{\partial_w}\, (x, w; a) \tag{5.1}$$

$$-iu_t = -u_{xx} + V(x,a)u + \varepsilon\, \frac{\partial\varphi}{\partial_{|u|^2}}\, \left(x, |u|^2; a\right) u. \tag{5.2}$$

Here $V(x,a)$ is a real periodic smooth potential, depending on n outer parameters $a = (a_1, \dots, a_n)$. Denote $\{\lambda_j(a)\}$ the Dirichlet spectrum of the Sturm-Liouville operator $-\frac{d^2}{dx^2} + V(x,a)$. Thus, $\lambda_j(a) = \pi^2 j^2 + 0(1)$ and we assume the following nondegeneracy condition

$$\det\{\partial\lambda_j(a)/\partial a_k \mid 1 \le j,k \le n\} \neq 0 \tag{5.3}$$

(this condition is a substitute for the classical "twist" condition). Denoting $\{\varphi_j\}$ the corresponding eigenfunctions, the $2n$-dimensional linear space

$$Z^0 = \text{span}\ \{\varphi_j, i\varphi_j \mid 1 \le j \le n\} \tag{5.4}$$

is invariant under the the flow of equation (5.2) for $\varepsilon = 0$ and foliated into invariant n-tori

$$T^n(I) = \left\{ \sum_{j=1}^n (x_j^+ + ix_j^-)\varphi_j \mid (x_j^+)^2 + (x_j^-)^2 = 2I_j\ ,\ j = 1, \dots, n \right\} \tag{5.5}$$

which are filled with quasi-periodic solutions of (5.2) for $\varepsilon = 0$. A typical result from [Kuk_1] is that under assumption (5.3), for most parameter values of a there is an invariant torus $\sum\limits_{a,I}^{\varepsilon} (\mathbb{T}^n)$ near the unperturbed torus $\sum\limits_{a,I}^{0}$ given by (5.5) and filled with quasi-periodic solutions of (5.2). The frequency vector ω_ε of a perturbed solution will be $c\varepsilon$ close to $\omega = (\lambda_1, \dots, \lambda_n)$ of the unperturbed one.

The methods in [Kuk_1] leave out the case of periodic boundary conditions, because of certain limitations of the KAM method (second Melnikov condition) excluding multiplicities in the normal frequencies. A different approach has been recently used by Craig and Wayne [$\text{C-W}_{1,2}$], based on the Lyapunov-Schmidt decomposition and leading to time periodic solutions of perturbed equations under periodic boundary conditions. This method consists in splitting the problem into a (finite-dimensional) resonant part (Q-equation) and an infinite-dimensional nonresonant part (P-equation). In the PDE-case (contrary to the case of a finite-dimensional phase space), small divisor problems appear when solving the P-equation by a Newton iteration method, also in the time periodic case. Writing u in the form

$$u = \sum_{m,k} \widehat{u}(m,k)\ e^{im\lambda t}\ \varphi_k(x) \tag{5.6}$$

and letting the linearized operator act on the Fourier coefficients $\widehat{u}(m,k)$, one gets operators of the form

$$(m\lambda - \lambda_k) + \varepsilon\, T \tag{5.7}$$

where the first term is diagonal and T is essentially given by Toeplitz operators with exponentially decreasing matrix elements. The main task is then to obtain reasonable bounds on their inverses. The problem is closely related to a line of research around localization in the Anderson model and in particular the works of Fröhlich, Spencer, and Surace with quasi-periodic potentials (see [F-S-W], [Sur]). In this case, the operator T in (5.7) is replaced by $-\Delta$, $\Delta =$ lattice Laplacian, and the first term plays the role of the potential.

The author succeeded very recently in dealing with the quasi-periodic case by the same methods [B_1]; giving thus a new proof of the KAM theorem where one avoids Melnikov's second condition. Also the case of periodic boundary conditions and quasi-periodic solutions for (5.1), (5.2) may be treated this way. Observe that in the quasi-periodic setting, the diagonal part of (5.7) becomes now $\langle m, \lambda\rangle - \lambda_k$ where for instance $\langle m, \lambda\rangle = m_1\lambda_1 + m_2\lambda_2$. The singularities here are more severe and a large part of the difficulty already appears in the classical finite-dimensional case.

The Lyapunov-Schmidt method is significantly more flexible than KAM, and other applications, possibly to the 2D problem, should be expected.

Added in Proof: The author succeeded more recently in developing a theory of quasi-periodic solutions for NLS equations of the form (5.2) in 2D (see [B9]). For the special case of time periodic solutions, the work of [C-W_1] may be extended to any dimension, leading for instance to periodic solutions of the NLW equation $u_{tt} - \Delta u + \rho u + u^3 = 0$, for typical ρ (cf. [B_{10}]).

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(strictly for the purpose of previous exposition)

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