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Transition to chaos in small-world dynamical network

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Abstract

The transition from a non-chaotic state to a chaotic state is an important issue in the study of coupled dynamical networks. In this paper, by using the theoretical analysis and numerical simulation, we study the dynamical behaviors of the NW small-world dynamical network consisting of nodes that are in non-chaotic states before they are coupled together. It is found that, for any given coupling strength and a sufficiently large number of nodes, the small-world dynamical network can be chaotic, even if the nearest-neighbor coupled network cannot be chaotic under the same condition. More interesting, the numerical results show that the measurement $\frac{1}{R}$ of the transition ability from non-chaos to chaos approximately obeys power-law forms as $\frac{1}{R} \sim p^{-r_1}$ and $\frac{1}{R} \sim N^{-r_2}$. Furthermore, based on dissipative system criteria, we obtain the relationship between the network topology parameters and the coupling strength when the network is stable in the sense of Lyapunov (i. s. L.).

1. Introduction

Much research interest has been directed recently towards the theory and applications of complex networks. In particular, collective motions of coupled dynamical networks have received a great deal of attention towards subjects, such as stabilization, synchronization, epidemic of disease, and transition from non-chaos to chaos [1–14]. Since we are now confronting not a single complex system, but a network of complex systems connected to form a large-scale ensemble, we should consider the collective emergence properties of a network. In the collective motions, chaos, as a very interesting nonlinear phenomenon, has been intensively studied in recent years [4–14]. It has been found that there are many useful and potential applications in many fields, such as in chaotic neural networks, collapse prevention of power systems, and secure communication technology.

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As to most existing literature about chaos in complex dynamical networks, the research works have been concerning with the transition from chaos to hyper-chaos [5,6] and chaos synchronization [7–11]. In recent two years, there are also reports of the transition from non-chaos to chaos (i.e., all the nodes are non-chaotic before they are coupled together, however, these nodes will be chaotic if they are connected through a certain type of network). For example, Ref. [12] found that the required coupling strength for achieving chaos can be decreased if the topology is more heterogenous; Ref. [13] introduced the emergence of chaos in several simple types of small-scale networks; and Ref. [14] studied evolvement from collective order to collective chaos by mapping a complex network of N coupled identical oscillators to a quantum system. However, in these references, the effect of small-world property (a small average distance as well as a high degree of local clustering) on the transition from non-chaos to chaos has not been discussed. A great deal of research interest in the theory and applications of small-world networks has arisen [4,11,15–17] since the pioneering work of Watts and Strogatz [18]. In this paper, we investigate the transition from non-chaos to chaos in a small-world dynamical network. Furthermore, we study the condition of stability of the small-world dynamical network in the sense of Lyapunov (i. s. L.), which has never been investigated in complex dynamical networks.

2. The theoretical analysis of the transition from non-chaos to chaos in complex dynamical networks

2.1. Condition of the transition from non-chaos to chaos

Here, we consider an isolated node being an n-dimensional nonlinear dynamical system, which is described by

$$\dot{X}(t) = f(X(t)), \tag{1}$$

where $X(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}} \in R^n$ are the state variables of the node, and $f(\cdot)$ is a given nonlinear vector-valued function describing the dynamics of the node. According to the theory of coupled dynamical networks, we consider a general complex dynamical network consisting of such N linearly coupled identical nodes. The network is specified by

$$\dot{X}_i(t) = f(X_i(t)) - c\sum_{j=1}^{N} a_{ij}X_j(t), \quad i = 1, 2, \ldots, N, \tag{2}$$

where $X_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^{\mathrm{T}} \in R^n$ are the state variables of node i, and c is the coupling strength. Let $A = (a_{ij})_{N\times N} \in R^{N\times N}$ represents the coupling configuration of the network, where $a_{ij} = a_{ji} = 1$ if there is a connection between node i and node j ($i \neq j$); otherwise, $a_{ij} = a_{ji} = 0$ ($i \neq j$) and

$$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij} = -\sum_{j=1, j\neq i}^{N} a_{ji}, \quad i = 1, 2, \ldots, N. \tag{3}$$

We assume that the parameters of node (1) are not located in chaotic regions and a solution of the isolated node (1) is $s(t)$, which satisfies

$$\dot{s}(t) = f(s(t)), \tag{4}$$

where $s(t) = [s_1(t), s_2(t), \ldots s_n(t)]^{\mathrm{T}} \in R^n$ can be an equilibrium point or a periodic orbit. Hence, all the Lyapunov exponents h_i ($i = 1, 2, \ldots, n$) of node (1) are non-positive. We let

$$0 \geqslant h_{\max} = h_1 > h_2 \geqslant \cdots \geqslant h_n, \tag{5}$$

where $h_{\max}$ is the largest Lyapunov exponent.

In the following, the transversal Lyapunov exponents [7] are calculated for studying the dynamical behavior of network (2). Let

$$X_i(t) = s(t) + \xi_i(t), \quad i = 1, 2, \ldots, N, \tag{6}$$

linearize Eq. (2) at the solution $s(t)$ of the isolated node (1). This leads to

$$\dot{\xi}(t) = \xi(t)[Df(s(t))] - cA\xi(t), \tag{7}$$

where $\xi(t) = [\xi_1(t), \xi_2(t), \ldots, \xi_N(t)]^{\mathrm{T}} \in R^{N\times n}$ is a matrix, and $Df(s(t)) \in R^{n\times n}$ is the Jacobian matrix of $f(\cdot)$ on $s(t)$. Using the method presented in Refs. [8,9], we get

$$\dot{\omega}(t) = [Df(s(t)) - c\lambda_k I]\omega, \quad k = 1, 2, \ldots, N, \tag{8}$$

where $\omega \in R^{n\times N}$ is a matrix, $I \in R^{n\times n}$ is diag$[1,1,\ldots,1]$, and all the λ_k are the eigenvalues of the coupling matrix A. Since $A = (a_{ij})_{N\times N}$ is a real symmetric and irreducible matrix, we have [19]

$$0 = \lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_N. \tag{9}$$

According to Refs. [8,13], for any given λ_k, the corresponding transversal Lyapunov exponents $\mu_i(\lambda_k)$ in Eq. (8) are given by

$$\mu_i(\lambda_k) = h_i - c\lambda_k, \quad i = 1,2,\ldots,n. \tag{10}$$

Generally, if network (2) is chaotic, then there is at least one positive transversal Lyapunov exponent in Eq. (10), so the maximum of $\mu_i(\lambda_k)$ is $\mu_1(\lambda_N) = h_{\max} - c\lambda_N > 0$. That is, the dynamical network (2) will be chaotic if

$$c > \frac{|h_{\max}|}{|\lambda_N|}. \tag{11}$$

From condition (11), we can conclude that

(1) for any given eigenvalue λ_N of a network coupling matrix A, there exists a critical coupling strength $c^* = \frac{|h_{\max}|}{|\lambda_N|}$ so that if $c > c^*$, then the network is chaotic, even if such isolated nodes of the network are non-chaotic;
(2) for any given coupling strength c, there exists a critical eigenvalue $\lambda_N^* = \frac{h_{\max}}{c}$ so that if $\lambda_N < \lambda_N^*$, then the network is chaotic, even if such isolated nodes of the network are non-chaotic.

2.2. The transition ability from non-chaos to chaos

From above analysis, we know that the topology of a network has some effects on the state transition of the network. Using Eq. (10), we can get $N \times n$ transversal Lyapunov exponents of network (2) and order the corresponding N transversal Lyapunov exponents with $h_{\max}$ as follows:

$$\mu_1(\lambda_N) = h_{\max} - c\lambda_N \geqslant \mu_1(\lambda_{N-1}) = h_{\max} - c\lambda_{N-1} \geqslant \cdots > \mu_1(\lambda_1) = h_{\max} \leqslant 0. \tag{12}$$

Here, we suppose that network (2) is chaotic and the above N transversal Lyapunov exponents satisfy

$$\mu_1(\lambda_N) \geqslant \mu_1(\lambda_{N-1}) \geqslant \cdots \geqslant \mu_1(\lambda_{M+1}) > 0 > \mu_1(\lambda_M) \geqslant \cdots \geqslant \mu_1(\lambda_2) > \mu_1(\lambda_1) = h_{\max}, \tag{13}$$

where M $(1 \leqslant M \leqslant N-1)$ is a positive integer. Substituting (10) into (13), we have

$$c_1 = \frac{|h_{\max}|}{|\lambda_N|} < c < \frac{|h_{\max}|}{|\lambda_M|} = c_2. \tag{14}$$

We introduce a measurement [12]

$$\frac{1}{R} = \frac{c_2 - c_1}{c_1} = \frac{|\lambda_N| - |\lambda_M|}{|\lambda_M|}. \tag{15}$$

Obviously, $\frac{1}{R}$ measures the relative region size of the required coupling strength c for generating chaos satisfying (13), which can describe the transition ability from non-chaos to chaos in network (2). Since the measurement $\frac{1}{R}$ dependents on the eigenvalues of network coupling matrix A, the different network topologies have different transition abilities from non-chaos to chaos.

3. The emergence of chaos in small-world dynamical network

In this section, we investigate the effect of a small-world dynamical network on the transition from non-chaos to chaos. Hear, we adopt the NW small-world coupled network [20]. In the NW model, we add with probability p a connection between each unconnected pair of nodes in a nearest-neighbor coupled network. For $p = 0$, it reduces to the originally nearest-neighbor coupled network; and for $0 < p < 1$, it is the NW small-world coupled network.

For the nearest-neighbor coupled network with 2-neighbors, there are N eigenvalues $\lambda(k) = -4\sin^2\left(\frac{k\pi}{N}\right)$, $(k = 0,1,\ldots,N-1)$ [11]. We get $\lambda_N = -4$ shown in inequality (9), when N is an even number; otherwise, $\lambda_N = -4\sin^2\left(\frac{N-1}{2N}\pi\right)$. Obviously, we have $-4 \leqslant \lambda_N < 0$. Thus, for any given coupling strength c with $c < \frac{|h_{\max}|}{4}$, the nearest-neighbor coupled network cannot be chaotic no matter what the network size is, because condition (11) cannot hold for sufficiently large N. In the following, we study the emergence of chaos in the NW small-world dynamical network starting from the nearest-neighbor coupled network with 2-neighbors.

Figs. 1 and 2 show the numerical values of $\lambda_2(p,N)$ and $\lambda_N(p,N)$ as functions of the connection-adding probability p and the number of nodes N. In these figures, for each pair of p and N, $\lambda_2(p,N)$ and $\lambda_N(p,N)$ are obtained by averaging the results of 20 runs. Here, we concern more the eigenvalue $\lambda_N(p,N)$, because it reflects the coupling strength threshold $c^* = c_1 = \frac{|h_{\max}|}{|\lambda_N(p,N)|}$ required to generate the first positive transversal Lyapunov exponent in network (2). From Figs. 1 and 2, we can see that

(1) for any given $N \geqslant 3$, $\lambda_N(p,N)$ decreases almost linearly from about -4 to $-N$ as p increases from 0 to 1;
(2) for any given $0 < p \leqslant 1$, $\lambda_N(p,N)$ decreases almost linearly to $-\infty$ as N increases to $+\infty$.

Combining condition (11), we know that, for any given p and N, there exists a critical value c^* such that if $c > c^*$, then the small-world dynamical network will be chaotic (in the sense of statistical average).

Next, we study the transition ability from non-chaos to chaos in the small-world dynamical network. We assume $M = 2$ in inequalities (13) and (14), then equality (15) becomes

$$\frac{1}{R} = \frac{c_2 - c_1}{c_1} = \frac{|\lambda_N| - |\lambda_2|}{|\lambda_2|}, \tag{16}$$

which can be regarded as the measurement of the transition ability from non-chaos to chaos with $M = 2$. From Figs. 3 and 4, we can see clearly that the measurement $\frac{1}{R}$ decreases sharply as the increase of p and N. More interesting, as shown in the insets of Figs. 3 and 4, the measurement $\frac{1}{R}$ approximately obeys power-law forms as $\frac{1}{R} \sim p^{-r_1}$ and $\frac{1}{R} \sim N^{-r_2}$ in wide intervals of p and N, respectively, where r_1 and r_2 are two positive constants. This implies that the transition ability from non-chaos to chaos in the small-world dynamical network becomes weak as the increase of p.

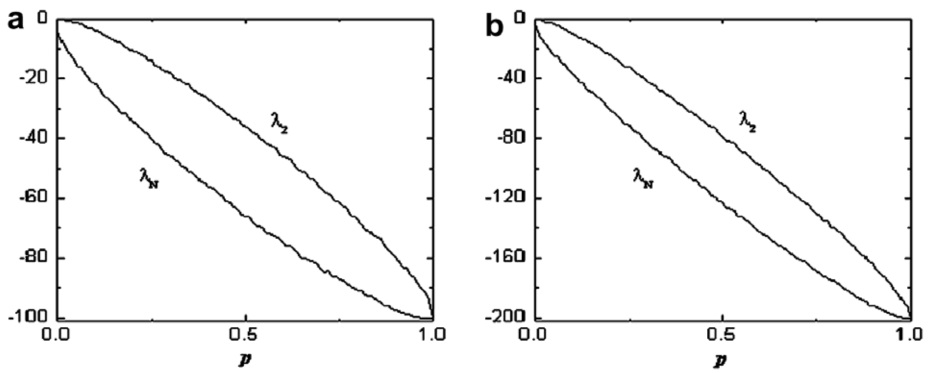

Fig. 1. Numerical values of λ_2 and λ_N versus the connection-adding probability p: (a) $N = 100$; (b) $N = 200$.

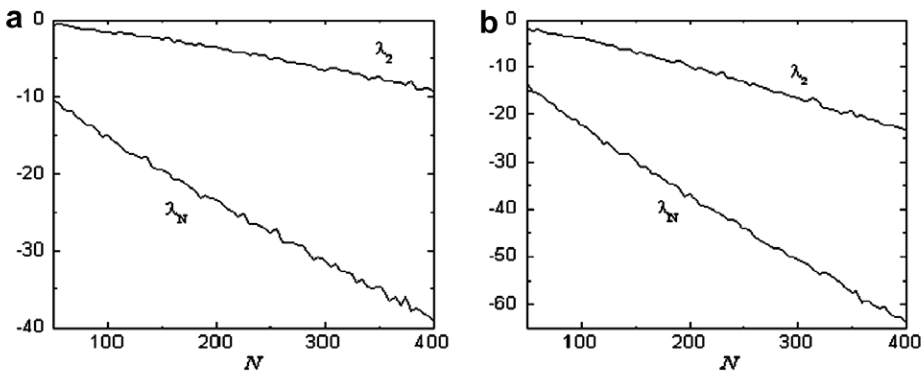

Fig. 2. Numerical values of λ_2 and λ_N versus the number of nodes N: (a) $p = 0.05$; (b) $p = 0.1$.

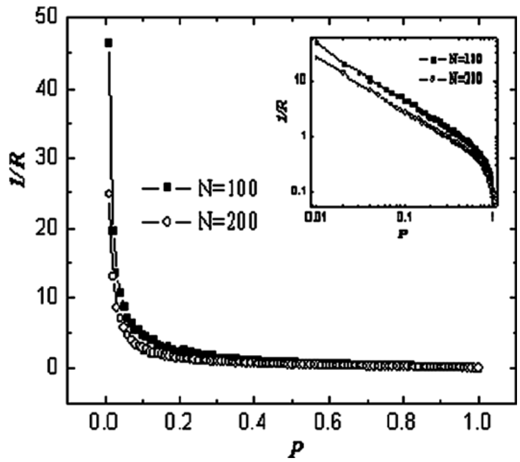

Fig. 3. Numerical value of $1/R$ versus the connection-adding probability P. The inset shows the same data in log–log plot, indicating that $1/R$ approximately obeys a power-law form.

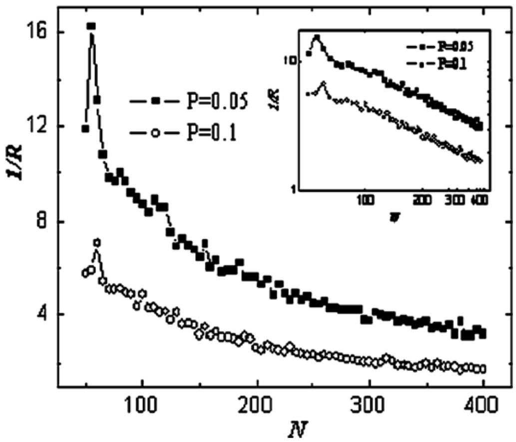

Fig. 4. Numerical value of $1/R$ versus the number of nodes N. The inset shows the same data in log–log plot, indicating that $1/R$ approximately obeys a power-law form.

4. A numerical example

As an example, we now study the transition from non-chaos to chaos in a network of small-world connected Lorenz systems. In the network, every node is a Lorenz system [21], which is described by

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = \gamma x - y - xz, \\ \dot{z} = xy - bz, \end{cases} \tag{17}$$

where the system parameters are chosen to be $\sigma = 10$, $\gamma = 0.5$, and $b = 8/3$. For these parameters, Lorenz system (17) has a stable equilibrium $(0, 0, 0)$ with the largest Lyapunov exponent $h_{\max} \approx -0.69$. According to condition (11), the small-world connected network will be chaotic if

$$c > \frac{0.69}{|\lambda_N|}. \tag{18}$$

Fig. 5(a) and (b) shows the Poincare section diagrams and chaotic attractors (see the insets of Fig. 5) of a random-chosen node in the small-world dynamical network with $p = 0.05$ and $p = 0.1$, respectively. Clearly, for $p = 0.05$ and $p = 0.1$, the small-world dynamical network can achieve chaos, for $c > 0.14$ and $c > 0.09$, respectively. From

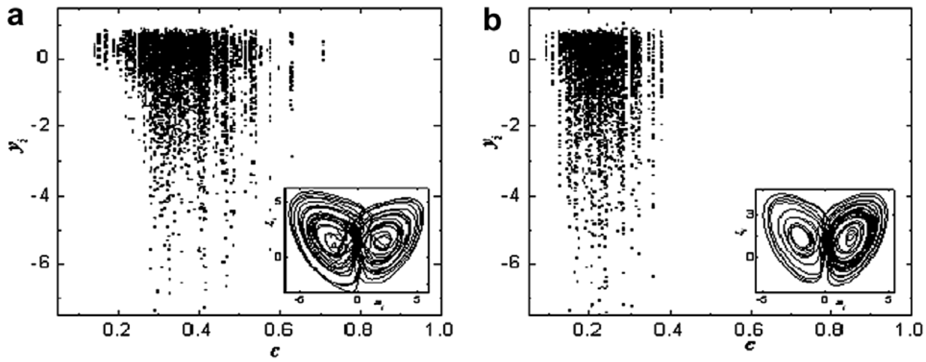

Fig. 5. The Poincare section diagrams of a random-chosen node i in the 100-node network of small-world connected Lorenz systems with (a) $p = 0.05$; and (b) $p = 0.1$. The insets give chaotic attractors of the corresponding node i (a) at $p = 0.05$, for $c = 0.3$; and (b) at $p = 0.1$, for $c = 0.15$.

comparison between Fig. 5(a) and (b), it is found that, the larger the connection-adding probability p is, the narrower the region of the required coupling strength c for achieving chaos is. This indicates that the transition ability from non-chaos to chaos in the small-world dynamical network becomes weak as the increase of p, as consisting with the result in Section 3. Furthermore, as the increase of c (see the right blank regions of Fig. 5(a) and (b)), these nodes are in unstable states i. s. L. (i.e., $\lim_{t\to\infty} y_i(t) = \infty$). These unstable states should be avoided in the applications of complex dynamical networks. In the following section, we study the condition of stability i. s. L. in the small-world dynamical network.

5. The condition of stability i. s. L. in small-world dynamical network

For the study on the theory and applications of complex dynamical networks, the stability i. s. L. in these networks plays a key role and is a precondition to construct a complex dynamical network. So, it has important significance to obtain the relationship between the network topology parameters and the coupling strength c when the complex dynamical network is stable i. s. L. In the following, from aspect of phase space volume, we study the stability i. s. L. in the foregoing small-world dynamical network based on dissipative system criteria. Phase space volume contraction rate R_{V_i} of node i in network (2) is calculated by

$$R_{V_i} = \frac{1}{\Delta V_i} \cdot \frac{\mathrm{d}(\Delta V_i)}{\mathrm{d}t} = \sum_{j=1}^{n} \frac{\partial}{\partial x_{ij}} \cdot \frac{\mathrm{d}x_{ij}}{\mathrm{d}t} = \sum_{j=1}^{n} \frac{\partial\left(f_j - c\sum_{k=1}^{N} a_{ik}x_{kj}\right)}{\partial x_{ij}} = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_{ij}} - nca_{ii} = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_{ij}} + nck_i,$$
$$i = 1, 2, \ldots, N, \tag{19}$$

where ΔV_i is cell of phase space volume of node i, and k_i is degree of node i. Based on dissipative system criteria, node i will be stable i. s. L. when

$$R_{V_i} = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_{ij}} + nck_i < 0. \tag{20}$$

For above stability condition, we need calculate the phase space volume contraction rate R_{V_i} of node i. Firstly, it is easy to calculate that the mathematical expectation of degree k_i of node i is $2 + p(N-3)$, where $N \geqslant 3$. Then, we can calculate the mathematical expectation $E(R_{V_i})$ of phase space volume contraction rate of node i, which is

$$E(R_{V_i}) = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_{ij}} + nc(2 + p(N-3)), \quad i = 1, 2, \ldots N, \tag{21}$$

where $2 + p(N-3)$ is also the average connectivity $\langle k \rangle$ of the small-world network. Substituting (21) into (20), we get the condition of stability i. s. L. in the small-world dynamical network, which is described by

$$\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_{ij}} + nc(2 + p(N-3)) < 0, \quad i = 1, 2, \ldots N. \tag{22}$$

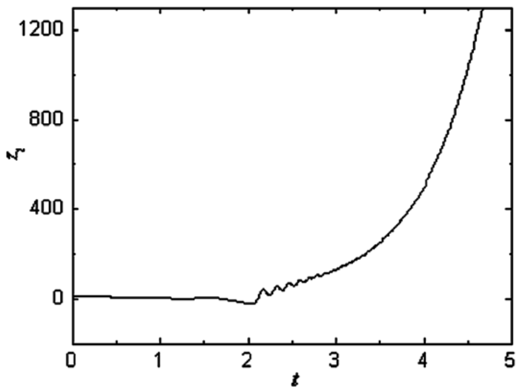

Fig. 6. Unstable phase of a random-chosen node i at the coupling strength, $c = 0.7$, in the 100-node network of small-world connected Lorenz systems with $p = 0.05$.

The stability condition implies that, there exists a coupling strength threshold c_e such that if $c > c_e$, then the network will be unstable i. s. L. (i.e., $\lim_{t\to\infty} y_i(t) = \infty$). According to inequality (22), We get

$$c_e = \frac{-\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_{ij}}}{n(2 + p(N-3))}. \tag{23}$$

For the small-world connected Lorenz systems in Section 4, by substituting (17) into (23), we have the coupling strength threshold

$$c_e = \frac{41}{9(2 + p(N-3))}. \tag{24}$$

For $N = 100$ and $p = 0.05$, we can calculate $c_e \approx 0.66$. Thus, the coupled Lorenz systems will be unstable i. s. L. when $c > 0.66$, shown in Fig. 6 .

6. Conclusion

In conclusion, by using both theoretical analysis and numerical example, we have studied the transition from non-chaos to chaos in the NW small-world dynamical network. It has been found that, for any given coupling strength and a sufficiently large number of nodes, the small-world dynamical network can be chaotic, even if the nearest-neighbor coupled network cannot be chaotic under the same condition. In other words, the ability of achieving chaos in an originally nearest-neighbor coupled network can be greatly enhanced by simply adding a small fraction of new connection, which reveals an advantage of small-world network for achieving chaos. In addition, the transition ability from non-chaos to chaos in the NW small-world dynamical network becomes weak as the increase of p. Furthermore, we have obtained the condition of stability i. s. L. in the small-world dynamical network, which is determined by the average connectivity. The stability condition will have guidance significance for the constructing of complex dynamical networks.

Acknowledgments

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