# 4 LECTURES ON JACOBI FORMS

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ABSTRACT.

## 1. Plan

This lecture series is intended for graduate students or motivated undergraduate students. We introduce a concept of Jacobi forms and try to explain a various connection with other fields, such as quasi-modular forms and Mock modular forms if time allows. More specifically, I am planning to cover:

1 (Lecture 1 ) After introducing a concept of Jacobi forms we look at two expansions of Jacobi forms ; its Taylor series expansions and theta series expansions. The first lecture will focus on theta-expansion and a map between the space of Jacobi forms and the vector valued modular form of half integral weight.

2 (Lecture 2) We discuss the correspondence between the space of Jacobi form of integral weight and index 1 and that of modular form of half integral weight. We discuss the correspondence between Jacobi Eisenstein series and Cohen Eisenstein series.

3 (Lecture 3)

The Taylor series expansion gives a map between the space Jacobi forms of integral weight and the space of modular forms of integral weight while the theta series expansion gives a map between the space

Keynote: Jacobi forms, Theta expansions, Taylor expansions, Quasimodular forms, Mock modular forms, Mock Jacobi forms .

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of Jacobi forms and that of modular forms of half integral weight. In particular we will derive the explicit isomorphism in the case of the space of Jacobi forms of weight $k, k$ integral, $J_{k,1}(\Gamma(1))$. Furthermore, as an application, we discuss Cohen-Rankin operator of ellipic modular form via Jacob form. We give examples.

4 (Lecture 4)

Quasi-modular forms generalize classical modular forms and were introduced by Kaneko and Zagier in [**?**], where they identified quasi-modular forms with generating functions counting maps of curves of genus $g > 1$ to curves of genus 1. More generally hey often appear as generating functions of counting functions in various problems such as Hurwitz enumeration problems, which include not only number theoretic problems but also those in certain areas of applied mathematics Unlike modular forms, derivatives of quasimodular forms are also quasi-modular forms. The goal of this talk is to study various properties of quasimodular forms by using their connections with Jacobi-like forms and modular forms. We will briefly mention about the connection with pseudodifferential operators. For more details see [7].

I will briefully introduce a mock modular form and mock Jacobi forms as a holomorphic part of harmonic weak Maass forms. There are vast of recent research results, due to K.Brigmann, K. Ono, J Bruinier, S. Zwegers, etc.

## 2. Lecture 1

2.1. **Definitions and Notations.** Let us set up the following notations. Let $\mathcal{H}$ be the usual complex upper half plane and $\tau \in \mathcal{H}, z \in \mathbb{C}$. Let $\Gamma$ be any discrete subgroup of $SL(2, \mathbb{R})$.

In this lecture we take

$$\Gamma = \Gamma(1) = SL(2, \mathbb{Z}) =< \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) >$$

First $\Gamma$ acts on $\mathcal{H}$ as a linear fractional transformation:

$$\Gamma \times \mathcal{H} \longrightarrow \mathcal{H}$$

$$(\tau, \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) \longrightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d}$$

Fix $k, m \in \mathbb{Z}$. Consider a holomorphic function $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ such that

(1) (modular) For all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

$$(c\tau + d)^{-k} e^{-2\pi i m \frac{cz^2}{c\tau+d}} \phi(\gamma\tau, \frac{z}{c\tau + d}) = \phi(\tau, z)$$

(2) (elliptic) For all $(\lambda, \mu) \in \mathbb{Z}^2$,

$$e^{2\pi i m(\lambda^2\tau + 2\lambda z)} \phi(\tau, z + \lambda\tau + \mu) = \phi(\tau, z)$$

(3) Since $\phi(\tau, z+1) = \phi(\tau, z)$ and $\phi(\tau, z+1) = \phi(\tau, z)$ implies that *phi* has a Fourier expansion $\phi(\tau, z) = \sum_{n,r \in \mathbb{Z}} c(n, r) q^n \xi^r, q = e^{2\pi i\tau}, \psi = e^{2\pi i z}$.

- The function $\phi(\tau, z)$ is called a holomorphic Jacobi form (or simply a Jacobi form) of weight $k$ and index $m$ if the coefficients $c(n, r) = 0$ unless $4mn - r^2 \geq 0$.
- It is called a Jacobi cusp form if it satisfies the stronger condition that $c(n, r) = 0$ unless $4mn - r^2 > 0$.
- Moreover, it is called a weak Jacobi form if it satisfies the weaker condition $c(n, r) = 0$ unless $n \geq 0$.

They form vector spaces and denote those by

$$J_{k,m}^{cusp}(\Gamma) \subset J_{k,m}(\Gamma) \subset J_{k,m}^{weak}(\Gamma)$$

We can summarize the above discussion as the following way: The Jacobi group $\Gamma^J$ is defined as follows:

**Definition 2.1.** Let

$$\Gamma^J := \Gamma \propto \mathbb{Z}^2 = \{[M, (\lambda, \mu)] | M \in \Gamma, \lambda, \mu \in \mathbb{Z}\}.$$

This set $\Gamma^J$ forms a group under a group law

$$[M_1, (\lambda_1, \mu_1)][M_2, (\lambda_2, \mu_2)] = [M_1M_2, (\lambda, \mu)) + (\lambda_2, \mu_2)],$$

where $\binom{\lambda'}{\mu'} = M_2^t \binom{\lambda_1}{\mu_1}$ and is called the **Jacobi group**. Note that the Jacobi group $\Gamma^J$ acts on $\mathcal{H} \times \mathbb{C}^j$ as, for each $\gamma = [\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda, \mu)] \in \Gamma^J, \lambda, \mu \in \mathbb{Z}^j$,

$$\gamma(\tau, z) = (\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}).$$

Furthermore, for $\gamma = [\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda, \mu)] \in \Gamma^J, k \in \frac{1}{2}\mathbb{Z}$ and $m \in \frac{1}{2}\mathbb{Z}$, let

$$j_{k,m}(\gamma, (\tau, z)) := (c\tau + d)^{-k} e^{2\pi i(-\frac{cz^2}{c\tau+d} + \lambda^2\tau\lambda + 2\lambda z + \mu)}.$$

Let us define the usual slash operator on a function $f : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ :

$$(f|_{k,m}\gamma)(\tau, z) := j_{k,m}(\gamma, (\tau, z))f(\gamma(\tau, z)), \ \gamma \in \Gamma^J,$$

Then one checks the following consistency condition(see also [**?**], Section I.1 ):

$$(f|_{k,m}\gamma|_{k,m}\gamma')(\tau, z) = (f|_{k,m}\gamma\gamma')(\tau, z), \ \gamma, \gamma' \in \Gamma^J.$$

A Jacobi form of weight $k$ and index $m$ is a holomorphic function $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ such that

$$(\phi|_{k,m}\gamma)(\tau, z) = \phi(\tau, z), \forall \gamma \in \Gamma^J$$

satisfying the proper condition of the Fourier expansion.

2.2. **Examples.**

**Example 2.2.** For $\tau \in \mathcal{H}$, The Weierstrass $\wp$-function

$$\wp(\tau, z) := \frac{1}{z^2} + \sum_{w \in \mathbb{Z} + \tau\mathbb{Z}, w \neq 0} \frac{1}{(z - w)^2} - \frac{1}{w^2}$$

$$= \frac{1}{z^2} + \sum_{m,n \in \mathbb{Z}, (m,n) \neq (0,0)} \frac{1}{(z - (m + n\tau))^2} - \frac{1}{(m + n\tau)^2}$$

Then

- $\wp(\tau + 1, z) = \wp(\tau, z)$
- $\wp(-\frac{1}{\tau}, \frac{z}{\tau}) = \tau^2(\frac{1}{z^2} + \frac{1}{(z-(m+n\tau))^2} - \frac{1}{(m+n\tau)^2} = \tau^2\wp(\tau, z)$
- $\wp(\tau, z + \lambda\tau + \mu) = \wp(\tau, z), \forall (\lambda, \mu) \in \mathbb{Z}^2$

So $\wp(\tau, z)$ is a (meromorphic) Jacobi form of weight $k = 2$ and index $m = 0$.

**Example 2.3.** Let $Q : \mathbb{Z}^N \longrightarrow \mathbb{Z}$ be a positive definite integer valued quadratic form and $B$ be the associated bilinear form. Then for any vector $y_0 \in \mathbb{Z}^N$, the theta series

$$\theta_{y_0}(\tau, z) := \sum_{x\in\mathbb{Z}^N} e^{2\pi i(Q(x)\tau+B(x,y_0)z)}$$

is a Jacobi form (on a congruence subgroup of $SL(2,\mathbb{Z})$ of weight $k = \frac{N}{2}$ and index $Q(y_0)$
(Hint: to check "modular" one uses Possion summation formula. To check "elliptic" it is immediate using relation $B(x,y) = Q(x+y) - Q(x) - Q(y)$.)

2.3. **Theta expansion.** This can be derived from elliptic property: take $\phi(\tau, z) = \sum_{n,r} c(n,r)q^n\xi^r \in J^{weak}_{k,m}(\Gamma)$

Then, for any $(\lambda, \mu) \in \mathbb{Z}^2$,

$$\begin{aligned}\phi(\tau, z) = \sum_{n,r} c(n,r)q^n\xi^r &= e^{2\pi im(\lambda^2+2\lambda)}\phi(\tau, z+\lambda\tau+\mu)\\ &= q^{\lambda^2m}\xi^{2\lambda m}\sum_{n,r} c(n,r)q^n\xi^r\\ &= \sum_{n,r} c(n,r)q^{n+\lambda^2m+\lambda r}\xi^{r+2\lambda m}\end{aligned}$$

So,

$$c(n,r) = c(n+\lambda^2m+\lambda r, r+2\lambda m),$$

i.e. $c(n,r) = c(n',r')$ whenever $r \equiv r' \pmod{2m}$ and $4n'm - r'^2 = 4mn - r^2$

That is to say that $c(n,r)$ depends on the discriminant $4mn - r^2$ and on the value of $r \pmod{2m}$.

So we let

$$c(n,r) = c_r(4mn - r^2),$$

Then

$$c_r(N) = c_{r'}(N) \text{for } r \equiv r' \pmod{2m}$$

This gives us coefficients $c_\mu(N)$ for all $\mu \in \mathbb{Z}/2m\mathbb{Z}$ and all integers $N$ satisfying $N \equiv \mu^2 \pmod{4m}$, namely,

$$c_\mu(N) := c(\frac{N+r^2}{4m}, r) \text{ for } r \in \mathbb{Z}, r \equiv \mu \pmod{2m}.$$

We further extend definition of $c_\mu(N)$ by

$$c_\mu(N) = \left\{ \begin{array}{c} c(\frac{N+r^2}{4m}, r), \ \text{ for } N \equiv -\nu^2 \pmod{4m} \\ 0, \ \text{ otherwise} \end{array} \right\}$$

So

$$\phi(\tau, z) = \sum_{n,r} c(n,r) q^n \xi^r = \sum_{\mu \pmod{2m}} \ \sum_{r \in \mathbb{Z}, r \equiv \mu \pmod{2m}} \ \sum_{n>0} c_\mu(4mn - r^2) q^n \xi^r$$

$$= \sum_{\mu \pmod{2m}} \ \sum_{r \in \mathbb{Z}, r \equiv \mu \pmod{2m}} \ \sum_{N} c_\mu(N) q^{\frac{N+r^2}{4m}} \xi^2$$

$$= \sum_{\mu \pmod{2m}} \ \sum_{N} c_\mu(N) q^{\frac{N}{4m}} \sum_{r \equiv \mu \pmod{2m}} q^{\frac{r^2}{4m}} \xi^r$$

Defining, for each $\mu \pmod{2m}$,

$$h_\mu(\tau) := \sum_{N} c_\mu(N) q^{\frac{N}{4m}}$$

$$\theta_{m,\mu}(\tau, z) := \sum_{r \equiv \mu \pmod{2m}, r \in \mathbb{Z}} q^{\frac{r^2}{4m}} \xi^r$$

we can write

**Proposition 2.4.** *(theta expansion)*

$$\phi(\tau, z) = \sum_{\mu \pmod{2m}} h_\mu(\tau) \theta_{m,\mu}(\tau, z)$$

**Remark 2.5.** Recall

$$h_\mu(\tau) := \sum_{N} c_\mu(N) q^{\frac{N}{4m}}$$

where

$$c_\mu(N) = \left\{ \begin{array}{c} c(\frac{N+r^2}{4m}, r), \ \text{ for } N \equiv -\mu^2 \pmod{4m} \\ 0, \ \text{ otherwise} \end{array} \right\}$$

So $N$ can be $N = 4mn - r^2 \geq 0$ or $N = 4mn - r^2 > 0$ or $N + r^2 > 0$ according to where $\phi(\tau, z)$ belongs to $J^{cusp}_{k,m}(\Gamma)$ or $J_{k,m}(\Gamma)$ or $J^{weak}_{k,m}(\Gamma)$.

On the other hand,

$$\theta_{m,\mu}(\tau+1,z) = e^{2\pi i\frac{\mu^2}{4m}}\theta_{m,\mu}(\tau,z)$$

and, using Poisson summation formula,

$$\theta_{m,\mu}(-\frac{1}{\tau},\frac{z}{\tau}) = (\frac{\tau}{2mi})^{\frac{1}{2}}e^{2\pi i m\frac{z^2}{\tau}}\sum_{\nu \pmod{2m}}\theta_{m,\nu}(\tau,z)e^{\frac{2\pi i(-\nu\mu)}{2m}}$$

So the transformation law of $h_\mu(\tau)$ with those of $\phi(\tau,z)$ and $\theta_{m,\mu}(\tau,z)$

$$h_\mu(\tau+1) = e^{-2\pi i\frac{\mu^2}{4m}}h_\mu(\tau) \tag{2.1}$$

$$h_\mu(-\frac{1}{\tau}) = \frac{\tau^k}{\sqrt{\frac{2m\tau}{i}}}\sum_{\nu \pmod{2m}} e^{\frac{2\pi i(\nu\mu)}{2m}}h_\nu(\tau) \tag{2.2}$$

**Theorem 2.6.**

$J_{k,m}(\Gamma) \simeq \{(h_0(\tau),h_1(\tau),..,h_{2m-1}(\tau)) |\, h_\mu : \mathcal{H}\to\mathbb{C}$ *satisfying* (2.1) *and* (2.2) *and proper growth*$\}$

Next we wish to identify the space of Jacobi forms with that of scalar valued modular forms, that is , usual elliptic modular forms. In the case when $m = 1$ this can be done easily and for general $m$ it is not easy, however when $m$ is prime this was done in Eichler-Zagier[8]. Note that the similar result exists in the case of harmonic weak Maass forms[6], but proofs are totally different since the space of Harmonic weak Maass forms is not finite dimensional. Hope we can discuss this in the last lecture.

## 3. **Lecture 2**

3.1. **In the case of $J_{k,1}(\Gamma)$.** Let $\phi(\tau,z)\in J_{k,1}(\Gamma)$, with $\Gamma=\Gamma(1)$.
Then

$$\begin{aligned}\phi(\tau,z) &= \sum_{\mu \pmod 2} h_\mu(\tau)\theta_{1,\mu}(\tau,z)\\ &= h_0(\tau)\theta_{1,0}(\tau,z)+h_1(\tau)\theta_{1,1}(\tau,z)\end{aligned}$$

Since

$$h_\mu(\tau) = \sum_\mu c_\mu(N) q^{\frac{N}{4}}$$

$$c_\mu(N) = \begin{Bmatrix} c(\frac{N+r^2}{4}, r) \ \text{ if } r \equiv \mu \pmod 2 \\ 0, \ \text{otherwise} \end{Bmatrix}$$

define

$$h(\tau) = \sum_{N \equiv 0 \pmod 4} c_0(N) q^N + \sum_{N \equiv -1 \pmod 4} c_1(N) q^N$$

So $h(\tau) = h_0(4\tau) + h_1(4\tau)$

Using the transformation formula given in (2.1) and (2.2) one checks

(1) $h(\tau+1) = h(\tau)$

(2) $h(\frac{\tau}{4\tau+1}) = (4\tau+1)^{k-\frac{1}{2}} h(\tau)$

Since $\Gamma_0(4) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma | \, c \equiv 0 \pmod 4\} =< \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 4 & 1 \end{smallmatrix}\right) >$

$$h(\tau) \in M_{k-\frac{1}{2}}(\Gamma_0(4))$$

Conversely, if $h \in M^+_{k-\frac{1}{2}}(\Gamma_0(4))$ (here $M^+_{k-\frac{1}{2}}(\Gamma_0(4))$ is Kohnen "plus" space) then the reversing the same calculation shows that $h_0$ an $h_1$, where $h(\tau) = h_0(4\tau) + h_1(4\tau)$, satisfies the proper transformation formula given in (2.1) and (2.2) to see

$$\phi(\tau, z) = h_0(\tau)\theta_{1,0}(\tau, z) + h_1(\tau)\theta_{1,1}(\tau, z) \in J_{k,1}(\Gamma).$$

This gives an idea of the following result:

**Theorem 3.1.** *Let $k$ be an even integer. Then*

$$M^+_{k-\frac{1}{2}}(\Gamma_0(4)) \simeq J_{k,1}(\Gamma(1))$$

*with isomorphism given by*

$$\sum_{N \equiv 0,3 \pmod 4, N \geq 0} a(N) q^N \longrightarrow \sum_{n,r \in \mathbb{Z}, 4n - r^2 \geq 0} a(4n - r^2) q^n \xi^r$$

*This isomorphism is compatible with the Petersson schlar products, with the action of Hecke operators, and with the structures of $M_{2*-\frac{1}{2}}$ and $J_{*,1}$ an modules over $M_*$ Here, $M_* = \bigoplus_{k \in \mathbb{Z}} M_k$ and $J_{*,1} = \bigoplus_{k \in \mathbb{Z}} J_{k,1}$ denote the graded ring of modular forms and the graded ring of Jacobi forms of index 1, respectively.*

**Proof** See page 64 in [8] □

**Example 3.2.** (see [8] Fix an integer $k > 3$. Consider Jacobi Eisenstein series $E_{k,1}(\tau, z) \in J_{k,1}(\Gamma(1))$;

$$E_{k,1}(\tau, z) = \frac{1}{2} \sum_{c,d\in\mathbb{Z},\gcd(c,d)=1} \sum_{\lambda\in\mathbb{Z}} (c\tau + d)^{-k} e^{2\pi i m(\lambda^2 \frac{a\tau+b}{c\tau+d} + 2\lambda \frac{z}{c\tau+d} - \frac{cz^2}{c\tau+d})}$$

It is known that this series converges absolutely and uniformly on every compact subset if $k > 3$ and zero if $k$ odd.

Consider its Fourier expansion, for $k > 2$,

$$E_{2k,1}(\tau, z) = \sum_{n,r\in\mathbb{Z},4n-r^2\geq 0} e_{2k,1}(n, r) q^n, \xi^r$$

Then it turns out that

$$e_{2k,1}(n, r) = \frac{H_{2k-1}(4n - r^2)}{\zeta(3 - 4k)}$$

with $H_{2k-1}(N) = L_{-N}(2 - 2k)$ (called Cohen's function).

Here,

$$L_D(s) := \left\{ \begin{array}{cc} 0 & \text{if } D \neq 0, 1 \pmod 4 \\ \zeta(2s - 1) & \text{if } D = 0 \\ L_{D_0}(s) \sum_{d|f} \mu(d)(\frac{D_0}{d}) d^{-s} \sigma_{1-2s}(\frac{f}{d}) & \text{if } D \equiv 0, 1 \pmod 4, D \neq 0 \end{array} \right\}$$

Where $D = D_0 f^2, f \in \mathbb{N}, D_0$ is the discriminant of $Q(\sqrt{D})$, that is, $D_0$ is $D \equiv 1$ or $0 \pmod 4$ otherwise $4D_0$, and

$\mu : \mathbb{N} \to \{0, -1, 1\}$ is the Möbius function defined by

$$\mu(n) = \left\{ \begin{array}{cc} 0 & \text{if } n \text{ is not square-free} \\ -1 & \text{if } n \text{ is square free and has a odd number of distinct prime factors} \\ 1 & \text{if } n \text{ is square free and has an even number of distinct prime factors} \end{array} \right\}$$

Further $(\frac{*}{n})$ is the Kronecker symbol defined by, with unique prime factorization of $n = 2^{e_0} \prod_i^{\ell} p_i^{e_i}$, distinct $p_i$ odd prime, $e_i \in \mathbb{Z}_{\geq 0}$,

$$(\frac{a}{n}) = (\frac{a}{2})^{e_0} \prod_{i=1}^{\ell} (\frac{a}{p_1})^{e_i}$$

where $(\frac{a}{p})$ is the Legendre symbol.

Theorem says that

$$M^+_{2k-\frac{1}{2}}(\Gamma_0(4)) \simeq J_{2k}(\Gamma(1))$$

So the correspondence for $E_{2k,1}(\tau,z)$ will be

$$\sum_{N\geq 0, N\equiv 0,3 \pmod 4} \frac{1}{\zeta(3-4k)} H_{2k-1}(N)q^N \longrightarrow \sum_{4n-r^2\geq 0} H_{2k-1}(4n-r^2)q^n\xi^r$$

So we conclude that

$$\sum_{N\geq 0, N\equiv 0,3 \pmod 4} \frac{1}{\zeta(3-4k)} H_{2k-1}(N)q^n \in M^+_{2k-\frac{1}{2}}(\Gamma_0(4))$$

## 4. **Lecture 3**

4.1. **Taylor expansion in $z$.** For simplicity let $k \equiv 0 \pmod 2$ and $m$ be any positive integer,

Take $\phi(\tau,z) = \sum_{n,r\in\mathbb{Z}, 4nm-r^2\geq 0} c(n,r)q^n\xi^r \in J_{k,m}(\Gamma)$. Since it is holomorphic on $\mathcal{H}\times\mathbb{C}$ we have

$$\phi(\tau,z) = \sum_{\nu\geq 0} \chi_\nu(\tau)z^\nu$$

Since $\phi(\tau,-z) = (-1)^k\phi(\tau,z)$ and $k$ even we may assume that

$$\phi(\tau,z) = \sum_{\nu\geq 0} \chi_{2\nu}(\tau)z^{2\nu}$$

From the modular transformation law of $\phi(\tau,z)$, for any$\gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma$,

$$\phi(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}) = (c\tau+d)^k e^{2\pi i \frac{cz^2}{c\tau+d}}\phi(\tau,z)$$

we get

$$\chi_0(\gamma\tau) = (c\tau+d)^k\chi_0(\tau)$$

$$\chi_2(\gamma\tau) = (c\tau+d)^{k+2}(\chi_2(\tau) + 2\pi i m\frac{c}{c\tau+d}\chi_0(\tau))$$

$$\cdots$$

(4.1)
$$\chi_\nu(\gamma\tau) = (c\tau+d)^{k+\nu}[\chi_\nu(\tau) + (2\pi m\frac{c}{c\tau+d})\chi_{\nu-2} + \frac{1}{2!}(2\pi m\frac{c}{c\tau+d})^2\chi_{\nu-4} + ..+]$$

Note that
$$g_0(\tau) := \chi_0(\tau) \in M_k(\Gamma)$$

$$g_2(\tau) := \chi_2(\tau) - \frac{2\pi im}{k}\chi_0'(\tau) \in M_{k+2}(\Gamma)$$

(why? First note that
$$\chi_0^{'}(\gamma\tau) = (c\tau+d)^{-k-2} = (c\tau+d)^{k+d}(\chi_0^{'}(\tau) + \frac{kc}{c\tau+d}\chi_0(\tau)$$

$$g_2(\gamma\tau) = \chi_2(\gamma\tau) - \frac{2\pi im}{k}\chi_0^{'}(\gamma\tau)$$
$$= (c\tau+d)^{k+2}(\chi_2(\tau) + 2\pi im\frac{c}{c\tau+d}\chi_0(\tau) - \frac{2\pi im}{k}\frac{kc}{c\tau+d}\chi_0(\tau) - \frac{2\pi im}{k}\chi_0^{'}(\tau))$$
$$= (c\tau+d)^{k+2}(\chi_2(\tau) - \frac{2\pi im}{k}\chi_0^{'}(\tau)) = g_2(\tau))$$

In general,
(4.2)
$$g_{2\nu}(\tau) := (2\pi i)^{2\nu}\sum_{n\geq 0}(\sum_{r\in\mathbb{Z}}(\sum_{0\leq\mu\leq\nu}\frac{(k+2\nu-\mu-2)!}{(k+\nu-2)!}\frac{(-mn)^\mu r^{\nu-2\mu}}{\mu!(\nu-2\mu)!})c(n,r))q^n$$

In terms of Fourier coefficients of $\phi$

$$(2\pi i)^{-dirnu}\frac{(k+2\nu-2)!}{(k+\nu-2)!}g_{2\nu}(\tau) = \sum_{n\geq 0}(\sum_{r\in\mathbb{Z}}P_{\nu,k}(mn,r^2)c(n,r))q^n$$

where $P_{2\nu,k}$ is a homogeneous polynomial of degree $\nu$ in $r^2$ an $n$ with coefficients depending on $k$ and $m$, the first few being

$$P_{0,k} = 1$$
$$P_{2,k} = kr^2 - 2mn$$
$$P_{4,k} = (k+1)(K+2)r^4 - 12(k+1)r^2mn + 12m^2n^2$$

By inverting the formula in (4.2), we have

$$\chi_{2\nu}(\tau) = \sum_{0\leq\mu\leq\nu} \frac{(2\pi i m)^{\nu}(k+2\nu-2\mu-1)!}{(k+2\nu-\mu-1)!\mu!} g^{(\mu)}_{2(\nu-\mu)}(\tau)$$

, that is

$$\phi(\tau, z) = \sum_{\nu\geq 0} \chi_{2\nu}(\tau) z^{2\nu}$$

$$= g_0(\tau)+(\frac{g_2(\tau)}{2}+\frac{mg_0^{'}(\tau)}{k})(2\pi iz)^2+(\frac{g_4(\tau)}{24}+\frac{mg_2^{'}(\tau)}{2(k+2)}+\frac{m^2g_0^{''}(\tau)}{2k(k+1)})(2\pi iz)^4+...$$

In fact, in the case of the weak Jacobi forms, first $m+1$ coefficients $g_0(\tau), .g_{2\nu}(\tau), .., g_{2m}(\tau)$ in $z$, where $g_{2\nu}(\tau) \in M_{k+2\nu}(\Gamma), \nu = 0, 1, .., 2m$:

So

**Theorem 4.1.** [8] *For all* $k \equiv 0 \pmod 2$ *and* $m$*,*

$$J^{weak}_{k,m}(\Gamma) \simeq M_k(\Gamma) \bigoplus M_{k+2}(\Gamma) \bigoplus \cdot\cdot \bigoplus M_{k+2m}(\Gamma)$$

*The isomorphism is given by*

$$\phi(\tau, z) \to (g_0, g_2, .., g_{2m})$$

**Remark 4.2.** The similar result holds for odd $k$

One can do it better when the index $m = 1$ using the explicit generators $E_{4,1}(\tau, z)$ and $E_{6,1}(\tau, z)$;

**Theorem 4.3.** *Let* $k$ *even. Then*

$$J_{k,1}(\Gamma) \backsimeq M_k(\Gamma) \bigoplus S_{k+2}(\Gamma)$$

*The isomorphism is given by*

$$\phi(\tau, z) \longrightarrow (g_0(\tau), g_2(\tau))$$

*Here,* $S_{k+2}(\Gamma)$ *is the space of cusp forms of weight* $k + 2$ *on* $\Gamma$.

4.2. **Application.** Now, we take $m = 1$, a nontrivial modular form $g_0 = g \in M_k(\Gamma)$ and $g_2 = 0$

Then $(g, 0, 0, 0, 0)$ gives

$$\chi_{2\nu}(\tau) = \frac{(2\pi i)^\nu (k-1)!}{(k+\nu-1)!\nu!} g^{(\nu)}(\tau)$$

so that

$$G(\tau, z) = \sum_{\nu \geq 0} \left(\frac{(2\pi i)^\nu (k-1)!}{(k+\nu-1)!\nu!} g^{(\nu)}(\tau)\right) z^{2\nu}$$

and this satisfies, for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

$$G(\gamma\tau, \frac{z}{c\tau + d}) = (c\tau + d)^k e^{2\pi i \frac{cz^2}{c\tau+d}} G(\tau, z) \tag{4.3}$$

Similarly, take $f \in M_\ell(\Gamma)$ and consider

$$F(\tau, z) = \sum_{\nu \geq 0} \left(\frac{(2\pi i)^\nu (\ell-1)!}{(\ell+\nu-1)!\nu!} f^{(\nu)}(\tau)\right) z^{2\nu}$$

so that, for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

$$F(\gamma\tau, \frac{z}{c\tau + d}) = (c\tau + d)^\ell e^{2\pi i \frac{cz^2}{c\tau+d}} F(\tau, z) \tag{4.4}$$

Define

$$H(\tau, z) := G(\tau, z) \cdot F(\tau, iz)$$

Then from (4.3) and (4.4) we check that, for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$,

$$H(\gamma\tau, \frac{z}{c\tau + d}) = (c\tau + d)^{k+\ell} H(\tau, z) \tag{4.5}$$

Again by comparing the coefficient of $z$ in $H(\tau, z) = \sum_{\nu \geq 0} h_\nu(\tau) z^{2\nu}$ using the transformation formula in (4.5)

$$h_\nu(\gamma\tau) = (c\tau + d)^{k+\ell+2\nu} = h_\nu(\tau), \forall \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

So,

$$h_\nu(\tau) = (2\pi i)^{-\nu} \sum_{r+s=\nu} (-1)^r \left( \begin{array}{c} k+\nu-1 \\ r \end{array} \right) \left( \begin{array}{c} \ell+\nu-1 \\ s \end{array} \right) f^{(r)}(\tau) g^{(s)}(\tau)$$

is in $M_{k+\ell+2\nu}(\Gamma)$
Usually,

$$[g,f]_\nu^{(k,\ell)}(\tau) := \sum_{r+s=\nu} (-1)^r \left( \begin{array}{c} k+\nu-1 \\ r \end{array} \right) \left( \begin{array}{c} \ell+\nu-1 \\ s \end{array} \right) f^{(r)}(\tau) g^{(s)}(\tau)$$

is called the $\nu$th **Rankin-Cohen's bracket**.

**Example 4.4.** Let $E_k(\tau) \in M_k(\Gamma)$ be an Eisenstein series.

$$E_4(\tau) = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n = 1 + 240q + ...$$

$$E_6(\tau) 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n = 1 - 504q + ...$$

and

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = q - 24q^2 + 252q^3 + ... \in S_{12}(\Gamma)$$

Then

$$[E_4, E_6]_1 = -3456\Delta$$

$$[E_4, E_4]_2 = 4800\Delta$$

## 5. Lecture 4

See the next separated slide

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