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Some Recent Transcendental Techniques in Algebraic and Complex Geometry*

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Abstract

This article discusses the recent transcendental techniques used in the proofs of the following three conjectures. (1) The plurigenera of a compact projective algebraic manifold are invariant under holomorphic deformation. (2) There exists no smooth Leviflat hypersurface in the complex projective plane. (3) A generic hypersurface of sufficiently high degree in the complex projective space is hyperbolic in the sense that there is no nonconstant holomorphic map from the complex Euclidean line to it.

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1. Introduction

Since the use of function theory in the study of algebraic curves as Riemann surfaces about two hundred years ago, transcendental methods such as harmonic forms in Hodge theory and curvature in the theory of Chern-Weil have been very important tools in complex algebraic geometry. Since the nineteen sixties very powerful techniques in the estimates of $\bar{\partial}$, especially L^2 estimates and regularity techniques, have been extensively developed by C. B. Morrey, J. J. Kohn, L. Hörmander, *et al.* (To avoid too lengthy a bibliography here, we refer to [2],[4],[7],[10],[15],[16],[18] for references not listed here.) During the last two decades these new transcendental techniques have been increasingly used in complex algebraic geometry. The most noteworthy among them is J. J. Kohn's method of multiplier ideals for $\bar{\partial}$ estimates [7] which holds the promise of applicability to general partial differential equations and global geometry. Nadel [11] introduced multiplier ideal sheaves dual to Kohn's. A number of longstanding problems in algebraic and complex geometry hitherto beyond the reach of known methods have been solved by

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the new techniques of $\bar\partial$ estimates. On the other hand, demands of geometric applications motivate new approaches to $\bar\partial$ estimates. We will discuss here some recent results in the following three topics in algebraic and complex geometry obtained by the new transcendental methods. (1) Invariance of plurigenera. (2) Nonexistence of smooth Levi-flat hypersurface in $\mathbf{P}_2$. (3) Hyperbolicity of generic hypersurface of high degree in $\mathbf{P}_n$. Though topic (3) is only peripherally linked to $\bar\partial$ estimates, a long outstanding problem there is solved by some recent transcendental techniques.

2. Invariance of plurigenera

Let $\Delta_r = \{t \in \mathbf{C} \,|\, |t| < r\}$ and $\Delta = \Delta_1$. Denote by K_Y the canonical line bundle of a complex manifold Y. The m-genus of a compact complex manifold X is the complex dimension of $\Gamma(X, m K_X)$. By Hodge theory the 1-genus of a compact Kähler manifold is a topological invariant and therefore is invariant under holomorphic deformation. For the general m-genus there is the following conjecture on its invariance under holomorphic deformation for a compact Kähler manifold.

Conjecture 2.1 *(on Deformational Invariance of Plurigenera for Kähler Manifolds). Let $\pi : X \to \Delta$ be a holomorphic family of compact Kähler manifolds with fiber X_t. Then for any positive integer m the complex dimension of $\Gamma(X_t, m K_{X_t})$ is independent of t for $t \in \Delta$.*

Conjecture (2.1) has been verified in [20] when X is a family of compact projective algebraic manifolds.

Theorem 2.2 [20]. *Let $\pi : X \to \Delta$ be a holomorphic family of compact complex projective algebraic manifolds. Then for any integer $m \geq 1$ the complex dimension of $\Gamma(X_t, m K_{X_t})$ is independent of t for $t \in \Delta$.*

The main techniques to solve the problem were first introduced in [17] where for technical reasons the assumption of each fiber being of general type is added. Because of the upper semicontinuity of $\dim_{\mathbf{C}} \Gamma(X_t, m K_{X_t})$, the conjecture is equivalent to extending every element of $\Gamma(X_t, m K_{X_t})$ to $\Gamma(X, m K_X)$. We can assume $t = 0$. The idea of the main techniques stemmed from the following naive motivation. If one could write an element $s^{(m)}$ of $\Gamma(X_0, m K_{X_0})$ as a sum of terms, each of which is the product of an element $s^{(1)}$ of $\Gamma(X_0, K_{X_0})$ and an element $s^{(m-1)}$ of $\Gamma(X_0, (m-1) K_{X_0})$, then one can extend $s^{(m)}$ to an element of $\Gamma(X, m K_X)$ by induction on m. Of course, in general it is clearly impossible to so express $s^{(m)}$ as a sum of such products. However, one could successfully implement a modified form of this naive motivation, in which $s^{(1)}$ is only a local holomorphic section and $s^{(m-1)}$ is an element of $\Gamma(X_0, (m-1) K_{X_0} + E)$ instead of $\Gamma(X_0, (m-1) K_{X_0})$, where E is a sufficiently ample line bundle on X independent of m. The implementation of the modified form depends on the following two ingredients.

Proposition 2.3 *(Global Generation of Multiplier Ideal Sheaves). Let L be a holomorphic line bundle over an n-dimensional compact complex manifold Y with a Hermitian metric which is locally of the form $e^{-\xi}$ with ξ plurisubharmonic. Let $\mathcal{I}_\xi$ be the multiplier ideal sheaf of the Hermitian metric $e^{-\xi}$ (i.e., the sheaf consisting of all holomorphic function germs f with $|f|^2 e^{-\xi}$ locally integrable). Let E be an ample holomorphic line bundle over Y such that for every point P of Y there are a* *finite number of elements of* $\Gamma(Y,E)$ *which all vanish to order at least* $n+1$ *at* P *and which do not simultaneously vanish outside* P. *Then* $\Gamma(Y,\mathcal{I}_\xi\otimes(L+E+K_Y))$ *generates* $\mathcal{I}_\xi\otimes(L+E+K_Y)$ *at every point of* Y.

Proposition 2.4 *(Extension Theorem of Ohsawa-Takegoshi Type). Let* $\gamma: Y\to\Delta$ *be a projective algebraic family of compact complex manifolds. Let* $Y_0=\gamma^{-1}(0)$ *and let* n *be the complex dimension of* Y_0. *Let* L *be a holomorphic line bundle with a Hermitian metric* $e^{-\chi}$ *with* χ *plurisubharmonic. Then for* $0<r<1$ *there exists a positive constant* A_r *with the following property. For any holomorphic* L*-valued* n*-form* f *on* Y_0 *with* $\int_{Y_0}|f|^2e^{-\chi}<\infty$, *there exists a holomorphic* L*-valued* $(n+1)$*-form* $\tilde f$ *on* $\gamma^{-1}(\Delta_r)$ *such that* $\tilde f|_{Y_0}=f\wedge\gamma^*(dt)$ *at points of* Y_0 *and* $\int_Y|\tilde f|^2e^{-\chi}\le A_r\int_{Y_0}|f|^2e^{-\chi}$.

Locally expressing an element $s^{(m)}$ of $\Gamma\left(X_0,m\,K_{X_0}\right)$ as a sum of terms, each of which is the product of a local holomorphic function $s^{(1)}$ and an element $s^{(m-1)}$ of $\Gamma\left(X_0,(m-1)\,K_{X_0}+E\right)$ is precisely Proposition 2.3, necessitating the use of E.

One constructs a metric of $(m-1)K_X+E$ by using the sum of absolute-value squares of elements of $\Gamma\left(X,(m-1)\,K_X+E\right)$ whose restrictions to X_0 form a basis of $\Gamma\left(X_0,(m-1)\,K_{X_0}+E\right)$. Proposition 2.4 is now applicable to show the surjectivity of $\Gamma\left(X,(m-1)\,K_X+E\right)\to\Gamma\left(X_0,(m-1)\,K_{X_0}+E\right)$ by induction on m. To get rid of E, for a sufficiently large ℓ one takes the ℓ-th power of an element of $\Gamma\left(X_0,m\,K_{X_0}\right)$ and multiplies it by an element of $\Gamma\left(X,E\right)$ and then takes the ℓ-th root of the absolute value after its extension. This process, together with Hölder's inequality, is used to produce a metric of $(m-1)K_X$ which we can use in the application of Proposition 2.4 to get the surjectivity of $\Gamma\left(X,m\,K_X\right)\to\Gamma\left(X_0,m\,K_{X_0}\right)$. The assumption of general type facilitates the last technical step of getting rid of E by writing $aK_X=E+D$ for some sufficiently large integer a and an effective divisor D.

Kawamata [6] translated the argument of [17] to a purely algebraic geometric setting and Nakayama [12] explored generalizations including results concerning $\lim_{m\to\infty}\frac{1}{m}\log\dim_{\mathbf{C}}\Gamma\left(X_t,m\,K_{X_t}+E\right)$ as a function of t. The case of non general type necessitates letting ℓ, which is used in taking the power and the root, go to infinity. One has to control the estimates in the limiting process.

Tsuji put on the web a preprint on the deformational invariance of the plurigenera for manifolds not necessarily of general type [26], in which, besides the techniques of [17], he uses his theory of analytic Zariski decomposition and generalized Bergman kernels. Tsuji's approach of generalized Bergman kernels naturally and elegantly reduces the problem of the deformational invariance of the plurigenera to a growth estimate on the generalized Bergman kernels. Unfortunately this crucial estimate is lacking and seems unlikely to be establishable, as explained in [20].

In [20] a metric as singular as possible is introduced for the limiting process, which, together with an estimation technique using the concavity of the logarithmic function, successfully removes the technical assumption of general type in [17].

The deformational invariance of the plurigenera for Kähler manifolds is still open. Only known results on the Kähler case are due to Levine's [9] with the assumption of some pluricanonical section with nonsingular divisor (or only mild singularities). To generalize the methods of [17] and [20] to the Kähler case, one possibility is to use the "absolute value" of a holomorphic line bundle constructed from the Kähler metric, because in the key argument only the absolute value of the constructed holomorphic section is used and not the section itself. There is still no method of implementing this possibility.

3. Nonexistence of smooth Levi-Flat hypersurface in $\mathbf{P}_2$

The problem of the nonexistence of smooth Levi-flat hypersurface in $\mathbf{P}_2$ has its origin in dynamical systems in $\mathbf{P}_2$ (see [8]). In terms of $\bar\partial$ estimates, its significance is that it gives a natural geometric setting for the understanding of $\bar\partial$ regularity for domains with Levi-flat boundary. The $\bar\partial$ regularity problem for a relatively compact domain Ω with smooth boundary $\partial\Omega$ in a complex manifold is to find a solution u on Ω, smooth up to $\partial\Omega$, to the equation $\bar\partial u = g$ with a given $\bar\partial$-closed $(0,1)$-form g on Ω, smooth up to of $\partial\Omega$. *Global regularity* is said to hold for Ω if regularity holds for the particular solution u, known as the *Kohn solution*, which is orthogonal to all the L^2 holomorphic functions on Ω. The problem of global regularity has been very extensively studied in the past couple of decades (see bibliographies in [2],[7]). Global regularity holds for strictly pseudoconvex domains and, more generally, for weakly pseudoconvex domains whose boundary points are all of finite type. Finite type means that local complex-analytic curves touch the boundary only to bounded finite (normalized) order. Global regularity holds also for weakly pseudoconvex domains defined by global smooth weakly plurisubharmonic functions. On the other hand, worm domains are counter-examples for global regularity for general weakly pseudoconvex domains [2]. Though the nonexistence of smooth Levi-flat hypersurface in $\mathbf{P}_2$ is connected with the regularity of any one solution of the $\bar\partial$-equation rather than the particular Kohn solution, its proof ushers in a new approach of using vector fields to obtain $\bar\partial$ regularity for domains with Levi-flat boundary. The following solution of the Levi-flat hypersurface problem was given in [21].

Theorem 3.1 [21]. *Let $q \geq 8$. Then there exists no C^q Levi-flat real hypersurface M in $\mathbf{P}_2$.*

The nonexistence of real-analytic Levi-flat hypersurface in $\mathbf{P}_3$ was proved by Lins-Neto [8]. Ohsawa [14] treated the nonexistence of real-analytic Levi-flat hypersurface in $\mathbf{P}_2$ (some points in the argument there not yet complete). The nonexistence of smooth Levi-flat hypersurface in $\mathbf{P}_3$ was proved in [19]. The real-analytic case is completely different in nature from the smooth case, because the structure is automatically extendible to a neighborhood for the real-analytic case. Nonexistence in $\mathbf{P}_2$ implies nonexistence in $\mathbf{P}_n$ $(n \geq 2)$ by slicing with a generic linear $\mathbf{P}_2$.

The following argument reduces the problem to a $\bar\partial$ regularity question. Suppose M exists. We seek a contradiction from the positivity of the $(1,0)$-normal bundle $N^{(1,0)}_{M,\mathbf{P}_2}$ of the Levi-flat hypersurface M. The curvature θ of $N^{(1,0)}_{M,\mathbf{P}_2}$ with the metric induced from the Fubini-Study metric is positive, because a quotient bundle cannot be less positive. On the other hand, M is the zero-set of a smooth $\mathbf{R}$-valued function f_M on $\mathbf{P}_2$ with df_M nowhere zero on M. Evaluation by ∂f_M shows that

$N^{(1,0)}_{M,\mathbf{P}_2}$ is smoothly trivial and θ must be d-exact on M, which means that $\theta = d\alpha$ for some smooth real 1-form α on M. Decompose $\alpha = \alpha^{(1,0)} + \alpha^{(0,1)}$ into its $(1,0)$ and $(0,1)$ components. If $\alpha^{(0,1)} = \bar\partial_b \psi$ for some smooth function ψ on M, then $\theta = \sqrt{-1}\partial_b\bar\partial_b\,(2\mathrm{Im}\,\psi)$. At a point of M where $\mathrm{Im}\,\psi$ assumes its maximum, the positivity of θ along the holomorphic foliation is contradicted. The problem is thus reduced to solving the $\bar\partial_b$ equation on M with regularity. By applying the Mayer-Vietoris sequence to $\mathbf{P}_2 - M = U_1 \cup U_2$ and using the vanishing of $H^2(\mathbf{P}_2, \mathcal{O}_{\mathbf{P}_2})$, the problem is reduced to whether, for any $\bar\partial$-closed $(0,1)$-form g on U_j smooth up to ∂U_j, the equation $\bar\partial u = g$ can be solved on U_j with u smooth up to ∂U_j.

The usual approach to $\bar\partial$ regularity is to use the Bochner-Kodaira formula with boundary $\|\bar\partial g\|^2_\Omega + \|\bar\partial^* g\|^2_\Omega = \int_{\partial\Omega} \langle \mathcal{L}, \bar g \wedge g\rangle + \|\bar\nabla g\|^2_\Omega + (\Theta_E, \bar g \wedge g)_\Omega$ (with g being a smooth E-valued $(n,1)$-form in the domain of $\bar\partial^*$), to solve the equation with L^2 estimates and then apply differential operators, integration by parts, and commutation relations to prove regularity. Here $n = \dim_{\mathbf{C}} \Omega$, $\|\cdot\|_\Omega$ is the L^2 norm over Ω, $\bar\partial^*$ is the adjoint of $\bar\partial$, $\bar\nabla$ means covariant differentiation in the $(0,1)$-direction, $\mathcal{L}$ is the Levi form of $\partial\Omega$, and Θ_E is the curvature form of the Hermitian holomorphic line bundle E (which is usually chosen to be trivial).

In our new approach to get $\bar\partial$ regularity for the Levi-flat domain Ω in $\mathbf{P}_2$ the use of holomorphic vector fields compensates for the complete lack of strict positivity for the Levi form of the boundary. We use a new norm to derive the Bochner-Kodaira formula with boundary. We choose a vector field ξ on $\mathbf{P}_2$ which generates biholomorphisms preserving the Fubini-Study metric. The new norm is the L^2 norm $L^2_m(\Omega, \xi)$ for Lie derivatives $(\mathcal{L}ie_\xi)^j\, g$ along ξ for order $j \leq m$ on Ω for $(0,1)$-form g. Since ξ generates metric-preserving biholomorphisms, the formal adjoint of $\bar\partial$ with respect to $L^2_m(\Omega,\xi)$ agrees with the one with respect to usual L^2. One usual difficulty with regularity is the error terms from the commutation of differential operators with $\bar\partial$ and $\bar\partial^*$. One advantage of using $\mathcal{L}ie_\xi$ is that there are no error terms from its commutation with $\bar\partial$ and $\bar\partial^*$.

There are two technical problems. One is how to establish, for such a norm, the Bochner-Kodaira formula with boundary. The other is that appropriate regularity for a solution of the $\bar\partial$ equation with finite $L^2_m(\Omega,\xi)$ norm can be obtained only at points where the real and imaginary parts of ξ are not both tangential to $\partial\Omega$. We handle the first problem as follows. We prove that, if g belongs to the domain of the adjoint of $\bar\partial$ with respect to $L^2_m(\Omega,\xi)$, then $(\mathcal{L}ie_\xi)^j\, g$ belongs to the domain of the adjoint of $\bar\partial$ with respect to the usual L^2 norm on Ω for $j \leq m$. The formula for the new norm is simply the sum, over $0 \leq j \leq m$, of such a formula for the usual L^2 norm on Ω for $(\mathcal{L}ie_\xi)^j\, g$. The proof for $(\mathcal{L}ie_\xi)^j\, g$ to belong to the domain of the adjoint of $\bar\partial$ with respect to the usual L^2 norm on Ω consists of two steps. One shows that this is locally true at points where ξ is not tangential to $\partial\Omega$. Then one uses a removable singularity argument to handle the other points when ξ has been chosen generic enough. For the second problem, to handle the other points for a generic ξ, we use the foliation of $\partial\Omega$ by local complex-analytic curves and the generalized Cauchy integral formula along the local complex-analytic curves.

4. Hyperbolicity of generic hypersurface of high degree in $\mathbf{P}_n$

A complex manifold X is *hyperbolic* if there exists no nonconstant holomorphic map $\mathbf{C} \to X$. For the last few decades the study of hyperbolicity has been focussed on hypersurfaces and their complements in two important settings: (1) inside an abelian variety and (2) inside $\mathbf{P}_n$. In the general setting hyperbolicity is conjectured to be linked to the positivity of canonical line bundle in the following formulation.

Conjecture 4.1 *(Conjecture of Green-Griffiths). In a compact algebraic manifold X of general type (or with positive canonical line bundle) there exists a proper subvariety Y containing the images of all nonconstant holomorphic maps $\mathbf{C} \to X$.*

The theory for the setting inside an abelian variety is very well developed (see [16],[18] for references). The Zariski closure of any holomorphic map from $\mathbf{C}$ to an abelian variety A is the translate of an abelian subvariety of A. In particular, a subvariety of an abelian variety A which does not contain any translate of an abelian subvariety of A is hyperbolic. The defect of an ample divisor in an abelian variety is zero. In particular, the complement of an ample divisor in an abelian variety is hyperbolic.

Except those motivated by methods of number theory due to McQuillan, practically all the major techniques for problems related to hyperbolicity in the setting of abelian varieties are due to Bloch [1] who introduced the use of holomorphic jet differentials and differential equations in conjunction with the jet differentials. Investigations on problems related to hyperbolicity in the setting of abelian varieties have essentially been completed. Only technical details such as getting an optimal lower bound for k_n in Theorem 4.2 below remain open. Theorem 4.2 (proved in Addendum of [24]) was added to [24] in response to a difficulty in the proof of Lemma 2 of the original paper [24] pointed out in [13]. The difficulty resulted from an attempt to use semi-continuity of cohomology groups in deformations to avoid employing Bloch's technique from [1] which involves the uniqueness part of the fundamental theorem of ordinary differential equations. Putting back Bloch's technique removes the difficulty and at the same time improves the zero defect statement in [24] to Theorem 4.2 on the second main theorem with truncated multiplicity.

Theorem 4.2 (Addendum, [24]). *Let D be an ample divisor of an abelian variety A of complex dimension n and let $k_0 = 0$, $k_1 = 1$, and $k_{\ell+1} = k_\ell + 3^{n-\ell-1}\left(4(k+1)\right)^\ell D^n$ ($1 \leq \ell < n$). Then for any holomorphic map $f : \mathbf{C} \to A$ whose image is not contained in any translate of D, the following second main theorem with truncated multiplicity holds: $m(r,f,D) + (N(r,f,D) - N_{k_n}(r,f,D)) = O(\log T(r,f,D) + \log r)$ for r outside some set whose measure with respect to $\frac{dr}{r}$ is finite. Here $T(r,f,D)$, $m(r,f,D)$, $N(r,f,D)$, $N_{k_n}(r,f,D)$ are respectively the characteristic, proximity, counting functions, and truncated counting functions.*

For the setting inside $\mathbf{P}_n$ there is the following outstanding conjecture.

Conjecture 4.3 (Kobayashi's Conjecture). (a) *The complement in $\mathbf{P}_n$ of a generic hypersurface of degree at least $2n+1$ is hyperbolic.* (b) *A generic hypersurface of degree at least $2n-1$ in $\mathbf{P}_n$ is hyperbolic for $n \geq 3$.*

For Conjecture 4.3(a) the complement in $\mathbf{P}_2$ of a generic curve of sufficiently high degree is known to be hyperbolic [23]. For Conjecture 4.3(b) a generic surface of degree ≥ 36 in $\mathbf{P}_3$ is known to be hyperbolic [10]. The degree bound is lowered to 21 in [4]. There are some constructions of examples of smooth hyperbolic hypersurfaces in $\mathbf{P}_n$ (see [15]). The hyperbolicity result we want to discuss here is the following.

Theorem 4.4 [22]. *There exists a positive integer δ_n such that a generic hypersurface in $\mathbf{P}_n$ of degree $\geq \delta_n$ is hyperbolic.*

We sketch its proof here. A central role will be played by jet differentials which we now define. A k-jet differential on a complex manifold X with local coordinates $x_1, \cdots, x_n$ is locally a polynomial in $d^\ell x_j$ $(1 \leq \ell \leq k, 1 \leq j \leq n)$.

Lemma 4.5 (Lemma of Jet Differentials). *If a holomorphic jet differential ω on a compact complex manifold X vanishes on an ample divisor of X and $\varphi : \mathbf{C} \to X$ is a holomorphic map, then $\varphi^*\omega$ is identically zero on $\mathbf{C}$.*

The intuitive reason for Lemma 4.5 is that $\mathbf{C}$ does not admit a metric (or not even a *k-jet metric*) with curvature bounded above by negative number. While a usual metric assigns a value to a tangent vector (which is a 1-jet), a k-jet metric assigns a value to a k-jet. A non identically zero $\varphi^*\omega$ defines a k-jet metric $|\varphi^*\omega|^2$ on $\mathbf{C}$ which, even with some degeneracy, still gives a contradiction by its negative curvature. A rigorous proof of Lemma 4.5 depends on the logarithmic derivative lemma of Nevanlinna theory. A consequence of Lemma 4.5 is that the image of the k-jet $d^k\varphi$ of any holomorphic map $\varphi : \mathbf{C} \to X$ satisfies the differential equation $\omega = 0$ on X. If there exist enough independent such ω on X, then the system of all equations $\omega = 0$ does not admit any local solution curve and X is hyperbolic.

In the setting of abelian varieties Bloch constructed jet differentials by comparing meromorphic functions on the image and the target of a map with finite fibers. For a holomorphic map φ from $\mathbf{C}$ to an abelian variety A, let X be the Zariski closure of the image of φ in A and $\mathcal{X}$ be the Zariski closure of $(d^k\varphi)(\mathbf{C})$ in $J_k(A) = A \times \mathbf{C}^{nk}$. Here $J_k(\cdot)$ means the space of k-jets. Let $\sigma_k : \mathcal{X} \to \mathbf{C}^{nk}$ be induced by the natural projection $J_k(A) = A \times \mathbf{C}^{nk} \to \mathbf{C}^{nk}$ which forgets the position and keeps the differentials. Let $\tau : J_k(X) \to X$ be the natural projection. Let F be a meromorphic function on X whose pole-set is some ample divisor D. Suppose $\sigma_k : \mathcal{X} \to \mathbf{C}^{nk}$ is generically finite. Let $x_1, \cdots, x_n$ be the coordinates of $\mathbf{C}^n$. Then $F \circ \tau$ belongs to a finite extension of the rational function field of $\mathbf{C}^{nk}$ and there exist polynomials P_j $(0 \leq j \leq p)$ with constant coefficients in the variables $d^\ell x_\nu$ $(1 \leq \ell \leq k, 1 \leq \nu \leq n)$ such that $\sum_{j=0}^p (\sigma_k^* P_j)(\tau^* F)^j = 0$ on $\mathcal{X}$ and $\sigma_k^* P_p$ is not identically zero on $\mathcal{X}$. The equation forces the holomorphic jet differential P_p on $\mathcal{X}$ to vanish on the ample divisor $\tau^{-1}(D)$. The assumption of generical finiteness of σ_k is tied to the translational invariance of X.

The idea of our method of construction of holomorphic jet differentials on a generic hypersurface X in $\mathbf{P}_n$ defined of by a polynomial f of degree δ is to use the theorem of Riemann-Roch and the lower bound of negativity of jet differential bundles of X. The theorem of Riemann-Roch was first used by Green-Griffiths to obtain holomorphic jet differentials and is applicable only for surfaces where the higher cohomology groups could be easily handled. We can handle the higher cohomology groups in our higher dimensional case because of the lower bound of negativity of jet differential bundles of X. Since the twisted cohomology groups of $\mathbf{P}_n$ comes from counting the number of monomials, in the actual proof direct counting of monomials is used. Let $x_1, \cdots, x_n$ (respectively $z_0, \cdots, z_n$) be the inhomogeneous (respectively homogeneous) coordinates of $\mathbf{P}_n$. Let Q be a non identically zero polynomial of degree m_0 in $x_1, \cdots, x_n$ and of homogeneous weight m in $d^j x_\ell$ $(1 \leq j \leq n-1, 1 \leq \ell \leq n)$ with the weight of $d^j x_\ell$ equal to j. If $m_0 + 2m < \delta$, then Q is not identically zero on $J_{n-1}(X)$. By counting the number of coefficients of Q and the number of equations needed for the jet differential on X defined by Q to vanish on an ample divisor in X of high degree defined by a polynomial $g = 0$ in $\mathbf{P}_n$ and using $f = df = \cdots = d^{n-1}f = 0$ to eliminate one coordinate and its differentials, we obtain a jet differential $\frac{Q}{g}$ on X which is holomorphic and vanishes on an ample divisor of high degree.

Proposition 4.6 (Existence of Holomorphic Jet Differentials). *If $0 < \theta_0, \theta, \theta' < 1-\epsilon$ with $n\theta_0 + \theta \geq n + \epsilon$, then there exists an explicit $A = A(n, \epsilon) > 0$ such that for $\delta \geq A$ there exists a non identically zero $\mathcal{O}_{\mathbf{P}_n}(-q)$-valued holomorphic $(n-1)$-jet differential ω on X of total weight m with $q \geq \delta^{\theta'}$ and $m \leq \delta^\theta$.*

To construct enough independent jet differentials, we use meromorphic vector fields of low pole order on the total space $\mathcal{X}$ of all hypersurfaces in $\mathbf{P}_n$ of degree δ. The total space $\mathcal{X}$ is defined by $f = \sum_{\nu \in \mathbf{N}^{n+1}, |\nu| = \delta} \alpha_\nu z^\nu$ of bidegree $(\delta, 1)$ in $\mathbf{P}_n \times \mathbf{P}_N$, where $N = \binom{\delta+n}{n} - 1$, $z^\nu = z_0^{\nu_0} \cdots z_n^{\nu_n}$, $|\nu| = \nu_0 + \cdots + \nu_n$, and $\mathbf{N}$ is the set of all nonnegative integers. Let $e_\ell = (0, \cdots, 0, 1, \cdots, 0) \in \mathbf{N}^{n+1}$ with 1 in the ℓ-th place. The $(1,0)$-twisted tangent bundle of $\mathcal{X}$ is globally generated by holomorphic sections of the forms $L\left(z_q\left(\frac{\partial}{\partial \alpha_{\lambda+e_p}}\right) - z_p\left(\frac{\partial}{\partial \alpha_{\lambda+e_q}}\right)\right)$ and $\sum_j B_j \frac{\partial}{\partial z_j} + \sum_\mu L_\mu \frac{\partial}{\partial \alpha_\mu}$, where $\lambda \in \mathbf{N}^{n+1}$ with $|\lambda| = \delta - 1$ and L, L_μ (respectively B_j) are homogeneous linear functions of $\{\alpha_\nu\}$ (respectively $z_0, \cdots, z_n$) with L_μ and B_j suitably chosen.

We introduce the space $J_{n-1}^{\text{vert}}(\mathcal{X})$ of vertical $(n-1)$-jets of $\mathcal{X}$ which is defined by $f = df = \cdots = d^{n-1}f = 0$ in $(J_{n-1}(\mathbf{P}_n)) \times \mathbf{P}_N$ with the coefficients α_ν of f regarded as constants when forming $d^j f$. By generalizing the above construction of vector fields on $\mathcal{X}$ to vector fields on $J_{n-1}^{\text{vert}}(\mathcal{X})$, one obtains the following.

Proposition 4.7 (Existence of Low Pole-Order Vector Fields). *There exist $c_n, c_n' \in \mathbf{N}$ such that the (c_n, c_n')-twisted tangent bundle of the projectivization of $J_{n-1}^{\text{vert}}(\mathcal{X})$ is globally generated. (To avoid considering the singularities of weighted projective spaces, one can interpret the statement by using functions which are polynomials of homogeneous weight along the fibers of $J_{n-1}^{\text{vert}}(\mathcal{X})$.)*

For a generic fiber X of $\mathcal{X}$ the constructed holomorphic $(n-1)$-jet differential ω on X with vanishing order at least q on the infinity divisor can be extended holomorphically to $\tilde{\omega}$ on all neighboring fibers with vanishing order at least q on the infinity divisor. We use vector fields $v_1, \cdots, v_p$ on $J_{n-1}^{\text{vert}}(\mathcal{X})$ with fiber pole order low relative to q and take successive Lie derivatives $\mathcal{L}ie_{v_1} \cdots \mathcal{L}ie_{v_p} \tilde{\omega}$ whose restrictions to X give holomorphic jet differentials on X vanishing on an ample divisor of X. Because of the bound on the weight m in the construction of ω, for δ sufficiently large the jet differentials from the Lie derivatives are independent enough to eliminate the derivatives from the differential equations they define. As a result, one concludes that for some proper subvariety Y in X the image of any nonconstant holomorphic map from $\mathbf{C}$ to X is contained in Y.

To get the full conclusion of hyperbolicity, for the constructed ω one has to control the vanishing order of the coefficients of the monomials of the differentials. For a generic X the construction process enables one to bound the vanishing order by $\delta^{2-\eta}$ for some $\eta > 0$. For hyperbolicity one needs the better bound of $\delta^{1-\eta}$ for some $\eta > 0$. To achieve it, one uses an appropriate embedding $\Phi : \mathbf{P}_n \to \mathbf{P}_{\hat{n}}$ of degree δ_1 so that a generic hypersurface X of degree $\delta := \delta_1\delta_2$ in $\mathbf{P}_n$ can be extended to a hypersurface $\hat{X}$ of degree δ_2 in $\mathbf{P}_{\hat{n}}$. For this step the method of multiplier ideal sheaves from $\bar{\partial}$ estimates is used. We deform Φ slightly and pull back the jet differential $\hat{\omega}$ constructed on $\hat{X}$ to get a differential ω on a slight deformation of X. When the image of the deformed Φ has appropriate transversality to the zero set of the coefficients of $\hat{\omega}$, an appropriate choice of δ_1 and δ_2 gives the required bound on the vanishing order of the coefficients of ω. This is at the expense of increasing the order of the jet differential from $n-1$ to $\hat{n}-1$, which does not affect the argument. For this additional step in the argument the degree δ must be a product. To remove this condition, one uses an embedding $\mathbf{P}_n \to \mathbf{P}_{\hat{n}_1} \times \mathbf{P}_{\hat{n}_2}$ instead of Φ.

The use and the construction of meromorphic vector fields on $J^{\text{vert}}_{n-1}(\mathcal{X})$ of low pole order along the fibers are motivated by Clemens's work [3] (with later generalizations and improvements by Ein [5] and Voisin [25]) on the nonexistence of regular rational and elliptic curves on generic hypersufaces of sufficiently high degree.

There is no way yet to handle Conjecture 4.1. Additional assumptions such as $K_X - mL > 0$ or $(K_X - mL)\, L^{n-1} > 0$ for some large m and L ample or very ample could facilitate the construction of holomorphic jet differentials vanishing on an ample divisor. One possibility to handle the question of enough independent jet differentials is to deform $d^k\varphi$ of φ for each $k \geq 1$ separately and use techniques analogous to the twisted difference maps in the Vojta-Faltings proof of the Mordell conjecture and to McQuillan's separate rescaling of an entire holomorphic curve in each factor of a product of several copies of an abelian variety (see pp.504-505,[16]).

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