"Sui principali resultati ottenuti nella teoria dei gruppi continui dopo la morte di Sophus Lie (1898-1907), *Congrès de la Société Italienne pour le progrès des Sciences*, Parma, 1907.

On the principal results obtained in the theory of continuous groups since the death of Sophus Lie (1898-1907) (*)

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Translated by D. H. Delphenich

Although it has been almost ten years since the death of SOPHUS LIE, only a few people have been able to appreciate the grand program of research into which he invested the full measure of his powers of invention and inquiry in a cohesive, well-coordinated, and connected theory. We shall choose not to consider the problems of integration completely, in order to consider just the theory of continuous groups. Here, we shall also show that the theory of finite continuous groups at the time LIE's death could now be regarded as essentially complete, but as for the theory of infinite groups (i.e., ones that depend upon more than a finite number of arbitrary parameters), LIE did not leave to us a summary treatise of the fundamentals of the theory, but only a scarce and fragmentary mention of some further general considerations that he also knew were hard to overcome in that field.

Be that as it may, as of this decade very few people have ventured into the field of infinite groups, while the theory of finite, continuous groups has taken on the appeal of an eminently brilliant concept and the attraction of a structure that is well-defined and complete by now, and can claim a veritable multitude of devotees.

However, while this host of researchers has left out almost all of the broader and more advanced problems, as they have endeavored, above all, to illustrate the theory of finite, continuous groups in its many geometric and analytical applications, almost to compensate, the much fewer devotees of the theory of infinite, continuous groups have confronted the more comprehensive and salient questions with a variety of expedients and with lofty viewpoints, so today it is in this field that we can observe the more significant progress in the theory of continuous groups.

Under these conditions, it will be very easy task for me to summarize the new results that have been obtained in theory of infinite groups; however, as far as finite continuous groups are concerned, I would not possibly presume that nothing from that vast body of research has escaped my attention. Neither, in other respects, on a very different subject, is it legitimate for me to hope to be able to quickly and clearly reach a synthesis that might serve to compensate for the suspicion and probable deficiencies in the information.

(*) Report read to the Congress of the "Società Italiana per il progresso della scienze," in Parma, September 1907.

Rather, I have sought to group together the latter research according to the affinity of the problems and results considered. However, this affinity will often be so weak and formal that I must enumerate, instead of classify.

GENERAL THEORY OF FINITE CONTINUOUS GROUPS

Considering the movement towards *work of a general nature on the theory of finite, continuous groups,* we also recall some of the research that was concerned with the formal, symbolic approach that Lie had given an entirely secondary status to in his constructions.

One should note that in a certain neighborhood of the identity transformation the transformations of a continuous group with r parameters can be regarded as having been obtained by the infinite iteration of certain ∞^{r-1} infinitesimal transformations that, when represented in the manner of LIE by means of linear differential operators, give rise to a linear family that contains, along with any copies of its operators, their POISSON brackets, as well. One can consider this family to be a system of numbers in r units that is *closed* with respect to all sums and to certain particular operations of multiplication that are non-commutative, but alternating. In their studies of such systems of complex numbers, J. E. CAMPBELL ([1]), POINCARÉ ([2]), PASCAL ([3]), and ultimately HAUSSDORFF ([4]) sought to arrive at LIE's fundamental theorems along different paths and with singular algorithmic virtuosity, and, above all, to attain the effective construction of r infinitesimal transformations of a group when the composition constants are fixed.

This analysis has clarified the operator and symbolic aspects of the theory of finite, continuous groups and made it more precise in its limits and its strengths. Even so, the constructive viewpoint of the theory implied that this theory, for the most part, led to conclusions whose validity was exclusively formal, and otherwise led one to very nearly repeat the procedures of LIE and SCHUR, other than that of the synthetic view and those of geometric representations that might make the path bright and pleasant. In substance, all of the aforementioned investigations seemed to be somewhat rooted in a generic

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([1]) "Proof of the third fundamental theorem in Lie's theory of continuous groups," Proc. math. Soc. London **33** (1901). This paper, which established CAMPBELL as the principal contributor along this line of research, was at the end of the decade that we occupy: "On a law of Combination of Operators," Proc. math. Soc. London, **28** (1907); it is reproduced in *Theory of Continuous Groups* (Oxford, 1903), Chapter IV.

([2]) "Sur les groupes continus," Comptes rendus **128** (1899); *in extenso* in *Memoirs presented to the Cambridge Philosophical Society on the occasion of the Jubilee of Sir G. G. Stokes*, Cambridge, 1900. "Quelques remarques sur les groupes continus," Rend. del Circ. mat. di Palermo **15** (1901).

([3]) "Sopra alcuna identità fra i simboli operativi rappresentanti trasformazioni infinitesime," Rend. Ist. Lombardi (2) **34** (1901). – "Sulla formola del prodotto di due trasformazioni finite e sulle dimostrazione dei cosidetto secondo teorema fondamentale di LIE nella teoria dei gruppi," *ibid.*, *ibid.* – "Sopra i numeri Bernoulliani," *ibid.*, **35** (1902). – "Del terzo teorema di LIE sull'esistenza di dara stuttura," *ibid.*, *ibid.* – "Altre ricerche sulla formola del prodotto di due trasformazioni finite, e sul gruppo parametrico di un dato," *ibid.*, *ibid.* – The preceding research is summarized in: "Resumé de quelques-uns de mes récents travaux sur la théorie des groupes de LIE," Prac matematyczno fizycznych **14** (1903), Warsaw.

([4]) "Die symbolische Exponentialformel in der Gruppentheorie," Leipz. Berichte **58** (1906).

interest along the lines of the algorithmic theory of complex numbers with more units, as well as its specific importance with respect to the theory of continuous groups ([1]).

Truly, POINCARÉ, taking his symbolic developments as a starting point, arrived at noteworthy expressions, in the form of curvilinear integrals, for the infinitesimal transformations of the adjoint group and the parameter groups; however, it is easy to see how all of this resulted almost immediately by starting with the equations of definition that are assigned to LIE groups.

Nonetheless, the idea that originated in POINCARÉ's paper in Circ. Mat. (1900) remains noteworthy. He pointed out how upon starting with the composition constants, all of the constructions of a group could be achieved either across (traverso) the adjoint group or across the parameter group, and proposed to study and interpret the relations that give rise to a confrontation between the results of these two procedures. However, as POINCARÉ himself warned, this path is, to be sure, hampered by major analytical difficulties, but he pressed onward, so that the results obtained would not differ substantially from conclusions that were known to KILLING, ENGEL, UMLAUFF, and CARTAN.

A second group of papers of a general nature come from all of the American mathematicians, such as TABER ([2]), NEWSON ([3]), SLOCUM ([4]), and RETTGER ([5]), and which were based in the observation of ENGEL that related to the possibility that a finite, continuous group might contain *singular transformations* that were not generated by means of the infinite repetition of an infinitesimal transformation. The problem of the existence of such singular transformations and their effective study obviously falls with the purview of the theory of functions, in which the determination of the singularity depends, in substance, on its effect on the general solutions of the equations of definition of the group considered, such that the exceptional difficulty in this line of questioning should seem obvious, which indeed, at the present state of our knowledge, seems to be posed in a form that is too indeterminate and general. Certainly, the work that we just now recalled sheds very little light on the problem, which has its roots in the fact that they provide no particular examples of groups that contain singular transformations that are substantial and uniform.

([1]) Along with the aforementioned research, we must compare that of H. F. BAKER, who has applied the symbolic calculus of matrices to the deduction of some results in the theory of finite, continuous groups: "On the exponential theorem for a simple transitive continuous group, and the calculation of the finite equations from the constants of structure," Proc. math. Soc. London **34** (1902). – "Further applications of matrix notation to integration problems," *ibid*., *ibid*. – "On the calculation of the finite equations of a continuous group," *ibid*., **35** (1903). – "Alternants and continuous groups," *ibid*., (2) **3** (1905). The result of the penultimate paper has already been asserted by KLEIN.

([2]) "On the singular transformations of groups generated by infinitesimal transformations," Proc. Amer. Acad. **35** (1900); Bull. Amer. Math. Soc. (2) **6** (1900).

([3]) "On singular transformations in real projective groups," Bull. Amer. Soc. (2) **6** (1900).

([4]) "Note on the chief theorem of Lie's theory of continuous groups," Proc. Amer. Acad. **35** (1900). – "Supplementary note on the chief theorem of Lie's theory of finite continuous groups," *ibid*., *ibid*. – "On the continuity of groups generated by infinitesimal transformations," *ibid*., **36** (1900). These last two notes are weak in content.

([5]) "On Lie's theory of continuous groups," Amer. J. Math. **22** (1900). This note contains some interesting considerations on the possibility of infinitesimal transformations with broken trajectories.

Concerning the research on the new *analogy between the substitution groups of finite order and finite continuous groups*, MAILLET was led to ultimately study the *decomposibility* of finite continuous groups, that was already known on the basis of LIE's fundamental theorems ([1]), and to generalize, in some sense, the concept of the composition series of a finite, continuous group ([2]).

BURNSIDE has proved that any discontinuous group of finite order g can be associated with a linear continuous group G, a consideration that facilitates the search for properties of discontinuous groups; in particular, the determination of what KLEIN called the *degree* of the *normal problem* that is connected with g, which is to find the minimum number of variables in which g is representable as a group of linear substitutions ([3]).

Finally, in the theory of substitution groups of finite order, BIANCHI has transported the concept of *complementary group* or *factor group* into the theory of finite, continuous groups, which illuminates the essential place that this concept occupies implicitly in the determination that was given by LIE of transitive groups of given composition ([4]).

In particular, he has established a new characteristic property of the derived group of a given group G, showing how it is the smallest invariant subgroup Σ of G for which the complementary group G / Σ is Abelian. This happens in the same way that the *commutator* subgroup ([5]) coincides with the derived group ([6]) for a continuous group.

E. E. LEVI has obtained noteworthy results for *transitive groups of a space of as many dimensions as one would desire*, for which he succeeded, in particular, in reducing to 4 the maximum order of $2n + 1$ that was given by LIE for the infinitesimal transformations of a generic group ([7]). After another step, one will ascertain the observation of LIE concerning when this maximum order will be equal to 2.

In a final group, we can collect *some research that is related to structure* ([8]).

([1]) "Sur la décomposition des groupes finis continus de transformations de Lie," Comptes rendus **130** (1900). A group G is called *decomposable* if it contains two subgroups G_1, G_2 such that any transformation of G is equal to the product of one in G_1 and one in G_2 . – An example of a singular group in the degree of decomposability of its *finite* transformations was highlighted by FRATTINI: "Di un gruppo continuo di trasformazioni decomponibili finitamente," Rend. della R. Acc. dei Lincei (5) **12** (1903).

([2]) "Sur des suites remarquables de sous-groupes d'un groupe de substitutions ou de trasformations de Lie," Comptes rendus **130** (1900). – "Sur de nouvelles analogies entre la théorie des groupes de substitutions et celle des groupes finis continus de transformations de Lie," J. de Math. (5) **7** (1901).

([3]) "On the continuous group that is defined by any given group of finite order," Proc. math. Soc. London **23** (1898).

([4]) "Sulla nozione di gruppo complementare e di gruppo derivato nella teoria dei gruppi finiti di transformazioni," Rend. dell'Acc. dei Lincei (5) **12** (1903).

([5]) This is the group that is defined by all transformations of the form $STS^{-1}T^{-1}$.

([6]) This also gives one a new proof of a known theorem of KILLING (LIE-ENGEL: *Theorie der Transformationsgruppen*, Bd. III, pp. 770.

([7]) "Sui gruppi transitivi dello spazio a n dimensioni," Rend. dell'Acc. dei Lincei (5) **14** (1905). For some theorems on transitive groups that are intended, above all, to make the concept of *class* more precise, in analogy to the substitution groups, see: MAILLET, "Sur la classe des groupes finie continus primitifs de transformations de Lie," Comptes rendus **130** (1900).

([8]) E. E. LEVI has rigorously established the noteworthy theorem of KILLING on the decomposability of any finite, continuous group that is not semi-simple that is integrable into an integrable invariant group and a semi-simple group: "Sulla struttura dei gruppi finiti e continui," Atti della R. Acc. delle Sc. di Torino **40** (1905). Also note the following *Dissertation* of MÜNSTER, which he could not have seen: W. BRÜSER, "Untersuchungen über die sechsgliedrige halbeinfache Transformationsgruppe" (1904).

A SÜSS, in his *Dissertation* at GREIFSWALD, has determined all of the point groups that have the structure of the full projective group of the plane, and has studied a class of groups of contact transformations that have the same structure, which sheds more light on some general considerations that permit one to say what type of group of contact transformations is represented by an arbitrary type of point-like transitive group of given composition ([1]).

KILLING responded with a brief note on the theory of composition in which, to a great degree, he classified the structure of the groups of rank 0, based upon a new invariant numerical character ([2]).

LOEWY has given a new characteristic property of groups of rank 0 and proved that for such groups – and only for them – any arbitrary subgroup belongs to some composition series ([3]).

AHRENS has determined the groups for which any subgroup is invariant ([4]). Finally, ZINDLER has studied the commutable groups (which are particular case of groups of rank 0) in general, and in particular, he has classified their types in the case of four parameters ([5]).

Upon taking a quick look at the investigations that were carried out since the death of LIE on the general theory of finite, continuous groups, one can see that in the past decade there have not been any substantial changes. When one thinks about that theory, which is already ten years old, what other opinion would be natural than that it constitutes a closed set of results and procedures?

On the other hand, it is also true that there exists a profound and singular contrast between the systematically analytical form in which LIE and his collaborators ultimately published the theory of finite, continuous groups and the essential, boldly synthetic, conception of it that LIE first had. Immediately after the initial fervid period of investigation and discovery, he felt upset with the state of isolation in which he lived scientifically, and reproached the mathematicians of the time for the coolness that they showed to his program of study, which was so bold, so new, and to be sure, so obscure. I think that perhaps, without a doubt, one should overlook the sin of his impatience and consider the ultimate scope of his work: the theory of integration. Certainly, he was induced to translate, and almost to disguise ([6]), his thoughts into the analytical form that seemed more appropriate and facilitated the publication of his discovery. Moreover, before his death, he could also be regarded as having answered to an ineluctable necessity, since only in the newer form of his theory does one recognize the approval and admiration that he had originally desired. However, it is certain that in the present theory of finite, continuous groups one can follow the simple and bold course that was first

([1]) "Die Gruppen, die mit der allgemeinen projectiven Gruppe die Ebene gleiche Zusammensetzen haben," Dissertation, Greifswald (1905).

([2]) "Der Bau einer besonderen Klasse von Transformationsgruppen," Festschrift dedicated to L. BOLTZMANN, Leipzig (1904).

([3]) "Zur Theorie der endlichen kontinuierlichen Transformationsgruppen," Math. Ann. **55** (1901).

([4]) "Ueber kontinuierlichen Involutionsgruppen," Wien. Berichte **110** (1901).

([5]) "Ueber Transformationsgruppen, deren sämtliche Untergruppen invariant sind," Mittheil. der math. Gesell. in Hamburg **4** (1902). These groups reduce to the groups of rank 0 with three parameters and commutable groups.

([6]) "Ueber kontinuierliche Involutionsgruppen," Wien. Berichte **110** (1901).

outlined by LIE, and feel the boredom and regret that he saw as spoiling and misinterpreting the restoration of the original beauty of a work of art.

I think that LIE, accepting the isolation that was initially appropriate to the originality of his thoughts and the enduring novelty of his methods, did not, in his passion to discover, deviate in his compulsion to follow a path that he had blazed, so by now we perhaps possess the complete and organic synthetic theory of finite, continuous groups. Today, the reconstruction, remounting, and traversing of the obscure hints in LIE's first papers, in the original form of his ideas, seems to be tedious work; however, I believe that the same impulse by which this idea became more ingenious and more alive, suggests that the synthetic theory must be constructed in such a form one day.

Meanwhile, one can, with good reason, presume that projective groups must occupy a fundamental and almost preponderant place in this, and in this sense a considerable step will be made when one manages to establish (as LIE was always convinced, even after recognizing the fallacy in his old proof) that *for any finite, continuous group there exists a projective group that has the same structure.*

Along this same order of ideas, it will be particularly desirable that, also only for projective groups, one can blaze a less indirect and obscure path than the notes that arrived at ENGEL's theorem, which *characterizes the non-integrable groups as the ones that contain a simple* ∞^3 *group* that is holohedrally isomorphic to the full projective group of the line.

Perhaps there has not be enough attention given to the fundamental place that this noteworthy theorem undoubtedly occupies in the theory of composition, which indeed remains isolated and weakly connected to the rest of the theory.

As far as I know, only KRATZI and KOWALEWSKI have taken the opportunity to study, at least, some particular kind of groups in relation to their simple ∞^3 subgroups, or as one can say, their *ENGEL subgroups.*

KRATZI, in his *Dissertation* at Greifswald ([1]), has systematically studied the groups of an n-dimensional space that contained an ENGEL subgroup, and did not belong to any larger subgroup, and proved that such a group has $n + 3$ parameters, in general, and that its stability subgroup with respect to a generic point leaves a normal rational cone of directions fixed, in such a way that the group is equivalent, by means of a point-like transformation, to a projective group that contains all of the simple ∞^n translations of S_n and leaves a normal rational curve in the improper plane fixed. There is another composition that satisfies the conditions above in just four cases: For $n = 3$, one has the group of the quadric. For $n = 5$, there is the ∞^n group of S_n , which is induced by the full planar projective group on the ∞^5 conics of the plane. For $n = 7$, there is the ∞^{10} group of S_7, which is such that the twisted (gobbe) cubics of a (non-special) linear complex are permuted by the projective group of the complex, and finally, for $n = 11$, one has a simple ∞^{14} group of S_{11} that has the structure that was discovered by KILLING in his memorable research on composition.

([1]) "Gruppen mit einer dreigliedrige Untergruppe, die in keiner grösseren Untergruppe steckt," Leipzig (1904).

Meanwhile, KOWALEWSKI ([1]) has considered the projective groups of S_n that do not leave any linear variety fixed and contain either the (simple, ∞^3) group of a normal rational C^n or the simple, ∞^3 group that admits a S_{n-h-1} with united points and a skew S_h in which a normal rational C^h is invariant ([2]). – For n even and h odd, there exists no group that satisfies the conditions above. However, in some other cases one has, in general, a unique type of projective group, which, for h even, is the full projective group of a non-special quadric in S_n, and for h (and n) odd it is the full projective group of a null system in S_n . Another group that satisfies the conditions above exists only in S_4, S_7, and S_{11}: For $n = 5$ (and $h = 4$), there is just the ∞^8 group that is induced by the planar projective group on the ∞^5 conics. For $n = 6$ (and $h = 6$), one has the ∞^{14} projective group that transforms a non-special quadric in it (CLIFFORD quadric of normal rational C^6) and on it, an ENGEL complex ([3]). Finally, for $n = 7$ (and $h = 6$), there is the ∞^{21} projective group that is induced by the full projective group of the quadric on the ∞^{21} ENGEL complexes that lie on a non-special quadric in S_6 . – To these results of KOWALEWSKI, one adds that, when evaluated by a "Gewichtsmethode," in its fundamental principles it can be traced back to LIE, and has its basis in a classification of the infinitesimal projective transformations (in homogeneous coordinates) of *weight* two, suitably defined.

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This only begins to review the *research that was directed to the determination of the particular classes of finite, continuous groups*.

We can now advance into this field; however, the multiplicity and variety of the questions considered will lead one to their enumeration, instead of a true classification.

PROJECTIVE GROUPS

We shall now address the projective groups.

In this field, we first review the research of NEWSON ([4]), who has constructed, on the basis of elementary synthetic considerations, all of the types of continuous, projective

([1]) "Ueber die projektive Gruppe der Normkurve und eine charakteristische Eigenschaft des sechsdimensionalen Raumes," Leipz. Ber. **54** (1902). – "Ueber projektive Transformationsgruppen," *ibid*. **55** (1903). – "Eine charackteristische Eigenschaft der projektiven Gruppe des Nullsystems," *ibid*. **58** (1906). – "Ueber die projektive Gruppe einer Mannigfaltigkeit zweiten Grades," *ibid*., *ibid*.

([2]) These are, in a certain sense, the less complex types of simple ∞^3 projective groups of S_n . In regard to this, one remembers the beautiful theorem of STUDY [LIE-ENGEL, *Theorie der Trans.-gr.*, t. III, pp. 785], which has not been published, except for the proof of FANO [Mem. della R. Acc. delle Sc.di Torino (2) **46** (1896).

([3]) "Ein neues, dem linearen Komplexe analoges Gebilde," Note 2, Leipz. Ber. **52** (1900). The ∞^{14} group that was just now pointed out has the simple composition that was discovered by KILLING.

([4]) "Continuous groups of projective transformations treated synthetically," Kansas Univ. Quart. **4** (1895-96). – "Supplementary notes," *ibid*. – "Projective Transformations in One Dimension and their Continuous Groups," The Kansas University Science Bulletin **1** (1902). In this last note, the complex projective groups of the complex line are classified in the same way as the real conformal groups of the groups of the line and plane ([1]) and some types of projective groups of space. He has grasped the majority of homology groups and, by combining them in all possible ways, has constructed larger groups by hand that are characterized by means of their respective invariant configurations ([2]).

As for the classification of the projective groups of space, which, as was noted, has not been successful, moreover (although, on the basis of LIE's method, this does not present any great difficulty), there has been a contribution to this in that *all of the types of line complexes that admit a projective, continuous group* have been determined ([3]); in particular, that determination is predicated on the classification of the three-parameter projective groups.

R. LE VAVASSEUR has classified the subgroups with one and two parameters of the homogeneous linear group in four variables ([4]).

BEMPORAD, E. E. LEVI, and MEDICI were occupied with the *groups of motions in linear spaces*. The first of them has classified them in S_3, S_4, and partially in S_5 ([5]). The second of them has applied some general considerations to the derived group of the group of motion in S_n that were, above all, directed to the determination of the groups of two and three parameters ([6]). Finally, MEDICI has determined all of the possible types of *rotation groups* ([7]) in a space of any desired dimension, and has shown the benefit that the knowledge of such groups can contribute to the general theory of groups of motions ([8]).

Along with this research on the projective groups, one can further mention the brief investigation that was carried out by EPSTEIN on the groups that coincide with their respective adjoint groups ([9]).

However, nothing substantially new has been published on the problem that has merited the attention of some geometers (and which already gave rise to some important

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plane. – "Types of Projective Transformations in the Plane and in Space," Kansas Univ. Quart. **6** (1899). – The five Types of Projective Transformations in the Plane," *ibid.*, **8** (1899). This, and other research, is discussed systematically in: "A New Theory of Collineations and their Lie Groups," Amer. J. Math. **24** (1904). For space, cf., "On the Group and Subgroups of real Collineations leaving a Tetrahedron invariant," Kansas Univ. Quart. (A) **10** (1901); "A new Theory of Collineations in Space," *ibid.*, *ibid.* Finally, cf., "Report on the Theory of Collineations," Proc. of the Amer. Assoc. for the Adv. of Sc. **51** (1902).

([1]) FANO (art. cited in the *Encyklopädie*) noted that the research of NEWSON does not lead to an effective classification of the projective groups in the plane insofar as he did not prove to have exhausted all of the possible types of groups. This result leads to a contradiction between his results and LIE's table.

([2]) By this same elementary synthetic method of NEWSON, H. B. BREWSTER has constructed the projective groups of S_3 that leave a non-special quadric fixed: "On Collineations in Space which leave invariant a Quadric Surface," Kans. Univ. Bull. (2) **1** (1903).

([3]) "Sui complessi di rette, che ammettono un gruppo continuo proiettivo," Rend. del Cir. mat. di Palermo **23** (1907).

([4]) "Sur l'enumération des sous-groupes du groupe linéaire, homogène, à quatre variables: sous-groupes à un et à deux paramétres," Bulletin des sc. math. (2) **29** (1905). – "Les sous-groupes du groupe linéaire à quatre variables: sous-groupes à un et à deux paramétres," Ann. de la Fac. des sc. de Toulouse (2) **8** (1906).

([5]) "Sui gruppi di movimenti e similitudini nello spazio a 3, 4 e 5 dimensioni," Ann. della R. Scuola Norm. Sup. di Pisa **8** (1898).

([6]) "Sui gruppi di movimenti," Rend. della R. Acc. dei Lincei (5) $\mathbf{14}_1$ (1905).

([7]) "Sui gruppi di rotazioni," Ann. della R. Scuola Norm. Sup. di Pisa **10** (1907).

([8]) "Sui gruppi di movimenti," Rend. della R. Acc. dei Lincei (5) $\mathbf{16}_2$ (1907).

([9]) "Les groupes qui coincident avec leurs groupes adjoints," Math. Ann. **56** (1902).

observations of STUDY) on projective groups that are projectively equivalent to their respective parameter groups ([1]).

On the *theory of invariants of projective groups* ([2]), we mention the contributions of KASNER, who has studied the invariants of a curve traced on a non-special quadric in S_3 with respect to the ∞^6 mixed projective group of that quadric ([3]), and, from a more general viewpoint, MAURER has extended the theorem on HILBERT on the finite number of invariants of a complete system of a linear, continuous group in the case where the variables are subject to a system of algebraic equations ([4]).

Before we leave the field of projective groups, we must give a nod to the research of STUDY on the (differential) elements of second order in the plane, each of which can be defined as the totality of all (analytic) curves that have three given, infinitely close, non-collinear points in common ([5]). STUDY made the set of these second-order elements closed in various ways by means of the agency of suitably chosen *improper* elements, and established that it results in a system of *projective* coordinates on a ∞^4 variety (namely, one that is characterized by being invariant with respect to the planar projective group), and based upon this, he has developed the invariant theory of the ∞^4 variety of second-order elements with respect to a ∞^9 group that is obtained by combining the ∞^8 group that is induced by the planar projective group with a ∞^1 group (which is linear in the coordinates of the second-order elements) that transforms any set of ∞^1 second-order elements that pertain to the same second-order element into itself ([6]).

Moreover, STUDY has considered the one, two, or three-dimensional varieties of second-order elements, which is obviously of interest in the theory of second-order, ordinary differential equations, and has determined the varieties V_4 in a space of any desired dimension that are representable as the set of all ∞^4 second-order elements in the plane, in such a way that the group on them that corresponds to the planar projective group is seen to be induced by a projective group of the respective space.

In this manner, it results that one establishes, in particular, a noteworthy relation between the projective geometry of second-order elements in the plane and the ordinary geometry of the line.

However, I will not go further in that direction. From the strictly group-theoretic viewpoint to which I must cleave, this almost cripples, or at least misrepresents, the

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([1]) Cf., BURNSIDE, "On groups which are linear and homogeneous in both variables and parameters," Proc. math. Soc. London **35** (1903). – "On reciprocal linear homogeneous groups," Quart. J. **34** (1903).

([2]) The work of ENRIQUES and FANO on the varieties that admit a projective group is previous to the decade that we are concerned with, except for the note of FANO" "Un teorema sulle varietà algebraiche a due dimensioni con infinite trasformazioni proiettive in sè," Rend. della R. Acc. dei Lincei (5) **12** (1899). In this note, he proves the remarkable theorem that any three-dimensional algebraic variety that admits a transitive, continuous group of projective transformations is rational.

([3]) "The invariant theory of the inversion group; Geometry upon a quadric surface," Trans. Amer. Math. Soc. **1** (1900).

([4]) "Ueber die Endlichkeit der Invarianten Systeme," Math. Ann. **57** (1903).

([5]) "Die Elemente zweiter Ordnung in die ebenen projectiven Geometrie," Leipz. Ber. **53** (1901). Regarding the systems of coordinates that are needed in order to determine the elements of order higher than two, see ENGEL: "Die höheren Differentialquotienten," Leipz. Ber. **46** (1893), **54** (1902).

([6]) Any second-order element admits three orientations, and accordingly, if one considers oriented or unoriented elements then one obtains two different ∞^2 groups that are in [3, 1] correspondence with each other.

interest and the importance of this research of STUDY, which cannot be separated from the other previous and subsequent investigations of that author, starting with his early research on complex numbers and continuous groups and the *Methoden der ternären Formen* (Leipzig, 1889) and concluding with the ponderous *Geometrie der Dynamen* (Leipzig, 1903) and his many papers on non-Euclidian geometry. For us, it will be enough to recall how the theory of continuous groups has been a fruitful "transport principle" for STUDY, by which he has, as it were, augmented the efficacy of the SEGRE process for the analytic continuation of the parameters and variables in a real field to ones in a complex field more extensively by hand, by superimposing it with the classical method of the construction of "fields" of geometrical entities with respect to a given group ([1]). It was from this precise origin that STUDY systematically presented the various geometries (dual and radial projective geometry, the projective and pseudo-conformal geometry of "somas") in the *Geometrie der Dynamen* that we cited ([2]).

However, we shall now turn our attention to the groups of birational transformations.

CREMONA GROUPS

In principle, it is possible to review the research of H. STENDER ([3]), myself ([4]), and NEWSON ([5]) on real conformal groups of space, as well as on curves and surfaces, which admit an infinitude of conformal transformations.

However, at a more elevated level of research that relates to this field, there is the work of FANO on finite continuous groups of Cremona transformations of space ([6]) and their classification, concluding with the generalized JONQUIÈRES groups, or groups of birational transformations that transform a sheaf of planes or a pencil of lines into itself ([7]).

These works complete the line of research that ENRIQUES initiated into the determination of the continuous Cremona groups of the plane and was continued with that of ENRIQUES and FANO in the masterly papers in which they proved that the continuous, birational groups of space are birationally equivalent to either projective groups, conformal groups, generalized Jonquières groups, or finally, to two well-defined

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([1]) We can also recall the following research: E. v. WEBER: "Zur Theorie der Kreisverwandschaften," Münch. Ber. **31** (1901). "Die komplexen Bewegungen," Leipz. Ber. **55** (1903).

([2]) A clear and complete review of STUDY's views and investigations can be found in nos. 16-20 of the cited article by FANO.

([3]) "Invariante Flächen und Kurven bei conformen Gruppen des Raumes," Disseration, Leipzig (1899). This dissertation was briefly brought to my attention, and I have only recently been informed, after the release of the manuscript of this report (19 January 1908), of the existence of a mention that was made of my cited paper in the subsequent note of ENGEL in the fascicle that was just now brought to light in the *Jahrbuch über die Fortschr. der Math.* **36** (1907).

([4]) "La superficie con infinite trasformazioni conformi in sè stesse," Rend. della R. Accad. dei Lincei (5) **10** (1901). – "I gruppi continui reali di trasformazioni conformi dello spazio," Mem. della R. Acc. delle Sc. di Torino (2) **55**) (1905).

([5]) "Types and continuous Groups of real conformal Transformations in S_2," Giorn. di Mat. **45** (1907).

([6]) "I gruppi continui primitivi di trasformazioni cremoniane dello spazio," Atti della R. Accad. delle Sc. di Torino **33** (1908). "Le trasformazioni infinitesime dei gruppi cremoniani tipici dello spazio," two notes in Rend. della R. Accad. dei Lincei (5) $\mathbf{7}_1$ (1898).

([7]) "I gruppi di Jonquières generalizzati," Mem. della R. Acc. delle Sc. di Torino **48** (1898).

∞^3 simple, transitive, groups for which the transformations that leave a generic point fixed constitute a group of finite order that is holohedrally isomorphic to the group of either the octahedron or the icosahedron.

In an affine context, ENRIQUES has, more recently, recovered the results of the profound analysis of PICARD, PAINLEVÉ, and CASTELNUOVO, and the same ENRIQUES has classified the surfaces with a continuous birational group into three families: ruled surfaces, elliptical surfaces that contain two sheaves of curves of equal moduli, and hyperelliptic surfaces that are coupled to the points of a curve of genus two, and he succeeded in characterizing these three families, either separately or collectively, by means of the values of its invariant characters (genera and plenigenera) ([1]).

These two cycles of results, thus completed, on the continuous, Cremona groups, on the one hand, and on surfaces that admit such a group, on the other, undoubtedly constitute one of the more beautiful chapters in the theory of continuous groups, not just for the importance and singular difficulty of the problem's solution, but also for the profundity of the vision and the richness and genius of the expedients that are so profuse. However, one can say that while the primary basis for this algebraic geometry of finite, continuous groups has been defined – at least, implicitly – by now, the work of LIE has been left completely in the shadows.

Along these same algebraic lines, one can even recall the systematic research of CARDA on the algebraic groups of the line and plane ([2]). He has confirmed that, other than a type of ∞^1 groups of the line ([3]), the algebraic groups that he considered already appear among LIE's representative types in his table of groups of the line and plane.

Moreover, CARDA has drawn attention to an interesting class of algebraically integrable differential equations. The equation $X\xi = 1$, where X is the symbol of an arbitrary algebraic infinitesimal transformation in just one variable, admits only algebraic integral curves of genus 0. The direct integration, which CARDA has carried out explicitly for $n = 3$, leads to the consideration of pseudo-Abelian integrals, which represent algebraic functions, for $n > 2$.

SPACES THAT ADMIT GROUPS OF MOTIONS

Another fruitful line of new and important results traces its origin to the research and conclusions of BIANCHI on *three-dimensional spaces with a continuous group of motions* ([4]). Such research is immediately linked with the *Dissertation* of RIMINI, in

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([1]) "Sulle superficie algebraiche che ammetono un gruppo continuo do trasformazioni birazionale in sè stesse," Rend. del Circ. nat. di Palermo **20** (1905).

([2]) "Zur Theorie der algebraischen Gruppen der Geraden und der Ebene," Monats. f. Math. u. Phys. **11** (1900). – "Ueber eine Schar dreigliedriger algebraischer Gruppen der Ebene," Monats. f. Math. u. Phys. **17** (1906).

([3]) As CARDA himself observed, the ∞^1 algebraic groups of the line were already determined implicitly by WEIERSTRASS: cf., BOHLMANN, "Ueber eine gewisse Klasse continuierlicher Gruppen und ihren Zusammenhang mit den Additionstheoremen," Diss. Halle a. S. (1892).

([4]) "Sugli spazi a tre dimensioni, che ammettono un gruppo continuo di movimenti," Mem. Soc. it. delle Sc. (3) **11** (1897).

which he pointed out some noteworthy and elegant properties of the three BIANCHI spaces with variable curvature that admit a four-parameter group of motions ([1]).

From another viewpoint, the same problem of three-dimensional spaces with a continuous group of motions was the object of the investigations of RICCI, on the one hand, who, based upon his methods of the *absolute differential calculus*, posed the following question: Given the linear elements of an arbitrary three-dimensional variety in general coordinates, determine whether rigid motions are possible in it, and in the affirmative case, determine the group by means of the defining equations ([2]). – More recently, RICCI has extended some of the fundamental results that were established for V_3 in his early research to a variety of as many dimensions as one desires ([3]).

From the same line of research that commenced with the cited papers of BIANCHI, one can further derive a large and complex body of research of FUBINI, of which I can only illustrate just one aspect, and by limiting oneself to conclusions that are of interest in the theory of continuous groups, one must leave unmentioned the very important arithmetic and analytical results to which the investigations have arrived.

FUBINI, attacking the general problem of n-dimensional spaces with a continuous group of motions, has given, in a remarkable synthetic form, the necessary and sufficient conditions for a group to be admitted as a group of motions of such a space, and upon the basis of that result, he has shown how for the solution to the algebraic equations, it suffices to determine all of the groups that can be considered to be groups of motions by means of their infinitesimal transformations, and how, with just one quadrature, one can obtain the linear elements of the corresponding space. Upon the basis of his general method and another quicker special procedure, he then exhausted the search for the groups of motions that contain no more than four independent infinitesimal transformations, and then, the four-dimensional spaces with groups of motions. Some observations on discontinuous groups that are contained in the continuous groups of motions of three-dimensional spaces ([4]) are noteworthy in this research, by their novelty and perhaps for their greater importance.

As a first generalization of the BIANCHI problem, FUBINI has studied the spaces that admit a conformal group, while splitting the problem into two:

1. Determine the groups that are susceptible to being considered as conformal groups of such spaces.
2. Effectively construct the corresponding linear elements.

He has thus, on the one hand, pointed to the intimate connection between the theory of conformal groups and the theory of groups of motions ([5]) and, on the other hand,

([1]) "Sugli spazi a tre dimensioni, che ammettono un gruppo a quattro parametri do movimenti," Ann. della R. Sc. Norm. Sup. di Pisa **9** (1907).

([2]) "Sui gruppi continui di movimenti in una varietà qualunque a tre dimensioni," Mem. della Soc. it. delle Sc. (3) **12** (1899). Cf., also, RICCI-LEVI CIVITA: "Méthodes de Calcul différentiel absolu et leurs applications," Math. Ann. **54** (1900).

([3]) "Sui gruppi continui di movimenti rigidi negli iperspazi," Rend. della R. Acc. dei Lincei (5) **14** (1905).

([4]) "Sugli spazi che ammetono un gruppo continuo do movimento," Rend. della R. Acc. dei Lincei (5) $\mathbf{11}_2$ (1902); Ann. di Mat. (3) **8** (1902). – "Sugli spazi a quattro dimensioni che ammetono un gruppo continuo di movimento," Ann. di Mat. (3) **9** (1903). – "Sulla teoria delle forme quadratiche Hermitiane e dei sistemi do tali forme," Atti dell'Acc. Gioenia (4) **17** (1904).

([5]) "Sulla teoria degli spazi che ammettono un gruppo conforme," Atti delle R. Acc. delle Sc. di Torino **38** (1903).

generalized those groups of point-like or contact transformations that preserve the hypersphere in an arbitrary metric ([1])in various directions.

However, the more important contribution that was made by FUBINI along this line of research was the determination of the spaces that admit a continuous group that permutes the geodesics. By means of an analysis that was truly perspicacious, he succeeded, so to speak, in decomposing the difficulty of the problem by reducing the discussion that pertained directly to the second-order partial differential equations to the successive study of systems of first-order partial differential equations or to just ordinary differential equations. He showed how one can rapidly solve the problem in the case of surfaces that was already treated, in part, by LIE and substantially exhausted by the research of KOENIGS and RAFFY. He explicitly arrived at the determination of the three-dimensional varieties that admitted a continuous group of geodesic transformations and has shown how – except for some special cases – his method of discussion also serves to exhaust the problem in a space of as many dimensions as one desires ([2]).

These group-theoretic questions that were solved by FUBINI come down, as particular cases, to the general questions of determining all of the dynamical problems for which there exists a continuous group that permutes the trajectories. Along with that dynamical question, FUBINI successfully addressed, studied in general, and determined, in the case of three free coordinates, the conservative dynamical problems that admit continuous groups of DARBOUX transformations (viz., transformation groups that permute an ∞^1 sheaf of trajectories that correspond to the various values of the constant of *vis viva*), and solved the problem in its generality for the dynamical problems in just two variables ([3]).

OTHER PARTICULAR FINITE CONTINUOUS GROUPS

Now, in order to complete our review of the results that were obtained in the last decade in the theory of finite, continuous groups, all that remains is for us to mention some widely-varied determinations of the arguments.

BIANCHI has studied the finite subgroups of two infinite, continuous groups of equivalence transformations and proportional transformations by abstracting from the determination in the planar case ([4]).

Then, in order to complete the classification of the point-like finite, continuous groups of space, based upon a procedure that was already pointed out by LIE, I have determined the groups that transform a congruence of curves into itself and operate on it imprimitively ([5]).

([1]) "Sulla teoria delle ipersfere e dei gruppi conformi in una metrica qualunque," Rend. del R. Ist. Lomb. (2) **38** (1905).

([2]) "Sui gruppi di trasformazioni geodetiche," Mem. della R. Acc. delle Sc. di Torino (2) **53** (1903).

([3]) "Ricerche gruppali relative alle equazione delle dinamica," Three notes in: Rend. della R. Acc. dei Lincei (5) $\mathbf{12}_1$ and $\mathbf{12}_2$ (1904). – "Sulle traiettorie di un problem dinamico," Rend. de Circ. mat. di Palermo **18** (1905).

([4]) "Sui gruppi continui finiti trasformazioni che conservano le aree od i columi," Atti della R. Acc. della Sc. di Torino **38** (1903). – "Sui gruppi continui finiti di trasformazioni proporzionali," *ibid.*, *ibid.*

([5]) "Contributo alla determinazione dei gruppi continui finiti dello spazio ordinario," Giorn. di Matematiche (2) **9** (1901).

We now note how LIE, in his classical research on the fundamental of geometry was led to classify those point-like groups of S_3, with respect to which, two points admit one and only one invariant and two or more points that do not have any essential invariants (viz., independent of the invariants of the pair). This group-theoretic problem has given rise to two different generalizations: On the one hand, BLICHFELDT has determined the groups in S_3 that operate transitively on the pair and possess one or more essential invariants for the trio of points ([1]). On the other hand, KOWALEWSKI has solved the same problem of LIE in S_4 and, without any limitations with regard to reality, in S_5 ([2]), and then, expanding upon the research, has taken the lead by determining all of the primitive groups in the spatial case of five dimensions ([3]).

The *finite, continuous groups of contact transformations* were already stated by LIE, all of the planar types and three groups that exist in any space, only one of which is primitive ([4]). ENGEL, searching for a representative of the simple 14-parameter composition that was discovered by KILLING, has constructed a noteworthy type of ∞^{14} groups of contact transformations of S_3 that operate primitively on the surface elements in S_5 ([5]), and SCHEFFERS, in his dissertation, has classified those finite, continuous groups of contact transformations of space that transform ∞^1 first-order partial differential equations into each other ([6]).

While determining the primitive groups of S_5 , KOWALEWSKI ([7]) then confirmed that in S_3 there exist no other primitive, finite, continuous groups of contact transformations than the two types of LIE and ENGEL, and OSEEN has determined some new classes of contact transformations of S_3 (that operate imprimitively only on S_3) ([8]). In order to complete the classification in S_3, we are now missing only the types of a category of groups that that is well-defined and, to be sure, not devoid of interest for the

([1]) "On a certain class of Groups of Transformations in Space of three Dimensions," Amer. J. Math. **22** (1900). We can recall that the same author has also repeated the determination of the primitive, finite, continuous groups of the plane (Trans. Amer. Math. Soc. **2** (1901).

([2]) "Ueber eine Kategorie von Transformationsgruppen einer vierdimensionalen Mannigfaltigkeit," Leipz. Ber **50** (1898).

([3]) "Die primitiven Transformationsgruppen in fünf Veranderlichen," Leipz. Ber. **51** (1899).

([4]) Cf., e.g., LIE-ENGEL, *Theorie der Transformationsgruppen*, v. II, chap. 23-25, which deserved to be mentioned in the note of SCHEFFERS, "Synthetische Bestimmung aller Berührungstransformationen der Kreise in die Ebene," Leipz. Ber. **51** (1899). – For other geometric considerations in regard to this, see W. DE TANNENBERG, "Sur quelques transformations de contact," Comptes rendus **134** (1902). – LUDWIG, "Ueber den Zusammenhang der Berührungstransformationen der Kreise einer Ebene mit der conformen Abbildungen des Raumes," Rend. del Circ. mat. di Palermo **23** (1907). LUDWIG has then also studied the ∞^{10} group of algebraic contact transformations on a sphere that transform circles into circles: "Ueber die Berührungstransformationen der Kreise auf eine Kugel," Deutsche Math. Ver. **14** (1905).

([5]) This group, which was pointed out by ENGEL at the end of 1888, was made known in mathematical publications only in 1893 ("Sur un groupe simple à quatorze paramétres," Comptes rendus, t. **116**). It was simultaneously discovered by CARTAN (Comptes rendus, *ibid.*). There is a reference to that group in the recent paper of ENGEL that was cited on pp. 7.

([6]) "Bestimmung einer Klasse von Berührungstransformationsgruppen des dreifach ausgedehnten Raumes," Acta math. **14** (1891).

([7]) "Die primitiven Transformationsgruppen in fünf Veränderlichen," Leipz. Ber. **51** (1899).

([8]) "Ueber einige irreducible Gruppen von Berührungstransformationsgruppen im Raume, Oefr. af k. Vetenskaps-Akad. Förhandl. (1901). – "Ueber die endlichen, continuierlichen, irreduciblen Berührungstransformationsgruppen im Raume," Disseratation, Lund, 1901.

problems of integration: It is the category of contact transformations that multiply the single Pfaffian:

$$dz - p\,dx - q\,dy$$

by a constant and transform the variables x, y, p, q between them imprimitively ([1]).

To these determinations of the groups of contact transformations, one can add the research of KOWALEWSKI on Pfaffian systems that admit a finite, continuous group of transformations. The consideration of such systems presents itself naturally in the investigations into the primitive point-like groups of an arbitrary space. – The results of ENGEL on the invariant theory of Pfaffian systems ([2]) are fundamental in this line of reasoning, results that perhaps have not attracted the attention of mathematicians that they deserve. On the basis of this, KOWALEWSKI has established some general conclusions that go beyond the scope of this survey on Pfaffian systems that are connected invariantly with indefinitely integrable systems. In regard to the theory of groups, she has determined all of the Pfaffian systems in ([3]) and six ([4]) variables that admit a primitive, finite, continuous groups of point-like transformations and has proved, as a corollary to her general theorem, that in eight variables there exists no Pfaffian system that is invariant with respect to a point-like group of transformations ([5]).

FINITE CONTINUOUS GROUPS AND DIFFERENTIAL EQUATIONS

We will now enter into that line of investigation that was directed towards determining and studying equations and differential systems that admit a finite, continuous group of transformations ([6]).

([1]) In regard to the theory of contact transformations, one can mention the research of LIEBMANN [“Zur Theorie der erweiterten Berührungstransformationen,” Leipz. Ber. **51** (1899)], in which he proved the fundamental theorem of BÄCKLUND and ENGEL on the osculatory transformations of arbitrary order of the variety V_m into some S_n . On the basis of that theorem, these transformations reduce to contact transformations when $m = n - 1$ and to point-like transformations when $m < n - 1$. PASCAL has also given a direct verification of BÄCKLUND’s theorem in the plane [Rend. del Circ. mat. di PALERMO **18** (1904). – Also in the theory of contact transformations, but along different lines, one can mention the disseratation (at GREIFSWALD) of F. J. DOHMEN (“Darstellung der Berührungstransformationen in Konnexkoordinaten,” Leipzig, 1905), in which he systematically studied the various way in which a contact transformation of S_n is representable as a transformation of the $2n + 2$ homogeneous coordinates of the point and the plane.

([2]) “Zur Invariantentheorie der Systeme von Pfaff’schen Gleichungen,” Leipz. Ber. **41** (1899) and **42** (1890).

([3]) “Die primitiven Transformationsgruppen in fünf Veränderlichen,” Leipz. Ber. **51** (1899).

([4]) “Ueber Systeme von Pfaff’schen Gleichungen mit einer primitiven Transformationsgruppen,” Leipz. Ber. **51** (1899).

([5]) “Ueber Systeme von Pfaff’schen Gleichungen,” Leipz. Ber. **53** (1901).

([6]) It would not be out of place here to mention the “Vorläufiger Bericht” of SCHEFFERS, “Ueber Integrationstheorien von Sophus Lie,” Jahresber. der deutschen Math.-Ver. **12** (1903), in which he discussed the gradual development of LIE’s views and conclusions relating to the integration of complete systems that admit distinguished infinitesimal transformations, so it was highly desirable that this definitive “Bericht” be published quickly.

CZUBER has systematically revised, along a somewhat different path from that of LIE, the theory of one-parameter groups of the plane and their application to the theory of the integration of first-order, ordinary differential equations ([1]).

BOULANGER has proved that the third-order, ordinary differential equations that have the form:

$$y''' = R(x, y, y', y''),$$

in which R is a rational function of y', y'' that is analytic in x and y, and that admit a continuous ∞^1 group:

$$x_1 = x, \qquad y_1 = F(x, b, a, b, c)$$

are either linear or reducible to linear by means of a transformation of the – possibly, associated – variable and a change of the function ([2]), such that one abandons any hope of being able to arrive that determination of new transcendental classes by taking this path.

ZINDLER ([3]) has instituted a direct line of systematic research on systems of ordinary differential equations that admit a finite, continuous group of point-like transformations, in which he arrived, in particular, at the determination of all differential systems in three unknown functions that admit a point-like ∞^2, ∞^3, or ∞^4 continuous group, where in the last case one is limited to groups that contain an Abelian or transitive invariant subgroup ([4]).

CAMPBELL ([5]) has classified the types of second-order, linear, partial differential equations in three independent variables that admit a continuous group of transformations with no more than three parameters ([6]).

([1]) "Zur Theorie der eingliedrigen Gruppen in der Ebene und ihre Beziehungen zu der gewöhnlichen Differentialgleichungen 1° Ordnung," Wien. Ber. **112** (1903). – "Zur Geometrie der gewöhnlichen Differentialgleichungen," Festschrift dedicated to L. Boltzmann (1904).

([2]) "Sur les équations différentielles du troisième ordre, qui admettant un groupe continu de transformations," Comptes rendus **136** (1903); Bull. de la Soc. math. de France **31** (1903).

([3]) In order to get back to the case of a first-order system or a unique differential equation in two variables, it is sufficient to increase the number of variables or the maximum order of derivation. However, it is clear that in this way one arrives at a differential system that is not completely equivalent to the given one.

([4]) "Ueber simultane gewöhnliche Differentialgleichungen, welche continuierliche Transformationsgruppen gestatten," Monats. f. Math. u. Phys. **11** (1900).

([5]) "On the Types of linear partial differential Equations of the second order in three independent variables which are unaltered by the Transformations of a continuous group," Trans. Amer. Math. Soc. **1** (1900).

([6]) On the basis of this group-theoretic concept, BISCONCINI has classified the dynamical problems that relate to holonomic systems with time-independent constraints that, when no forces act, admit a group of fundamental integrals ("Di una classificazione dei problemi dinamici," Nuovo Cim. (5) **1** (1901). By tracing the research of LEVI-CIVITA on binary potentials, the same author has applied group-theoretic considerations to the determination of the types of solutions of the equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2} = 0$$

Finally, WILCZINSKI, generalizing, in a certain sense, differential systems with fundamental solutions, has considered ([1]) ordinary differential systems in n unknown functions whose general solution η_i (i = 1, 2, …, n) is obtained from an arbitrary particular solution y_i (i = 1, 2, …, n) by means of relations of the form ([2]):

$$\eta_i = \sum_k \varphi_{ik}(x, a_1, a_2, \ldots, a_r)\, y_k \,,$$

where the φ_{ik} are uniform functions of x.

These relations, in which one considers the x to be an untransformed variable, define a continuous ∞^r group that WILCZINSKI called *linearoid*, while referring to the differential systems above by the same name. On the analytical nature of the solutions of these systems, which have critical fixed points, WILCZINSKI has arrived at an investigation of a general character, and then proceeded to a complete determination of the linearoid groups in two variables and the corresponding differential systems.

On the new contributions to the theory of the *classification* and *integration* of differential equations and systems on the basis of their respective *rationality groups*, one must mention that later on this will account for the investigations into infinite, continuous groups.

Meanwhile, here we must mention the research of MAROTTE, who has, above all, shed light upon the analytical study of the singularities of a linear differential equation with rational coefficients and the classification of the transcendents, and shown that its integration naturally and advantageously presents the consideration of those subgroups of PICARD's rationality groups that he called *meromorphy groups* relative to the various singular points of the equation and that each, in substance, enjoy the characteristic property in the neighborhood of a singular point that is analogous to the one that defines the rationality group with to all of the complex plane ([3]).

In another direction, FANO has, with great profit, treated the theory of rationality groups in some investigations that, upon the basis of geometric considerations, has arrived at homogeneous, linear, differential equations whose fundamental solutions are constrained by algebraic relations ([4]).

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that can depend upon just two parameters. ("Sulle vibrazioni di una membrana che si possono far dipendere da due soli parametri," Mem. della R. Acc. delle Sc. di Torino (2) **54** (1903). My paper "Tipi di potenziali che, divisi per una funzione fissa, si possono far dipendere da due sole variabili," Rend. del Circ. mat. di Palermo **16** (1902) also belongs to this same line of reasoning.

([1]) "On Systems of multiform functions belonging to a group of linear Substitutions with uniform coefficients," Amer. J. Math. **21** (1899). – "On Linearoid Differential Equations," *ibid.*, *ibid.* – "On continuous Binary Linearoid Groups and the corresponding Differential Equations and A Functions," *ibid.*, **22** (1900).

([2]) He treated (particular) differential systems with fundamental solutions where the φ_a were independent of x.

([3]) "Les équations différentielles linéaires et la théorie des groupes," Ann. de la Fac. Sc. de Toulouse **12** (1898).

([4]) This research is, on the one hand, previous to the decade that we are concerned with. We shall confine ourselves to mentioning the note on differential equations that belong to the same species as them additionally [Atti della R. Acc. delle Sc. di Torino **34** (1899); Rend. Lincei (5) **8** (1899)] and the voluminous paper in which all of the preceding research of FANO is summarized and coordinated: "Ueber lineare homogene Differentialgleichungen mit algebraischen Relationen zwischen der

Finally, along this line of thinking, the research of LOEWY ([1]) on the reducibility of linear differential equations in relation to the respective rationality groups deserves some mention, along with his thoughts regarding the fundamental principles of PICARD-VESSIOT theory ([2]).

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In order to complete our summary regarding the work on the theory of finite, continuous groups, before passing on to the infinite groups in the LIE sense, we point out two species of groups that were spontaneously brought to the attention of mathematicians and might one day be the object of worthwhile investigations.

S. KANTOR, in his last paper, while referring to the group of birational transformations of an arbitrary space, has, by considerations and deductions that were, so to speak, very nebulous, taken the opportunity to study the mixed groups of infinite rank, also in the case in which the transformations that belong to the product of two such ranks are not given either in a unique rank or a finite number of them ([3]).

On the other hand, J. LE ROUX has introduced analytic functions of an infinite number of variables into the study of systems of partial differential equations and considered the general solutions to be functions of the independent variables, the initial values, and the initial values of the fundamental parameters (which are infinite in number, except for the MAYER systems) and showed how the study of the variations that such solutions are subjected to by varying the initial values are naturally connected to certain continuous groups of transformations with a finite number of parameters, but which operate on an infinite number of variables. With regard to the research of LE ROUX along these group-theoretic lines, only the first part of which has been published, to my knowledge, it does not go beyond the first definitions ([4]).

SPECIAL INFINITE CONTINUOUS GROUPS

Passing to the infinite, continuous groups, in the sense of LIE, we rapidly enumerate the various particular classes of infinite groups that have been the object of inquiry during the last decade in that field.

Fundamentallösungen," Math. Ann. **53** (1900). To the note of FANO that was cited just now, we directly attach the two notes of LOEWY [Compted rendus **133** (1901); Münch. Ber. **32** (1902)].

([1]) "Ueber die irreduciblen Factoren eines linearen homogenen Differentialausdrückes," Leipz. Ber. **54** (1902). – "Ueber reduzible lineare homogene Differentialgleichungen," Gött. Nachrichten (1902); Math. Ann. **58** (1903). – A noteworthy theorem on the various possible decompositions into factors of a linear differential form (a theorem was also proved in the first of the two preceding notes) is found in LANDAU: "Ein Satz über die Zerlegung homogener linearer Differentialausdrücke in irreducible Factoren," J. f. d. reine u. angew. Math. **124** (1901).

([2]) "Sur les groupes de transformations des équations différentielles linéaires," Bull. des Sc. math. (2) **26** (1902). – "Ueber die Adjunction von Integralen linearer homogener Differentialgleichungen," Math. Ann. **59** (1904). Cf., also, "Zur Gruppentheorie mit Anwendungen auf die Theorie der linearen homogenen Differentialgleichungen," Trans. Amer. Math. Soc. **5** (1904).

([3]) "Ueber eine neue Klasse gemischter Gruppen und eine Frage über die birationelen Transformationen," Wien. Ber. **112** (1903).

([4]) "Recherches sur les équations aux dérivées partielles," J. de math. (5) **9** (1903).

KOWALEWSKI has proved that in five variables there exist no other primitive, infinite, continuous groups except for three types that were noted by LIE: the total point group, the group of proportional transformations, and the group of equivalence transformations ([1]).

I will take this occasion to observe that the same thing happens in S_4 that happens in S_3 ; viz., along with the three primitive groups of LIE, there is also the infinite point group that transforms the Pfaffian equation:

$$dz - y\,dx = 0$$

(namely, the group of contact transformations in the plane ([2])).

ENGEL, continuing a search that was initiated by VIVANTI ([3]) and completed by E. v. WEBER ([4]), has determined, analogous to what he had done for the infinitesimal contact transformations ([5]), all of the infinitesimal transformations that leave an arbitrary Pfaffian equation unaltered ([6]). Following the path of ENGEL, F. WINKLER, in his dissertation at Lipsia, determined the infinitesimal transformation that transform an arbitrary Pfaffian into itself, up to a numerical factor and an additive term that is an exact differential ([7]).

PASCAL ([8]) has made noteworthy contributions to the analogous problems for the form of differentials of order higher than the first; in particular, ones of second order.

FORSYTH, continuing and modifying the process of ZORAWSKI ([9]), determined and discussed, in conformity with the methods and basis for the views of LIE, differential invariants of curves in the plane ([10]), of an arbitrary surface ([11]), and of the curves and surfaces in space ([12]), ([13]).

([1]) "Die primitiven Transformationsgruppen in fünf Veränderlichen," Leip. Ber., **51** (1899).

([2]) "I gruppi continui infiniti primitivi in tre e quattro variabilii," Atti della R. Accad. di Sc. L. e A. in Modena (3) **7** (1906).

([3]) "Sulle trasformazioni infinitesime che lasciano invariata un'equazione pfaffiana," Rend. del Circ. mat. di Palermo **12** (1898).

([4]) (DHD: previous footnote duplicated in original)

([5]) "Kleinere Beiträge u. s. w., III. Die infinitesimalen Berührungstransformationen," Leipz. Ber. **43** (1891).

([6]) "Die infinitesimalen Transformationen einer Pfaff'schen Gleichung," Leipz. Ber. **51** (1899).

([7]) "Die infinitesimalen Transformationen, welche einen Pfaff'schen Ausdruck absolut der modulo eines vollständigen Differentials invariant lassen," Dissertation, Leipzig (1905).

([8]) "Le trasformazioni infinitesime applicate ad una forma differenziale di ordine *r*." Rend. Lincei (5) $\mathbf{12}_2$ (1903). – "Sulle trasformazioni infinitesime che lasciano invariata una forma o una equazione ai differenziali totali," *ibid., ibid.*

([9]) "Ueber Biegungsinvarianten. Eine Anwendung der Lie'schen Gruppentheorie," Acta math. **16** (1892).

([10]) "Differential invariants of a Plane and of Curves in the Plane," Rend. del Circ. mat. di Palermo **21** (1906).

([11]) "The differential invariants of a surface, and their geometric significance," Phil. Trans. **201** (A) (1903); Proc. of the R. Math. Soc. **71**.

([12]) "The differential invariants of space," Phil. Trans. **202** (A) (1904); Proc. of the R. Math. Soc. **72** (1904).

([13]) Research into integral invariants, also in the context of the theory of continuous groups, has been carried out by TH. DE DONDER, "Étude sur les invariants intégraux," Rend. del Circ. math. di Palermo **15** (1901), **16** (1902): "Application nouvelle des invariants intégraux," Mém. couronnés, etc., publiés par l'Acad. de Belgique; nuov. séries; **1** (1905).

WILCZINSKI, generalizing the classical work of LAGUERRE, HALPHEN, and BRIOSCI ([1]), has determined the infinite point group that leaves unaltered the form of an arbitrary system of linear, ordinary differential equations in several unknown functions and has studied its differential invariants, and explicitly constructed the complete system in the particular case of two equations of second order in just two functions.

Just as any linear differential equation corresponds to a projective type of integral curve, any system of two second-order ordinary linear differential equations in two functions may be associated with projective type of ruled surface; this is why WILCZINSKI has applied his results to the study of the differential projective properties of the ruled surface ([2]).

Finally, MEDOLAGHI has considered the infinite, continuous groups that depend upon only one arbitrary function and one argument and has classified the second-order partial differential equations in two independent variables that admit such a point group ([3]).

As far as the *infinite, continuous groups of contact transformations* are concerned, it results from the work that was cited above by KOWALEWSKI ([4]) that any such group of S_3 that does not coincide with the total group operates imprimitively on the five-dimensional space of surface elements. That is why I determined the infinite groups that are analogous to the finite groups of SCHEFFERS, namely, the irreducibile, infinite, continuous groups of S_3 that transform one of the ∞^1 first-order partial differential equations ([5]).

LEBESGUE, recalling a problem that LIE had already occupied himself with, without actually publishing the solution, determined the contact transformations that make *any* minimal surface correspond to a minimal surface. They are transformations of the plane, and the most general contact transformation of this group is obtained, except for a similitude, by arbitrarily fixing a minimal surface Σ_0 and making any plane P correspond to the plane that is parallel and equidistant to P and to the tangent plane to Σ_0 that is parallel to P. This same contact transformation transforms all of the surfaces that are parallel to a minimal surface amongst themselves ([6]).

([1]) The invariant theory of homogeneous, linear, differential equations has also been systematically elaborated using the methods of LIE, by BOUTON, "Invariants of the general linear differential equation and their relation to the theory of continuous groups," Amer. J. Math. **21** (1899).

([2]) The numerous memoirs of WILCZINSKI on the subject in question are published, for the most part, in the *Transactions of the Amer. Math. Society*, starting in 1901. The results were systematically presented by WILCZINSKI in their totality, with a personal revision of the theory of HALPHEN on the projective invariants of plane and bent curves, in: *Projective differential Geometry of Curves and ruled surfaces* (Leipzig, 1906).

([3]) "Sui gruppi isomorfi al gruppo di tutte le trasformazioni di una variabile (Mem. I)," Rend. del Circ. Mat. di Palermo **12** (1898). – "Classificazione delle equazioni alle derivate parziali del secondo ordine, che ammettano un gruppo infinito di trasformazioni puntuali," Ann. di. Math. (3), t. 1 (1898).

([4]) "Die primitiven Transformationsgruppen in fünf Veränderlichen," Leipz. Ber. **51** (1899).

([5]) "Sui gruppi continui infiniti di trasformazioni do contatto dello spazio," Mem. della R. Acc. delle Sc. di Torino (2) **57** (1906).

([6]) "Sui gruppi isomorfi al gruppo di tutte le trasformazioni di una variabile (Mem. I)," Rend. del Circ. Mat. di Palermo **12** (1898). – "Classificazione delle equazioni alle derivate parziali del secondo ordine, che ammettono un gruppo infinito so trasformazioni puntuali," Ann. di. Math. (3) **1** (1898).

Lastly, we mention the reducible, infinite, continuous group of *equidistant* contact transformations of the plane, to which SCHEFFERS was led in his work on *isogonal* and *equi-tangential curves* ([1]). This group of equidistant contact transformations is composed of all the contact transformations that transform lines into lines ([2]) *in such a way that corresponding segments are equal*. – Such infinite continuous groups compare perfectly to the planar conformal group. They also contain, as maximal point-like subgroups, the group of motions and similitudes, and transform one system of ∞^2 equi-tangential curves into another one in precisely the same way that the conformal group operates on systems of ∞^2 isogonal curves. Moreover, if in the plane the line assumes the coefficients u, v for its coordinates that appear in the normal form for the equation of the line:

$$x \cos u + y \sin a - v = 0,$$

and along with the real unit we introduce another unit j such that $j^2 = 0$, then the most general equidistant contact transformation is represented in the form:

$$\bar{u} + j\,\bar{v} = f(u + j\,v),$$

where f denotes an arbitrary analytic function of the complex field $[1, j]$ ([3]).

As in the case of the conformal group, the equidistant transformations that transform circles into circles constitute a mixed algebraic group ∞^6 that is analytically representable in the complex field $[1, j]$ by means of linear transformations and constitutes a fundamental group of the "geometry of direction" of LAGUERRE ([4]).

STUDY has shown how there exists a group of equidistant contact transformations for any surface that is characterized by the properties of transforming geodesics into other ones and of leaving the geodetic distance between the points of two arbitrary linear elements of a geodesic unaltered. In particular, for a surface of constant curvature, such equidistant contact transformations admit an analytic representation that is analogous to that of the conformal group and the SCHEFFERS group, on the basis of convenient systems of complex numbers. Moreover, STUDY has also extended his considerations to three-dimensional spaces and to ones of constant curvature and it is noteworthy that whereas in non-Euclidian S_3 the equidistant contact transformations, like the conformal point transformations, constitute a finite group with ten parameters, in Euclidian S_3 they give rise to an infinite, continuous group ([5]).

([1]) "Die primitiven Transformationsgruppen in fünf Veränderlichen," Leipz. Ber. **51** (1899).

([2]) Which certainly results in the reducibility of the group.

([3]) "Isogonalkurven, Aequitangentialkurven, und komplexe Zahlen," Verh. des 3. intern. Math.-Kongr., Heidelberg, 1904; Math. Ann. **60** (1905).

([4]) GRÜNWALD, "Ueber duale Zahlen und ihre Anwendungin der Geometrie," "Monats. f. Math. u. Phys. **17** (1906).

([5]) "Ueber mehrere Probleme der Geometrie, die dem Problem der konformen Abbildung analog sind," Sitzber. d. Miederrhein. Ges. f. Natur.-u. Heilkunde, Bonn (1904).

GENERAL THEORY OF INFINITE CONTINUOUS GROUPS

Due to its broader and more elevated level of ideas and results, we have finally seen fewer publications in this decade on the *general theory of infinite continuous groups.*

Only VESSIOT, CARTAN, and MEDOLAGHI have worked in that field.

VESSIOT, in the first part of his *Memoir couronné* in 1903, had chiefly the distinction of recalling, explaining, and completing the work of ENGEL and MEDOLAGHI in some respects, which has not attained a certain circulation and following that would be equal to its importance.

ENGEL, in his *Dissertation* that appeared in 1885) ([1]) and in a complementary note of 1894 ([2]), has given us a method that permits us to construct the defining equations of infinitesimal transformations for *any* continuous group – finite or infinite – in n variables when one knows the infinitesimal transformations of certain finite continuous groups with a particular composition. This singular correspondence between groups in n finite or infinite variables and certain special finite groups – or *characteristic groups* – of Engel was studied and discussed in 1897 by MEDOLAGHI ([3]), and he succeeded, in particular, in identifying the characteristic groups of ENGEL with two series of transitive, simple groups that were pair-wise reciprocal, and whose finite equations are obtained in the following way: One imagines three sequences of n variables:

$$z_i, y_i, x_i \qquad (i = 1, 2, \ldots, n)$$

and if one thinks of the z_i as composite functions of the x_i , by means of the y_i , then one expresses the derivative, which is finite to a certain order, of z with respect to x as a function of the derivative of z with respect to y and that of y with respect x. Finally, in the equations thus obtained one considers the derivative of the z with respect to x as new variables, the derivative of z with respect to y (or of y with respect to x) as old variables, and the derivative of y with respect to x (or of z with respect to y) as parameters. These are the indicated groups.

One striking result of the analysis of MEDOLAGHI it that it has led to a well-defined and direct procedure that permits one to pass to the defining equations of the infinitesimal transformations of any group from the defining equations of the respective finite transformations without having to know the corresponding characteristic group of ENGEL; at the moment, these defining equations in the MEDOLAGHI form represent the simplest and most manageable instrument for research that we possess in that field.

All of the continuous groups that belong to the same LIE type, namely, ones that have the similar mediating point transformations of the space in question, have the same characteristic group of ENGEL; however, conversely, to the same ENGEL group there generally corresponds more than one type of group in the LIE sense, which we, like MEDOLAGHI, say have *ENGEL type*.

([1]) "Ueber die Definitionsgleichungen der continuirlichen Transformationsgruppen," Math. Ann. **27** (1886).

([2]) "Kleinere Beiträge, etc.; I: Definitionsgleichungen der continuirlichen Transformationsgruppen," Leipz. Ber. **46** (1894).

([3]) "Sulla teoria dei gruppi infiniti continui," Ann. di Mat. **25** (1897).

In 1899, MEDOLAGHI, motivated by his defining equations, sketched out an admittedly perfunctory direct method for separating the various LIE types that were contained in the same ENGEL type ([1]).

Now, as we already alluded, in the first part of the *Memoir couronné* ([2]) VESSIOT revised and recast the analytical deductions of ENGEL and MEDOLAGHI, and was led to conclude the aforementioned discussion relative to the various LIE types that belong to the same ENGEL type in detail. On the basis of an analogous argument, he showed how, starting from the defining equations of a group in MEDOLAGHI form, one may determine the various subgroups, and in particular, the invariant subgroups, and finally made the concept of isomorphism more precise for infinite groups, which preceded CARTAN in the work that he did shortly afterwards.

The second part of the *Memoir couronné* was dedicated to the integration of differential systems that admit a continuous group of transformations ([3]).

Such an integration problem, on the basis of LIE's principle, may always be decomposed into a series of problems that lead to a series of differential *resolvents*, which must naturally be arbitrary, while considering, among other things, systems whose more general solutions may be deduced from a particular solution by means of the transformations of a simple continuous group. Now, VESSIOT has proved that one can always proceed in such a way that these ultimate differential systems are *automorphic*, namely, in such a way that the most general solution is obtained from an arbitrary solution by means of a continuous group of point-like transformations that act upon only the unknown functions; despite it formal nature, this simplification has a noteworthy importance.

In the third and final part ([4]), VESSIOT occupied himself with the extension of GALOIS theory to linear partial differential equations and their complete systems, an extension that had already been first indicated by DRACH ([5]), and for which VESSIOT completely abandoned the algebraic methods of GALOIS, while generalizing that of PICARD for the case of ordinary differential equations, and substituted, by reason of the diverse nature of the problem being considered, an analytical method that was useful for the examination of both the problems of GALOIS and PICARD.

VESSIOT, in the paper that we just pointed out, discovered a gap in the proof of a theorem of MEDOLAGHI: It was occasioned by the recent research of MEDOLAGHI ([6]), in which his theorem was well-established on a solid basis, at least in a special case of obvious interest.

MEDOLAGHI ([*****]) had already first noted how those infinite groups whose defining equations PICARD had arrived at in 1891 by generalizing the partial differential

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([1]) "Contributo alla determinazione dei gruppi continui in une spazio ad *N* dimensioni," Rend. della R. Accad. dei Lincei (5) **8** (1899).

([2]) "Sur la théorie des groupes continus," Ann. de l'Éc. N. Sup. (3) **20** (1903).

([3]) "Sur l'integration des systèmes différentielles qui admettant des groupes continus de transformations," Acta math. **28** (1903).

([4]) "Sur la théorie de Galois et ses diverses généralisations," Ann. de l'Éc. N. Sup. (3) **21** (1904).

([5]) Ann. de l'Éc. Norm. Sup. (3), vol. ? (1898).

([6]) "Sopra i gruppi definiti da equazioni differenziali del 1.[*] ordine," Rend. del Circ. mat. di Palermo, **24** (1907).

([*****])

equations of the theory of functions in one complex variable, or, if one prefers, the defining equations of the planar conformal group ([1]), could be characterized in the set of all infinite groups as the ones that contained the commutative group of all translations of the related space. The PICARD groups are such groups, and MEDOLAGHI proved that *any* ENGEL *type of first order* (viz., one defined by a first-order equation) *contains a* PICARD *group*. Also note that, as MEDOLAGHI has observed, when one abstracts from the total projective group or the infinite group of proportional transformations, which are defined by only second-order equations, any group is either finite or infinite and of first order, or it contains a group of first order.

Now, MEDOLAGHI, on the basis of his theorem and the fact that the characteristic groups of ENGEL of groups of first order are all contained in the parametric groups of the linear homogeneous group in n variables, has established a correspondence for the infinite groups of PICARD, and for various types of homogeneous, linear groups, of which already the first and most obvious consequence has proved its singular importance, for the time being, if one limits oneself as MEDOLAGHI did. – It is, in particular, noteworthy that the link between this method of research into the PICARD groups and the classical method that was given by LIE is developed in a series around a generic point.

While MEDOLAGHI and VESSIOT, like ENGEL before, had attached their deductions to the first principles that were previously established by LIE for the theory of infinite groups, the research of CARTAN completely excluded the consideration of infinitesimal transformations of the group, and it dealt systematically with the theory of integration of the system of total differential equations that he himself had illustriously constructed, while elaborating and generalizing some viewpoints that go back to GRASSMANN ([2]).

At the basis of his deductions one finds the concept of holohedral isomorphism – or the equivalence of composition – for infinite groups, which he defined in the following way: Among other things, he said that, given a group G that operates on n variables x_1, …, x_n, another group G' that operates on the x, along with m other variables y_1, …, y_n, is obtained by prolonging G if G' transforms the x amongst themselves in exactly the same way as the given group G; he called this prolongation *holohedral* when the identity transformation of the group G corresponds to only the identity transformation in the prolonged group. Having assumed this, CARTAN called two groups *holohedrally isomorphic* if it is possible to holohedrally prolong them in such a way as to obtain two groups that operate on the same number of variables and are similar to each other.

Now, the most important problem consists in the search for the criteria that permit one to recognize that equivalence of composition, and CARTAN solved it by an analytic method that appears to be, to a certain degree, obscure and artificial, but, when one understands it, has led the author to some very precise and important results. Assuming the definition of the finite transformations of a group G in the form of a system of total

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([1]) "Sur les groupes que se présentent dans la généralisation des fonctions analytiques," Comptes rendus **126** (1898).

([2]) "Sur l'intégration des systèmes d'équations aux différentiels totales," Ann. sc. l'Éc. N. Sup. (3) **18** (1901). – "Sur l'intégration des systèmes différentiels complètement intégrables," Comptes rendus **134** (1902). – "Sur l'equivalence des systèmes différentiels," *ibid.,* **135** (1902).

differential equations, he proved how it is always possible to holohedrally prolong the given group in such a way as to obtain a group G' that is the *maximal* group with respect to the invariants of h particular functions U_i (which naturally omits the case of transitive groups) and $r + p$ particular Pfaffians:

$$\omega_1, \ldots, \omega_r \,;\, \varpi_1, \ldots, \varpi_p ,$$

such that if:

$$dU_i = V_{i1}\omega_1 + \ldots + V_{ir}\omega_r$$

and the bilinear covariants of ω_k ($k = 1, \ldots, r$) have the form:

$$\sum_{ij} c_{ijk}\omega_i\omega_j + \sum_{i\rho} a_{i\rho k}\omega_i\varpi_j ,$$

in which the V_{ik}, c_{ijk}, $a_{i\rho k}$ are functions of the U (that reduce to numerical constants for transitive groups) then the $a_{i\rho k}$ constitute an *involutory* system (1). These coefficients V_{ik}, c_{ijk}, $a_{i\rho k}$ characterize the structure of the group, such that two groups for which those coefficients coincide are holohedrally isomorphic to each other.

CARTAN demanded that these structure coefficients (which generalize the composition constants of finite groups) must satisfy necessary and sufficient conditions, and this had the more noteworthy consequence that he mentioned some considerations about the possibility of reducing the degree of intransitivity in certain cases without altering the structure of the group and the observation that, contrary to what happens for finite groups, the minimum degree of intransitivity for the structure of an infinite group must be non-zero, or, in other words, that *there exist intransitive infinite groups that are not holohedrally isomorphic to any transitive group.*

CARTAN discussed in detail the problem of the determination of all groups that are holohedrally isomorphic to a given group, if one accepts the new definition of a group that it is characterized by all of the transformations that induce a certain linear group on certain Pfaffians (which is interesting for its possible relations with the characteristic group of ENGEL). In particular, it resulted from this discussion that, contrary to what happens for finite groups, *an infinite group may be merohedrally isomorphic to itself*, which gives rise to the necessity of distinguishing two types of merohedral isomorphisms: *proper and improper*. One says that the merohedral isomorphism between two groups is *proper* if it is not possible to regard them as holohedrally isomorphic in some manner, which again implies that the simple groups can be distinguished as proper and improper. The foremost examples are the ones that admit no merohedrally isomorphic group, whether proper or improper; other ones might admit improper merohedral isomorphisms, but not proper merohedral isomorphisms.

This reveals the exceptional difficulty in the problem of the decomposition of an infinite group into a normal series of subgroups, and indeed, as CARTAN pointed out, in certain cases one is led to doubt the possibility of finding a decomposition into a *finite* series of simple subgroups. Finally, CARTAN applied his general theory to the study of

(1) Which implies certain linear relations between the $a_{i\rho k}$ (which depend, in part, upon possible linear relations between ω_i, ϖ_i) that are pointless to specify.

the groups that depend upon arbitrary functions of each argument; in particular, he proved that such transitive and simple groups have the same structure as the total group in just one variable.

More recently, CARTAN, in a note to the *Comptes Rendus*, has resolved another problem of more advanced interest, chiefly in the theory of integration, namely, the determination of all *infinite, simple, continuous groups*, which, in contrast to the analogous problem for finite groups, offers a new level of difficulty for the existence of intransitive groups that are not holohedrally isomorphic to any transitive group ([1]).

Now, CARTAN has proved that, in the first place, the simple transitive groups all reduce to the four types that were already pointed out by LIE: The full point group, the group of proportional transformations, the full group of contact transformations, and the group of all contact transformations of the x_i, p_i .

In the second place, he determined all of the simple groups that are not isomorphic to any transitive group, proving, in particular, that the simple, intransitive groups, properly speaking, are obtained from transitive, simple groups, given that the arbitrary elements in them depend upon an arbitrary number of non-transformed variables of the group in a possibly more general way.

The consequences of this are truly beautiful and important!

In the field of infinite groups, which had been scarcely noticed until a few years ago, two viewpoints are discussed as of now: That of ENGEL-MEDOLAGHI-VESSIOT and that of CARTAN; both of them have pointed to paths that might lead to new and unexpected conclusions.

However, given the tribute that has been devoted to the intrinsic merit and lasting value of such results, others might beg to differ.

The two directions that were mentioned just now are mutually divergent, and contain, as it were, a regime in which it is, above all, important to carry out the investigations diligently until one has established a system of relations – if not, of communication – between the two viewpoints.

That might raise the fundamental problems of the theory of infinite groups to a new conception, perhaps a more synthetic conception, and one that would better respond to the general view of LIE that in any problem everything is subordinate to the untiring search for the intimate harmony between methods and results that inspires the character and esthetic value of all mathematical inquiry.

([1]) "Les groupes de transformations continus, infinis, simples," Comptes rendus **144** (1907).