H-Measures and Applications

Luc Tartar

Department of Mathematics, Carnegie Mellon University, Pittsburgh, PA 15213, USA

Introduction

Historians interested in the evolution of Science will probably be very surprised when they will analyse all the strange fashions that have struck the scientific community in the second part of this century. They will certainly be aware of the political orientation of those who had launched many of these fake new ideas and wonder why mathematicians had not been more rational in their behaviour. Will they find another example of political censorship than the one that had suppressed two pages of the introduction of an article where I had described my scientific ideas [1]? Will everyone agree that it was indeed slanderous that I had thanked there two of my teachers, Laurent Schwartz and Jacques-Louis Lions and that this part should definitely be cut? Had the censors thought that they could suppress the mention of other political facts and avoid me describing them elsewhere [2]? Historians may wonder at the stupidity of these censors and ponder if they had even understood the meaning of the few lines that they had spared at the beginning: "Il y a une différence énorme entre l'étude des singularités d'équations aux dérivées partielles (linéaires ou non) et celle de leurs oscillations: c'est la différence entre la physique classique et la physique quantique".

In this article I had described my point of view that the study of oscillating solutions of partial differential equations was the key mathematical question to investigate in order to shed some light on the strange rules invented by physicists for explaining natural phenomena. Every specialist of differential equations is aware of the distinction between finite and infinite dimensional effects and it is only the result of an intensive propaganda that so many have adopted a point of view about mechanics which was adequate in the eighteenth century when partial differential equations had not yet found their place and that continuum mechanics and electromagnetism were not even thought of. However, even if all the extensive knowledge about linear partial differential equations contained in the treatise of L. Hörmander [3] had been available at the beginning of the century, it would not have helped so much the physicists puzzled as they were by the spectroscopic measurements of light absorbed and emitted in some gases. One cannot blame then those who have invented the strange rules of quantum

[1] There is a huge difference between the study of singularities of partial differential equations (be them linear or not) and that of their oscillations: it is the difference between classical and quantum physics.

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mechanics for their lack of knowledge of partial differential equations but one should blame those who have transformed these rules into dogma. As R. Penrose once wrote "Quantum theory, it may be said, has two things in its favour and only one against. First, it agrees with all the experiments. Second, it is a theory of astonishing and profound mathematical beauty. The only thing to be said against the theory is that it makes absolutely no sense" . In order to give a rational explanation of the puzzling measurements made in these experiments one needs an increased knowledge of partial differential equations and there are a few new mathematical questions that should be understood for that purpose.

As light is involved we know that there will be some hyperbolic equations, the wave equation or Maxwell's system or some even larger system, and we expect that the linearised system will only have the velocity of light as characteristic speed so that we may reasonably restrict our attention to semilinear systems. Even standard questions as the relation between the wave equation and geometrical optics needs to be thought again. Fifteen years ago it was already clear why the mathematical results now found in [3] were not adapted to the goal that I was pointing at and a first reason was that one cannot expect to understand the partial differential equations of continuum mechanics without accepting discontinuous coefficients; even if one was ready to make smoothness assumptions and stay away from interfaces one definitely had to avoid assuming the coefficients to be infinitely differentiable or analytic. There is however a more important drawback of the linear theory of propagation of singularities which became more apparent once I had obtained my personal version of propagation using the tool of H-measures [4] which I will describe in a moment. What the linear theory is really interested in is the propagation of regularity and this leads to a quite negative concept of a singularity which is not defined as a quantitative object; of course, measuring the propagation of H^s regularity instead of C^∞ regularity does not correct this defect in any way. The physically intuitive idea of a beam of light is then absolutely not described by the theory of "propagation of singularities" for partial differential equations and the inadequacy is hidden by the fact that the bicharacteristic rays which have appeared in the linear theory are precisely those which physicists had thought important in their formal computations. One should then criticise this approach of propagation of singularities for describing the properties of light as not making more sense than some physicists' rules; a better test for a mathematical tool than making the bicharacteristic rays appear is to be able to measure what is transported along them and tell what happens along the bicharacteristic rays to important quantities for physics like energy and momentum.

As matter is also involved we face much more trouble because the question of what matter could be is at stake anyway and, even if the rules of quantum mechanics are indeed wrong, one cannot forget about the real defects of the classical concepts of light and matter. A probably good mathematical model to understand is the coupled Maxwell-Dirac system where matter is described by a complex four dimensional vector field and light is described by the electromagnetic field, the coupling involving quadratic terms with the famous Planck constant h appearing as a coupling parameter between light and matter and not as this mysterious parameter that the dogma wants to attach to every hamiltonian.

[2] This is the first paragraph of a review by R. Penrose of a book by J.C. Polkinghorne "The Quantum World" in The Times Higher Education Supplement, March 23, 1984.

If we were to follow this indication we would be interested in understanding some mathematical properties of semilinear hyperbolic systems with quadratic interaction and the special role played by the four dimensional space-time where we apparently live would probably be linked to Sobolev's embedding theorem, but before playing such a game which accepts too much of the physicists' dogmatic postulates, we should look at a more rational explanation of the mathematical difficulties encountered in the spectroscopic measurements.

Even if we do not know what atoms really are they do appear as tiny obstacles and we have then to face the difficulty of working with at least two scales, a microscopic one and a macroscopic one. In some way the practical goal of quantum physics is to compute corrections in the effective equations satisfied by the macroscopic quantities from a fine and possibly wrong description of what equation the microscopic quantities do satisfy. As mathematicians we should describe a general framework in order to understand more of this question and there are indeed choices to make and difficulties to overcome.

The first obvious choice that we have already made is to work with partial differential equations and not with ordinary differential equations; this can be considered a lesson learned from A. Einstein about the defects of I. Newton's classical approach. There has been much propaganda in recent years for works emphasising finite dimensional effects in partial differential equations and one may indeed be attracted by some of the difficult and interesting mathematical questions which had led H. Poincaré to introduce so many tools and ideas before the development of quantum physics. It was another great mathematician who formalised some of the rules followed by physicists in their quantic games but it is surprising that J. Von Neuman would show that no ordinary differential equation could produce the same results as the rules of quantum mechanics and forget to question the very nature of that set of rules. Certainly if one believed that one should create a game that will generate a sequence of numbers one might be tempted by the mathematical properties of spectra of linear operators. Was then the dogma already well accepted before it became obvious that the spectroscopic experiments were not generating mere lists of numbers? Were mathematicians so impressed by this apparent success of functional analysis? Had there been an intentional effort of propaganda around functional analysis in order to avoid that mathematicians study more relevant partial differential equations of continuum physics in the spirit of what S. Sobolev and J. Leray had already been doing in the 1930s? Certainly, and L. Hörmander [3] is right in pointing at some misconceptions created by L. Schwartz's approach, but some other misconceptions have been propagated by his own approach to partial differential equations. Is there indeed a classical treatise on partial differential equations which does mention these properties of partial differential equations related to the strange effects observed in spectroscopy or more simply which does quote the relation between microscopic and macroscopic levels which is such a crucial question in physics?

The mathematical tool of H-measures which I have introduced [4, 5] is a new step toward a better understanding of these questions and I have chosen the prefix H as a reminder that these objects had arisen naturally in the theory of homogenisation, a term to which I attribute a more general meaning than which is usually given in the rare books related to the subject like those of A. Bensoussan, J.-L. Lions and G. Papanicolaou [6] or of E. Sanchez-Palencia [7] where periodicity assumptions often obscure the methods which I had developed for more general situations [8], partly in collaboration with F. Murat [9] and as an extension of earlier results of S. Spagnolo [10, 11]. The H-measures added to my previous description of the role of oscillations in partial differential equations [1] that of concentration effects whose importance in continuum physics I had not foreseen before the work of P.-L. Lions, R. DiPerna and A. Majda [12-16]. My initial purpose for introducing H-measures was to derive small amplitude homogenisation theorems [4, 5, 17] in order to explain why some particular formula obtained by physicists [18] was indeed accurate despite the fact that the arguments used in its derivation did not make any sense. I can trace back my intuitive understanding about these objects to some formula for computing an exact quadratic correction term [19] appearing in a model which I had introduced earlier in order to understand some averaging question in hydrodynamics. It was only later that I found a way to use the same H-measures for describing the propagation of oscillations and concentration effects in some partial differential equations [4, 5] obtaining then a quantitative transport property in the form of partial differential equations in x and ξ satisfied by the H-measures. I wanted to avoid the standard theory of pseudo-differential operators [3] and construct what I needed for my quadratic microlocal tool of H-measures in order to be able to study partial differential equations of continuum mechanics without making spurious hypotheses of smoothness for the coefficients. However even for those who have devoted a long time reading [3] H-measures may still appear to be natural as they have been introduced independently by P. Gérard [20, 21] although the name of microlocal defect measures which he has chosen for them may reflect a negative attitude inherent in [3].

Of course H-measures are only a step toward the mathematical understanding of these questions of physics which I had sketched at the beginning and there are other pieces of that scientific puzzle which should not be left aside like the apparition of memory effects by homogenisation, which seems the mathematical explanation of what physicists attribute to their strange rules of spontaneous absorption and emission; it must be emphasised that these homogenisation results are obtained without any postulate of a probabilistic nature. There are some more or less classical cases of memory effects induced by homogenisation like viscoelasticity which can be found in Sanchez-Palencia [7] but the effects which I was mentioning are related to hyperbolic situations and have not received much attention apart from my own tentatives [22, 23] and that of Y. Amirat, K. Hamdache and A. Ziani [24, 25] and so a lot remains to be done.

H-Measures

Contrary to wave front sets which can be attached to general distributions but are mere geometric sets endowed with a negative property of lack of smoothness, H-measures are only defined for sequences of functions converging weakly to zero in $L^2(R^N)$ and express in a quantitative way the limit of quadratic quantities, the H-measure being zero in the case of strong convergence in $L^2(R^N)$ and, because they are measures on $R^N \times S^{N-1}$, they can see the action of a class of pseudo-differential operators of order zero.

Definition 1. An *admissible symbol* s is a continuous function on $R^N \times S^{N-1}$ admitting a decomposition $s(x,\xi) = \Sigma_n a_n(\xi) b_n(x)$ with the functions a_n being continuous on S^{N-1}, the functions b_n being continuous on R^N and converging to zero at infinity, and such that $\Sigma_n ||a_n||.||b_n|| < \infty$ where the norms are sup norms. The *standard operator* S with symbol s is the continuous operator on $L^2(R^N)$ defined by $\mathbf{F}(Su)(\xi) = \Sigma_n a_n(\xi/|\xi|)\mathbf{F}(b_n u)(\xi)$ where $\mathbf{F}$ denotes the Fourier transform . A continuous operator L on $L^2(R^N)$ is said to have symbol s if $L - S$ is a compact operator on $L^2(R^N)$.

The only technical point to check is that the commutator $L_1L_2 - L_2L_1$ of two such operators is a compact operator on $L^2(R^N)$.

Proposition 2. *If U^n is a sequence converging weakly to zero in $(L^2(R^N))^p$, then there is a subsequence and measures $\mu^{i,j}$ on $R^N \times S^{N-1}$, $i, j = 1,..,p$, such that for every operators L_1, L_2 with symbols s_1, s_2 the limit of $L_1(U_i^n)L_2(U_j^n)*$ is a measure ν on R^N defined by $< \nu, \phi > = < \mu^{i,j}, \phi s_1 s_2 * >$ for every test function ϕ continuous with compact support in R^N .*

One immediately finds that μ is hermitian nonnegative and has a few other obvious properties, one of them being the following localization principle for H-measures, which is analogous to the information on the wave front sets derived from application of the stationary phase method.

Proposition 3. *If a sequence U^n converges weakly to zero in $(L^2(R^N))^p$, corresponds to a H-measure μ, and is such that $\Sigma_{ij}\partial_i(b_{ij}U_j^n)$ converges strongly to zero in $H^{-1}_{\mathrm{loc}}(R^N)$ where the functions b_{ij} are continuous, then one has $\Sigma_{ij}\xi_i b_{ij}\mu^{jk} = 0$ for $k = 1,..,p$.*

Before describing the more technical property of propagation let us give a few examples of what was just mentioned.

Example 4. Let $u^n(x) = v(x, x/\varepsilon)$ where ε is a sequence converging to zero with v defined on $R^N \times R^N$ and $v(x,y)$ having period 1 in each component y_j, j = 1,..,N; denoting by Y the unit cube, we assume that v is continous in x with values in $L^2(Y)$ and decompose v in Fourier series in y, $v(x,y) = \Sigma_m v_m(x)e^{2i\pi(m.y)}$, assuming moreover that v_0 is zero. Under these hypotheses, without extraction of a subsequence, the H-measure μ associated to u^n is defined by $< \mu, \Phi > = \Sigma_m \int |v_m(x)|^2 \Phi(x, m/|m|)dx$ for every continuous function Φ on $R^N \times S^{N-1}$ with compact support in x.

Example 5. Let $u^n(x) = \varepsilon^{-N/2}v(x/\varepsilon)$ where ε is a sequence converging to zero and v belongs to $L^2(R^N)$. Without extraction of a subsequence the H-measure μ associated to u^n is defined by $< \mu, \Phi > = \int |\mathbf{F}v(\xi)|^2\Phi(0, \xi/|\xi|)d\xi$ for every continuous function Φ on $R^N \times S^{N-1}$ with compact support in x.

Example 6. Let u^n be a sequence converging weakly to zero in $L^2(R^N)$ and corresponding to a H-measure μ; assume moreover that for some continuous

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[3] I use L. Schwartz's notations so that $\mathbf{F}(Su)(\xi) = \int s(x, \xi/|\xi|)e^{-2i\pi(x.\xi)}u(x)dx$ for u smooth with compact support.

[4] z^* denotes the complex conjugate of z.

functions b_j, j = 1,..,N, $\Sigma_j \partial_j (b_j u^n)$ converges to zero in $H^{-1}_{loc}(R^N)$ strong. Then μ satisfies $P(x,\xi)\mu = 0$ with P defined by $P(x,\xi) = \Sigma_j b_j(x)\xi_j$.

Example 7. Assume that the sequence U^n converges weakly to zero in $(L^2(R^N))^N$ corresponds to a H-measure μ and satisfies $\partial_i U^n_j = \partial_j U^n_i$ for i, j = 1,..,N. Then there exists a scalar nonnegative measure ν on $R^N \times S^{N-1}$ such that $\mu^{i,j} = \xi_i \xi_j \nu$ for i, j = 1,..,N.

Example 8. Let u^n be a sequence converging weakly to zero in $H^1(R^{N+1})$ and assume that for some continuous functions ϱ and a_{ij}, i, j = 1,..,N independent of x_0, $\varrho\partial_0^2 u^n$ - $\Sigma_{ij}\partial_i(a_{ij}\partial_j u^n)$ converges to zero in $H^{-1}_{loc}(R^{N+1})$ strong; assume moreover that U^n defined by $U^n_i = \partial_i u^n$ for i = 0,..,N, corresponds to a H-measure μ. Then $\mu^{i,j} = \xi_i\xi_j\nu$ for i, j = 0,..,N and ν satisfies $Q(x,\xi)\nu = 0$ with Q defined by $Q(x,\xi) = \varrho(x)\xi_0^2$ - $\Sigma_{ij}a_{ij}(x)\xi_i\xi_j$.

The propagation effects for H-measures are related to the existence of quadratic balance laws and they take the form of partial differential equations in (x,ξ) satisfied by the H-measures. This is more quantitative than what can be said for wave front sets where it is the complementary property of regularity which is actually propagated. A precise commutation lemma is needed.

Proposition 9. *Under additional regularity hypotheses, if S_1 and S_2 are the standard operators of symbols s_1 and s_2 then $\partial_j(S_1S_2 - S_2S_1)$ is a continuous operator on $L^2(R^N)$ with symbol $\xi_j\{s_1,s_2\}$ where $\{,\}$ denotes the usual Poisson bracket.*

In particular if $s_1 = a(\xi)$ and $s_2 = b(x)$ then the formula is valid for a smooth and b merely of class C^1, thanks to a result of A. Calderon [26]. In the case of the scalar equation of Example 6 one can obtain then a propagation result under some natural regularity hypotheses.

Proposition 10. *Let u^n be a sequence converging weakly to zero in $L^2(R^N)$ and assume that $\Sigma_j b_j \partial_j u^n + cu^n = f^n$ with f^n converging weakly to zero in $L^2(R^N)$, the coefficients b_j being assumed to be real and of class C^1 while c is only assumed to be continuous. Assume moreover that (u^n, f^n) corresponds to a H-measure μ. Then $\mu^{1,1}$ satisfies the following transport equation $< \mu^{1,1}, \{\Phi, P\} - \Phi\,\mathrm{div}\, b + 2\Phi\,\mathrm{Re}\, c >$ $= < 2\,\mathrm{Re}\,\mu^{1,2}, \Phi >$ for every function Φ of class C^1 in (x,ξ) with compact support in x, with P defined by $P(x,\xi) = \Sigma_j b_j(x)\xi_j$.*

In the case of the wave equation of Example 8 one finds a similar result.

Proposition 11. *Let u^n be a sequence converging weakly to zero in $H^1(R^{N+1})$ and assume that $\varrho\partial_0^2 u^n$ - $\Sigma_{ij}\partial_i(a_{ij}\partial_j u^n) = f^n$ with f^n converging weakly to zero in $L^2(R^N)$, the coefficient ϱ being real positive independent of x_0 and of class C^1, the matrix a with entries a_{ij}, i, j = 1,..,N being hermitian positive independent of x_0 and of class C^1. Let U^n be defined by $U^n_i = \partial_i u^n$ for i = 0,..,N, and assume that (U^n, f^n) corresponds to a H-measure μ. Then $\mu^{i,j} = \xi_i\xi_j\nu^{1,1}$ for i, j = 0,..,N, and $\mu^{i,N+1} = \xi_i\nu^{1,2}$ for i = 0,..,N and $\nu^{1,1}$ satisfies the following transport equation $< \nu^{1,1}, \{\Phi, Q\} > =$ $< 2\,\mathrm{Re}\,\nu^{1,2}, \Phi >$ for every function Φ of class C^1 in (x,ξ) with compact support in x, with Q defined by $Q(x,\xi) = \varrho(x)\xi_0^2$ - $\Sigma_{ij}a_{ij}(x)\xi_i\xi_j$.*

One can complete Proposition 10 as in [4] by proving a trace theorem on a noncharacteristic hyperplane and from that deduce a result of change of variables for H-measures under local C^1 diffeomorphism, so that a theory on manifolds could be developed. A more interesting question lies in understanding the effect of semilinearity, that is when f^n does depend upon u^n or U^n in the framework of Proposition 10 or 11. The difficulty lies in the fact that H-measures are intrinsically quadratic objects and do not then predict anything about the limits of trilinear quantities for instance. H-measures do provide an improvement on the method of compensated compactness that I had developed with F. Murat [27, 28] but so far have not provided an alternative approach for quasilinear hyperbolic systems of conservation laws in order to replace the method that I had introduced [28] based on Young measures and compensated compactness, a method which had been successfully applied by R. DiPerna [29, 30, 31, 32]. Ron DiPerna had pointed out many years ago the defects of that old method and the need for a dynamic way of describing oscillations; H-measures is still the best answer to that quest and it is obviously not sufficient. It is important to point out that results like Proposition 11 correspond to the possibility of preparing initial data as a beam concentrated at a point and pointing in some direction and then follow where the energy goes; in the spirit of Example 4 one can also prepare initial data that correspond to a H-measure concentrated at a point in space and charging a countable number of points of the unit sphere and still follow the energy along each of these countably many small beams of light; as wave front sets are closed they cannot even see only a countable dense set of the sphere.

Other Results

It would be unfair not to point out that P. Gérard has introduced quite interesting variants which I cannot cover here [20, 21]. I cannot either discuss of other applications like the relation with homogenisation [4, 17].

Conclusion

In conclusion I was quite wrong in that small paragraph spared by the political censors [1] where I associated the study of "propagation of singularities" with classical physics as much better mathematical results connected to the propagation of light are like the above Proposition 11. I was also partly wrong in my previous ideas on quantum physics and oscillations as I had forgotten to include concentration effects in my description. Both these conclusions were learned from the possibilities created by the new mathematical tool of H-measures but a lot remains to be done on the way to a better understanding of physics through increased knowledge of some precise aspects of partial differential equations.

Obviously none of these new results are difficult and they could have been proved a long time ago by any of the best specialists of partial differential equations had they been interested in Science.

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