Group Theory

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1 Cartan Subalgebra and the Roots

1.1 Cartan Subalgebra

Let G be the Lie algebra, if $h \subset G$ it is called a subalgebra of G. Now we seek a basis in which $[x, T^a] = \zeta^a T^a$ (T^a being the generators of the group in the adjoint representation) . This means that the characteristic polynomial $det(\xi - 1 \cdot \zeta) = 0$ (ξ represents here the matrix which we wish to diagonalize) has a solution. This will be always the case over a complex field (central theorem of Algebra). The maximal set of such generators, which are linearly independent and commute mutually we call the Cartan generators $\{H_i\}$. The linear hull (means same as span, just all possible sums of the Cartan generators multiplied with numbers) is called the Cartan subalgebra$G_0 = span\{H_i,\, i = 1, ..., r\}$.
Note: The rank r is the dimensionality of the Cartan subalgebra and is a basis independent quantity.

1.2 Root system

The group G is spanned by the elements y that are simultaneous eigenvectors for any h being a Cartan generator.

$$[h,\, y] := ad_h(y) = \alpha_y(h)\, y$$

We see that $\alpha_y : G_0 \to \mathbb{C}$ is a map from the Cartan subalgebra to the complex numbers i.e. $\alpha_y(h) \in \mathbb{C}$. Such maps are elements of a so called dual space G_0^*. We call the alphas from now on roots. Now one can decompose the algebra

$$G = G_0 \bigoplus_{\alpha \neq 0} G_\alpha$$

where

$$G_\alpha = \{x \in G \,|\, [h, x] = \alpha(h)\, x \,\forall h \in G_0\}.$$

One associates with an element $y \in G$ a subalgebra G_α, so called root space decomposition of G. The G_α's are called the root subspaces and $\Phi = \Phi(G)$ the root system. The root system has following properties:

1. The root system spans G_0^* ; $\text{span}_{\mathbb{C}}(\Phi) = G_0^*$
2. For any $\alpha \in \Phi$ there is $x_{-\alpha}$ s.t. $K(x_\alpha, x_{-\alpha}) \neq 0$
3. The only multiples of $\alpha \in \Phi$ which are roots, are $\pm\alpha$
4. The root spaces G_α are one dimensional

Note: $K(\cdot,\cdot)$ is the Killing form (in the adjoint representation it's the trace over the commutator of two elements). There is also a Theorem which says that if an algebra is semi-simlpe i.e. a direct sum of algebras without a proper Ideal (Ideal is a subgroup for which all commutators of its elements with elements of the full group are again in this subgroup) then the killing form is not degenerate i.e. if the elements you put in there are non zero the number which comes out is also non zero.

Now coming to basis choices. We can always choose the so called Cartan-Weyl basis (Baby, which mathematician did you fancy, Cartan? Or was it Klein? I'm still jealous you know.).

In the Cartan-Weyl basis the E^α are the generators of the one dimensional root spaces G_α and:

$$[h,\, E^\alpha] = \alpha(h)\, E^\alpha. \tag{1}$$

We have a simple algebra and hence the Killing form is non degenerate, therefore we can associate to any α an element H^α such that for all $h \in G_0 : \alpha(h) = c_\alpha K(H^\alpha, h)$. We conclude from that that there exist a non degenerate inner product on G^* (the dual space of the algebra G):

$$(\alpha, \beta) := c_\alpha c_\beta\, K(H^\alpha, H^\beta) = c_\alpha \beta(H^\alpha) = c_\beta \alpha(H^\beta)$$

To any root α we can associate a subalgebra of G isometric to $SL(2)$ (3 dimensional) generated by $\{E^\alpha, E^{-\alpha}, H^\alpha\}$. Now for all $h \in G_0$:

$$K(h,\, [E^\alpha, E^{-\alpha}]) = K([h, E^\alpha], E^{-\alpha}) = \alpha(h) K(E^\alpha, E^{-\alpha})$$

and for all α there is $h \in G_0$ such that $\alpha(h') \neq 0$. We have seen that $K(E^\alpha, E^{-\alpha}) \neq 0$ and that implies $[E^\alpha, E^{-\alpha}] \neq 0$ and together with (1) we conclude that:

$$K(h, [E^\alpha, E^{-\alpha}]) = c_\alpha K(E^\alpha, E^{-\alpha}) K(h, H^\alpha). \tag{2}$$

Since K is non degenerate and (2) holds for all h is follows that:

$$[E^\alpha, E^{-\alpha}] = c_\alpha K(E^\alpha, E^{-a}) H^\alpha,$$

and

$$[H^\alpha, E^{\pm\alpha}] = \pm\alpha(H^\alpha) E^{\pm\alpha} = \pm\frac{(\alpha,\alpha)}{c_\alpha} E^{\pm\alpha}.$$

Comparing this result with the $SL(2)$ algebra where $[H, E^{\pm}] = 2(\pm E)$ we find $c_\alpha = \frac{1}{2}(\alpha, \alpha)$ and get is the end the relations:

$$[H^\alpha, E^\beta] = \beta(H^\alpha)E^\beta \text{ and } \quad \beta(H^\alpha) = \frac{(\beta, \alpha)}{c_\alpha} = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}.$$

The important message of this analysis is, that the structure of the $SL(2)$ algebra appears multiple times inside a simple (without proper ideals) Lie algebra. That's quite remarkable and useful.

To summarize, in the Cartan-Weyl basis the following relations hold:

$$[H^i, E^\alpha] = \alpha^i E^\alpha$$

$$[H^i, H^j] = 0$$

$$[E^\alpha, E^{-\alpha}] = H^\alpha = \sum_i \alpha_i^\nu H^i$$

$$[E^\alpha, E^\beta] = \begin{cases} e_{\alpha,\beta} E^{\alpha+\beta} & \alpha + \beta \in \Phi \\ 0 & \alpha + \beta \notin \Phi \end{cases}$$

Remark: The inner product can be written as: $(\alpha, \beta) = \alpha^i G_{ij} \beta^j$, where G_{ij} we will call the metric for similarity reasons. Now we will define the notion of the co-root α_i^ν as follows:

$$\alpha_i^\nu = \sum_j G_{ij}(\alpha^\nu)^j; \quad (\alpha^\nu)^j = \frac{2\alpha^j}{(\alpha, \alpha)}$$

Compute now:

$$E^\beta \beta(H^\alpha) = [E^\beta, H^\alpha] = [E^\beta, \alpha_i^\nu H^i] = \alpha_i^\nu [E^\beta, H^i] = \alpha_i^\nu \beta(H^i) E^\beta = \alpha_i^\nu \beta^i E^\beta$$

This gives:

$$\sum_i \beta^i \alpha_i^\nu = \beta(H^\alpha) = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \sum_{ij} \beta^j G_{ij} \frac{2\alpha^i}{(\alpha, \alpha)}$$

Which leaves us with the useful relation:

$$\beta(H^\alpha) = (\beta, \alpha^\nu)$$

1.3 Root Strings

The generators $\{E^\alpha, E^{-\alpha}, H^\alpha\}$ span an $SL(2)$ for any root $\alpha \in \Phi$, hence $\beta(H^\alpha) \in \mathbb{N}$. So $\beta(H^\alpha) = (\beta^\nu, \alpha)$ are just the integral weights of the corresponding $SL(2, \mathbb{C})$.

- We know that only $\pm\alpha$ are multiples of α.
- Regard: $[E^{\pm\alpha}, E^{\beta+m\alpha}] = e_{\pm\alpha,\beta+m\alpha} E^{\beta+(m\pm1)\alpha}$ (analogy to $SL(2, \mathbb{C})$ where $E^{\pm\alpha}$ are the ladder operators)

This allows us to define a new object, the root string:

$$S_{\alpha,\beta} = \{\beta + m\alpha,\ m = -n_-, -n_- - 1, ..., 0, ..., n_+ - 1,\ n_+\}$$

The root string is characterized by the numbers n_- and n_+, since they tell you how many roots α you can add to the root β and still get a root.

From the $SL(2)$ group we know, that (with v_n being the eigenvector of the Cartan with Eigenvalue n and $\Lambda = n_+ + n_-$ the highest weight):

$$E_- E_+ v_{\Lambda-2n} = n(\Lambda + 1 - n) v_{\Lambda-2n}.$$

Now we identify $E^{\pm\alpha} = E^\pm$ and $E^\beta = v_{\Lambda-2n}$ and calculate:

$$[E^{-\alpha}, [E^\alpha, E^\beta]] = n_+(n_+ + n_- + 1 - n_+)E^\beta = n_+(n_- + 1)E^\beta$$

Since highest and lowest weight are equal but have opposite sign we get:

$$(\beta + n_+ + \alpha)(H^\alpha) = n_+ + n_- = -(\beta - n_- - \alpha)(H^\alpha)$$

$$\Rightarrow \beta(H^\alpha) + n_+\alpha(H^\alpha) = -\beta(H^\alpha) - n_-\alpha(H^\alpha)$$

$$\Leftrightarrow (\alpha^\nu, \beta) + 2n_+ = -(\alpha^\nu, \beta) - 2n_-$$

Implying:

$$(\alpha^\nu, \beta) = n_- - n_+ = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$$

Note: If α, β are simple roots (i.e. $n_- = 0$) then $n_+ = -\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ gives you the number of roots α which you can add to β such that $\alpha + \beta$ is still a root. This is an important result and can be used to construct the root system from the simple roots (see below).

1.4 Simple roots

Since the Lie algebras have a finite number of roots there is a hyperplane without roots, which we can draw in the root space (this choice is arbitrary as a basis choice). All the roots above the hyperplane called positive, below negative.

$$\Phi_+ = \{\alpha \in \Phi | \, \alpha > 0\} \text{ and } \Phi_- = \{\alpha \in \Phi | \, \alpha < 0\}$$

So if $\alpha \in \Phi_+$ i.e $-\alpha \in \Phi_-$ and we have with respect to this particular split raising and lowering operators $E^{\pm\alpha}$. The Lie algebra can be split (Gauss decomposition) into following subalgebras:

$$G = G_+ \oplus G_0 \oplus G_-$$

Definition: Simple root

A positive root that can not be written as a sum of two or more positive roots with only positive coefficients.
There are $r = rank(G)$ simple roots and they form the basis of the root space.

$$span_{\mathbb{R}} \Phi_s = span_{\mathbb{R}} \Phi$$

Where the simple root space is:

$$\Phi_s = \{\alpha^{(i)} | i = 1, ..., r\}.$$

Note on the basis choice (Dynkin basis):

We can make a choice of the Cartan basis and co-roots such that:

$$(\alpha^{(i)\nu})_j = \delta^i_j$$

Then:

$$H^{\alpha^{(i)}} = H^i \quad \text{and} \quad \beta^\nu = \beta_i \alpha^{(i)\nu} \quad i.e. \; (\beta^\nu)_j = \beta_j$$

Any root β can be expanded in the fundamental weights $\Lambda_{(i)}$ such that $\beta^i = \beta(H^i)$ that means $\beta = \beta^j \Lambda_{(j)}$ in the Dynkin basis.

$$\Rightarrow \beta^i = \beta(H^i) = \beta^j \Lambda_{(j)}(H^i) \; \Rightarrow \delta^i_j = \Lambda_{(j)}(H^i) = (\Lambda_{(j)})^k (\alpha^{(i)\nu})_k$$

I think those pages show what we were talking about yesterday, that the choice of the Cartan basis and the fundamental roots might be arbitrary but influences the relations among the roots. Also the choice of the positive and negative roots depends on where one places the “hyperplane” in the root space. Also I think this explains how to construct the root space from the simple roots. I hope this summary can be of use to you. Have a nice day.

Love you.