Constant Term Identities, Orthogonal Polynomials, and Affine Hecke Algebras

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The main aim of this lecture is to survey a theory of orthogonal polynomials in several variables which has developed over the last ten years or so. We shall concentrate on the purely algebraic aspects of the theory, and for lack of time and competence we shall say nothing about its physical applications (completely integrable systems, KZ equations, etc.)

These polynomials include as special cases, on the one hand all the classical orthogonal polynomials in one variable (Legendre, Jacobi, Hermite, $\cdots$), and on the other hand polynomials that arise in the representation theory of Lie groups (characters of compact Lie groups, spherical functions on real and p-adic symmetric spaces and their quantum analogues). The underlying notion is that of a *root system*, to which I shall turn first.

1 Root systems

Root systems and their Weyl groups constitute the combinatorial infrastructure of much of the theory of Lie groups and Lie algebras. Thus a complex semisimple Lie algebra or a compact connected Lie group with trivial centre, is determined up to isomorphism by its root system. Moreover, and quite apart from their Lie-theoretic origin, the geometry and algebra of root systems presents an apparently inexhaustible source of beautiful combinatorics.

It is time for definitions and examples. Let V be a real vector space of finite dimension, endowed with a positive definite scalar product $\langle u, v\rangle$. For each non-zero $\alpha \in V$ let s_α denote the orthogonal reflection in the hyperplane H_α through the origin perpendicular to α. Explicitly,

(1.1) $$s_\alpha(v) = v - \langle v, \alpha^\vee\rangle\alpha$$

for $v \in V$, where $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$.

A *root system* R in V is a finite non-empty set of non-zero vectors (called *roots*) that span V and are such that for each pair $\alpha, \beta \in R$ we have

(1.2) $$\langle\alpha^\vee, \beta\rangle \in \mathbb{Z},$$

$$s_\alpha(\beta) \in R. \tag{1.3}$$

Thus each reflection $s_\alpha (\alpha \in R)$ permutes R, and the group of orthogonal transformations of V generated by the s_α is a finite group W_0, called the *Weyl group* of R.

We may remark straightaway that the integrality condition (1.2) by itself is extremely restrictive. Let $\alpha, \beta \in R$ and let Θ be the angle between the vectors α and β. Then

$$4\cos^2\Theta = \frac{4\langle\alpha,\beta\rangle^2}{\langle\alpha,\alpha\rangle\langle\beta,\beta\rangle} = \langle\alpha^\vee,\beta\rangle\langle\alpha,\beta^\vee\rangle$$

is an *integer*, hence can only take the values 0, 1, 2, 3, 4. It follows that the only possibilities for Θ are π/m or $\pi - (\pi/m)$, where $m = 1, 2, 3, 4$ or 6.

The vectors $\alpha^\vee$ for $\alpha \in R$ form a root system $R^\vee$, the *dual* of R. If $\alpha \in R$, then also $-\alpha \in R$ (because $-\alpha = s_\alpha(\alpha)$). The root system R is said to be *reduced* if the only scalar multiples of α in R are $\pm\alpha$. Furthermore, R is said to be *irreducible* if it is not possible to partition R into two non-empty subsets R_1 and R_2 such that each root in R_1 is orthogonal to each root in R_2 (which would imply that R_1 and R_2 are themselves root systems). We shall assume throughout that R is both reduced and irreducible.

For those to whom these notions are unfamiliar, some examples to bear in mind are the following. Let $\varepsilon_1, \cdots, \varepsilon_n$ be the standard basis of $\mathbb{R}^n (n \geq 2)$, with the usual scalar product, for which $\langle\varepsilon_i, \varepsilon_j\rangle = \delta_{ij}$. Then the vectors

$$\pm\ \varepsilon_i - \varepsilon_j \tag{A_{n-1}}$$

where $i \neq j$, form a root system (and V is the hyperplane in $\mathbb{R}^n$ orthogonal to $\varepsilon_1 + \cdots + \varepsilon_n$). The Weyl group is the symmetric group S_n, acting on V by permuting the ε_i.

Moreover, each of the sets of vectors

$$\pm\ \varepsilon_i \quad (1 \leq i \leq n), \quad \pm\varepsilon_i \pm \varepsilon_j \quad (1 \leq i < j \leq n), \tag{B_n}$$

$$\pm 2\varepsilon_i \quad (1 \leq i \leq n), \quad \pm\varepsilon_i \pm \varepsilon_j \quad (1 \leq i < j \leq n), \tag{C_n}$$

$$\varepsilon_i \pm \varepsilon_j \quad (1 \leq i < j \leq n) \tag{D_n}$$

is a root system. For B_n and C_n, the Weyl group is the group of all signed permutations of the ε_i, of order $2^n n!$ (the hyperoctahedral group). For D_n, it is a subgroup of index 2 in this group. The root systems B_n and C_n are duals of each other, and A_{n-1}, D_n are each self-dual.

In fact, the root systems $A_n (n \geq 1), B_n (n \geq 2), C_n (n \geq 3)$ and $D_n (n \geq 4)$ almost exhaust the catalogue of reduced irreducible root systems (up to isomorphism). Apart from these, there are just five others, the "exceptional" root systems, denoted by E_6, E_7, E_8, F_4 and G_2. (In each case the numerical suffix is the dimension of the space V spanned by R, which is also called the *rank* of R.)

Let R be any (reduced, irreducible) root system in V and consider the complement

$$X = V - \bigcup_{\alpha \in R} H_\alpha$$

of the union of the reflecting hyperplanes $H_\alpha, \alpha \in R$. The connected components of X are open simplicial cones which are permuted simply transitively by the Weyl group W_0. Let Γ be one of these components, chosen once and for all; it is bounded by $n = \dim V$ hyperplanes $H_{\alpha_i} (1 \leq i \leq n)$, and

$$\Gamma = \{x \in V : \langle \alpha_i, x \rangle > 0 \qquad (1 \leq i \leq n)\}.$$

The α_i are the *simple roots* determined by Γ, and each root $\alpha \in R$ is of the form

$$\alpha = \sum_1^r r_i \alpha_i \tag{1.4}$$

with integral coefficients r_i all of the same sign. A root $\alpha \in R$ is *positive* (resp. *negative*) relative to Γ if $\langle \alpha, x \rangle > 0$(resp. < 0) for all $x \in \Gamma$. Equivalently, $\alpha \in R$ is positive (resp. negative) if the coefficients r_i in (1.4) are all ≥ 0(resp. ≤ 0). Let R^+ (resp. R^-) denote the set of positive (resp. negative) roots. Then $R^- = -R^+$, and $R = R^+ \cup R^-$. Moreover, there is a unique root $\varphi \in R^+$, called the *highest root*, for which the sum of the coefficients $\sum r_i$ in (1.4) is maximal. In A_{n-1}, for example, we may take the simple roots to be $\alpha_i = \varepsilon_i - \varepsilon_{i+1} (1 \leq i \leq n-1)$; the positive roots are then $\varepsilon_i - \varepsilon_j$ with $i < j$, and the highest root is $\varepsilon_1 - \varepsilon_n$.

The abelian group Q generated by R, whose elements are the integral linear combinations of the roots, is a lattice in V (i. e. a free abelian group of rank $n = \dim V$) called the *root lattice*. Clearly the simple roots $\alpha_1, \cdots \alpha_n$ form a basis of Q. We denote by Q^+ the subsemigroup of Q consisting of all sums $\sum r_i \alpha_i$ where the coefficients are non negative integers.

Next, the set P of all $\lambda \in V$ such that $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\alpha \in R$ is another lattice, called the *weight lattice*. It has a basis consisting of the *fundamental weights* $\pi_1, \cdots, \pi_n$, defined by the equations $\langle \pi_i, \alpha_j^\vee \rangle = \delta_{ij}$. We denote by P^+ the set of *dominant weights* (i. e. $\lambda \in P$ such that $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in R^+$). We have $P \supset Q$ (by (1.2)) but $P^+ \not\supset Q^+$ (unless $n = 1$, i. e. $R = A_1$). The quotient P/Q is a finite group, since both P and G are lattices of the same rank n. Clearly, both P and Q are stable under the action of the Weyl group W_0. Each W_0-orbit in P contains exactly one dominant weight, i. e. P^+ is a fundamental region for the action of W_0 on P.

Finally, the Weyl group W_0 acts on V and therefore also on the algebra $S(V)$ of polynomial functions on V. It can be shown that the subring $S(V)^{W_0}$ of W_0-invariant polynomial functions in generated by $n = \dim V$ algebraically independent homogeneous polynomial functions, of degrees say $d_1, \cdots, d_n$. The functions themselves are not uniquely determined, but their degrees are: they are called the *degrees of* W_0. For example, if R is A_{n-1}, so that W_0 is the symmetric group S_n, we may take as generators of $S(V)^{W_0}$ the power sums

$$x_1^r + \cdots + x_n^r \qquad (2 \leq r \leq n)$$

where $x_1, \cdots, x_n$ are coordinates in $\mathbb{R}^n$. Thus in this case the degrees are $2, 3, \cdots, n$.

2 Constant term identities

Let F be a field of characteristic zero and let $A = F[P]$ be the group algebra over F of the weight lattice P. Since the group operation in P is addition, we shall use an exponential notation in A, and denote by e^λ the element of A corresponding to $\lambda \in P$. These "formal exponentials" e^λ form an F-basis of A, such that $e^\lambda \cdot e^\mu = e^{\lambda+\mu}$ and $(e^\lambda)^{-1} = e^{-\lambda}$. In particular, $e^0 = 1$ is the identity element of A. The ring A is an algebra of Laurent polynomials, namely $A = F\left[u_1^{\pm 1}, \cdots u_n^{\pm 1}\right]$ where $u_i = e^{\pi_i}$ (π_i the fundamental weights).

If

$$f = \sum_{\lambda \in P} f_\lambda e^\lambda$$

is an element of A, with coefficients $f_\lambda \in F$, the *constant term* of f is f_0, the coefficient of $e^0 = 1$ in f. We can now state two constant term identities that generalize those of Dyson and Andrews described in the abstract to this lecture. As before, R is a reduced irreducible root system and k a non negative integer.

(2.1) *The constant term in*

$$\prod_{\alpha \in R} (1 - e^\alpha)^k$$

is equal to

$$\prod_{i=1}^{n} \binom{kd_i}{k}$$

where $d_1, \cdots d_n$ *are the degrees of the Weyl group of* R.

When R is A_{n-1}, the roots are $\alpha = \varepsilon_i - \varepsilon_j$ where $i \neq j$, so that $e^\alpha = x_i x_j^{-1}$ where $x_i = e^{\varepsilon_i}$. Moreover, as we have seen, the degrees of the Weyl group in this case are $2, 3, \cdots, n$; and

$$\binom{2k}{k}\binom{3k}{k}\cdots\binom{nk}{k} = \frac{(nk)!}{k!^n}$$

Thus we recover Dyson's original conjecture [5].

Next, in order to state the generalization of Andrew's conjecture we introduce the q-analogue of the binomial coefficient $\binom{r}{s}$, namely the *Gaussian polynomial*

$$\begin{bmatrix} r \\ s \end{bmatrix} = \frac{(1-q^r)(1-q^{r-1})\cdots(1-q^{r-s+1})}{(1-q)(1-q^2)\cdots(1-q^s)}$$

which tends to $\binom{r}{s}$ as $q \to 1$.

(2.2) *The constant term in*

$$\prod_{\alpha \in R^+} \prod_{i=0}^{k-1} (1 - q^i e^{\alpha}) \ (1 - q^{i+1} e^{-\alpha})$$

is equal to

$$\prod_{i=1}^{n} \begin{bmatrix} kd_i \\ k \end{bmatrix}.$$

When R is A_{n-1}, the positive roots are $\alpha = \varepsilon_i - \varepsilon_j$ with $i < j$, so that we recover Andrews' conjecture. Clearly, also, (2.2) reduces to (2.1) when we let $q \to 1$.

When these conjectures and others like them were first put forward ([12], [18]), they appeared as isolated curiosities, and it was not clear what, if anything, lay behind them. Later [13] it became clear that they could be considered as a special case of a conjectured norm fomula for orthogonal polynomials, as we shall explain in the next section.

The identity (2.1) was first proved uniformly for all R by Opdam [20], using the technique of shift operators developed by Heckman and Opdam in the context of their theory of hypergeometric functions and Jacobi polynomials [8]. The q-version (2.2) took longer to resolve, and was finally proved in full generality by Cherednik [3], although by that time all the root systems with the exception of E_6, E_7 and E_8 has been dealt with one by one ([2], [9], [6], [7]).

3 Orthogonal polynomials

As in §2, let A be the group algebra $F[P]$ where F is a field of characteristic 0. The Weyl group W_0 acts on P and therefore also on $A: \ w(e^{\lambda}) = e^{w\lambda}(\lambda \in P, w \in W_0)$. Let A_0 denote the subalgebra of W_0-invariants.

Since each W_0-orbit in P meets P^+ exactly once, it follows that the *orbit-sums*

(3.1) $$m_{\lambda} = \sum_{\mu \in W_0 \lambda} e^{\mu}$$

where $\lambda \in P^+$ and $W_0\lambda$ is the W_0-orbit of λ, form an F-basis of A. Another basis of A_0 is obtained as follows. Let

(3.2 $$p = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$$

and let

(3.3) $$\delta = \prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2}).$$

In fact, $p \in P^+$ and $\delta \in A$: we have

$$\delta = \sum_{w \in W_0} \varepsilon(w) e^{wp} \tag{3.4}$$

where $\varepsilon(w) = \det(w) = \pm 1$. Thus δ is skew-symmetric for W_0, i. e. we have $w\delta = \varepsilon(w)\delta$ for each $w \in W_0$. For each $\lambda \in P^+$, the sum

$$\sum_{w \in W_0} \varepsilon(w) e^{w(\lambda+p)}$$

is likewise skew-symmetric, and is divisible by δ in A. The quotient

$$\mathcal{X}_\lambda = \delta^{-1} \sum_{w \in W_0} \varepsilon(w) e^{w(\lambda+p)} \tag{3.5}$$

is an element of A_0 called the *Weyl character* corresponding to λ. In terms of the orbit-sums we have

$$\mathcal{X}_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu \tag{3.6}$$

where the coefficients $K_{\lambda\mu}$ are integers (indeed positive integers) and $\mu < \lambda$ means that $\lambda - \mu \in Q^+$ and $\lambda \neq \mu$.

From (3.6) it follows that the χ_λ form another F-basis of A_0. From now on we shall take F to be the field $\mathbb{Q}(q,t)$ of rational functions in two indeterminates q, t. Let

$$\Delta = \Delta(q,t) = \prod_{\alpha \in R^+} \prod_{r=0}^{\infty} \frac{(1 - q^r e^\alpha)(1 - q^{r+1} e^{-\alpha})}{(1 - q^r t e^\alpha)(1 - q^{r+1} t e^{-\alpha})}. \tag{3.7}$$

Suppose first that $t = q^k$ where k is a non-negative integer. Then Δ is a finite product, namely the polynomial whose constant term was the subject of (2.2). (In the general case, Δ can be expanded as a formal power series in the $n+1$ variables $u_0, u_1, \cdots, u_n$, where $u_i = e^{\alpha_i} (1 \leq i \leq n)$ and $u_0 = qe^{-\varphi}, \varphi$ the highest root of R.

We shall use Δ to define a scalar product on A, as follows. If $f \in A$, say

$$f = \sum_{\lambda \in P} f_\lambda e^\lambda,$$

let

$$f^* = \textstyle\sum_{\lambda \in P} f_\lambda^* e^{-\lambda} \tag{3.8}$$

where f_λ^* is the image of f_λ under the automorphism $(q,t) \mapsto (q^{-1}, t^{-1})$ of F. We now define, for $f, g \in A$,

$$(f,g) = \text{ constant term in } fg^*\Delta. \tag{3.9}$$

We can now state

(3.10) *There is a unique F-basis $(P_\lambda)_{\lambda\in P^+}$ of A_0 such that*

(i) $P_\lambda = m_\lambda + \sum_{\mu<\lambda} u_{\lambda\mu} m_\mu$ *with coefficients* $u_{\lambda\mu} \in F$;

(ii) $(P_\lambda, P_\mu) = 0$ *if* $\lambda \neq \mu$.

It is easy to see that the P_λ, if they exist, are uniquely determined by (i) and (ii). Their existence, however, requires proof. If the partial order $\lambda > \mu$ on P^+ were a total ordering, existence would follow directly from the Gram-Schmidt orthogonalization process. But it is not a total ordering (unless $R = A_1$) and we should therefore have to extend it to a total ordering before applying the Gram-Schmidt mechanism. Thus the content of (3.10) is that however we extend the partial order $\lambda > \mu$ to a total order, we always obtain the same basis.

We shall not reproduce the original proof ([13] [16]) of (3.10) here, since if will arise more naturally later in the context of affine Hecke algebras. Instead, let us look at some special cases:

(1) When $t = 1$, we have $\Delta = 1$ and P_λ is the orbit-sum m_λ (3.1).

(2) When $t = q$, P_λ is the Weyl character $\mathcal{X}_\lambda$ (3.5).

(3) When $q \to 0$, t being arbitrary, the P_λ (suitably normalized) occur as the values of spherical functions on a p-adic symmetric space, when t^{-1} is a prime power.

(4) Let $t = q^k$ and fix k (which need not be an integer) and let $q \to 1$, so that $t \to 1$ also. In the limit we have $\Delta = \prod_{\alpha\in R}(1 - e^\alpha)^k$. In this limiting case the polynomials P_λ are the "Jacobi polynomials" of Heckman and Opdam [8]. For particular values of k these polynomials occur as values of spherical functions, but this time on a *real* symmetric space.

(5) Finally, when R is A_{n-1}, the P_λ are the symmetric functions of ([15], chapter VI), restricted to n variables $x_1, \cdots, x_n$ such that $x_1 \cdots x_n = 1$.

To conclude this section, we shall record some properties of the polynomials P_λ. For simplicity of statement, we shall assume that $t = q^k$ where k is a positive integer.

a.) *Norms*
The squared norm of P_λ is given by the formula

$$(3.11) \qquad (P_\lambda, P_\lambda) = W_0(t) \prod_{\alpha\in R^+} \prod_{i=0}^{k-1} \frac{1 - q^{\langle \lambda + kp, \alpha^\vee\rangle + i}}{1 - q^{\langle \lambda + kp, \alpha^\vee\rangle - i}}$$

where p is given by (3.2) and $W_0(t)$ is the Poincaré polynomial of the Weyl group W_0:

$$W_0(t) = \sum_{w\in W_0} t^{\ell(w)}$$

where $l(w)$ is the *length* of w, i. e. the number of $\alpha \in R^+$ such that $w\alpha \in R^-$.

Notice that when $\lambda = 0$ we have $P_\lambda = 1$, so that in this case (3.11) gives the constant term of Δ, i. e. it gives the constant term identity (2.2) (though a little work is required to recast it in that form). The formula (3.11) was originally conjectured in [13], and verified there in some cases. In the limiting case $q \to 1$, it was first proved for all root systems R by Opdam [20], and then in full generality by Cherednik [3]. We shall indicate a proof later, in §5.

b.) *Specialization*
Let $P^\vee$ be the weight lattice of the dual root system $R^\vee$: it consists of all $\lambda \in V$ such that $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in R$. It will be convenient to regard each $f \in A$ as a function on $P^\vee$, as follows: if $\mu \in P^\vee$ and $f = \sum f_\lambda e^\lambda$, then

$$f(\mu) = \sum f_\lambda q^{\langle \lambda, \mu \rangle}.$$

Then we have

$$P_\lambda(kp^\vee) = q^{-\langle \lambda, kp^\vee \rangle} \prod_{\alpha \in R^+} \prod_{i=0}^{k-1} \frac{1 - q^{\langle \lambda + kp, \alpha^\vee \rangle + i}}{1 - q^{\langle kp, \alpha^\vee \rangle + i}} \tag{3.12}$$

where

$$p^\vee = \frac{1}{2} \sum_{\alpha \in R^+} \alpha^\vee$$

(warning: $p^\vee \neq 2p/\langle p, p \rangle$).

When $k = 1$ and $q \to 1$, this reduces to Weyl's formula for the dimension of an irreducible representation of a compact Lie group. The formula (3.12) was originally conjectured in [13]. As with (3.11), it was first proved for all R in the limiting case $q \to 1$ by Opdam [20], and then in full generality by Cherednik [4].

c.) *Symmetry*
For $\lambda \in P$ let

$$\tilde{P}_\lambda = P_\lambda / P_\lambda(kp^\vee).$$

Then we have

$$\tilde{P}_\lambda(\mu + kp^\vee) = \tilde{P}_\mu(\lambda + kp) \tag{3.13}$$

for all $\lambda \in P^+$ and $\mu \in (P^\vee)^+$, and on the right-hand side of (3.13), P_μ is an orthogonal polynomial for $R^\vee$, so that $\tilde{P}_\mu = P_\mu / P_\mu(kp)$. When R is of type A_{n-1}, (3.13) is due to Koornwinder ([15], chapter VI, §6). The general case is due to Cherednik [4].

4 The affine root system and the extended affine Weyl group

The root systems and Weyl groups of §1 have affine counterparts, to which we now turn. As before, R is a reduced, irreducible root system spanning a real vector space V of dimension $n \geq 1$. Let $Q^\vee, P^\vee$ respectively denote the root lattice and the weight lattice of the dual root system $R^\vee$.

We shall regard each $\alpha \in R$ as a linear function on V: $\alpha(x) = \langle \alpha, x \rangle$ for $x \in V$. Let c denote the constant function 1 on V. Then

$$S = S(R) = \{\alpha + nc : \alpha \in R, n \in \mathbb{Z}\} \tag{4.1}$$

is the *affine root system* associated with R. The elements of S are affine-linear functions on V, called *affine roots*, and we shall denote them by italic letters, $a, b, \ldots$.

For each $a \in S$, let H_a denote the affine hyperplane in V on which a vanishes, and let s_a denote the orthogonal reflection in this hyperplane. The affine Weyl group W_S is the group of affine isometries of V generated by these reflections. For each $\alpha \in R$, the mapping $s_\alpha \circ s_{\alpha+c}$ takes $x \in V$ to $x + \alpha^\vee$, so that

$$\tau(\alpha^\vee) = s_\alpha \circ s_{\alpha+c}$$

is translation by $\alpha^\vee$. It follows that W_S contains a subgroup of translations isomorphic to $Q^\vee$, and we have

$$W_S = W_0 \ltimes \tau(Q^\vee) \tag{4.2}$$

(semidirect product).

The *extended affine Weyl group* is

$$W = W_0 \ltimes \tau(P^\vee). \tag{4.3}$$

It acts on V as a discrete group of isometries, and hence by transposition on functions on V. As such, it permutes the affine roots $a \in S$.

As in §1, let R^+ be a system of positive roots in R and $\alpha_1, \cdots, \alpha_n$ the simple roots, φ the highest root. Correspondingly, the affine roots $a_0, a_1, \cdots, a_n$, where $a_0 = -\varphi + c$ and $\alpha_i = \alpha_i$ $(1 \leq i \leq n)$ form a set of simple roots for S : each $a \in S$ is of the form

$$a = \sum_{i=0}^{n} r_i a_i \tag{4.4}$$

where the r_i are integers, all of the same sign. Let

$$C = \{x \in V : a_i(x) > 0 \ (0 \leq i \leq n)\}$$

so that C is an open n-simplex bounded by the hyperplanes H_{a_i} $(0 \leq i \leq n)$. The group W_S is generated by the reflections $s_i = s_{a_i} (0 \leq i \leq n)$, subject to the relations

$$s_i^2 = 1, \tag{4.5}$$

$$s_i s_j s_i \cdots = s_j s_i s_j \cdots \tag{4.6}$$

whenever $i \neq j$ and $s_i s_j$ has finite order m_{ij} in W_S, there being m_{ij} terms on either side of (4.6). In other words, W_S is a Coxeter group on the generators $s_0.s_1, \cdots, s_n$.

The connected components of $V - \bigcup_{a \in S} H_a$ are open simplexes, each congruent to C, and each component is of the form wC for a unique element $w \in W_S$. Thus, for example, when R is of type A_2 we obtain the familiar tessellation of the Euclidean plane by congruent equilateral triangles.

An affine root $a \in S$ is *positive* (resp. *negative*) relative to C if $a(x) > 0$ (resp. $a(x) < 0$) for $x \in C$. Equivalently, $a \in S$ is positive or negative according as the coefficients r_i in (4.4) are all ≥ 0 or all ≤ 0. Let S^+ (resp. S^-) denote the set of positive (resp. negative) affine roots. Then $S^- = -S^+$, and $S = S^+ \cup S^-$.

Explicitly, the positive affine roots are $\alpha + rc$ where $r \geq 0$ if $\alpha \in R^+$, and $r \geq 1$ if $\alpha \in R^-$. It follows that the product Δ (3.7) may be written in the form

$$\Delta = \prod_{a \in S^+} \frac{1 - e^a}{1 - te^a} \tag{4.7}$$

where for $a = \alpha + rc \in S$, $e^a = e^{\alpha + rc} = q^r e^\alpha$ (i. e. we define $e^c = q$).

We shall now define a length function on the extended group W. If $w \in W$, let

$$\ell(w) = \mathrm{card}(S^+ \cap wS^-),$$

the number of positive affine roots made negative by w. Equivalently, $\ell(w)$ is the number of hyperplanes $H_a, a \in S$, that separate C from wC.

Now W, unlike W_S, is not in general a Coxeter group (unless $P^\vee = Q^\vee$) and may contain elements $\neq 1$ of length zero. Let

$$\Omega = \{w \in W : \ell(w) = 0\}$$

The elements of Ω stabilize the simplex C, and hence permute the simple affine roots. For each $w \in W$ there is a unique $w' \in W_S$ such that $wC = w'C$, and hence w factorizes uniquely as $w = w'v$, with $w' \in W_S$ and $v \in \Omega$. Consequently we have

$$W = W_S \rtimes \Omega \tag{4.8}$$

(semidirect product). From (4.2), (4.3) and (4.8) it follows that $\Omega \cong W/W_S \cong P^\vee/Q^\vee$, hence is a finite abelian group.

Next, the *braid group* B of W is the group with generators $T(w), w \in W$, and relations

$$T(v)T(w) = T(vw)$$

whenever $\ell(vw) = \ell(v) + \ell(w)$. We shall denote $T(s_i)$ by T_i $(0 \leq i \leq n)$ and $T(\omega)$ $(\omega \in \Omega)$ simply by ω. Then B is generated by $T_0, T_1, \cdots, T_n$ and Ω subject to the following relations:

(a) the counterparts of (4.6), namely the *braid relations*

$$T_i T_j T_i \cdots = T_j T_i T_j \cdots \tag{4.9}$$

where $i \neq j$ and there are m_{ij} terms on either side;

(b) the relations

$$\omega T_i \omega^{-1} = T_j \tag{4.10}$$

for $\omega \in \Omega$, where $\omega(a_i) = a_j$.

Let $\lambda \in (P^\vee)^+$ be a dominant weight for $R^\vee$, and define

$$Y^\lambda = T(\tau(\lambda))$$

where $\tau(\lambda)$ is translation by λ. If λ and μ are both dominant, we have

$$Y^\lambda \cdot Y^\mu = Y^{\lambda+\mu} \tag{4.11}$$

in B. If now λ is any element in $P^\vee$, we can write $\lambda = \mu - \nu$ where μ, ν are both dominant, and we define

$$Y^\lambda = Y^\mu (Y^\nu)^{-1}. \tag{4.12}$$

In view of (4.10), this definition is unambiguous. The elements $Y^\lambda, \lambda \in P^\vee$, form a commutative subgroup of B, isomorphic to $P^\vee$.

5 The affine Hecke algebra

The *Hecke algebra* H of W is the quotient of the group algebra $F[B]$ of the braid group by the ideal generated by the elements $(T_i - t^{1/2})(T_i + t^{-1/2})$ $(0 \leq i \leq n)$. (The field F should now include $t^{1/2}$ as well as q and t.) For each $w \in W$, we denote the image of $T(w)$ in H by the same symbol $T(w)$: these elements form an F-basis of H. Thus H is generated over F by $T_0, T_1, \cdots, T_n$ and Ω subject to the relations (4.9), (4.10), together with the *Hecke relations*

$$(T_i - t^{1/2})(T_i + t^{-1/2}) = 0. \tag{5.1}$$

When $t = 1$, H is the group algebra of W.

The following proposition is due to Cherednik [3].

(5.2) *The Hecke algebra H acts on $A = F[P]$ as follows:*

$$\begin{aligned} T_i e^\mu &= t^{1/2} e^{s_i \mu} + \left(t^{1/2} - t^{-1/2}\right)(1 - e^{a_i})^{-1}\left(e^\mu - e^{s_i\mu}\right), \\ \omega e^\mu &= e^{\omega\mu}. \end{aligned}$$

where $0 \leq i \leq n$ and $\omega \in \Omega$. Moreover, this representation is faithful.

A proof of (5.2) is sketched in [14]. (In the formulas above, recall that $e^{a_0} = e^{-\varphi+c} = qe^{-\varphi}$.)

The elements $Y^\lambda, \lambda \in P^\vee$, span a commutative subalgebra of H, isomorphic to $A^\vee = F[P^\vee]$. If $u \in A^\vee$, say

$$u = \sum u_\lambda e^\lambda$$

let

$$u(Y) = \sum u_\lambda Y^\lambda \in H.$$

(5.3) *For each $w \in W$, the adjoint of $T(w)$ for the scalar product* (3.9) *on A is $T(w)^{-1}$, i. e., we have*

$$(T(w)f, g) = (f, T(w)^{-1}g)$$

for all $f, g \in A$. In particular, the adjoint of Y^λ is $Y^{-\lambda}$, and the adjoint of $u(Y)$, where $u \in A^\vee$, is $u^*(Y)$ (3.8).

It is enough to show that the adjoint of T_i (resp. $\omega \in \Omega$) is T_i^{-1} (resp. ω^{-1}), and this may be verified directly from the definitions.

From (5.2) we have an action of $A^\vee$ on A, with $u \in A$ acting as $u(Y)$. One shows that $A_0 = A^{W_0}$ is stable under the action of $A_0^\vee = (A^\vee)^{W_0}$, so that we have an action of $A_0^\vee$ on A_0. It turns out (see, e.g. [16] chapter III) that this action is diagonalized by the polynomials $P_\lambda (\lambda \in P^+)$, and more precisely that

$$u(Y)P_\lambda = u(-\lambda - kp)P_\lambda \tag{5.4}$$

for all $u \in A^\vee$. The pairwise orthogonality of the P_λ then follows immediately from (5.3) and (5.4).

Likewise, the action of $A^\vee$ on A can be diagonalized, and this gives rise to a family of non-symmetric orthogonal polynomials:

(5.5) *There is a unique F-basis $(E_\lambda)_{\lambda \in P}$ of A such that*

(i) $E_\lambda = e^\lambda +$ *lower terms*,

(ii) $(E_\lambda, E_\mu) = 0$ if $\lambda \neq \mu$.

(By "lower terms" is meant a linear combination of exponentials e^μ where $\mu < \lambda$ in a certain partial ordering on P.)

The polynomials E_λ are simultaneous eigenfunctions of all operators $u(Y)$, $u \in A^\vee$. (See [19] or [16], Ch. III.)

Consider now the operators

$$U^+ = \sum_{w \in W_0} t^{\ell(w)/2} T(w),$$
$$U^- = \sum_{w \in W_0} \varepsilon(w) t^{-\ell(w)/2} T(w),$$

on A. The operator U^+ maps A onto A_0, and in particular if $\lambda \in P^+$ then $U^+ E_\lambda$ is a scalar multiple of P_λ.

Next consider, again for $\lambda \in P^+$,

$$Q_\lambda = U^- E_\lambda.$$

If λ is not regular (i. e. if $\langle \lambda, \alpha_i \rangle = 0$ for some i) then $Q_\lambda = 0$.

Both P_λ and Q_λ are linear combinations of the $E_\mu, \mu \in W_0\lambda$, with coefficients that can be computed explicitly. Hence both (P_λ, P_λ) and (Q_λ, Q_λ) can be expressed in terms of (E_λ, E_λ). In this way we obtain [14]

$$\frac{(Q_\lambda, Q_\lambda)}{(P_\lambda, P_\lambda)} = q^{-Nk} \prod_{\alpha \in R^+} \frac{1 - q^{\langle \lambda + kp, \alpha^\vee \rangle + k}}{1 - q^{\langle \lambda + kp, \alpha^\vee \rangle - k}} \tag{5.6}$$

where as usual $t = q^k$, and $N =$ card(R^+).

To conclude, we shall sketch a proof of Cherednik's norm formula (3.1). The proof will be by induction on k, the cases $k = 0$ and $k = 1$ being trivial. From now on we shall write $P_{\lambda,k}$ and $Q_{\lambda,k}$ in place of P_λ and Q_λ, to stress the dependence on the parameter k, and likewise for the scalar product: $(f, g)_k$ in place of (f, g). Let

$$\pi_k = \prod_{\alpha \in R^+} (e^{\alpha/2} - q^{-k} e^{-\alpha/2}).$$

Then the P's and Q's are related as follows:

(5.7) *For all* $\lambda \in P^+$, *we have*

$$P_{\lambda,k+1} = \pi_k^{-1} Q_{\lambda+p,k}.$$

Taking $\lambda = 0$, it follows that $Q_{p,k} = \pi_k$. The formula (5.7) may be regarded as a generalization of Weyl's character formula (3.5), which is the case $k = 0$.

From (5.7) we obtain

$$\frac{(P_{\lambda,k+1}, P_{\lambda,k+1})_{k+1}}{(Q_{\lambda+p,k}, Q_{\lambda+p,k})_k} = q^{Nk} \frac{W_0(q^{k+1})}{W_0(q^k)}. \tag{5.8}$$

Coupled with (5.6) (with λ replaced by $\lambda + p$) this gives

$$\frac{(P_{\lambda,k+1}, P_{\lambda,k+1})_{k+1}}{(P_{\lambda+p,k}, P_{\kappa+p,k})_k} = \frac{W_0(q^{k+1})}{W_0(q^k)} \prod_{\alpha \in R^+} \frac{1 - q^{\langle \lambda + (k+1)p, \alpha^\vee \rangle + k}}{1 - q^{\langle \lambda + (k+1)p, \alpha^\vee \rangle - k}}$$

and (3.11) follows by induction on k.

For simplicity of exposition we have restricted ourselves in this survey to affine root systems of the type $S(R)$ (4.1). The general picture is that one can attach to any irreducible affine root system S, reduced or not, families of orthogonal polynomials P_λ, Q_λ and E_λ as above. These depend (apart from q) on as many parameters t_i as there are orbits in S under the affine Weyl group W_S, and the whole theory can be developed in this more general context. For an irreducible S, the maximum number of orbits is 5, and is attained by the (non-reduced) affine root systems denoted by $C^\vee C_n (n \geq 2)$ in the tables at the end of [11]. Correspondingly, we have orthogonal polynomials P_λ, Q_λ and E_λ depending on q and five parameters t_i. These P_λ are the orthogonal polynomials defined by Koornwinder [10], which are therefore amenable to the Hecke algebra techniques described here. A full account will (eventually) appear in the book [17].

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