Lecture notes on Topology

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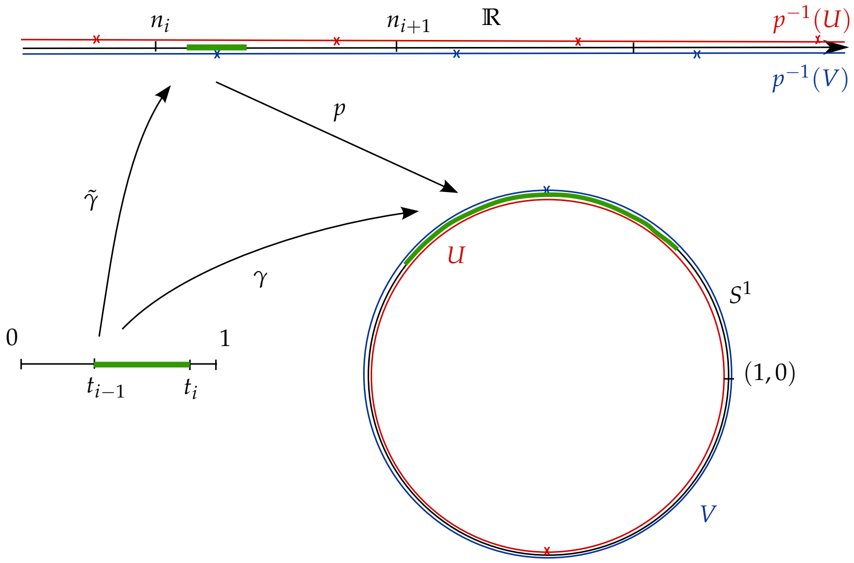

ABSTRACT. This is a set of lecture notes for a series of introductory courses in Topology for undergraduate students at the University of Science, Vietnam National University–Ho Chi Minh City. It is written to be delivered by myself, tailored to my students. I did not write it with other lecturers or self-study readers in mind.

In my experience many things here are much better explained in oral form than in written form. Therefore in writing these notes I intend that more explanations and discussions will be carried out in class. I hope by presenting only the essentials these notes will be more suitable for classroom use. Some details are left for students to fill in or to be discussed in class.

Since students in my department are required to take a course in Functional Analysis, I try not to duplicate material in that course.

A sign $\surd$ in front of a problem notifies the reader that this is an important one although it might not appear to be so initially. A sign * indicates a relatively more difficult problem.

This is a draft under development. The latest version is available on my web page at

http://www.math.hcmus.edu.vn/∼hqvu.

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Contents

General Topology

1. Set

In General Topology we often work in very general settings, in particular we often deal with infinite sets. We will not define what a set is. That means we will work on the level of "naive set theory". We will use familiar notions such as maps, Cartesian product of two sets, ... without giving precise definitions. We will not go back to definitions of the natural numbers or the real numbers.

Even so we should be aware of certain problems in naive set theory. Until the beginning of the 20th century, the set theory of George Cantor, in which set is not defined, was thought to be a good basis for mathematics. Then some critical problems were discovered.

Example (Russell's paradox). Consider the set $S = \{x \mid x \notin x\}$ (the set of all sets which are not members of themselves). Then whether $S \in S$ or not is undecidable, because answering yes or no to this question leads to contradiction. [1]

Axiomatic systems for the theory of sets have been developed since then. In the Von Neumann-Bernays-Godel system a more general notion than set, called *class* (lớp), is used ([**Dug66**, p. 32]). In this course, in occasions where we deal with "set of sets" we often replace the term set by the terms class or *collection* (họ).

Indexed collection. Suppose that A is a collection, I is a set and $f : I \to A$ is a map. The map f is called an *indexed collection*, or *indexed family* (họ được đánh chỉ số). We often write $f_i = f(i)$, and denote the indexed collection f by $\{f_i\}_{i \in I}$. Notice that it can happen that $f_i = f_j$ for some $i \neq j$.

Example. A sequence in a set A is a collection of elements of A indexed by the set $\mathbb{Z}^+$ of positive integer numbers.

Relation. A *relation* (quan hệ) R on a set S is a non-empty subset of the set $S \times S$.

When $(a, b) \in R$ we often say that a is related to b.

A relation is:

(a) reflexive (phản xạ) if $\forall a \in S, (a, a) \in R$.
(b) symmetric (đối xứng) if $\forall a, b \in S, (a, b) \in R \Rightarrow (b, a) \in R$.

[1] Discovered in 1901 by Bertrand Russell. A famous version of this paradox is the barber paradox: In a village there is a barber; his job is to do hair cut for a villager if and only if the villager does not cut his hair himself. Consider the set of all villagers who had their hairs cut by the barber. Is the barber himself a member of that set?

(c) antisymmetric (phản đối xứng) if $\forall a, b \in S, ((a,b) \in R \wedge (b,a) \in R) \Rightarrow a = b$.

(d) transitive (bắc cầu) if $\forall a, b, c \in S, ((a,b) \in R \wedge (b,c) \in R) \Rightarrow (a,c) \in R$.

An *equivalence relation* on S is a relation that is reflexive, symmetric and transitive.

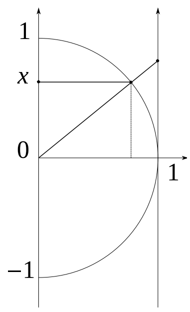
If R is an equivalence relation on S then an *equivalence class* (lớp tương đương) represented by $a \in S$ is the subset $[a] = \{b \in S \mid (a,b) \in R\}$. Two equivalence classes are either coincident or disjoint. The set S is partitioned (phân hoạch) into the disjoint union of its equivalence classes.

Equivalent sets. Two sets are said to be *equivalent* if there is a bijection from one to the other.

Example. Two intervals $[a,b]$ and $[c,d]$ on the real number line are equivalent. The bijection can be given by a linear map $x \mapsto \frac{d-c}{b-a}(x-a)+c$. Similarly, two intervals (a,b) and (c,d) are equivalent.

The interval $(-1,1)$ is equivalent to $\mathbb{R}$ via a map related to the tan function:

$$x \mapsto \frac{x}{\sqrt{1-x^2}}.$$



Countable sets.

Definition. A set is called *countably infinite* (vô hạn đếm được) if it is equivalent to the set of all positive integers. A set is called *countable* if it is either finite or countably infinite.

Intuitively, a countably infinite set can be "counted" by the positive integers. The elements of such a set can be indexed by the set of all positive integers as a sequence $a_1, a_2, a_3, \ldots$.

Example. The set $\mathbb{Z}$ of all integer numbers is countable.

Proposition 1.1. *A subset of a countable set is countable.*

PROOF. The statement is equivalent to the statement that a subset of $\mathbb{Z}^+$ is countable. Suppose that A is an infinite subset of $\mathbb{Z}^+$. Let a_1 be the smallest number in A. Let a_n be the smallest number in $A \setminus \{a_1, a_2, \ldots, a_{n-1}\}$. Then $a_{n-1} < a_n$ and the set $B = \{a_n \mid n \in \mathbb{Z}^+\}$ is a countably infinite subset of A.

We show that any element m of A is an a_n for some n, and therefore $B = A$.

Let $C = \{a_n \mid a_n \geq m\}$. Then $C \neq \emptyset$ since B is infinite. Let $a_{n_0} = \min C$. Then $a_{n_0} \geq m$. Further, since $a_{n_0-1} < a_{n_0}$ we have $a_{n_0-1} < m$. This implies $m \in A \setminus \{a_1, a_2, \dots, a_{n_0-1}\}$. Since $a_{n_0} = \min\left(A \setminus \{a_1, a_2, \dots, a_{n_0-1}\}\right)$ we must have $a_{n_0} \leq m$. Thus $a_{n_0} = m$. $\square$

Proposition 1.2. *If there is a surjective map from $\mathbb{Z}^+$ to a set S then S is countable.*

PROOF. Suppose that there is a surjective map $\phi : \mathbb{Z}^+ \to S$. For each $s \in S$ the set $\phi^{-1}(s)$ is non-empty. Let $n_s = \min \phi^{-1}(s)$. The map $s \mapsto n_s$ is an injective map from S to a subset of $\mathbb{Z}^+$, therefore S is countable, by 1.1. $\square$

Theorem 1.3. *The union of a countable collection of countable sets is a countable set.*

PROOF. The collection can be indexed as $A_1, A_2, \dots, A_i, \dots$ (if the collection is finite we can let A_i be the same set for all i starting from a certain index). The elements of each set A_i can be indexed as $a_{i,1}, a_{i,2}, \dots, a_{i,j}, \dots$ (if A_i is finite we can let $a_{i,j}$ be the same element for all j starting from a certain index).

This means there is a surjective map from the index set $\mathbb{Z}^+ \times \mathbb{Z}^+$ to the union $\bigcup_{i \in I} A_i$ by $(i, j) \mapsto a_{i,j}$.

Thus it is sufficient for us, by 1.2, to prove that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

We can index $\mathbb{Z}^+ \times \mathbb{Z}^+$ by the method shown in the following diagram:

$$\begin{tikzcd}
(1,1) \arrow[r] & (1,2) \arrow[dl] & (1,3) \arrow[r] & (1,4) \arrow[dl] \\
(2,1) \arrow[d] & (2,2) \arrow[ur] & (2,3) \arrow[dl] & \\
(3,1) \arrow[ur] & (3,2) \arrow[dl] & \dots & \\
(4,1) \arrow[d] & (4,2) & & \\
(5,1) \arrow[ur] & & &
\end{tikzcd}$$

$\square$

Theorem. *The set $\mathbb{Q}$ of all rational numbers is countable.*

PROOF. One way to prove this result is to write $\mathbb{Q} = \bigcup_{q=1}^{\infty} \left\{ \frac{p}{q} \mid p \in \mathbb{Z} \right\}$, then use 1.3.

Another way is to observe that if we write each rational number in the form $\frac{p}{q}$ with $q > 0$ and $\gcd(p, q) = 1$ then the map $\frac{p}{q} \mapsto (p, q)$ from $\mathbb{Q}$ to $\mathbb{Z} \times \mathbb{Z}$ is injective. $\square$

Theorem 1.4. *The set $\mathbb{R}$ of all real numbers is uncountable.*

PROOF. The proof uses the *Cantor diagonal argument*.

Suppose that set of all real numbers in decimal form in the interval $[0,1]$ is countable, and is enumerated as a sequence $\{a_i \mid i \in \mathbb{Z}^+\}$. Let us write

$$a_1 = 0.a_{1,1}a_{1,2}a_{1,3}\ldots$$

$$a_2 = 0.a_{2,1}a_{2,2}a_{2,3}\ldots$$

$$a_3 = 0.a_{3,1}a_{3,2}a_{3,3}\ldots$$

$$\vdots$$

There are real numbers whose decimal presentations are not unique, such as $\frac{1}{2} = 0.5000\ldots = 0.4999\ldots$. Choose a number $b = 0.b_1b_2b_3\ldots$ such that $b_n \neq 0, 9$ and $b_n \neq a_{n,n}$. Choosing b_n differing from 0 and 9 will guarantee that $b \neq a_n$ for all n (see more at 1.20). Thus the number b is not in the above table, a contradiction. □

Theorem 1.5 (Cantor-Bernstein-Schroeder). *If A is equivalent to a subset of B and B is equivalent to a subset of A then A and B are equivalent.*

PROOF. Suppose that $f : A \mapsto B$ and $g : B \mapsto A$ are injective maps. Let $A_1 = g(B)$, we will show that A is equivalent to A_1.

Let $A_0 = A$ and $B_0 = B$. Define $B_{n+1} = f(A_n)$ and $A_{n+1} = g(B_n)$. Then $A_{n+1} \subset A_n$. Furthermore via the map $g \circ f$ we have A_{n+2} is equivalent to A_n, and $A_n \setminus A_{n+1}$ is equivalent to $A_{n+1} \setminus A_{n+2}$.

Using the following identities

$$A = (A \setminus A_1) \cup (A_1 \setminus A_2) \cup \cdots \cup (A_n \setminus A_{n+1}) \cup \ldots \cup (\bigcap_{n=1}^{\infty} A_n),$$

$$A_1 = (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup \cdots \cup (A_n \setminus A_{n+1}) \cup \ldots \cup (\bigcap_{n=1}^{\infty} A_n),$$

we see that A is equivalent to A_1. □

Order. An *order* (thứ tự) on a set S is a relation R on S that is reflexive, anti-symmetric and transitive.

Note that two arbitrary elements a and b do not need to be comparable; that is, the pair (a,b) may not belong to R. For this reason an order is often called a partial order.

When $(a,b) \in R$ we often write $a \leq b$. When $a \leq b$ and $a \neq b$ we write $a < b$.

If any two elements of S are related then the order is called a *total order* (thứ tự toàn phần) and $(S, \leq)$ is called a *totally ordered set*.

Example. The set $\mathbb{R}$ of all real numbers with the usual order $\leq$ is totally ordered.

Example. Let S be a set. Denote by 2^S the collection of all subsets of S. Then $(2^S, \subseteq)$ is a partially ordered set, but is not totally ordered if S has more than one element.

Example (Dictionary order). Let $(S_1, \leq_1)$ and $(S_2, \leq_2)$ be two ordered sets. The following is an order on $S_1 \times S_2$: $(a_1, b_1) \leq (a_2, b_2)$ if $(a_1 < a_2)$ or $((a_1 = a_2) \wedge (b_1 \leq b_2))$. This is called the *dictionary order* (thứ tự từ điển).

In an ordered set, the *smallest element* (phần tử nhỏ nhất) is the element that is smaller than all other elements. More concisely, if S is an ordered set, the smallest element of S is an element $a \in S$ such that $\forall b \in S, a \leq b$. The smallest element, if exists, is unique.

A *minimal element* (phần tử cực tiểu) is an element which no element is smaller than. More concisely, a minimal element of S is an element $a \in S$ such that $\forall b \in S, b \leq a \Rightarrow b = a$. There can be more than one minimal element.

A *lower bound* (chặn dưới) of a subset of an ordered set is an element of the set that is smaller than or equal to any element of the subset. More concisely, if $A \subset S$ then a lower bound of A in S is an element $a \in S$ such that $\forall b \in A, a \leq b$.

The definitions of largest element, maximal element, and upper bound are similar.

Cardinality. A genuine definition of cardinality of sets requires an axiomatic treatment of set theory. Here we accept that for each set A there exists an object called its *cardinal* (lực lượng, bản số) $|A|$, and there is a relation $\leq$ on the set of cardinals such that:

(a) If a set is finite then its cardinal is its number of elements.

(b) Two sets have the same cardinals if and only if they are equivalent:

$$|A| = |B| \iff (A \sim B).$$

(c) $|A| \leq |B|$ if and only if there is an injective map from A to B.

Theorem 1.5 says that $(|A| \leq |B| \wedge |B| \leq |A|) \Rightarrow |A| = |B|$.

The cardinal of $\mathbb{Z}^+$ is denoted by $\aleph_0$,[2] while the cardinal of $\mathbb{R}$ is denoted by c (continuum). Since any infinite set contains a countably infinite subset, $\aleph_0$ is the smallest infinite cardinal. Since $\mathbb{R}$ is uncountable, we have $\aleph_0 < c$.[3]

Theorem (No maximal cardinal). *The cardinal of a set is strictly less than the cardinal of the set of all of its subsets, i.e.* $|A| < |2^A|$.

This implies that there is no maximal cardinal. There is no "universal set", "the set which contains everything", or "the set of all sets".

PROOF. Let $A \neq \emptyset$ and denote by 2^A the set of all of its subsets.

(a) $|A| \leq |2^A|$: The map from A to 2^A: $a \mapsto \{a\}$ is injective.

[2] $\aleph$ is read "aleph", a character in the Hebrew alphabet.

[3] Georg Cantor put forward the *Continuum hypothesis*: There is no cardinal between $\aleph_0$ and c.

(b) $|A| \neq |2^A|$: Let ϕ be any map from A to 2^A. Let $X = \{a \in A \mid a \notin \phi(a)\}$. Suppose that there is $x \in A$ such that $\phi(x) = X$. Then the question whether x belongs to X or not is undecidable. Therefore ϕ is not surjective.

□

The Axiom of choice.

Theorem. *The following statements are equivalent:*

(a) *Axiom of choice: Given a collection of non-empty sets, there is a function defined on this collection, called a choice function, associating each set in the collection with an element of that set.*

(b) *Zorn lemma: If any totally ordered subset of an ordered set X has an upper bound then X has a maximal element.*

Intuitively, a choice function "chooses" an element from each set in a given collection of non-empty sets. The Axiom of choice allows us to make infinitely many arbitrary choices in order to define a function. [4]

The Axiom of choice is needed for many important results in mathematics, such as the Tikhonov theorem in Topology, the Hahn-Banach theorem and Banach-Alaoglu theorem in Functional analysis, the existence of a Lebesgue unmeasurable set in Real analysis, ...

There are cases where this axiom could be avoided. For example in the proof of 1.2 we used the well-ordered property of $\mathbb{Z}^+$ instead. See for instance [**End77**, p. 151] for further material on this subject.

Zorn lemma is often a convenient form of the Axiom of choice.

Cartesian product. Let $\{A_i\}_{i\in I}$ be a family of sets indexed by a set I. The *Cartesian product* (tích Decartes) $\prod_{i\in I} A_i$ of this family is defined to be the collection of all maps $a : I \to \bigcup_{i\in I} A_i$ such that if $i \in I$ then $a(i) \in A_i$. The statement "the Cartesian product of a family of non-empty sets is non-empty" is therefore equivalent to the Axiom of choice.

An element a of $\prod_{i\in I} A_i$ is often denoted by $(a_i)_{i\in I}$, with $a_i = a(i) \in A_i$ being the coordinate of index i, in analog to the finite product case.

Problems.

1.6. Check that $(\bigcup_{i\in I} A_i) \cap (\bigcup_{j\in J} B_j) = \bigcup_{i\in I, j\in J} A_i \cap B_j$.

1.7. Which of the following formulas are correct?

(a) $(\bigcup_{i\in I} A_i) \cap (\bigcup_{i\in I} B_i) = \bigcup_{i\in I}(A_i \cap B_i)$.

(b) $\bigcap_{i\in I}(\bigcup_{j\in J} A_{i,j}) = \bigcup_{i\in I}(\bigcap_{j\in J} A_{i,j})$.

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[4] Bertrand Russell said that choosing one shoe from each pair of shoes from an infinite collection of pairs of shoes does not need the Axiom of choice (because in a pair of shoes the left shoe is different from the right one so we can define our choice), but if in a pair of socks the two socks are same, then choosing one sock from each pair of socks from an infinite collection of pairs of socks needs the Axiom of choice.

1.8. Let f be a function. Show that:

(a) $f(\bigcup_i A_i) = \bigcup_i f(A_i)$.
(b) $f(\bigcap_i A_i) \subset \bigcap_i f(A_i)$. If f is injective (one-one) then equality happens.
(c) $f^{-1}(\bigcup_i A_i) = \bigcup_i f^{-1}(A_i)$.
(d) $f^{-1}(\bigcap_i A_i) = \bigcap_i f^{-1}(A_i)$.

1.9. Let f be a function. Show that:

(a) $f(f^{-1}(A)) \subset A$. If f is surjective (onto) then equality happens.
(b) $f^{-1}(f(A)) \supset A$. If f is injective then equality happens.

1.10. Show that a union between a countable set and a finite set is countable.

1.11. If A is finite and B is infinite then $A \cup B$ is equivalent to B.

1.12. Show that two planes with finitely many points removed are equivalent.

1.13. Give another proof of 1.3 by checking that the map $\mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$, $(m,n) \mapsto 2^m 3^n$ is injective.

1.14. Show that the set of points in $\mathbb{R}^n$ with rational coordinates is countable.

1.15. Show that if A has n elements then $|2^A| = 2^n$.

1.16. Show that the set of all functions $f : A \to \{0,1\}$ is equivalent to 2^A.

1.17. A real number α is called an algebraic number if it is a root of a polynomial with integer coefficients. Show that the set of all algebraic numbers is countable.

A real number which is not algebraic is called transcendental. For example it is known that π and e are transcendental. Show that the set of all transcendental numbers is uncountable.

1.18. Show that the intervals $[a,b]$, $[a,b)$ and (a,b) are equivalent.

1.19. A continuum set is a set whose cardinal is c. Show that a countable union of continuum sets is a continuum set.

1.20. Show that any real number could be written in base d with any $d \in \mathbb{Z}$, $d \geq 2$. However two forms in base d could represent the same real number, as seen in 1.4. This happens only if starting from certain digits, all digits of one form are 0 and all digits of the other form are $d-1$. (This result is used in 1.4.)

1.21 ($2^{\aleph_0} = c$). We prove that $2^{\mathbb{N}}$ is equivalent to $\mathbb{R}$.

(a) Show that $2^{\mathbb{N}}$ is equivalent to the set of all sequences of binary digits.
(b) Using 1.20, deduce that $|[0,1]| \leq |2^{\mathbb{N}}|$.
(c) Consider a map $f : 2^{\mathbb{N}} \to [0,2]$, for each binary sequence $a = a_1a_2a_3\cdots$ define $f(a)$ as follows. If starting from a certain digit, all digits are 1, then let $f(a) = 1.a_1a_2a_3\cdots$. Otherwise let $f(a) = 0.a_1a_2a_3\cdots$. Show that f is injective.

Deduce that $|2^{\mathbb{N}}| \leq |[0,2]|$.

1.22 ($\mathbb{R}^2$ is equivalent to $\mathbb{R}$). * Here we prove that $\mathbb{R}^2$ is equivalent to $\mathbb{R}$, in other words, a plane is equivalent to a line. As a corollary, $\mathbb{R}^n$ is equivalent to $\mathbb{R}$.

(a) First method: Construct a map from $[0,1) \times [0,1)$ to $[0,1)$ as follows. In view of 1.20, we only allow decimal presentations in which not all digits are 9 starting from a certain digit. The pair of two real numbers $0.a_1a_2\dots$ and $0.b_1b_2\dots$ corresponds to the real number $0.a_1b_1a_2b_2\dots$. Check that this map is injective.

(b) Second method: Construct a map from $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ to $2^{\mathbb{N}}$ as follows. The pair of two binary sequences $a_1a_2\dots$ and $b_1b_2\dots$ corresponds to the binary sequence $a_1b_1a_2b_2\dots$. Check that this map is injective. Then use 1.21.

Note: In fact for all infinite cardinal ω we have $\omega^2 = \omega$, see [**Dug66**, p. 52], [**Lan93**, p. 888].

1.23 (Transfinite induction principle). An ordered set S is *well-ordered* (được sắp tốt) if every non-empty subset A of S has a smallest element, i.e. $\exists a \in A,\ \forall b \in A,\ a \leq b$.

For example with the usual order, $\mathbb{N}$ is well-ordered while $\mathbb{R}$ is not.

Ernst Zermelo proved in 1904 that any set can be well-ordered, based on the Axiom of choice.

The following is a generalization of the Principle of induction.

Let A be a well-ordered set. Let $P(a)$ be a statement whose truth depends on $a \in A$. If

(a) $P(a)$ is true when a is the smallest element of A

(b) if $P(a)$ is true for all $a < b$ then $P(b)$ is true

then $P(a)$ is true for all $a \in A$.

2. Topological space

Briefly, a topology is a system of open sets.

Definition. A *topology* on a set X is a collection τ of subsets of X satisfying:

(a) The sets $\emptyset$ and X are elements of τ.
(b) A union of elements of τ is an element of τ.
(c) A finite intersection of elements of τ is an element of τ.

Elements of τ are called *open sets* of X in the topology τ.

In short, a topology on a set X is a collection of subsets of X which includes $\emptyset$ and X and is closed under unions and finite intersections.

A set X together with a topology τ is called a *topological space*, denoted by (X, τ) or X alone if we do not need to specify the topology. An element of X is often called a *point*.

Example. On any set X there is the *trivial topology* (tôpô hiển nhiên) $\{\emptyset, X\}$. There is also the *discrete topology* (tôpô rời rạc) whereas any subset of X is open. Thus on a set there can be many topologies.

Remark. The statement "intersection of finitely many open sets is open" is equivalent to the statement "intersection of two open sets is open".

Metric space. Recall that, briefly, a metric space is a set equipped with a distance between every two points. Namely, a metric space is a set X with a map $d : X \times X \mapsto \mathbb{R}$ such that for all $x, y, z \in X$:

(a) $d(x, y) \geq 0$ (distance is non-negative),
(b) $d(x, y) = 0 \iff x = y$ (distance is zero if and only if the two points coincide),
(c) $d(x, y) = d(y, x)$ (distance is symmetric),
(d) $d(x, y) + d(y, z) \geq d(x, z)$ (triangular inequality).

A ball is a set of the form $B(x, r) = \{y \in X \mid d(y, x) < r\}$ where $r \in \mathbb{R}$, $r > 0$.

In the theory of metric spaces, a subset U of X is said to be open if for all x in U there is $\epsilon > 0$ such that $B(x, \epsilon)$ is contained in U. This is equivalent to saying that a non-empty open set is a union of balls.

To check that this is indeed a topology, we only need to check that the intersection of two balls is a union of balls. Let $z \in B(x, r_x) \cap B(y, r_y)$, let $r_z = \min\{r_x - d(z, x),\ r_y - d(z, y)\}$. Then the ball $B(z, r_z)$ will be inside both $B(x, r_x)$ and $B(y, r_y)$.

Thus a metric space is canonically a topological space with the topology generated by the metric. *When we speak about topology on a metric space we mean this topology*.

Example (Normed spaces). Recall that a normed space (không gian định chuẩn) is briefly a vector spaces equipped with lengths of vectors. Namely, a normed space is a set X with a structure of vector space over the real numbers and a real function $X \to \mathbb{R}$, $x \mapsto ||x||$, called a *norm* (chuẩn), satisfying:

(a) $||x|| \geq 0$ and $||x|| = 0 \iff x = 0$ (length is non-negative),
(b) $||cx|| = |c|||x||$ for $c \in \mathbb{R}$ (length is proportionate to vector),
(c) $||x + y|| \leq ||x|| + ||y||$ (triangle inequality).

A normed space is canonically a metric space with metric $d(x, y) = ||x - y||$. Therefore a normed space is canonically a topological space with the topology generated by the norm.

Example (Euclidean topology). In $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R}\}$, the Euclidean norm of a point $x = (x_1, x_2, \ldots, x_n)$ is $\|x\| = \left[\sum_{i=1}^n x_i^2\right]^{1/2}$. The topology generated by this norm is called the *Euclidean topology* (tôpô Euclid) of $\mathbb{R}^n$.

A complement of an open set is called a *closed set*.

Proposition (Dual description of topology). *In a topological space X:*

(a) *$\emptyset$ and X are closed.*
(b) *A finite union of closed sets is closed.*
(c) *An intersection of closed sets is closed.*

Bases of a topology.

Definition. Given a topology, a collection of open sets is a *basis* (cơ sở) for that topology if every non-empty open set is a union of members of that collection.

More concisely, let τ be a topology of X, then a collection $B \subset \tau$ is called a basis for τ if for any $\emptyset \neq V \in \tau$ there is $C \subset B$ such that $V = \bigcup_{D \in C} D$.

So a basis of a topology is a subset of the topology that generates the entire topology via unions. Specifying a basis is a more "efficient" way to describe a topology.

Example. In a metric space the collection of all balls is a basis for the topology.

Definition. A collection $S \subset \tau$ is called a *subbasis* (tiền cơ sở) for the topology τ if the collection of finite intersections of members of S is a basis for τ.

Clearly a basis for a topology is also a subbasis for that topology.

Briefly, given a topology, a subbasis is a subset of the topology that can generate the entire topology by unions and finite intersections.

Example. Let $X = \{1, 2, 3\}$. The topology $\tau = \{\emptyset, \{1, 2\}, \{2, 3\}, \{2\}, \{1, 2, 3\}\}$ has a basis $\{\{1, 2\}, \{2, 3\} \{2\}\}$ and a subbasis $\{\{1, 2\}, \{2, 3\}\}$.

Example 2.1. The collection of all open rays, that are, sets of the forms (a, ∞) and $(-\infty, a)$, is a subbasis for the Euclidean topology of $\mathbb{R}$.

Comparing topologies.

Definition. Let τ_1 and τ_2 be two topologies on X. If $\tau_1 \subset \tau_2$ we say that τ_2 is *finer* (mịn hơn) (or stronger, bigger) than τ_1 and τ_1 is *coarser* (thô hơn) (or weaker, smaller) than τ_2.

Example. On a set the trivial topology is the coarsest topology and the discrete topology is the finest one.

Generating topologies. Suppose that we have a set and we want certain subsets of that set to be open, how do find a topology for that purpose?

Theorem. *Let S be a collection of subsets of X. The collection τ consisting of $\emptyset$, X, and all unions of finite intersections of members of S is the coarsest topology on X that contains S, called the topology generated by S. The collection $S \cup \{X\}$ is a subbasis for this topology*

Remark. In several textbooks to avoid adding the element X to S it is required that the union of all members of S is X.

PROOF. Clearly τ is closed under unions. We only need to check that τ is closed under intersections of two elements, that is, to check that the intersection of two unions of finite intersections of members of S is a union of finite intersections of members of S. Let A and B be two collections of finite intersections of members of S. We have $(\bigcup_{C \in A} C) \cap (\bigcup_{D \in B} D) = \bigcup_{C \in A, D \in B}(C \cap D)$. Since each $C \cap D$ is a finite intersection of elements of S we get the desired conclusion. $\square$

By this theorem, *given a set, any collection of subsets generates a topology.*

Example. Let $X = \{1, 2, 3, 4\}$. The set $\{\{1\}, \{2, 3\}, \{3, 4\}\}$ generates the topology $\{\emptyset, \{1\}, \{3\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$. A basis for this topology is $\{\{1\}, \{3\}, \{2, 3\}, \{3, 4\}\}$.

Example (Ordering topology). Let $(X, \leq)$ be a totally ordered set. The collection of subsets of the forms $\{\beta \in X \mid \beta < \alpha\}$ and $\{\beta \in X \mid \beta > \alpha\}$ generates a topology on X, called the *ordering topology*.

Example. The Euclidean topology on $\mathbb{R}$ is the ordering topology with respect to the usual order of real numbers. (This is just a different way to state 2.1.)

Problems.

2.2 (Finite complement topology). The *finite complement topology* on X consists of the empty set and all subsets of X whose complements are finite. Check that this is indeed a topology.

2.3. Let X be a set and $p \in X$. Show that the collection consisting of $\emptyset$ and all subsets of X containing p is a topology on X. This topology is called the Particular Point Topology on X, denoted by PPX_p. Describe the closed sets in this space.

2.4. A collection B of open sets is a basis if for each point x and each open set O containing x there is a U in B such that U contains x and U is contained in O.

2.5. Show that two bases generate the same topology if and only if each member of one basis is a union of members of the other basis.

2.6. Let B be a collection of subsets of X. Then $B \cup \{X\}$ is a basis for a topology on X if and only if the intersection of two members of B is either empty or is a union of some members of B. (In several textbooks to avoid adding the element X to B it is required that the union of all members of B is X.)

2.7. In a metric space the set of all balls with rational radii is a basis for the topology.

2.8. In a metric space the set of all balls with radii $\frac{1}{2^m}$, $m \geq 1$ is a basis.

2.9 ($\mathbb{R}^n$ has a countable basis). $\checkmark$ The set of all balls each with rational radius whose center has rational coordinates forms a basis for the Euclidean topology of $\mathbb{R}^n$.

2.10. In $\mathbb{R}^n$ let $x = (x_1, x_2, \ldots, x_n)$ and consider the norms $\|x\|_1 = \sum_{i=1}^n |x_i|$, $\|x\|_2 = (\sum_{i=1}^n {x_i}^2)^{1/2}$, and $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$. Draw the unit ball for each norm. Show that these norms generate same topologies.

2.11. Let d_1 and d_2 be two metrics on X. If there are $\alpha, \beta > 0$ such that for all $x, y \in X$, $\alpha d_1(x,y) \leq d_2(x,y) \leq \beta d_1(x,y)$ then the two metrics are said to be equivalent. Show that two equivalent metrics generate same topologies.

Hint: Show that each ball in one metric contains a ball in the other metric with the same center.

2.12. Let (X, d) be a metric space. Let $d_1(x,y) = \min\{d(x,y), 1\}$. Show that d_1 is a metric on X generating the same topology as that generated by d.

2.13. Let (X, d) be a metric space. Let $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$. Show that d_1 is a metric on X generating the same topology as that generated by d.

2.14 (All norms in $\mathbb{R}^n$ generate the Euclidean topology). In $\mathbb{R}^n$ denote by $\|\cdot\|_2$ the Euclidean norm, and let $\|\cdot\|$ be any norm.

(a) Check that the map $x \mapsto \|x\|$ from $(\mathbb{R}^n, \|\cdot\|_2)$ to $(\mathbb{R}, \|\cdot\|_2)$ is continuous.

(b) Let S^n be the unit sphere under the Euclidean norm. Show that the restriction of the map above to S^n has a maximum value β and a minimum value α. Hence $\alpha \leq \left\| \frac{x}{\|x\|_2} \right\| \leq \beta$ for all $x \neq 0$.

Deduce that any two norms in $\mathbb{R}^n$ are equivalent, hence all norms in $\mathbb{R}^n$ generate the Euclidean topology.

2.15. Is the Euclidean topology on $\mathbb{R}^2$ the same as the ordering topology on $\mathbb{R}^2$ with respect to the dictionary order? If it is not the same, can the two be compared?

2.16. Show that an open set in $\mathbb{R}$ is a countable union of open intervals.

2.17. The collection of all intervals of the form $[a, b)$ generates a topology on $\mathbb{R}$. Is it the Euclidean topology?

2.18. On the set of all integer numbers $\mathbb{Z}$, consider all arithmetic progressions

$$S_{a,b} = a + b\mathbb{Z},$$

where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$.

(a) Show that these sets form a basis for a topology on $\mathbb{Z}$.

(b) Show that with this topology each set $S_{a,b}$ is closed.
(c) Show that if there are only finitely many prime numbers then the set $\{\pm 1\}$ is open.
(d) Conclude that *there are infinitely many prime numbers*. (This proof was given by Hillel Furstenberg in 1955.)

3. Continuity

Continuous function.

Definition. Let X and Y be topological spaces. We say a map $f : X \to Y$ is *continuous* at a point x in X if for any open set U of Y containing $f(x)$ there is an open set V of X containing x such that $f(V)$ is contained in U.

We say that f is continuous on X if it is continuous at every point in X.

A *neighborhood* (lân cận) of a point $x \in X$ is a subset of X which contains an open set containing x. Note that a neighborhood does not need to be open. [5]

Equivalently, f is continuous at x if for any open set U containing $f(x)$, the set $f^{-1}(U)$ is a neighborhood of x.

Theorem. *A map is continuous if and only if the inverse image of an open set is an open set.*

PROOF. ($\Rightarrow$) Suppose that $f : X \to Y$ is continuous. Let U be an open set in Y. Let $x \in f^{-1}(U)$. Since f is continuous at x and U is an open neighborhood of $f(x)$, there is an open set V_x containing x such that V_x is contained in $f^{-1}(U)$. Therefore $f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} V_x$ is open.

($\Leftarrow$) Suppose that the inverse image of any open set is an open set. Let $x \in X$. Let U be an open neighborhood of $f(x)$. Then $V = f^{-1}(U)$ is an open set containing x, and $f(V)$ is contained in U. Therefore f is continuous at x. $\square$

Example. Let X and Y be topological spaces.

(a) The identity function, $\mathrm{id}_X : X \to X$, $x \mapsto x$, is continuous.

(b) The constant function, with given $a \in Y$, $x \mapsto a$, is continuous.

Proposition. *A map is continuous if and only if the inverse image of a closed set is a closed set.*

Example (Metric space). Let (X, d_1) and (Y, d_2) be metric spaces. Recall that in the theory of metric spaces, a map $f : (X, d_1) \to (Y, d_2)$ is continuous at $x \in X$ if and only if

$$\forall \epsilon > 0, \exists \delta > 0, d_1(y, x) < \delta \Rightarrow d_2(f(y), f(x)) < \epsilon.$$

In other words, given any ball $B(f(x), \epsilon)$ centered at $f(x)$, there is a ball $B(x, \delta)$ centered at x such that f brings $B(x, \delta)$ into $B(f(x), \epsilon)$.

It is apparent that this definition is equivalent to the definition of continuity in topological spaces where the topologies are generated by the metrics.

In other words, if we look at a metric space as a topological space then continuity in the metric space is the same as continuity in the topological space. Therefore *we inherit all results concerning continuity in metric spaces*.

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[5] Be careful that not everyone uses this convention. For instance Kelley **[Kel55]** uses this convention but Munkres **[Mun00]** requires a neighborhood to be open.

Homeomorphism. A map from one topological space to another is said to be a *homeomorphism* (phép đồng phôi) if it is a bijection, is continuous and its inverse map is also continuous.

Two spaces X and Y are said to be *homeomorphic* (đồng phôi), sometimes written $X \approx Y$, if there is a homeomorphism from one to the other.

Example. Any two open intervals in the real number line $\mathbb{R}$ under the Euclidean topology are homeomorphic.

Proposition. *If $f : (X, \tau_X) \to (Y, \tau_Y)$ is a homeomorphism then it induces a bijection between τ_X and τ_Y.*

PROOF. The map

$$\begin{aligned} \tilde{f} : \tau_X &\to \tau_Y \\ O &\mapsto f(O) \end{aligned}$$

is a bijection. □

Roughly speaking, in the field of Topology, when two spaces are homeomorphic they are the same.

Topology generated by maps. Let (X, τ_X) be a topological space, Y be a set, and $f : X \to Y$ be a map. We want to find a topology on Y such that f is continuous.

The requirement for such a topology τ_Y is that if $U \in \tau_Y$ then $f^{-1}(U) \in \tau_X$.

The trivial topology on Y satisfies that requirement. It is the coarsest topology satisfying that requirement.

On the other hand the collection $\{U \subset Y \mid f^{-1}(U) \in \tau_X\}$ is actually a topology on Y. This is the finest topology satisfying that requirement.

In another situation, let X be a set, (Y, τ_Y) be a topological space, and $f : X \to Y$ be a map. We want to find a topology on X such that f is continuous.

The requirement for such a topology τ_X is that if $U \in \tau_Y$ then $f^{-1}(U) \in \tau_X$.

The discrete topology on X is the finest topology satisfying that requirement. The collection $\tau_X = \{f^{-1}(U) \mid U \in \tau_Y\}$ is the coarsest topology satisfying that requirement. We can observe further that if the collection S_Y generates τ_Y then τ_X is generated by the collection $\{f^{-1}(U) \mid U \in S_Y\}$.

Problems.

3.1. If $f : X \to Y$ and $g : Y \to Z$ are continuous then $g \circ f$ is continuous.

3.2. √ Suppose that $f : X \to Y$ and S is a subbasis for the topology of Y. Show that f is continuous if and only if the inverse image of any element of S is an open set in X.

3.3. Define an *open map* to be a map such that the image of an open set is an open set. A *closed map* is a map such that the image of a closed set is a closed set.

Show that a homeomorphism is both an open map and a closed map.

3.4. A continuous bijection is a homeomorphism if and only if it is an open map.

3.5. Show that (X, PPX_p) and (X, PPX_q) (see 2.3) are homeomorphic.

3.6. √ Let X be a set and (Y, τ) be a topological space. Let $f_i : X \to Y$, $i \in I$ be a collection of maps. Find the coarsest topology on X such that all maps f_i, $i \in I$ are continuous.

Note: In Functional Analysis this construction is used to construct the weak topology on a normed space. It is the coarsest topology such that all linear functionals which are continuous under the norm are still continuous under the topology. See for instance [**Con90**].

3.7. Suppose that X is a normed space. Prove that the topology generated by the norm is exactly the coarsest topology on X such that the norm and the translations (maps of the form $x \mapsto x + a$) are continuous.

4. Subspace

Subspace topology. Let (X, τ) be a topological space and let A be a subset of X. The *subspace topology* on A, also called the *relative topology* (tôpô tương đối), is defined to be the collection $\{A \cap O \mid O \in \tau\}$. With this topology we say that A is a *subspace* (không gian con) of X.

Thus a subset of a subspace A of X is open in A if and only if it is a restriction of a open set in X to A.

Proposition. *A subset of a subspace A of X is closed in A if and only if it is a restriction of a closed set in X to A.*

Remark. An open or a closed subset of a subspace A of a space X is not necessarily open or closed in X. For example, under the Euclidean topology of $\mathbb{R}$, the set $[0, 1/2)$ is open in the subspace $[0, 1]$, but is not open in $\mathbb{R}$.

When we say that a set is open, we must know which space we are talking about.

Example. For $n \in \mathbb{Z}^+$ define the *sphere* S^n to be the subspace of the Euclidean space $\mathbb{R}^{n+1}$ given by $\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$.

Proposition 4.1. *Suppose that X is a topological space and $Z \subset Y \subset X$. Then the relative topology of Z with respect to Y is the same as the relative topology of Z with respect to X.*

Embedding. An *embedding* (or *imbedding*) (phép nhúng) from the topological space X to the topological space Y is a map $f : X \to Y$ such that its restriction $\tilde{f} : X \to f(X)$ is a homeomorphism. This means f maps X homeomorphically onto its image. If there is an imbedding from X to Y, i.e. if X is homeomorphic to a subspace of Y then we say that X can be *embedded* in Y.

Example. The Euclidean line $\mathbb{R}$ can be embedded in the Euclidean plane $\mathbb{R}^2$ as a line in the plane.

Example. Suppose that $f : \mathbb{R} \to \mathbb{R}^2$ is continuous under the Euclidean topology. Then $\mathbb{R}$ can be embedded into the plane as the graph of f.

Example (Stereographic projection). The space $S^n \setminus \{(0, 0, \ldots, 0, 1)\}$ is homeomorphic to the Euclidean space $\mathbb{R}^n$ via the *stereographic projection* (phép chiếu nổi). Each point x on the sphere minus the North Pole corresponds to the intersection between the straight line from the North Pole to x with the plane through the equator. We can easily find the formula for this projection to be:

$$\begin{aligned} S^n \setminus \{(0, 0, \ldots, 0, 1)\} &\to \mathbb{R}^n \times \{0\} \\ (x_1, x_2, \ldots, x_{n+1}) &\mapsto (y_1, y_2, \ldots, y_n, 0) \end{aligned}$$

where $y_i = \dfrac{1}{1 - x_{n+1}} x_i$. The inverse map is given by $x_i = \dfrac{2y_i}{1 + \sum_{i=1}^n y_i^2}$, $1 \le i \le n$, and $x_{n+1} = \dfrac{-1 + \sum_{i=1}^n y_i^2}{1 + \sum_{i=1}^n y_i^2}$. Both maps are continuous. Thus the Euclidean space $\mathbb{R}^n$ can be embedded onto the n-sphere minus one point.

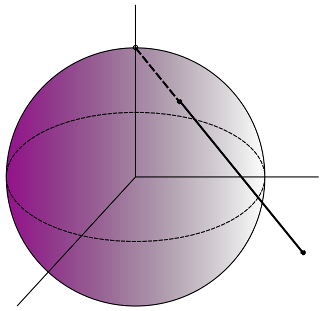


FIGURE 4.1. The stereographic projection.

Interior – Closure – Boundary. Let X be a topological space and let A be a subset of X. Let x be a point in X.

The point x is said to be an *interior point* of A in X if there is a neighborhood of x that is contained in A.

The point x is said to be a *contact point* (điểm dính) (or point of closure) of A in X if any neighborhood of x contains a point of A.

The point x is said to be a *limit point* (điểm tụ) (or cluster point, or accumulation point) of A in X if any neighborhood of x contains a point of A other than x.

Of course a limit point is a contact point. We can see that a contact point of A which is not a point of A is a limit point of A.

A point x is said to be a *boundary point* (điểm biên) of A in X if every neighborhood of x contains a point of A and a point of the complement of A.

In other words, a boundary point of A is a contact point of both A and the complement of A.

The set of all interior points of A is called the *interior* (phần trong) of A in X, denoted by $\mathring{A}$ or $\mathrm{int}(A)$.

The set of all contact points of A in X is called the *closure* (bao đóng) of A in X, denoted by $\overline{A}$ or $\mathrm{cl}(A)$.

The set of all boundary points of A in X is called the *boundary* (biên) of A in X, denoted by ∂A.

Example. On the Euclidean line $\mathbb{R}$, consider the subspace $A = [0,1) \cup \{2\}$. Its interior is $\mathrm{int}A = (0,1)$, the closure is $\mathrm{cl}A = [0,1] \cup \{2\}$, the boundary is $\partial A = \{0,1,2\}$, the set of all limit points is $[0,1]$.

Remark. It is crucial to understand that the notions of interior, closure, boundary, contact points, and limit points of a space A only make sense relative to a certain space X containing A as a subspace (there must be a "mother space").

Proposition. *The interior of A in X is the largest open subset of X that is contained in A. A subspace is open if all of its points are interior points.*

Proposition. *The closure of A in X is the smallest closed subset of X containing A. A subspace is closed if and only if it contains all of its contact points.*

Problems.

4.2. Let X be a topological space and let $A \subset X$. Then the subspace topology on A is exactly the coarsest topology on A such that the inclusion map $i : A \mapsto X$, $x \mapsto x$ is continuous.

4.3. $\surd$ Let X and Y be topological spaces and let $f : X \to Y$.

(a) If Z is a subspace of X, denote by $f|_Z$ the restriction of f to Z. Show that if f is continuous then $f|_Z$ is continuous.

(b) Let Z be a space containing Y as a subspace. Consider f as a function from X to Z, that is, let $\tilde{f} : X \to Z$, $\tilde{f}(x) = f(x)$. Show that f is continuous if and only if $\tilde{f}$ is continuous.

4.4 (Gluing continuous functions). $\surd$ Let $X = A \cup B$ where A and B are both open or are both closed in X. Suppose $f : X \to Y$, and $f|_A$ and $f|_B$ are both continuous. Then f is continuous.

Another way to phrase this is the following. Let $g : A \to Y$ and $h : B \to Y$ be continuous and $g(x) = h(x)$ on $A \cap B$. Define

$$f(x) = \begin{cases} g(x), & x \in A \\ h(x), & x \in B. \end{cases}$$

Then f is continuous.

Is it still true if the restriction that A and B are both open or are both closed in X is removed?

4.5. $\surd$ Any two balls in a normed space are homeomorphic.

4.6. $\surd$ A ball in a normed space is homeomorphic to the whole space.
Hint: Consider a map from the unit ball to the space, such as: $x \mapsto \dfrac{1}{\sqrt{1 - ||x||^2}} x$.

4.7. Two finte-dimensional normed spaces of same dimensions are homeomorphic.

4.8. Is it true that any two balls in a metric space homeomorphic?

4.9. In the Euclidean plane an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is homeomorphic to a circle.

4.10. In the Euclidean plane the upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ is homeomorphic to the plane.

4.11. In the Euclidean plane:

(a) A square and a circle are homeomorphic.

(b) The region bounded by a square and the region bounded by a the circle are homeomorphic.

4.12. $\surd$ If $f : X \to Y$ is a homeomorphism and $Z \subset X$ then $X \setminus Z$ and $Y \setminus f(Z)$ are homeomorphic.

4.13. On the Euclidean plane $\mathbb{R}^2$, show that:

(a) $\mathbb{R}^2 \setminus \{(0,0)\}$ and $\mathbb{R}^2 \setminus \{(1,1)\}$ are homeomorphic.

(b) $\mathbb{R}^2 \setminus \{(0,0),(1,1)\}$ and $\mathbb{R}^2 \setminus \{(1,0),(0,1)\}$ are homeomorphic.

4.14. Show that $\mathbb{N}$ and $\mathbb{Z}$ are homeomorphic under the Euclidean topology.

Further, prove that any two discrete spaces having the same cardinalities are homeomorphic.

4.15. Among the following spaces, which one is homeomorphic to another? $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, each with the Euclidean topology, and $\mathbb{R}$ with the finite complement topology.

4.16. Show that any homeomorphism from S^{n-1} onto S^{n-1} can be extended to a homeomorphism from the unit disk $D^n = B'(0,1)$ onto D^n.

4.17. Show that $\overline{A}$ is the disjoint union of $\mathring{A}$ and ∂A.

4.18. Show that X is the disjoint union of $\mathring{A}$, ∂A, and $X \setminus \overline{A}$.

4.19. The set $\{x \in \mathbb{Q} \mid -\sqrt{2} \le x \le \sqrt{2}\}$ is both closed and open in $\mathbb{Q}$ under the Euclidean topology of $\mathbb{R}$.

4.20. The map $\varphi : [0, 2\pi) \to S^1$ given by $t \mapsto (\cos t, \sin t)$ is a bijection but is not a homeomorphism, under the Euclidean topology.

4.21. Find the closures, interiors and the boundaries of the interval $[0, 1)$ under the Euclidean, discrete and trivial topologies of $\mathbb{R}$.

4.22. In a metric space X, a point $x \in X$ is a limit point of the subset A of X if and only if there is a sequence in $A \setminus \{x\}$ converging to x.

Note: This is not true in general topological spaces.

4.23. In a normed space, show that the boundary of the ball $B(x, r)$ is the sphere $\{y \mid d(x, y) = r\}$, and so the ball $B'(x, r) = \{y \mid d(x, y) \le r\}$ is the closure of $B(x, r)$.

4.24. In a metric space, show that the boundary of the ball $B(x, r)$ is a subset of the sphere $\{y \mid d(x, y) = r\}$. Is the ball $B'(x, r) = \{y \mid d(x, y) \le r\}$ the closure of $B(x, r)$?

4.25. Suppose that $A \subset Y \subset X$. Show that $\overline{A}^Y = \overline{A}^X \cap Y$. Furthermore show that if Y is closed in X then $\overline{A}^Y = \overline{A}^X$.

4.26. Let $O_n = \{k \in \mathbb{Z}^+ \mid k \ge n\}$. Check that $\{\emptyset\} \cup \{O_n \mid n \in \mathbb{Z}^+\}$ is a topology on $\mathbb{Z}^+$. Find the closure of the set $\{5\}$. Find the closure of the set of all even positive integers.

4.27. Verify the following properties.

(a) $X \setminus \mathring{A} = \overline{X \setminus A}$.
(b) $X \setminus \overline{A} = \mathring{\overline{X \setminus A}}$.
(c) If $A \subset B$ then $\mathring{A} \subset \mathring{B}$.

4.28. Which of the following equalities are correct?

(a) $\mathring{\overline{A \cup B}} = \mathring{A} \cup \mathring{B}$.
(b) $\mathring{\overline{A \cap B}} = \mathring{A} \cap \mathring{B}$.
(c) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(d) $\overline{A \cap B} = \overline{A} \cap \overline{B}$.

5. Connectedness

A topological space is said to be *connected* (liên thông) if it is not a union of two non-empty disjoint open subsets.

Equivalently, a topological space is connected if and only if its only subsets which are both closed and open are the empty set and the space itself.

Remark. When we say that a subset of a topological space is connected we implicitly mean that the subset under the subspace topology is a connected space.

Example 5.1. The Euclidean real number line minus a point is not connected.

Proposition (Continuous image of connected space is connected). *If $f : X \to Y$ is continuous and X is connected then $f(X)$ is connected.*

PROOF. Suppose that U and V are non-empty disjoint open subset of $f(X)$. Since $f : X \to f(X)$ is continuous (4.3), $f^{-1}(U)$ and $f^{-1}(V)$ are open in X, and are non-empty and disjoint. This contradicts the connectedness of X. □

Connected component.

Proposition 5.2. *If a collection of connected subspaces of a space has non-empty intersection then its union is connected.*

PROOF. Consider a topological space and let F be a collection of connected subspaces whose intersection is non-empty. Let A be the union of the collection, $A = \bigcup_{D \in F} D$. Suppose that C is subset of A that is both open and closed in A. If $C \neq \emptyset$ then there is $D \in F$ such that $C \cap D \neq \emptyset$. Then $C \cap D$ is a subset of D, both open and closed in D (we are using 4.1 here). Since D is connected and $C \cap D = \emptyset$, we must have $C \cap D = D$. This implies C contains the intersection of F. Therefore $C \cap D \neq \emptyset$ for all $D \in F$. The argument above shows that C contains all D in F, that is, $C = A$. We conclude that A is connected. □

Let X be a topological space. Define a relation on X whereas two points are related if both belong to a connected subspace of X (we say that the two points are connected). Then this relation is an equivalence relation, by 5.2.

Proposition. *Any equivalence class under the above equivalence relation is connected.*

PROOF. Consider the equivalence class $[a]$ represented by a point a. By the definition, $b \in [a]$ if and only if there is a connected set O_b containing both a and b. Thus $[a] = \bigcup_{b \in [a]} O_b$. By 5.2, $[a]$ is connected. □

Definition. Under the above equivalence relation, the equivalence classes are called the *connected components* of the space.

Thus a space is a disjoint union of its connected components.

The following is another characterization of connected components:

Proposition. *The connected component containing a point x is the union of all connected subspaces containing x, thus it is the largest connected subspace containing x.*

Theorem 5.3. *If two spaces are homeomorphic then there is a bijection between the collections of connected components of the two spaces. In particular, if two spaces are homeomorphic and one space is connected then the other space is also connected.*

For the above reason we say that *connectedness is a topological property*. We also say that *the number of connected components is a topological invariant*. If two spaces have different numbers of connected components then they must be different (not homeomorphic).

PROOF. Let $f : X \to Y$ be a homeomorphism. Since $f([x])$ is connected, we have $f([x]) \subset [f(x)]$. For the same reason, $f^{-1}([f(x)]) \subset [f^{-1}(f(x))] = [x]$. Apply f to both sides we get $[f(x)] \subset f([x])$. Therefore $f([x]) = [f(x)]$. Similarly $f^{-1}([f(x)]) = [x]$. Thus f brings connected components to connected components, inducing a bijection on the collections of connected components. □

Proposition 5.4. *A connected subspace with a limit point added is still connected. As a consequence the closure of a connected subspace is connected.*

PROOF. Let A be a connected subspace of a space X and let $a \notin A$ be a limit point of A, we show that $A \cup \{a\}$ is connected. Suppose that $A \cup \{a\} = U \cup V$ where U and V are non-empty disjoint open subsets of $A \cup \{a\}$. Suppose that $a \in U$. Then $a \notin V$, so $V \subset A$. Since a is a limit point of A, $U \cap A$ is non-empty. Then $U \cap A$ and V are open subsets of A, by 4.1, which are non-empty and disjoint. This contradicts the assumption that A is connected. □

Corollary. *A connected component must be closed.*

Connected sets in the Euclidean real number line.

Proposition. *A connected subspace of the Euclidean real number line must be an interval.*

PROOF. Suppose that a subset A of $\mathbb{R}$ is connected. Suppose that $x, y \in A$ and $x < y$. If $x < z < y$ we must have $z \in A$, otherwise the set $\{a \in A \mid a < z\} = \{a \in A \mid a \leq z\}$ will be both closed and open in A. Thus A contains the interval $[x, y]$.

Let $a = \inf A$ if A is bounded from below and $a = -\infty$ otherwise. Similarly let $b = \sup A$ if A is bounded from above and $b = \infty$ otherwise. Suppose that A contains more than one element. There are sequences $\{a_n\}_{n\in\mathbb{Z}^+}$ and $\{b_n\}_{n\in\mathbb{Z}^+}$ of elements in A such that $a < a_n < b_n < b$, and $a_n \to a$ while $b_n \to b$. By the above argument, $[a_n, b_n] \subset A$ for all n. So $(a, b) = \bigcup_{n=1}^{\infty}[a_n, b_n] \subset A \subset [a, b]$. It follows that A is either (a, b) or $[a, b)$ or $(a, b]$ or $[a, b]$. □

Proposition 5.5. *The Euclidean real number line is connected.*

PROOF. Suppose that $\mathbb{R}$ contains a non-empty, proper, open and closed subset C.

Let $x \notin C$ and let $D = C \cap (-\infty, x) = C \cap (-\infty, x]$. Then D is both open and closed in $\mathbb{R}$, and is bounded from above.

If $D \neq \emptyset$, consider $s = \sup D$. Since D is closed and s is a contact point of D, $s \in D$. Since D is open s must belong to an open interval contained in D. But then there are points in D which are bigger than s, a contradiction.

If $D = \emptyset$ we let $E = C \cap (x, \infty)$, consider $t = \inf E$ and proceed similarly. □

Theorem. *A subspace of the Euclidean real number line is connected if and only if it is an interval.*

PROOF. We prove that any interval is connected. By homeomorphisms we just need to consider the intervals $(0, 1)$, $(0, 1]$, and $[0, 1]$. Note that $[0, 1]$ is the closure of $(0, 1)$, and $(0, 1] = (0, 3/4) \cup [1/2, 1]$.

Or we can modify the proof of 5.5 to show that any interval is connected. □

Example. The Euclidean line $\mathbb{R}$ is connected. Since the Euclidean $\mathbb{R}^n$ is the union of all lines passing through the origin, it is connected.

Path-connected space. Let X be a topological space and let a and b be two points of X. A *path* (đường đi) in X from x to y is a continuous map $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$, where the interval $[a, b]$ has the Euclidean topology.

If α is a path defined on $[a, b]$ then there is a path β defined on $[0, 1]$ with the same images (also called the traces of the paths), we can just use the linear homeomorphism $(1 - t)a + tb$ from $[0, 1]$ to $[a, b]$ and let $\beta(t) = \alpha((1 - t)a + tb)$. For this reason for convenience we often assume the domains of paths to be $[0, 1]$.

The space X is said to be *path-connected* (liên thông đường) if for any two different points x and y in X there is a path in X from x to y.

Example. A normed space is path-connected, and so is any convex subspace of that space: any two points x and y are connected by a straight line segment $x + t(y - x)$, $t \in [0, 1]$.

Example. In a normed space, the sphere $S = \{x \mid ||x|| = 1\}$ is path-connected. One way to show this is as follow. If two points x and y are not opposite then they can be connected by the arc $\frac{x+t(y-x)}{||x+t(y-x)||}$, $t \in [0, 1]$. If x and y are opposite, we can take a third point z, then compose a path from x to z with a path from z to y.

Lemma. *Let X be a topological space. Define a relation on X whereas a point x is related to a point y if there is a path in X from x to y. Then this is an equivalence relation.*

PROOF. If there is a path $\alpha : [0, 1] \to X$ from x to y then there is a path from y to x, for example $\beta : [0, 1] \to X$, $\beta(t) = \alpha(1 - t)$.

If $\alpha : [0,1] \to X$ is a path from x to y and $\beta : [0,1] \to X$ is a path from y to z then there is a path from x to z, for example

$$\gamma(t) = \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2}, \\ \beta(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

This path follows α at twice the speed, then follow β at twice the speed and at half of a unit time later. It is continuous by 4.4. □

An equivalence class is called a *path-connected component*.

Proposition. *The path-connected component containing a point x is the union of all path-connected subspaces containing x, thus it is the largest path-connected subspace containing x.*

Theorem 5.6. *A path-connected space is connected.*

PROOF. This is a consequence of the fact that an interval on the Euclidean real number line is connected. Let X be path-connected. Let $x, y \in X$. There is a path from x to y. The image of this path is a connected subspace of X. That means every point y belongs to the connected component containing x. Therefore X has only one connected component. □

A topological space is said to be *locally path-connected* if every neighborhood of a point contains an open path-connected neighborhood of that point.

Example. Open sets in a normed space are locally path-connected.

Generally, the reverse statement of 5.6 is not correct. However we have:

Proposition 5.7. *A connected, locally path-connected space is path-connected.*

PROOF. Suppose that X is connected and is locally path-connected. Let C be a path-connected component of X. If $x \in X$ is a contact point of C then there is a path-connected neighborhood U in X of x such that $U \cap C \neq \emptyset$. By 5.23, $U \cup C$ is path-connected , thus $U \subset C$. This implies that C is open and C is closed in X. Hence $C = X$. □

Topologist's sine curve. The closure in the Euclidean plane of the graph of the function $y = \sin \frac{1}{x}$, $x > 0$ is often called the *Topologist's sine curve*. This is a classic example of a space which is connected but is not path-connected.

Denote $A = \{(x, \sin \frac{1}{x}) \mid x > 0\}$ and $B = \{0\} \times [-1, 1]$. Then the Topologist's sine curve is $X = A \cup B$.

Proposition. *The Topologist's sine curve is connected but is not path-connected.*

PROOF. By 5.12 the set A is connected. Each point of B is a limit point of A, so by 5.4 X is connected.

Suppose that there is a path $\gamma(t) = (x(t), y(t))$, $t \in [0,1]$ from the origin $(0,0)$ on B to a point on A, we show that there is a contradiction.

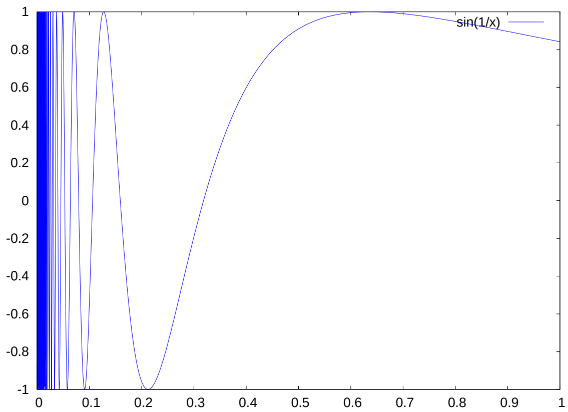


FIGURE 5.1. Topologist's sine curve.

Let $t_0 = \sup\{t \in [0,1] \mid x(t) = 0\}$. Then $x(t_0) = 0$, $t_0 < 1$, and $x(t) > 0$ for all $t > t_0$. Thus t_0 is the moment when the path γ departs from B. We can see that the path jumps immediately when it departs from B. Thus we will show that $\gamma(t)$ cannot be continuous at t_0 by showing that for any $\delta > 0$ there are $t_1, t_2 \in (t_0, t_0 + \delta)$ such that $y(t_1) = 1$ and $y(t_2) = -1$.

To find t_1, note that the set $x([t_0, t_0 + \frac{\delta}{2}])$ is an interval $[0, x_0]$ where $x_0 > 0$. There exists an $x_1 \in (0, x_0)$ such that $\sin \frac{1}{x_1} = 1$: we just need to take $x_1 = \frac{1}{\frac{\pi}{2} + k2\pi}$ with sufficiently large k. There is $t_1 \in (t_0, t_0 + \frac{\delta}{2}]$ such that $x(t_1) = x_1$. Then $y(t_1) = \sin \frac{1}{x(t_1)} = 1$. We can find t_2 similarly. $\square$

The Borsuk-Ulam theorem. Below is a simple version of the Borsuk-Ulam theorem:

Theorem (Borsuk-Ulam theorem). *For any continuous real function on a sphere S^n there must be two antipodal points on the sphere where the values of the function are same.* [6]

PROOF. Let $f : S^n \to \mathbb{R}$ be continuous. Let $g(x) = f(x) - f(-x)$. Then g is continuous and $g(-x) = -g(x)$. If there is an x such that $g(x) \neq 0$ then $g(x)$ and $g(-x)$ have opposite signs. Since S^n is connected (see 5.10), the range $g(S^n)$ is a connected subset of the Euclidean $\mathbb{R}$, and so is an interval, containing the interval between $g(x)$ and $g(-x)$. Thererfore 0 is in the range of g (this is a form of Intermediate value theorem). $\square$

Problems.

5.8. A space is connected if whenever it is a union of two non-empty disjoint subsets, then one of them must contain a contact point of the other one.

5.9. Here is a different proof of 5.5. Suppose that A and B are non-empty, disjoint subsets of $(0,1)$ whose union is $(0,1)$. Let $a \in A$ and $b \in B$. Let $a_0 = a$, $b_0 = b$, and for each $n \geq 1$

[6] On the surface of the Earth at any moment there are two opposite places where temperatures are same!

consider the middle point of the segment from a_n to b_n. If $\frac{a_n+b_n}{2} \in A$ then let $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = b_n$; otherwise let $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2}$. Then:

(a) The sequence $\{a_n \mid n \geq 1\}$ is a Cauchy sequence, hence is convergent to a number a.

(b) The sequence $\{b_n \mid n \geq 1\}$ is also convergent to a. This implies that $(0,1)$ is connected.

5.10. Show that the sphere S^n is connected.

5.11 (Intermediate value theorem). If X is a connected space and $f : X \to \mathbb{R}$ is continuous, where $\mathbb{R}$ has the Euclidean topology, then the image $f(X)$ is an interval.

A consequence is the following familiar theorem in Calculus: Let $f : [a,b] \to \mathbb{R}$ be continuous under the Euclidean topology. If $f(a)$ and $f(b)$ have opposite signs then the equation $f(x) = 0$ has a solution.

5.12. $\checkmark$ If $f : \mathbb{R} \to \mathbb{R}$ is continuous under the Euclidean topology then its graph is connected in the Euclidean plane. Moreover the graph is homeomorphic to $\mathbb{R}$.

5.13. Let X be a topological space and let A_i, $i \in I$ be connected subspaces. If $A_i \cap A_j \neq \emptyset$ for all $i, j \in I$ then $\bigcup_{i \in I} A_i$ is connected.

5.14. Let X be a topological space and let A_i, $i \in \mathbb{Z}^+$ be connected susbsets. If $A_i \cap A_{i+1} \neq \emptyset$ for all $i \geq 1$ then $\bigcup_{i=1}^{\infty} A_i$ is connected.

5.15. Is an intersection of connected subspaces of a space connected?

5.16. Let A be a subspace of X with the particular point topology (X, PPX_p) (see 2.3). Find the connected components of A.

5.17. Let X be connected and let $f : X \to Y$ be continuous. If f is locally constant on X (meaning that every point has a neighborhood on which f is a constant map) then f is constant on X.

5.18. Let X be a topological space. A map $f : X \to Y$ is called a *discrete map* if Y has the discrete topology and f is continuous. Show that X is connected if and only if all discrete maps on X are constant.

5.19. What are the connected components of $\mathbb{N}$ and $\mathbb{Q}$ on the Euclidean real number line?

5.20. What are the connected components of $\mathbb{Q}^2$ as a subspace of the Euclidean plane?

5.21. Show that if a space has finitely many components then each component is both open and closed. Is it still true if there are infinitely many components?

5.22. Suppose that a space X has finitely many connected components. Show that a map defined on X is continuous if and only if it is continuous on each components. Is it still true if X has infinitely many components?

5.23. If a collection of path-connected subspaces of a space has non-empty intersection then its union is path-connected.

5.24. If $f : X \to Y$ is continuous and X is path-connected then $f(X)$ is path-connected.

5.25. If two space are homeomorphic then there is a bijection between the collections of path-connected components of the two spaces. In particular, if one space is path-connected then the other space is also path-connected.

5.26. The plane with a point removed is path-connected under the Euclidean topology.

5.27. The plane with countably many points removed is path-connected under the Euclidean topology.

5.28. $\surd$ The Euclidean line and the Euclidean plane are not homeomorphic.

5.29. Show that $\mathbb{R}$ with the finite complement topology (see and $\mathbb{R}^2$ with the finite complement topology are homeomorphic.

5.30. Find as many ways as you can to prove that S^n is path-connected.

5.31. A topological space is locally path-connected if and only if the collection of all open path-connected subsets is a basis for the topology.

5.32. The Topologist's sine curve is not locally path-connected.

5.33. * Classify the alphabetical characters up to homeomorphisms, that is, which of the following characters are homeomorphic to each other as subspaces of the Euclidean plane?

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

Note that the result depends on the font you use!

Do the same for the Vietnamese alphabetical characters:

A Ă Â B C D Đ E Ê G H I K L M N O Ô Ơ P Q R S T U Ư V X Y

Hint: Use 4.4 to modify each letter part by part.

Further readings

Invariance of dimension. That the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic is not easy. It is a consequence of the following difficult theorem of L. Brouwer in 1912:

Theorem 5.34 (Invariance of dimension). *If two subsets of the Euclidean $\mathbb{R}^n$ are homeomorphic and one set is open then the other is also open.*

This theorem is often proved using Algebraic Topology, see for instance [**Mun00**, p. 381], [**Vic94**, p. 34], [**Hat01**, p. 126].

Corollary. *The Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic if $m \neq n$.*

PROOF. Suppose that $m < n$. It is easy to check that the inclusion map $\mathbb{R}^m \to \mathbb{R}^n$, $(x_1, x_2, \ldots, x_m) \mapsto (x_1, x_2, \ldots, x_m, 0, \ldots, 0)$ is a homeomorphism onto its image $A \subset \mathbb{R}^n$. If A is homeomorphic to $\mathbb{R}^n$ then by Invariance of dimension, A is open in $\mathbb{R}^n$. But A is clearly not open in $\mathbb{R}^n$. $\square$

This result allows us to talk about topological dimension.

Jordan curve theorem. The following is an important and deep result of plane topology.

Theorem (Jordan curve theorem). *A simple, continuous, closed curve separates the plane into two disconnected regions. More concisely, if C is a subset of the Euclidean plane homeomorphic to the circle then $\mathbb{R}^2 \setminus C$ has two connected components.*

Nowadays this theorem is usually proved in a course in Algebraic Topology.

Space filling curves. A rather curious and surprising result is:

Theorem 5.35. *There is a continuous curve filling a rectangle on the plane. More concisely, there is a continuous map from the interval* $[0,1]$ *onto the square* $[0,1]^2$ *under the Euclidean topology.*

Note that this map cannot be injective, in other words the curve cannot be simple.

Such a curve is called a *Peano curve*. It could be constructed as a limit of an iteration of piecewise linear curves.

6. Separation

Definition. We define:

- T_1: A topological space is called a T_1-space if for any two points $x \neq y$ there is an open set containing x but not y and an open set containing y but not x.
- T_2: A topological space is called a T_2-space or *Hausdorff* if for any two points $x \neq y$ there are disjoint open sets U and V such that $x \in U$ and $y \in V$.
- T_3: A T_1-space is called a T_3-space or *regular* (chính tắc) if for any point x and a closed set F not containing x there are disjoint open sets U and V such that $x \in U$ and $F \subset V$.[7]
- T_4: A T_1-space is called a T_4-space or *normal* (chuẩn tắc) if for any two disjoint closed sets F and G there are disjoint open sets U and V such that $F \subset U$ and $G \subset V$.

These definitions are often called separation axioms because they involve "separating" certain sets by open sets.

Proposition. *A space is a T_1 space if and only if any subset containing exactly one point is a closed set.*

Corollary ($T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1$). *If a space is T_i then it is T_{i-1}, for $2 \leq i \leq 4$.*

Example. Any space with the discrete topology is normal.

Example. Let $X = \{a, b\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}\}$. Then X is T_0 but is not T_1.

Example 6.1. The real number line under the finite complement topology is T_1 but is not T_2.

Remark. There are examples of a T_2-space which is not T_3, and a T_3-space which is not T_4, but they are rather difficult, see 6.10, 11.11, [**Mun00**, p. 197] and [**SJ70**].

Proposition. *Any metric space is normal.*

PROOF. We introduce the notion of distance between two sets in a metric space X. If A and B are two subsets of X then we define the distance between A and B as $d(A, B) = \inf\{d(x, y) \mid x \in A, y \in B\}$. In particular if $x \in X$ then $d(x, A) = \inf\{d(x, y) \mid y \in A\}$. Using the triangle inequality we can check that $d(x, A)$ is a continuous function with respect to x.

Now suppose that A and B are disjoint closed sets. Let $U = \{x \mid d(x, A) < d(x, B)\}$ and $V = \{x \mid d(x, A) > d(x, B)\}$. Then $A \subset U$, $B \subset V$ (using the fact that A and B are closed), $U \cap V = \emptyset$, and both U and V are open. $\square$

Proposition 6.2. *A T_1-space X is regular if and only if given a point x and an open set U containing x there is an open set V such that $x \in V \subset \overline{V} \subset U$.*

[7] We include T_1 requirement for regular and normal spaces, as in Munkres [**Mun00**]. Some authors such as Kelley [**Kel55**] do not include the T_1 requirement.

PROOF. Suppose that X is regular. Since $X \setminus U$ is closed and disjoint from $\{x\}$ there is an open set V containing x and an open set W containing $X \setminus U$ such that V and W are disjoint. Then $V \subset (X \setminus W)$, so $\overline{V} \subset (X \setminus W) \subset U$.

Now suppose that X is T_1 and the condition is satisfied. Given a point x and a closed set C disjoint from x, let $U = X \setminus C$. There is an open set V containing x such that $V \subset \overline{V} \subset U$. Then V and $X \setminus \overline{V}$ separate x and C. $\square$

Similarly we have:

Proposition 6.3. *A T_1-space X is normal if and only if given a closed set C and an open set U containing C there is an open set V such that $C \subset V \subset \overline{V} \subset U$.*

Problems.

6.4. If a finite set is a T_1-space then the topology is the discrete topology.

6.5. Is the space (X, PPX_p) (see 2.3) a Hausdorff space?

6.6. Prove directly that any metric space is a regular space.

6.7. A subspace of a Hausdorff space is a Hausdorff space.

6.8. Let X be Hausdorff and let $f : X \to Y$ be continuous. Is $f(X)$ a Hausdorff space?

6.9. A closed subspace of a normal space is normal.

6.10. Show that the set $\mathbb{R}$ with the topology generated by all the subsets of the form (a, b) and $(a, b) \cap \mathbb{Q}$ is Hausdorff but is not a regular space.

6.11. * Let X be normal, let $f : X \to Y$ be a surjective, continuous, and closed map. Prove that Y is a normal space.

7. Convergence

In metric spaces we can study continuity of functions via convergence of sequences. In general topological spaces, we need to use a notion more general than sequences, called nets. Roughly speaking, sequences (having countable indexes) might not be sufficient for describing the neighborhood systems at a point, we need something of arbitrary index.

A *directed set* (tập được định hướng) is a (partially) ordered set such that for any two indices there is an index greater or equal to both. In symbols: $\forall i, j \in I, \exists k \in I, k \geq i \wedge k \geq j$.

A *net* (lưới) (also called a generalized sequence) is a map from a directed set to a space. In other words, a net on a space X with index set a directed set I is a map $x : I \to X$. It is an element of the set $\prod_{i \in I} X$. Thus, writing $x_i = x(i)$, we often denote the net as $(x_i)_{i \in I}$. The notation $\{x_i\}_{i \in I}$ is also used.

Example. Nets with index set $I = \mathbb{N}$ with the usual order are exactly sequences.

Example. Let X be a topological space and $x \in X$. Let I be the family of open neighborhoods of x. Define an order on I by $U \leq V \iff U \supset V$. Then I becomes a directed set.

Convergence.

Definition. A net $(x_i)_{i \in I}$ is said to be *convergent* (hội tụ) to $x \in X$ if for each neighborhood U of x there is an index $i \in I$ such that if $j \geq i$ then x_j belongs to U. The point x is called a limit of the net $(x_i)_{i \in I}$ and we often write $x_i \to x$.

Example. Convergence of nets with index set $I = \mathbb{N}$ with the usual order is exactly convergence of sequences.

Example. Let $X = \{x_1, x_2, x_3\}$ with topology $\{\emptyset, X, \{x_1, x_3\}, \{x_2, x_3\}, \{x_3\}\}$. The net (x_3) converges to x_1, x_2, x_3. The net (x_1, x_2) converges to x_2.

Example. If X has the trivial topology then any net in X is convergent to any point in X.

Proposition 7.1. *A point $x \in X$ is a limit point of a subset $A \subset X$ if and only if there is a net in $A \setminus \{x\}$ convergent to x.*

This proposition allows us to describe topologies in terms of convergences. With it many statements about convergence in metric spaces could be carried to topological spaces by simply replacing sequences by nets.

PROOF. ($\Leftarrow$) Suppose that there is a net $(x_i)_{i \in I}$ in $A \setminus \{x\}$ convergent to x. Let U be an open neighborhood of x. There is an $i \in I$ such that for $j \geq i$ we have $x_j \in U$, in particular $x_i \in U \cap (A \setminus \{x\})$.

($\Rightarrow$) Suppose that x is a limit point of A. Consider the directed set I consisting of all the open neighborhoods of x with the partial order $U \leq V$ if $U \supset V$.

For any open neighborhood U of x there is an element $x_U \in U \cap A$, $x_U \neq x$. Consider the net $\{x_U\}_{U \in I}$. It is a net in $A \setminus \{x\}$ convergent to x. Indeed, given an open neighborhood U of x, for all $V \geq U$, $x_V \in V \subset U$. $\square$

Remark. When can nets be replaced by sequences? By examining the above proof we can see that the term net can be replaced by the term sequence if there is a countable collection F of neighborhoods of x such that any neighborhood of x contains a member of F. In this case the point x is said to have a countable *neighborhood basis*. A space having this property at every point is said to be a *first countable space*. A metric space is such a space, where for example each point has a neighborhood basis consisting of balls of rational radii. See also 7.10.

Similarly to the case of metric spaces, we have:

Theorem. *Let X and Y be topological spaces. Then $f : X \to Y$ is continuous at x if and only if whenever a net n in X converges to x, the net $f \circ n$ converges to $f(x)$.*

In more familiar notations, f is continuous at x if and only if for all nets (x_i), $x_i \to x \Rightarrow f(x_i) \to f(x)$.

PROOF. The proof is simply a repeat of the proof for the case of metric spaces.

($\Rightarrow$) Suppose that f is continuous at x. Let U is a neighborhood of $f(x)$. Then $f^{-1}(U)$ is a neighborhood of x in X. Since (x_i) is convergent to x, there is an $i \in I$ such that for all $j \geq i$ we have $x_j \in f^{-1}(U)$, which implies $f(x_j) \in U$.

($\Leftarrow$) We will show that if U is an open neighborhood in Y of $f(x)$ then $f^{-1}(U)$ is a neighborhood in X of x. Suppose the contrary, then x is not an interior point of $f^{-1}(U)$, so it is a limit point of $X \setminus f^{-1}(U)$. By 7.1 there is a net (x_i) in $X \setminus f^{-1}(U)$ convergent to x. Since f is continuous, $f(x_i) \in Y \setminus U$ is convergent to $f(x) \in U$. This contradicts the assumption that U is open. $\square$

Proposition 7.2. *Suppose that τ_1 and τ_2 are two topologies on X. If for all nets x_i and all points x, $x_i \xrightarrow{\tau_1} x \Rightarrow x_i \xrightarrow{\tau_2} x$, then $\tau_2 \subset \tau_1$. In other words, if convergence in τ_1 implies convergence in τ_2 then τ_1 is finer than τ_2. As a consequence, if convergence are same then topologies are same.*

PROOF. Consider the identity map on X. $\square$

Proposition 7.3. *If a space is Hausdorff then a net has at most one limit.*

PROOF. Suppose that a net (x_i) is convergent to two different points x and y. Since the space is Hausdorff, there are disjoint open neighborhoods U and V of x and y. There is $i \in I$ such that for $\gamma \geq i$ we have $x_\gamma \in U$, and there is $j \in I$ such that for $\gamma \geq j$ we have $x_\gamma \in U$. Since there is a $\gamma \in I$ such that $\gamma \geq i$ and $\gamma \geq j$, the point x_γ will be in $U \cap V$, a contradiction. $\square$

Problems.

7.4. Let $I = (0, \infty) \subset \mathbb{R}$. For $i, j \in I$, define $i \leq_I j$ if $i \geq_{\mathbb{R}} j$ (i less than or equal to j as indexes if i is greater than or equal to j as real numbers). On $\mathbb{R}$ with the Euclidean topology, consider the net $(x_i = i)_{i \in I}$. Is this net convergent?

7.5. On $\mathbb{R}$ with the finite complement topology, consider the net $(x_i = i)_{i \in \mathbb{R}}$. Where does this net converge to?

7.6. Reconsider Problems 2.11, 2.12, and 2.13, using 7.2.

7.7. Let X be a topological space, $\mathbb{R}$ have the Euclidean topology and $f : X \to \mathbb{R}$ be continuous. Suppose that $A \subset X$ and $f(x) = 0$ on A. Show that $f(x) = 0$ on $\overline{A}$, by:

(a) using nets.
(b) not using nets.

7.8. Let Y be Hausdorff and let $f, g : X \to Y$ be continuous. Show that the set $\{x \in X \mid f(x) = g(x)\}$ is closed in X, by:

(a) using nets.
(b) not using nets.

Show that, as a consequence, if f and g agree on a *dense* (trù mật) subspace of X (meaning the closure of that subspace is X) then they agree on X.

7.9. The converse statement of 7.3 is also true. A space is Hausdorff if and only if a net has at most one limit.

7.10 (Sequence is not adequate for convergence). * Let $(A, \leq)$ be a well-ordered uncountable set (see 1.23). If A does not have a biggest element then add an element to A and define that element to be the biggest one. Thus we can assume now that A has a biggest element, denoted by ∞. For $a, b \in A$ denote $[a, b] = \{x \in A \mid a \leq x \leq b\}$ and $[a, b) = \{x \in A \mid a \leq x < b\}$. For example we can write $A = [0, \infty]$.

Let Ω be the smallest element of the set $\{a \in A \mid [0, a] \text{ is uncountable}\}$ (this set is non-empty since it contains ∞).

(a) Show that $[0, \Omega)$ is uncountable, and for all $a \in A$, $a < \Omega$ the set $[0, a]$ is countable.
(b) Show that every countable subset of $[0, \Omega)$ is bounded in $[0, \Omega)$.
(c) Consider $[0, \Omega]$ with the order topology. Show that Ω is a limit point of $[0, \Omega]$.
(d) However, show that a sequence in $[0, \Omega)$ cannot converge to Ω.

7.11 (Filter). A *filter* (lọc) on a set X is a collection F of non-empty subsets of X such that:

(a) if $A, B \in F$ then $A \cap B \in F$,
(b) if $A \subset B$ and $A \in F$ then $B \in F$.

For example, given a point, the collection of all neighborhoods of that point is a filter.

A filter is said to be convergent to a point if any neighborhood of that point is an element of the filter.

A *filter-base* (cơ sở lọc) is a collection G of non-empty subsets of X such that if $A, B \in G$ then there is $C \in G$ such that $G \subset (A \cap B)$.

If G is a filter-base in X then the filter generated by G is defined to be the collection of all subsets of X each containing an element of G: $\{A \subset X \mid \exists B \in G, B \subset A\}$.

For example, in a metric space, the collection of all open balls centered at a point is the filter-base for the filter consisting of all neighborhoods of that point.

A filter-base is said to be convergent to a point if the filter generated by it converges to that point.

(a) Show that a filter-base is convergent to x if and only if every neighborhood of x contains an element of the filter-base.

(b) Show that a point $x \in X$ is a limit point of a subspace A of X if and only if there is a filter-base in $A \setminus \{x\}$ convergent to x.

(c) Show that a map $f : X \to Y$ is continuous at x if and only if for any filter-base F that is convergent to x, the filter-base $f(F)$ is convergent to $f(x)$.

Filter gives an alternative way to net for describing convergence. For more see [**Dug66**, p. 209], [**Eng89**, p. 49], [**Kel55**, p. 83].

8. Compact space

A cover of a set X is a collection of subsets of X whose union is X. A cover is said to be an open cover if each member of the cover is an open subset of X. A subset of a cover which is itself a cover is called a subcover.

Definition. A space is *compact* if every open cover has a finite subcover.

Example. Any finite space is compact. Any space whose topology is finite (that is, the space has finitely many open sets) is compact.

Example. On the Euclidean line $\mathbb{R}$ the collection $\{(-n, n) \mid n \in \mathbb{Z}^+\}$ is an open cover without a finite subcover. Therefore the Euclidean line $\mathbb{R}$ is not compact.

Remark. Let A be a subspace of a topological space X. Let I be an open cover of A. Each $O \in I$ is an open set of A, so it is the restriction of an open set U_O of X. Thus we have a collection $\{U_O \mid O \in I\}$ of open sets of X whose union contains A. On the other hand if we have a collection I of open sets of X whose union contains A then the collection $\{U \cap A \mid U \in I\}$ is an open cover of A. For this reason we often use the term open cover of a subspace A of X in both senses: either as an open cover of A or as a collection of open subsets of the space X whose union contains A.

Theorem (Continuous image of compact space is compact). *If X is compact and $f : X \to Y$ is continuous then $f(X)$ is compact.*

In particular, compactness is preserved under homeomorphism. We say that *compactness is a topological property*.

Proposition. *A closed subspace of a compact space is compact.*

PROOF. Suppose that X is compact and $A \subset X$ is closed. Let I be an open cover of A. By adding the open set $X \setminus A$ to I we get an open cover of X. This open cover has a finite subcover. This subcover of X must contain $X \setminus A$, thus omitting this set we get a finite subcover of A from I. $\square$

Proposition 8.1. *A compact subspace of a Hausdorff space is closed.*

PROOF. Let A be a compact set in a Hausdorff space X. We show that $X \setminus A$ is open.

Let $x \in X \setminus A$. For each $a \in A$ there are disjoint open sets U_a containing x and V_a containing a. The collection $\{V_a \mid a \in A\}$ covers A, so there is a finite subcover $\{V_{a_i} \mid 1 \le i \le n\}$. Let $U = \bigcap_{i=1}^n U_{a_i}$ and $V = \bigcup_{i=1}^n V_{a_i}$. Then U is an open neighborhood of x disjoint from V, a neighborhood of A. $\square$

Example. Any subspace of $\mathbb{R}$ with the finite complement topology is compact. Note that this space is not Hausdorff (6.1).

Characterization of compact spaces in terms of closed subsets. In the definition of compact spaces, writing open sets as complements of closed sets, we get a dual statement: A space is compact if for every collection of closed subsets whose intersection is empty there is a a finite subcollection whose intersection is empty.

A collection of subsets of a set is said to have the *finite intersection property* if the intersection of every finite subcollection is non-empty.

Theorem 8.2. *A space is compact if and only if every collection of closed subsets with the finite intersection property has non-empty intersection.*

Compact metric spaces. A space is called *sequentially compact* if every sequence has a convergent subsequence.

Theorem 8.3 (Lebesgue's number). *In a sequentially compact metric space, for any open cover there exists a number $\epsilon > 0$ such that any ball of radius ϵ is contained in a member of the cover.*

PROOF. Let O be a cover of a sequentiallly compact metric space X. Suppose the opposite of the conclusion, that is for any number $\epsilon > 0$ there is a ball $B(x, \epsilon)$ not contained in any of the element of O. Take a sequence of such balls $B(x_n, 1/n)$. The sequence $\{x_n\}_{n\in\mathbb{Z}^+}$ has a subsequence $\{x_{n_k}\}_{k\in\mathbb{Z}^+}$ converging to x. There is $\epsilon > 0$ such that $B(x, 2\epsilon)$ is contained in an element U of O. Take k sufficiently large such that $n_k > 1/\epsilon$ and x_{n_k} is in $B(x, \epsilon)$. Then $B(x_{n_k}, 1/n_k) \subset B(x_{n_k}, \epsilon) \subset U$, a contradiction. $\square$

Theorem. *A metric space is compact if and only if it is sequentially compact.*

PROOF. ($\Rightarrow$) Let $\{x_n\}_{n\in\mathbb{Z}^+}$ be a sequence in a compact metric space X. Suppose that this sequence has no convergent subsequence. This implies that for any point $x \in X$ there is an open neighborhood U_x of x and $N_x \in \mathbb{Z}^+$ such that if $n \geq N_x$ then $x_n \notin U_x$. Because the collection $\{U_x \mid x \in X\}$ covers X, it has a finite subcover $\{U_{x_k} \mid 1 \leq k \leq m\}$. Let $N = \max\{N_{x_k} \mid 1 \leq k \leq m\}$. If $n \geq N$ then $x_n \notin U_{x_k}$ for all k, a contradiction.

($\Leftarrow$) First we show that for any $\epsilon > 0$ the space X can be covered by finitely many balls of radii ϵ (a property called *total boundedness* or *pre-compact* (tiền compắc)). Suppose the contrary. Let $x_1 \in X$, and inductively let $x_{n+1} \notin \bigcup_{1\leq i\leq n} B(x_i, \epsilon)$. Since $d(x_m, x_n) \geq \epsilon$ if $m \neq n$, the sequence $\{x_n\}_{n\geq 1}$ cannot have any convergent subsequence, a contradiction.

Now let O be any open cover of X. By 8.3 there is a corresponding Lebesgue's number ϵ such that a ball of radius ϵ is contained in an element of O. The space X is covered by finitely many balls of radii ϵ. The collection of finitely many corresponding elements of O covers X. Thus O has a finite subcover. $\square$

The above theorem shows that compactness in metric space as defined in previous courses agrees with compactness in topological spaces. We inherit all results obtained previously on compactness in metric spaces. In particular we have the following results, which were proved using sequential compactness (it should be helpful to review the previous proofs).

Proposition. *If a subspace of a metric space is compact then it is closed and bounded.*

PROOF. We give a proof using compactness. Suppose that X is a metric space and suppose that Y is a compact subspace of X. Let $x \in Y$. Consider the open cover of Y by balls centered at x, that is, $\{B(x,r) \mid r > 0\}$. Since there is a finite subcover, there is an $r > 0$ such that $Y \subset B(x,r)$, thus Y is bounded. That Y is closed in X follows from 8.1. □

Theorem (Heine-Borel). *A subspace of the Euclidean space $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

PROOF. It is sufficient to prove that the unit rectangle $I = [0,1]^n$ is compact. Suppose that O is an open cover of I. Suppose that no finite subset of O can cover I. Divide each dimension of I by half, we get 2^n subrectangles. Let I_1 be one of these rectangles that cannot be covered by a finite subset of O. Inductively, divide I_k to 2^n equal subrectangles and let I_{k+1} be a subrectangle that is not covered by a finite subset of O. We have a family of descending rectangles $\{I_k\}_{k\in\mathbb{Z}^+}$. The dimension of I_k is $1/2^k$, going to 0 as k goes to infinity.

We claim that the intersection of this family is non-empty. Let $I_k = \prod_{i=1}^n [a_k^i, b_k^i]$. For each i, the sequence $\{a_i^k\}_{k\in\mathbb{Z}^+}$ is increasing and is bounded from above. Let $x^i = \lim_{k\to\infty} a_k^i = \sup\{a_k^i \mid k \in \mathbb{Z}^+\}$. Then $a_k^i \le x^i \le b_k^i$ for all $k \ge 1$. Thus the point $x = (x^i)_{1\le i\le n}$ is in the intersection of $\{I_k\}_{k\in\mathbb{Z}^+}$.

There is $U \in O$ that contains x. There is a number $\epsilon > 0$ such that $B(x,\epsilon) \subset U$. Then for k sufficiently large $I_k \subset B(x,\epsilon) \subset U$. This is a contradiction. □

Example. In the Euclidean space $\mathbb{R}^n$ the closed ball $B'(a,r)$ is compact.

Problems.

8.4. A discrete compact topological space is finite.

8.5. In a topological space a finite unions of compact subsets is compact.

8.6. In a Hausdorff space an intersection of compact subsets is compact.

8.7 (An extension of Cantor lemma in Calculus). Let X be compact and $X \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \ldots$ be a descending sequence of closed, non-empty sets. Then $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$.

8.8 (The extreme value theorem). $\surd$ If X is a compact space and $f : X \to (\mathbb{R}, \text{Euclidean})$ is continuous then f has a maximum value and a minimum value.

8.9 (Uniformly continuous). $\surd$ A function f from a metric space to a metric space is uniformly continuous if for any $\epsilon > 0$, there is $\delta > 0$ such that if $d(x,y) < \delta$ then $d(f(x), f(y)) < \epsilon$. Show that a continuous function from a compact metric space to a metric space is uniformly continuous.

8.10. $\surd$ If X is compact, Y is Hausdorff, $f : X \to Y$ is bijective and continuous, then f is a homeomorphism.

8.11. In a compact space any infinite set has a limit point.

8.12. In a Hausdorff space a point and a disjoint compact set can be separated by open sets.

8.13. In a regular space a closed set and a disjoint compact set can be separated by open sets.

8.14. In a Hausdorff space two disjoint compact sets can be separated by open sets.

8.15. A compact Hausdorff space is normal.

8.16. The set of $n \times n$-matrix with real coefficients, denoted by $M(n;\mathbb{R})$, could be naturally considered as a subset of the Euclidean space $\mathbb{R}^{n^2}$ by considering entries of a matrix as coordinates, via the map

$$(a_{i,j}) \longmapsto (a_{1,1}, a_{2,1}, \ldots, a_{n,1}, a_{1,2}, a_{2,2}, \ldots, a_{n,2}, a_{1,3}, \ldots, a_{n-1,n}, a_{n,n}).$$

The *General Linear Group* $\mathrm{GL}(n;\mathbb{R})$ is the group of all invertible $n \times n$-matrices with real coefficients.

(a) Is $\mathrm{GL}(n;\mathbb{R})$ compact?
(b) Find the number of connected components of $\mathrm{GL}(n;\mathbb{R})$.
(c) Show that the product of two matrices is a continuous map.
(d) Show that taking inverse of a matrix is a continuous map.

A set with both a group structure and a topology such that the group operations are continuous is called a *topological group*. Thus $\mathrm{GL}(n;\mathbb{R})$ is a topological group.

8.17. The *Orthogonal Group* $\mathrm{O}(n)$ is defined to be the group of matrices representing orthogonal linear maps of $\mathbb{R}^n$, that is, linear maps that preserve inner product. Thus

$$\mathrm{O}(n) = \{A \in M(n;\mathbb{R}) |\ A \cdot A^T = I_n\}.$$

The *Special Orthogonal Group* $\mathrm{SO}(n)$ is the subgroup of $\mathrm{O}(n)$ consisting of all orthogonal matrices with determinant 1.

(a) Show that any element of $\mathrm{SO}(2)$ is of the form

$$\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}.$$

This is a rotation in the plane around the origin with an angle φ. Thus $\mathrm{SO}(2)$ is the group of rotations on the plane around the origin.
(b) Is $\mathrm{SO}(2)$ path-connected?
(c) How many connected components does $\mathrm{O}(2)$ have?
(d) Is $\mathrm{SO}(n)$ compact?

9. Product of spaces

Finite products of spaces. Let X and Y be two topological spaces, and consider the Cartesian product $X \times Y$. The *product topology* on $X \times Y$ is the topology generated by the collection F of sets of the form $U \times V$ where U is an open set of X and V is an open set of Y.

Since the intersection of two members of F is also a member of F, the collection F is a basis for the product topology. Thus every open set in the product topology is a union of products of open sets of X with open sets of Y.

Similarly, the product topology on $\prod_{i=1}^{n}(X_i, \tau_i)$ is defined to be the topology generated by the collection $\{\prod_{i=1}^{n} U_i \mid U_i \in \tau_i\}$.

Remark. Note that, as sets:

(a) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.
(b) $(A \times B) \cup (C \times D) \subsetneq (A \cup C) \times (B \cup D) = (A \times B) \cup (A \times D) \cup (C \times B) \cup (C \times D)$.

Proposition. *If each b_i is a basis for X_i then $\prod_{i=1}^{n} b_i$ is a basis for the product topology on $\prod_{i=1}^{n} X_i$.*

Example (Euclidean topology). Recall that $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ copies of } \mathbb{R}}$. Let $\mathbb{R}$ have Euclidean topology, generated by open intervals. An open set in the product topology of $\mathbb{R}^n$ is a union of products of open intervals.

Since a product of open intervals is an open rectangle, and an open rectangle is a union of open balls and vice versa, *the product topology on $\mathbb{R}^n$ is exactly the Euclidean topology*.

Infinite products of spaces.

Definition. Let $\{(X_i, \tau_i)\}_{i \in I}$ be a family of topological spaces. The *product topology* on the set $\prod_{i \in I} X_i$ is the topology generated by the collection F consisting of all sets of the form $\prod_{i \in I} U_i$, where $U_i \in \tau_i$ and $U_i = X_i$ for all except finitely many $i \in I$.

Notice that the collection F above is a basis of the product topology. The subcollection of all sets of the form $\prod_{i \in I} U_i$, where $U_i \in \tau_i$ and $U_i = X_i$ for all except one $i \in I$ is a subbasis for the product topology.

Recall that an element of the set $\prod_{i \in I} X_i$ is written $(x_i)_{i \in I}$. For $j \in I$ the *projection to the j-coordinate* $p_j : \prod_{i \in I} X_i \to X_j$ is defined by $p_j((x_i)) = x_j$.

The definition of the product topology is explained in the following:

Theorem 9.1 (Product topology is the topology such that projections are continuous). *The product topology is the coarsest topology on $\prod_{i \in I} X_i$ such that all the projection maps p_i are continuous. In other words, the product topology is the topology generated by the projection maps.*

PROOF. Notice that if $O_j \in X_j$ then $p_j^{-1}(O_j) = \prod_{i\in I} U_i$ with $U_i = X_i$ for all i except j, and $U_j = O_j$. The topology generated by all the maps p_i is the topology generated by all sets of the form $p_i^{-1}(O_i)$ with $O_i \in \tau_i$, see 3.6. $\square$

Theorem 9.2 (Map to product space is continuous if and only if each component map is continuous). *A map $f : Y \to \prod_{i\in I} X_i$ is continuous if and only if each component $f_i = p_i \circ f$ is continuous.*

Remark 9.3. However continuity of a map from a product space is not the same as continuity with respect to each variable, as we have seen in Calculus.

Theorem 9.4 (Convergence in product topology is coordinate-wise convergence). *A net $n : J \to \prod_{i\in I} X_i$ is convergent if and only if all of its projections $p_i \circ n$ are convergent.*

PROOF. ($\Leftarrow$) Suppose that each $p_i \circ n$ is convergent to a_i, we show that n is convergent to $a = (a_i)_{i\in I}$.

A neighborhood of a contains an open set of the form $U = \prod_{i\in I} O_i$ with O_i are open sets of X_i and $O_i = X_i$ except for $i \in K$, where K is a finite subset of I.

For each $i \in K$, $p_i \circ n$ is convergent to a_i, therefore there exists an index $j_i \in J$ such that for $j \geq j_i$ we have $p_i(n(j)) \in O_i$. Take an index j_0 such that $j_0 \geq j_i$ for all $i \in K$. Then for $j \geq j_0$ we have $n(j) \in U$. $\square$

Tikhonov Theorem.

Theorem (Tikhonov theorem). *A product of compact spaces is compact.* [8]

Example. Let $[0,1]$ have the Euclidean topology. The space $\prod_{i\in\mathbb{Z}^+}[0,1]$ is called the *Hilbert cube*. By Tikhonov theorem the Hilbert cube is compact.

Applications of Tikhonov theorem include the Banach-Alaoglu theorem in Functional Analysis and the Stone-Cech compactification.

Tikhonov theorem is equivalent to the Axiom of choice. The proofs we have are rather difficult. However in the case of finite product it can be proved more easily (9.19). Different techniques can be used in special cases of this theorem (9.23 and 9.7).

PROOF OF TIKHONOV THEOREM. Let X_i be compact for all $i \in I$. We will show that $X = \prod_{i\in I} X_i$ is compact by showing that if a collection of closed subsets of X has the finite intersection property then it has non-empty intersection (see 8.2).[9]

Let F be a collection of closed subsets of X that has the finite intersection property. We will show that $\bigcap_{A\in F} A \neq \emptyset$.

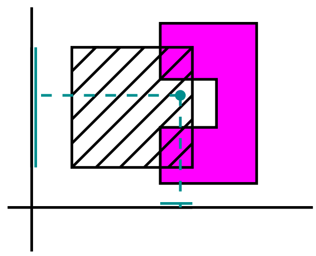
Have a look at the following argument, which suggests that proving the Tikhonov theorem might not be easy. If we take the closures of the projections of the collection F to the i-coordinate then we get a collection $\{\overline{p_i(A)}, A \in F\}$ of closed subsets

[8]Proved by Andrei Nicolaievich Tikhonov around 1926. The product topology was defined by him. His name is also spelled as Tychonoff.

[9]A proof based on open covers is also possible, see **[Kel55**, p. 143].

of X_i having the finite intersection property. Since X_i is compact, this collection has non-empty intersection.

From this it is tempting to conclude that F must have non-empty intersection itself. But that is not true, see the figure.



In what follows we will overcome this difficulty by first enlarging the collection F.

(a) We show that there is a maximal collection $\tilde{F}$ of subsets of X such that $\tilde{F}$ contains F and still has the finite intersection property. We will use Zorn lemma for this purpose.[10]

Let K be the collection of collections G of subsets of X such that G contains F and has the finite intersection property. On K we define an order by the usual set inclusion.

Now suppose that L is a totally ordered subcollection of K. Let $H = \bigcup_{G\in L} G$. We will show that $H \in K$, therefore H is an upper bound of L.

First H contains F. We need to show that H has the finite intersection property. Suppose that $H_i \in H$, $1 \le i \le n$. Then $H_i \in G_i$ for some $G_i \in L$. Since L is totally ordered, there is an i_0, $1 \le i_0 \le n$ such that G_{i_0} contains all G_i, $1 \le i \le n$. Then $H_i \in G_{i_0}$ for all $1 \le i \le n$, and since G_{i_0} has the finite intersection property, we have $\bigcap_{i=1}^{n} H_i \neq \emptyset$.

(b) Since $\tilde{F}$ is maximal, it is closed under finite intersection. Moreover if a subset of X has non-empty intersection with every element of $\tilde{F}$ then it belongs to $\tilde{F}$.

(c) Since $\tilde{F}$ has the finite intersection property, for each $i \in I$ the collection $\{p_i(A) \mid A \in \tilde{F}\}$ also has the finite intersection property, and so does the collection $\{\overline{p_i(A)} \mid A \in \tilde{F}\}$. Since X_i is compact, $\bigcap_{A\in\tilde{F}} \overline{p_i(A)}$ is non-empty.

(d) Let $x_i \in \bigcap_{A\in\tilde{F}} \overline{p_i(A)}$ and let $x = (x_i)_{i\in I} \in \prod_{i\in I}[\bigcap_{A\in\tilde{F}} \overline{p_i(A)}]$. We will show that $x \in \overline{A}$ for all $A \in \tilde{F}$, in particular $x \in A$ for all $A \in F$.

We need to show that any neighborhood of x has non-empty intersection with every $A \in \tilde{F}$. It is sufficient to prove this for neighborhoods of x belonging to the basis of X, namely finite intersections of sets of the form $p_i^{-1}(O_i)$ where O_i is an open neighborhood of $x_i = p_i(x)$. For any $A \in \tilde{F}$, since $x_i \in \overline{p_i(A)}$ we have $O_i \cap p_i(A) \neq \emptyset$. Therefore $p_i^{-1}(O_i) \cap A \neq \emptyset$. By the maximality of $\tilde{F}$ we have $p_i^{-1}(O_i) \in \tilde{F}$, and the desired result follows.

[10]This is a routine step; it might be easier for the reader to carry it out instead of reading.

$\square$

Problems.

9.5. Check that in topological sense (i.e. up to homeomorphisms):

(a) $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

(b) More generally, is the product associative? Namely, is $(X \times Y) \times Z = X \times (Y \times Z)$? Is $(X \times Y) \times Z = X \times Y \times Z$?

9.6. Show that the sphere S^2 with the North Pole and the South Pole removed is homeomorphic to the infinite cylinder $S^1 \times \mathbb{R}$.

9.7. Let (X_i, d_i), $1 \le i \le n$ be metric spaces. Let $X = \prod_{i=1}^n X_i$. For $x = (x_1, x_2, \dots, x_n) \in X$ and $y = (y_1, y_2, \dots, y_n) \in X$, define

$$\delta_1(x, y) = \max\{d_i(x_i, y_i) \mid 1 \le i \le n\},$$

$$\delta_2(x, y) = \left(\sum_{i=1}^n d_i(x_i, y_i)^2\right)^{1/2}.$$

Show that δ_1 and δ_2 are metrics on X generating the product topology.

9.8. Show that a space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \in X \times X\}$ is closed in $X \times X$, by:

(a) using nets,

(b) not using nets.

9.9. Show that if Y is Hausdorff and $f : X \to Y$ is continuous then the graph of f (the set $\{(x, f(x)) \mid x \in X\}$) is closed in $X \times Y$.

9.10. If for each $i \in I$ the space X_i is homeomorphic to the space Y_i then $\prod_{i \in I} X_i$ is homeomorphic to $\prod_{i \in I} Y_i$.

9.11. $\surd$ Show that each projection map p_i is a an open map, mapping an open set onto an open set.

9.12. Is the projection p_i a closed map?

9.13. Is it true that a map on a product space is continuous if it is continuous on each variable?

9.14 (Disjoint union). $\surd$ Let A and B be topological spaces. On the set $(A \times \{0\}) \cup (B \times \{1\})$ consider the topology generated by subsets of the form $U \times \{0\}$ and $V \times \{1\}$ where U is open in A and V is open in B. Show that $A \times \{0\}$ is homeomorphic to A, while $B \times \{1\}$ is homeomorphic to B. Notice that $(A \times \{0\}) \cap (B \times \{1\}) = \emptyset$. The space $(A \times \{0\}) \cup (B \times \{1\})$ is called the *disjoint union* (hội rời) of A and B, denoted by $A \sqcup B$.

We use this construction when for example we want to consider a space consisting of two disjoint circles.

9.15. Fix a point $O = (O_i) \in \prod_{i \in I} X_i$. Define the inclusion map $f : X_i \to \prod_{i \in I} X_i$ by

$$x \mapsto f(x) \text{ with } f(x)_j = \begin{cases} O_j & \text{if } j \neq i \\ x & \text{if } j = i \end{cases}.$$

Show that f is a homeomorphism onto its image $\tilde{X}_i$ (an embedding of X_i).

Thus $\tilde{X}_i$ is a copy of X_i in $\prod_{i\in I} X_i$. The spaces $\tilde{X}_i$ have O as the common point. This is an analog of the coordinate system Oxy on $\mathbb{R}^2$.

9.16. Show that

(a) If each X_i, $i \in I$ is a Hausdorff space then $\prod_{i\in I} X_i$ is a Hausdorff space.
(b) If $\prod_{i\in I} X_i$ is a Hausdorff space then each X_i is a Hausdorff space.

9.17. Show that

(a) If $\prod_{i\in I} X_i$ is path-connected then each X_i is path-connected.
(b) If each X_i, $i \in I$ is path-connected then $\prod_{i\in I} X_i$ is path-connected.

9.18. Show that

(a) If $\prod_{i\in I} X_i$ is connected then each X_i is connected.
(b) If X and Y are connected then $X \times Y$ is connected.
(c) * If each X_i, $i \in I$ is connected then $\prod_{i\in I} X_i$ is connected.

9.19. Show that

(a) If $\prod_{i\in I} X_i$ is compact then each X_i is compact.
(b) * If X and Y are compact then $X \times Y$ is compact.

9.20. Consider $PP\mathbb{R}^2_{(0,0)}$ (the plane with the particular point topology at the origin, see 2.3). Is it homeomorphic to $PP\mathbb{R}_0 \times PP\mathbb{R}_0$? In other words, is it true that $PP\mathbb{R}^2_{(0,0)} = (PP\mathbb{R}_0)^2$?

9.21. Consider the Euclidean space $\mathbb{R}^n$. Check that the usual addition $(x, y) \mapsto x + y$ is a continuous map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$, while the usual scalar multiplication $(c, x) \mapsto c \cdot x$ is a continuous map from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$. This is an example of a *topological vector space*.

9.22 (Zariski topology). * Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

A polynomial in n variables on $\mathbb{F}$ is a function from $\mathbb{F}^n$ to $\mathbb{F}$ that is a finite sum of terms of the form $ax_1^{m_1}x_2^{m_2}\cdots x_n^{m_n}$, where a, $x_i \in \mathbb{F}$ and $m_i \in \mathbb{N}$. Let P be the set of all polynomials in n variables on $\mathbb{F}$.

If $S \subset P$ then define $Z(S)$ to be the set of all common zeros of all polynomials in S, thus $Z(S) = \{x \in \mathbb{F}^n \mid \forall p \in S,\ p(x) = 0\}$. Such a set is called an *algebraic set*.

(a) Show that if we define that a subset of $\mathbb{F}^n$ is closed if it is algebraic, then this gives a topology on $\mathbb{F}^n$, called the *Zariski topology*.
(b) Show that the Zariski topology on $\mathbb{F}$ is exactly the finite complement topology.
(c) Show that if both $\mathbb{F}$ and $\mathbb{F}^n$ have the Zariski topology then all polynomials on $\mathbb{F}^n$ are continuous.
(d) Is the Zariski topology on $\mathbb{F}^n$ the product topology?

The Zariski topology is used in Algebraic Geometry.

9.23. Using the characterization of compact subsets of Euclidean spaces, prove the Tikhonov theorem for finite products of compact subsets of Euclidean spaces.

9.24. Using the characterization of compact metric spaces in terms of sequences, prove the Tikhonov theorem for finite products of compact metric spaces.

Further readings

Strategy for a proof of Tikhonov theorem based on net. The proof that we will outline here is based on further developments of the theory of nets and a characterization of compactness in terms of nets.

Definition 9.25 (Subnet). Let I and I' be directed sets, and let $h : I' \to I$ be a map such that

$$\forall k \in I, \exists k' \in I', (i' \geq k' \Rightarrow h(i') \geq k).$$

If $n : I \to X$ is a net then $n \circ h$ is called a *subnet* of n.

The notion of subnet is an extension of the notion of subsequence. If we take $n_i \in \mathbb{Z}^+$ such that $n_i < n_{i+1}$ then (x_{n_i}) is a subsequence of (x_n). In this case the map $h : \mathbb{Z}^+ \to \mathbb{Z}^+$ given by $h(i) = n_i$ is a strictly increasing function. Thus a subsequence of a sequence is a subnet of that sequence. On the other hand a subnet of a sequence does not need to be a subsequence, since for a subnet the map h is only required to satisfy $\lim_{i \to \infty} h(i) = \infty$.

A net $(x_i)_{i \in I}$ is called *eventually* in $A \subset X$ if there is $j \in I$ such that $i \geq j \Rightarrow x_i \in A$.

Definition 9.26. Universal net A net n in X is *universal* if for any subset A of X either n is eventually in A or n is eventually in $X \setminus A$.

Proposition 9.27. *If $f : X \to Y$ is continuous and n is a universal net in X then $f(n)$ is a universal net.*

Theorem 9.28. *The following statements are equivalent:*

(a) *X is compact.*
(b) *Every universal net in X is convergent.*
(c) *Every net in X has a convergent subnet.*

The proof of the last two propositions above could be found in **[Bre93]**.

Then we finish the proof of Tikhonov theorem as follows.

PROOF OF TIKHONOV THEOREM. Let $X = \prod_{i \in I} X_i$ where each X_i is compact. Suppose that $(x_j)_{j \in J}$ is a universal net in X. By 9.4 the net (x_j) is convergent if and only if the projection $(p_i(x_j))$ is convergent for all i. But that is true since $(p_i(x_j))$ is a universal net in the compact set X_i. $\square$

10. Compactification

A *compactification* (compắc hóa) of a space X is a compact space Y such that X is homeomorphic to a dense subspace of Y.

Example. A compactification of the Euclidean interval $(0,1)$ is the Euclidean interval $[0,1]$. Another is the circle S^1. Yet another is the Topologist's sine curve $\{(x, \sin\frac{1}{x}) \mid 0 < x \leq 1\} \cup \{(0,y) \mid -1 \leq y \leq 1\}$ (see 5.1).

Example. A compactification of the Euclidean plane $\mathbb{R}^2$ is the sphere S^2. When $\mathbb{R}^2$ is identified with the complex plane $\mathbb{C}$ then S^2 is often called the *Riemann sphere*.

One-point compactification. In some cases it is possible to compactify a non-compact space by adding just one point, obtaining a *one-point compactification*. For example the Euclidean interval $[0,1]$ is a one-point compactification of the Euclidean interval $[0,1)$.

Let X be a space, let ∞ be not in X, and let $X^\infty = X \cup \{\infty\}$. Let us see what a topology on X^∞ should be in order for X^∞ to contain X as a subspace and to be compact. If an open subset U of X^∞ does not contain ∞ then U is contained in X, therefore U is an open subset of X in the subspace topology of X, which is the same as the original topology of X. If U contains ∞ then its complement $X^\infty \setminus U$ must be a closed subset of X^∞, hence is compact, furthermore $X^\infty \setminus U$ is contained in X and is therefore a closed subset of X.

Theorem (Alexandroff compactification). *The collection consisting of all open subsets of X and all complements in X^∞ of closed compact subsets of X is the finest topology on X^∞ such that X^∞ is compact and contains X as a subspace. If X is not compact then X is dense in X^∞, and X^∞ is called the Alexandroff compactification of X.* [11]

PROOF. We go through several steps.

(a) We check that we really have a topology.

Let I be a collection of closed compact sets in X. Then $\bigcup_{C \in I}(X^\infty \setminus C) = X^\infty \setminus \bigcap_{C \in I} C$, where $\bigcap_{C \in I} C$ is closed compact.

If O is open in X and C is closed compact in X then $O \cup (X^\infty \setminus C) = X^\infty \setminus (C \setminus O)$, where $C \setminus O$ is a closed and compact subset of X.

Also $O \cap (X^\infty \setminus C) = O \cap (X \setminus C)$ is open in X.

If C_1 and C_2 are closed compact in X then $(X^\infty \setminus C_1) \cap (X^\infty \setminus C_2) = X^\infty \setminus (C_1 \cup C_2)$, where $C_1 \cup C_2$ is closed compact.

So we do have a topology. With this topology X is a subspace of X^∞.

(b) We show that X^∞ is compact. Let F be an open cover of X^∞. Then an element $O \in F$ will cover ∞. The complement of O in X^∞ is a closed compact set C in X.

Then $F \setminus \{O\}$ is an open cover of C. From this cover there is a finite cover. This finite cover together with O is a finite cover of X^∞.

[11] Proved in the early 1920s by Pavel Sergeyevich Alexandrov. Alexandroff is another way to spell his name.

(c) Since X is not compact and X^∞ is compact, X cannot be closed in X^∞, therefore the closure of X in X^∞ is X^∞.

$\square$

A space X is called *locally compact* if every point has a compact neighborhood.

Example. The Euclidean space $\mathbb{R}^n$ is locally compact.

Proposition. *The Alexandroff compactification of a locally compact Hausdorff space is Hausdorff.*

PROOF. Suppose that X is locally compact and is Hausdorff. We check that ∞ and $x \in X$ can be separated by open sets. Since X is locally compact there is a compact set C containing an open neighborhood O of x. Since X is Hausdorff, C is closed in X. Then $X^\infty \setminus C$ is open in the Alexandroff compactification X^∞. So O and $X^\infty \setminus C$ separate x and ∞. $\square$

The need for the locally compact assumption is discussed in 10.11.

Proposition. *If X is homeomorphic to Y then a Hausdorff one-point compactification of X is homeomorphic to a Hausdorff one-point compactification of Y.*

In particular, Hausdorff one-point compactification is unique up to homeomorphisms. For this reason we can talk about *the* one-point compactification of a locally compact Hausdorff space.

PROOF. Suppose that $h : X \to Y$ is a homeomorphism. Let $X \cup \{a\}$ and $Y \cup \{b\}$ be Hausdorff one-point compactifications of X and Y. Let $\tilde{h} : X \cup \{a\} \to Y \cup \{b\}$ be defined by $\tilde{h}(x) = h(x)$ if $x \neq a$ and $\tilde{h}(a) = b$. We show that $\tilde{h}$ is a homeomorphism. We will prove that $\tilde{h}$ is continuous, that the inverse map is continuous is similar, or we can use 8.10 instead.

Let U be an open subset of $Y \cup \{b\}$. If U does not contain b then U is open in Y, so $h^{-1}(U)$ is open in X, and so is open in $X \cup \{a\}$. If U contains b then $(Y \cup \{b\}) \setminus U$ is closed in $Y \cup \{b\}$, which is compact, so $(Y \cup \{b\}) \setminus U = Y \setminus U$ is compact. Then $\tilde{h}^{-1}((Y \cup \{b\}) \setminus U) = h^{-1}(Y \setminus U)$ is a compact subspace of X and therefore of $X \cup \{a\}$. Since $X \cup \{a\}$ is Hausdorff, $\tilde{h}^{-1}((Y \cup \{b\}) \setminus U)$ is closed in $X \cup \{a\}$. Thus $\tilde{h}^{-1}(U)$ must be open in $X \cup \{a\}$. $\square$

Example. The Euclidean line $\mathbb{R}$ is homeomorphic to the circle S^1 minus a point. The circle is of course a Hausdorff one-point compactification of the circle minus a point. Thus a Hausdorff one-point compactification (in particular, the Alexandroff compactification) of the Euclidean line is homeomorphic to the circle.

Stone-Cech compactification. Let X be a topological space. Denote by $C(X)$ the set of all bounded continuous functions from X to $\mathbb{R}$ where $\mathbb{R}$ has the Euclidean topology. Define

$$\begin{aligned} \Phi : X &\to \prod_{f\in C(X)} [\inf f, \sup f] \\ x &\mapsto (f(x))_{f\in C(X)}. \end{aligned}$$

Thus for each $x \in X$ and each $f \in C(X)$, the f-coordinate of the point $\Phi(x)$ is $\Phi(x)_f = f(x)$. This means the f-component of Φ is f, i.e. $p_f \circ \Phi = f$, where p_f is the projection to the f-coordinate.

Since $\prod_{f\in C(X)}[\inf f, \sup f]$ is compact, the subspace $\overline{\Phi(X)}$ is compact.

Definition. A space is said to be *completely regular* (also called a $T_{3\frac{1}{2}}$-space) if it is a T_1-space and for each point x and each closed set A with $x \notin A$ there is a map $f \in C(X)$ such that $f(x) = a$ and $f(A) = \{b\}$ where $a \neq b$.

Thus in a completely regular space a point and a closed set disjoint from it can be separated by a continuous real function.

Theorem 10.1. *If X is completely regular then $\Phi : X \to \Phi(X)$ is a homeomorphism, i.e. Φ is an embedding. In this case $\overline{\Phi(X)}$ is called the Stone-Cech compactification of X. It is a Hausdorff space.*

PROOF. We go through several steps.

(a) Φ is injective: If $x \neq y$ then since X is completely regular there is $f \in C(X)$ such that $f(x) \neq f(y)$, therefore $\Phi(x) \neq \Phi(y)$.
(b) Φ is continuous: Since the f-component of Φ is f, which is continuous, the result follows from 9.2.
(c) Φ^{-1} is continuous: We prove that Φ brings an open set onto an open set. Let U be an open subset of X and let $x \in U$. There is a function $f \in C(X)$ that separates x and $X \setminus U$. In particular there is an interval (a, b) containing such that $f^{-1}((a, b)) \cap (X \setminus U) = \emptyset$. We have $f^{-1}((a, b)) = (p_f \circ \Phi)^{-1}((a, b)) = \Phi^{-1}(p_f^{-1}((a, b))) \subset U$. Apply Φ to both sides, we get $p_f^{-1}((a, b)) \cap \Phi(X) \subset \Phi(U)$. Since $p_f^{-1}((a, b)) \cap \Phi(X)$ is an open set in $\Phi(X)$ containing $\Phi(x)$, we see that $\Phi(x)$ is an interior point of $\Phi(U)$. We conclude that $\Phi(U)$ is open.

That $\overline{\Phi(X)}$ is a Hausdorff space follows from that Y is a Hausdorff space, by 9.16, and 6.7. $\square$

Theorem. *A bounded continuous real function on a completely regular space has a unique extension to the Stone-Cech compactification of the space.*

More concisely, if X is a completely regular space and $f \in C(X)$ then there is a unique function $\tilde{f} \in C(\overline{\Phi(X)})$ such that $f = \tilde{f} \circ \Phi$.

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "f"'] & \overline{\Phi(X)} \arrow[dl, "\tilde{f}"] \\ \mathbb{R} & \end{tikzcd}$$

PROOF. A continuous extension of f, if exists, is unique, by 7.8.
Since $p_f \circ \Phi = f$ the obvious choice for $\tilde{f}$ is the projection p_f. □

Problems.

10.2. Find the one-point compactification of $(0,1) \cup (2,3)$ with the Euclidean topology, that is, describe this space more concretely.

10.3. Find the one-point compactification of $\{\frac{1}{n} \mid n \in \mathbb{Z}^+\}$ under the Euclidean topology?

10.4. Find the one-point compactification of $\mathbb{Z}^+$ under the Euclidean topology? How about $\mathbb{Z}$?

10.5. Show that $\mathbb{Q}$ is not locally compact (under the Euclidean topology of $\mathbb{R}$). Is its Alexandroff compactification Hausdorff?

10.6. What is the one-point compactification of the Euclidean open ball $B(0,1)$? Find the one-point compactification of the Euclidean space $\mathbb{R}^n$.

10.7. What is the one-point compactification of the Euclidean annulus $\{(x,y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 2\}$?

10.8. Define a topology on $\mathbb{R} \cup \{\pm\infty\}$ such that it is a compactification of the Euclidean $\mathbb{R}$.

10.9. Consider $\mathbb{R}$ with the Euclidean topology. Find a necessary and sufficient condition for a continuous function from $\mathbb{R}$ to $\mathbb{R}$ to have an extension to a continuous function from the one-point compactification $\mathbb{R} \cup \{\infty\}$ to $\mathbb{R}$.

10.10. If a subset of X is closed will it be closed in the Alexandroff compactification of X?

10.11. If there is a topology on the set $X^\infty = X \cup \{\infty\}$ such that it is compact, Hausdorff, and containing X as a subspace, then X must be Hausdorff, locally compact, and there is only one such topology – the topology of the Alexandroff compactification.

10.12. We could have noticed that the notion of local compactness as we have defined is not apparently a local property. For a property to be local, every neighborhood of any point must contain a neighborhood of that point with the given property (as in the cases of local connectedness and local path-connectedness). Show that for Hausdorff spaces local compactness is indeed a local property, i.e., every neighborhood of any point contains a compact neighborhood of that point.

10.13. Any locally compact Hausdorff space is a regular space.

10.14. In a locally compact Hausdorff space, if K is compact, U is open, and $K \subset U$, then there is an open set V such that $\overline{V}$ is compact and $K \subset V \subset \overline{V} \subset U$. (Compare with 6.3.)

10.15. A space is locally compact Hausdorff if and only if it is homeomorphic to an open subspace of a compact Hausdorff space.

10.16. Any completely regular space is a regular space.

10.17. Prove 10.1 using nets.

10.18. A space is completely regular if and only if it is homeomorphic to a subspace of a compact Hausdorff space. As a corollary, a locally compact Hausdorff space is completely regular.

Further readings

By 10.18 if a space has a Hausdorff Alexandroff compactification then it also has a Hausdorff Stone-Cech compactification.

In a certain sense, for a noncompact space the Alexandroff compactification is the "smallest" Hausdorff compactification of the space and the Stone-Cech compactification is the "largest" one. For more discussions on this topic see for instance [**Mun00**, p. 237].

11. Real functions and spaces of functions

Urysohn lemma. Here we consider real functions, i.e. maps to the Euclidean $\mathbb{R}$.

Theorem 11.1 (Urysohn lemma). *If X is normal, F is closed, U is open, and $F \subset U$, then there exists a continuous map $f : X \to [0,1]$ such that $f(x) = 0$ on F and $f(x) = 1$ on $X \setminus U$.*

Equivalently, if X is normal, A and B are two disjoint closed subsets of X, then there is a continuous function f from X to $[0,1]$ such that $f(x) = 0$ on A and $f(x) = 1$ on B.

Thus in a normal space two disjoint closed subsets can be separated by a continuous real function.

Example. It is much easier to prove Urysohn lemma for metric space, using the function

$$f(x) = \frac{d(x,A)}{d(x,A) + d(x,B)}.$$

PROOF OF URYSOHN LEMMA. Recall 6.3: Because X is normal, if F is closed, U is open, and $F \subset U$ then there is an open set V such that $F \subset V \subset \overline{V} \subset U$.

(a) We construct a family of open sets in the following manner.

Let $U_1 = U$.

$n = 0$: $F \subset U_0 \subset \overline{U_0} \subset U_1$.

$n = 1$: $\overline{U_0} \subset U_{\frac{1}{2}} \subset \overline{U_{\frac{1}{2}}} \subset U_1$.

$n = 2$: $\overline{U_0} \subset U_{\frac{1}{4}} \subset \overline{U_{\frac{1}{4}}} \subset U_{\frac{2}{4}} = U_{\frac{1}{2}} \subset \overline{U_{\frac{2}{4}}} \subset U_{\frac{3}{4}} \subset \overline{U_{\frac{3}{4}}} \subset U_{\frac{4}{4}} = U_1$.

Inductively we have a family of open sets:

$$F \subset U_0 \subset \overline{U_0} \subset U_{\frac{1}{2^n}} \subset \overline{U_{\frac{1}{2^n}}} \subset U_{\frac{2}{2^n}} \subset \overline{U_{\frac{2}{2^n}}} \subset U_{\frac{3}{2^n}} \subset \overline{U_{\frac{3}{2^n}}} \subset \cdots \subset$$
$$\subset U_{\frac{2^n-1}{2^n}} \subset \overline{U_{\frac{2^n-1}{2^n}}} \subset U_{\frac{2^n}{2^n}} = U_1.$$

Let $I = \{\frac{m}{2^n} \mid m, n \in \mathbb{N}; 0 \le m \le 2^n\}$. We have a family of open sets $\{U_r \mid r \in I\}$ having the property $r < s \Rightarrow \overline{U_r} \subset U_s$.

(b) We can check that I is dense in $[0,1]$ (this is really the same thing as that any real number in $[0,1]$ can be written in binary form, compare 1.20).

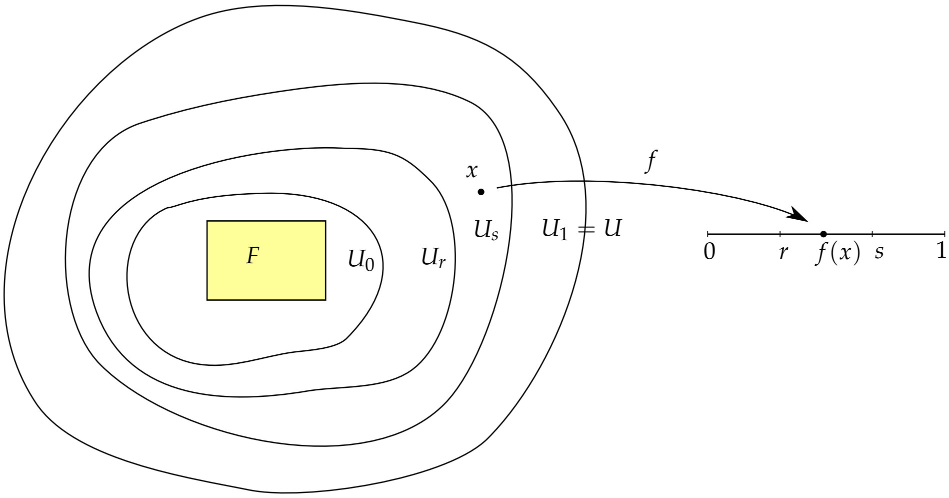
(c) Define $f : X \to [0,1]$,

$$f(x) = \begin{cases} \inf\{r \in I \mid x \in U_r\} & \text{if } x \in U, \\ 1 & \text{if } x \notin U. \end{cases}$$

In this way if $x \in U_r$ then $f(x) \le r$, while if $x \notin U_r$ then $f(x) \ge r$. So $f(x)$ gives the "level" of x on the scale from 0 to 1, while U_r is like a sublevel set of f.

We prove that f is continuous, then this is the function we are looking for. It is enough to prove that sets of the form $\{x \mid f(x) < a\}$ and $\{x \mid f(x) > a\}$ are open.

(d) If $a \le 1$ then $f(x) < a$ if and only if there is $r \in I$ such that $r < a$ and $x \in U_r$. Thus $\{x \mid f(x) < a\} = \{x \in U_r \mid r < a\} = \bigcup_{r<a} U_r$ is open.



(e) If $a < 1$ then $f(x) > a$ if and only if there is $r \in I$ such that $r > a$ and $x \notin U_r$. Thus $\{x \mid f(x) > a\} = \{x \in X \setminus U_r \mid r > a\} = \bigcup_{r>a} X \setminus U_r$.

Now we show that $\bigcup_{r>a} X \setminus U_r = \bigcup_{r>a} X \setminus \overline{U_r}$, which implies that $\bigcup_{r>a} X \setminus U_r$ is open. Indeed, if $r \in I$ and $r > a$ then there is $s \in I$ such that $r > s > a$. Then $\overline{U_s} \subset U_r$, therefore $X \setminus U_r \subset X \setminus \overline{U_s}$.

□

Uniform convergences. Let X and Y be two topological spaces. We say that a net $(f_i)_{i\in I}$ of functions from X to Y *converges point-wise* to a function $f : X \to Y$ if for each $x \in X$ the net $(f_i(x))_{i\in I}$ converges to $f(x)$.

Now let Y be a metric space. Recall that a function $f : X \to Y$ is said to be *bounded* if the set of values $f(X)$ is a bounded subset of Y. We consider the set $B(X,Y)$ of all bounded functions from X to Y. If $f, g \in B(X,Y)$ then we define a metric on $B(X,Y)$ by $d(f,g) = \sup\{d(f(x), g(x)) \mid x \in X\}$. The topology generated by this metric is called the *topology of uniform convergence*. If a net $(f_i)_{i\in I}$ converges to f in the metric space $B(X,Y)$ then we say that $(f_i)_{i\in I}$ converges to f *uniformly*.

Proposition. *Suppose that $(f_i)_{i\in I}$ converges to f uniformly. Then:*

(a) *$(f_i)_{i\in I}$ converges to f point-wise.*

(b) *If each f_i is continuous then f is continuous.*

PROOF. The proof of part (2) is the same as the one for the case of metric spaces. Suppose that each f_i is continuous. Let $x \in X$, we prove that f is continuous at x. The key step is the following inequality:

$$d(f(x), f(y)) \leq d(f(x), f_i(x)) + d(f_i(x), f_i(y)) + d(f_i(y), f(y)).$$

Given $\epsilon > 0$, fix an $i \in I$ such that $d(f_i, f) < \epsilon$. For this i, there is a neighborhood U of x such that if $y \in U$ then $d(f_i(x), f_i(y)) < \epsilon$. The above inequality implies that for $y \in U$ we have $d(f(x), f(y)) < 3\epsilon$. □

Let X and Y be two topological spaces. Let $C(X,Y)$ be the set of all continuous functions from X to Y. The topology on generated by all sets of the form

$$S(A,U) = \{f \in C(X,Y) \mid f(A) \subset U\}$$

where $A \subset X$ is compact and $U \subset Y$ is open is called the *compact-open topology* on $C(X,Y)$.

Proposition. *Let X be compact and Y be a metric space, then on $C(X,Y)$ the compact-open topology is the same as the uniform convergence topology.*

Tiestze extension theorem. We consider real functions again.

Theorem (Tiestze extension theorem). *Let X be a normal space. Let F be closed in X. Let $f : F \to \mathbb{R}$ be continuous. Then there is a continuous map $g : X \to \mathbb{R}$ such that $g|_F = f$.*

Thus *in a normal space a continuous real function on a closed subspace can be extended continuously to the whole space*.

PROOF. First consider the case where f is bounded.

(a) The general case can be reduced to the case when $\inf_F f = 0$ and $\sup_F f = 1$. We will restrict our attention to this case.

(b) By Urysohn lemma, there is a continuous function $g_1 : X \to [0, \frac{1}{3}]$ such that

$$g_1(x) = \begin{cases} 0 & \text{if } x \in f^{-1}([0, \frac{1}{3}]) \\ \frac{1}{3} & \text{if } x \in f^{-1}([\frac{2}{3}, 1]). \end{cases}$$

Let $f_1 = f - g_1$. Then $\sup_X g_1 = \frac{1}{3}$, $\sup_F f_1 = \frac{2}{3}$, and $\inf_F f_1 = 0$.

(c) Inductively, once we have a function $f_n : F \to \mathbb{R}$, for a certain $n \geq 1$ we will obtain a function $g_{n+1} : X \to [0, \frac{1}{3}\left(\frac{2}{3}\right)^n]$ such that

$$g_{n+1}(x) = \begin{cases} 0 & \text{if } x \in f_n^{-1}([0, \frac{1}{3}\left(\frac{2}{3}\right)^n]) \\ \frac{1}{3}\left(\frac{2}{3}\right)^n & \text{if } x \in f_n^{-1}([\left(\frac{2}{3}\right)^{n+1}, \left(\frac{2}{3}\right)^n]). \end{cases}$$

Let $f_{n+1} = f_n - g_{n+1}$. Then $\sup_X g_{n+1} = \frac{1}{3}\left(\frac{2}{3}\right)^n$, $\sup_F f_{n+1} = \left(\frac{2}{3}\right)^{n+1}$, and $\inf_F f_{n+1} = 0$.

(d) The series $\sum_{n=1}^{\infty} g_n$ converges uniformly to a continuous function g.

(e) Since $f_n = f - \sum_{i=1}^{n} g_i$, the series $\sum_{n=1}^{n} g_n|_F$ converges uniformly to f. Therefore $g|_F = f$.

(f) Note that with this construction $\inf_X g = 0$ and $\sup_X g = 1$.

Now consider the case when f is not bounded.

(a) Suppose that f is neither bounded from below nor bounded from above. Let h be a homeomorphism from $(-\infty, \infty)$ to $(0,1)$. Then the range of $f_1 = h \circ f$ is a subset of $(0,1)$, therefore it can be extended as in the previous case to a continuous function g_1 such that $\inf_{x \in X} g_1(x) = \inf_{x \in F} f_1(x) = 0$ and $\sup_{x \in X} g_1(x) = \sup_{x \in F} f_1(x) = 1$.

If the range of g_1 includes neither 0 nor 1 then $g = h^{-1} \circ g_1$ will be the desired function.

It may happens that the range of g_1 includes either 0 or 1. In this case let $C = g_1^{-1}(\{0,1\})$. Note that $C \cap F = \emptyset$. By Urysohn lemma, there is a continuous function $k : X \to [0,1]$ such that $k|_C = 0$ and $k|_F = 1$. Let $g_2 = kg_1 + (1-k)\frac{1}{2}$. Then $g_2|_F = g_1|_F$ and the range of g_2 is a subset of $(0,1)$ ($g_2(x)$ is a certain convex combination of $g_1(x)$ and $\frac{1}{2}$). Then $g = h^{-1} \circ g_2$ will be the desired function.

(b) If f is bounded from below then similarly to the previous case we can use a homeomorphism $h : [a, \infty) \to [0,1)$, and we let $C = g_1^{-1}(\{1\})$.

The case when f is bounded from above is similar.

$\square$

Problems.

11.2. A normal space is completely regular. So: normal $\Rightarrow$ completely regular $\Rightarrow$ regular. In other words: $T_4 \Rightarrow T_{3\frac{1}{2}} \Rightarrow T_3$.

11.3. A space is completely regular if and only if it is homeomorphic to a subspace of a compact Hausdorff space. As a corollary, a locally compact Hausdorff space is completely regular.

11.4. Show that the Tiestze extension theorem implies the Urysohn lemma.

11.5. The Tiestze extension theorem is not true without the condition that the set F is closed.

11.6. Show that the Tiestze extension theorem can be extended to maps to the space $\prod_{i \in I} \mathbb{R}$ where $\mathbb{R}$ has the Euclidean topology.

11.7. Let X be a normal space and F be a closed subset of X. Then any continuous map $f : F \to S^n$ can be extended to an open set containing F.

11.8. Prove the following version of Urysohn lemma, as stated in **[Rud86]**. Suppose that X is a locally compact Hausdorff space, V is open in X, $K \subset V$, and K is compact. Then there is a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ for $x \in K$ and $\mathrm{supp}(f) \subset V$, where $\mathrm{supp}(f)$ is the closure of the set $\{x \in X \mid f(x) \neq 0\}$, called the *support* of f.

Hint: Use 10.13 and 8.13.

11.9 (Point-wise convergence topology). Now we view a function from X to Y as an element of the set $Y^X = \prod_{x \in X} Y$. In this view a function $f : X \to Y$ is an element $f \in Y^X$, and for each $x \in X$ the value $f(x)$ is the x-coordinate of the element f.

(a) Let $(f_i)_{i \in I}$ be a net of functions from X to Y, i.e. a net of points in Y^X. Show that $(f_i)_{i \in I}$ converges to a function $f : X \to Y$ point-wise if and only if the net of points $(f_i)_{i \in I}$ converges to the point f in the product topology of Y^X.

(b) Define the *point-wise convergence topology* on the set Y^X of functions from X to Y as the topology generated by sets of the form

$$S(x, U) = \{f \in Y^X \mid f(x) \in U\}$$

with $x \in X$ and $U \subset Y$ is open. Show that the point-wise convergence topology is exactly the product topology on Y^X.

11.10. Let X and Y be two topological spaces. Let $C(X,Y)$ be the set of all continuous functions from X to Y. Show that if a net $(f_i)_{i\in I}$ converges to f in the compact-open topology of $C(X,Y)$ then it converges to f point-wise.

11.11 (Niemytzki space). * Let $\mathbb{H} = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$ be the upper half-plane. Equip $\mathbb{H}$ with the topology generated by the Euclidean open disks (i.e. open balls) in $K = \{(x,y) \in \mathbb{R}^2 \mid y > 0\}$, together with sets of the form $\{p\} \cup D$ where p is a point on the line $L = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$ and D is an open disk in K tangent to L at p. This is called the *Niemytzki space*.

(a) Check that this is a topological space.
(b) What is the subspace topology on L?
(c) What are the closed sets in $\mathbb{H}$?
(d) Show that $\mathbb{H}$ is Hausdorff.
(e) Show that $\mathbb{H}$ is regular.
(f) Show that $\mathbb{H}$ is not normal.

Further readings

Metrizability. A space is said to be metrizable if its topology can be generated by a metric.

Theorem 11.12 (Urysohn Metrizability Theorem). *A regular space with a countable basis is metrizable.*

The proof uses the Urysohn lemma **[Mun00]**.

12. Quotient space

In this section we consider the operation of gluing parts of a space to form a new space. For example when we glue the two endpoints of a line segment together we get a circle.

Mathematically, gluing points mean to let them be equivalent. For a set X and an equivalence relation $\sim$ on X, the quotient set $X/\sim$ is exactly what we get byidentifying equivalent elements of X into one element.

We also want the gluing to be continuous. That means we equip the quotient set $X/\sim$ with the finest topology such that the projection map (the gluing map) $p : X \to X/\sim, x \mapsto [x]$ is continuous. Namely, a subset U of $X/\sim$ is open if and only if the preimage $p^{-1}(U)$ is open in X (see 3.6). The set $X/\sim$ with this topology is called the *quotient space* of X by $\sim$.

In a special case, if A is a subspace of X then there is this equivalence relation on X: $x \sim x$ if $x \notin A$, and $x \sim y$ if $x, y \in A$. The quotient space $X/\sim$ is often written as X/A, and we can think of it as being obtained from X by collapsing the whole subspace A to one point.

Theorem 12.1. *Let Y be a topological space. A map $f : X/\sim \to Y$ is continuous if and only if $f \circ p$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "f\circ p"'] & X/\sim \arrow[d, "f"] \\ & Y \end{tikzcd}$$

PROOF. The map $f \circ p$ is continuous if and only if for each open subset U of Y, the set $(f \circ p)^{-1}(U) = p^{-1}(f^{-1}(U))$ is open in X. The latter statement is equivalent to that $f^{-1}(U)$ is open for every U, that is, f is continuous. $\square$

The following result will provide us a tool for identifying many quotient spaces:

Theorem. *Suppose that X is compact and $\sim$ is an equivalence relation on X. Suppose that Y is Hausdorff, and $f : X \to Y$ is continuous and onto. Suppose that $f(x_1) = f(x_2)$ if and only if $x_1 \sim x_2$. Then f induces a homeomorphism from $X/\sim$ onto Y.*

PROOF. Define $h : X/\sim \to Y$ by $h([x]) = f(x)$. Then h is onto and is injective, thus it is a bijection.

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "f"'] & X/\sim \arrow[d, "h"] \\ & Y \end{tikzcd}$$

Notice that $f = h \circ p$ (in such a case people often say that the above diagram is *commutative*, and the map f can be factored). By 12.1 h is continuous. By 8.10, h is a homeomorphism. $\square$

Example (Gluing the two end-points of a line segment gives a circle). More precisely $[0,1]/0\sim 1$ is homeomorphic to S^1:

$$\begin{tikzcd}
{[0,1]} \arrow[r, "p"] \arrow[dr, "f"'] & {[0,1]/0\sim 1} \arrow[d, "h"] \\
& S^1
\end{tikzcd}$$

Here f is the map $t\mapsto(\cos(2\pi t),\sin(2\pi t))$.

Example (Gluing a pair of opposite edges of a square gives a cylinder). Let $X=[0,1]\times[0,1]/\sim$ where $(0,t)\sim(1,t)$ for all $0\le t\le 1$. Then X is homeomorphic to the cylinder $[0,1]\times S^1$. The homeomorphism is induced by the map $(s,t)\mapsto(s,\cos(2\pi t),\sin(2\pi t))$.

Example (Gluing opposite edges of a square gives a torus). Let $X=[0,1]\times[0,1]/\sim$ where $(s,0)\sim(s,1)$ and $(0,t)\sim(1,t)$ for all $0\le s,t\le 1$, then X is homeomorphic to the *torus*[12] (mặt xuyến) $T^2=S^1\times S^1$.

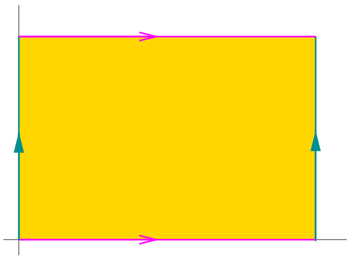


FIGURE 12.1. The torus.

The torus T^2 is homeomorphic to a subspace of $\mathbb{R}^3$, in other words, the torus can be embedded in $\mathbb{R}^3$. The subspace is the surface of revolution obtained by revolving a circle around a line not intersecting it.

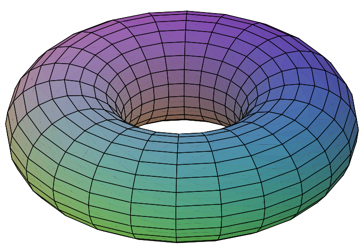


FIGURE 12.2. The torus embedded in $\mathbb{R}^3$.

Suppose that the circle is on the Oyz-plane, the center is on the y-axis and the axis for the rotation is the z-axis. Let a be the radius of the circle, b be the distance

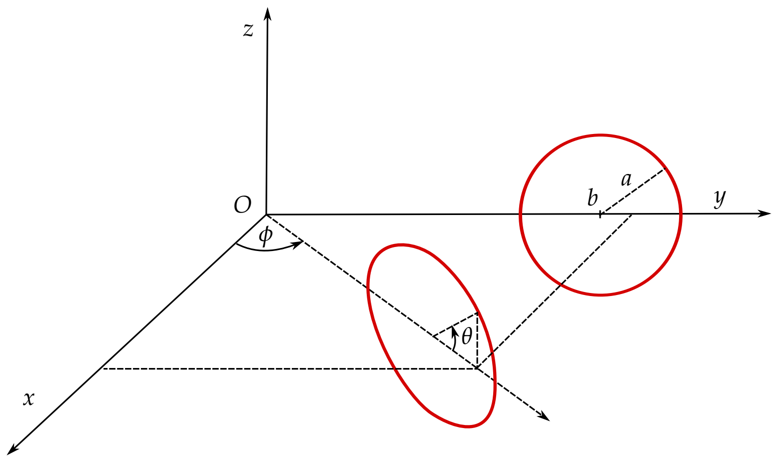
[12]The plural form of the word torus is tori.

from the center of the circle to O, $(a < b)$. Let S be the surface of revolution, then the embedding can be given by

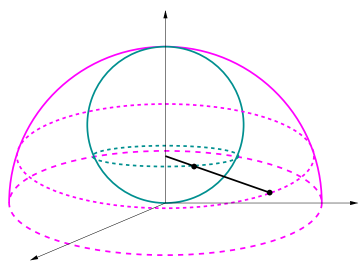
$$\begin{tikzcd}
{[0,2\pi] \times [0,2\pi]} \arrow[r] \arrow[dr, "f"'] & T^2 = ([0,2\pi]/0 \sim 2\pi) \times ([0,2\pi]/0 \sim 2\pi) \arrow[d, "h"] \\
& S
\end{tikzcd}$$

where $f(\phi, \theta) = ((b + a\cos\theta)\cos\phi, (b + a\cos\theta)\sin\phi, a\sin\theta)$.

We can also obtain an implicit equation: $(\sqrt{x^2+y^2} - b)^2 + z^2 = a^2$.



Example (Gluing the boundary circle of a disk together gives a sphere). More precisely $D^2/\partial D^2$ is homeomorphic to S^2. We only need to construct a continuous map from D^2 onto S^2 such that after quotient out by the boundary ∂D^2 it becomes injective.



Example (The Mobius band). Gluing a pair of opposite edges of a square in opposite directions gives the *Mobius band* (dải, lá Mobius). More precisely the Mobius band is $X = [0,1] \times [0,1]/\sim$ where $(0,t) \sim (1, 1-t)$ for all $0 \le t \le 1$.

The Mobius band could be embedded in $\mathbb{R}^3$. It is homeomorphic to a subspace of $\mathbb{R}^3$ obtained by rotating a straight segment around the z-axis while also turning that segment "up side down". The embedding can be induced by the map $(s,t) \mapsto ((a + t\cos(s/2))\cos(s), (a + t\cos(s/2))\sin(s), t\sin(s/2))$, with $0 \le s \le 2\pi$ and $-1 \le t \le 1$.

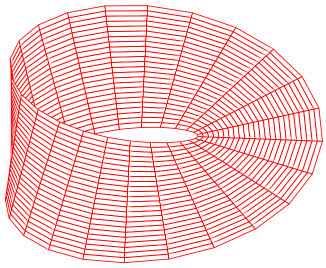


FIGURE 12.3. The Mobius band embedded in $\mathbb{R}^3$.

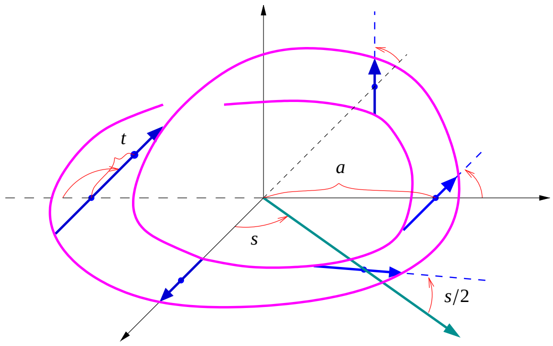


FIGURE 12.4. The embedding of the Mobius band in $\mathbb{R}^3$.

Example (The projective plane). Identifying opposite points on the boundary of a disk (they are called *antipodal points*) we get a topological space called the *projective plane* (mặt phẳng xạ ảnh) $\mathbb{R}P^2$. The real projective plane cannot be embedded in $\mathbb{R}^3$. It can be embedded in $\mathbb{R}^4$.

More generally, identifying antipodal boundary points of D^n gives us *the projective space* (không gian xạ ảnh) $\mathbb{R}P^n$. With this definition $\mathbb{R}P^1$ is homeomorphic to S^1. See also 12.10.

Example (Gluing a disk to the Mobius band gives the projective plane). In other words, deleting a disk from the projective plane gives the Mobius band. See Figure 12.5

Example (The Klein bottle). Identifying one pair of opposite edges of a square and the other pair in opposite directions gives a topological space called the *Klein bottle* . More precisely it is $[0,1] \times [0,1]/\sim$ with $(0,t) \sim (1,t)$ and $(s,0) \sim (1-s,1)$.

This space cannot be embedded in $\mathbb{R}^3$, but it can be *immersed* in $\mathbb{R}^3$. An *immersion* (phép nhúng chìm) is a local embedding. More concisely, $f : X \to Y$ is an immersion if each point in X has a neighborhood U such that $f|_U : U \to f(U)$ is a homeomorphism. Intuitively, an *immersion allows self-intersection* (tự cắt).

Problems.

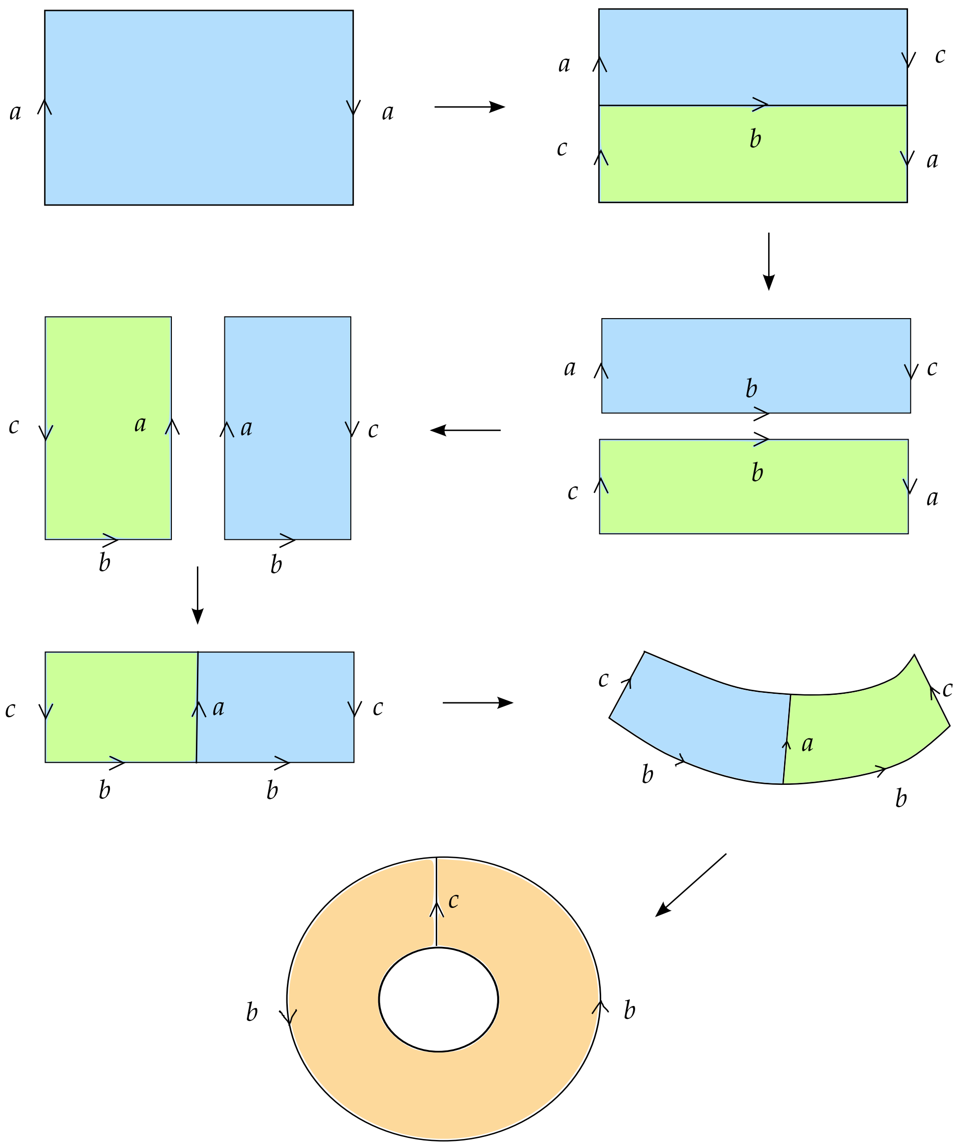


FIGURE 12.5. Gluing a disk to the Mobius band gives the projective plane.

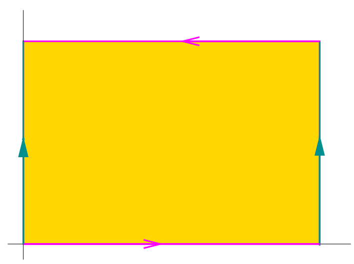


FIGURE 12.6. The Klein bottle.

12.2. Describe the space $[0,1]/\frac{1}{2} \sim 1$.

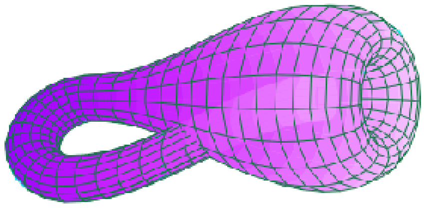
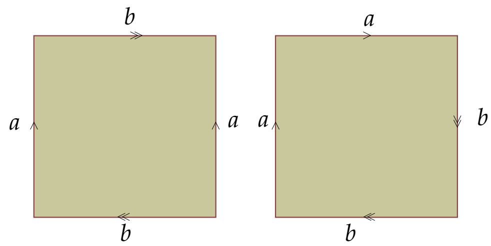


FIGURE 12.7. The Klein bottle immersed in $\mathbb{R}^3$.

12.3. On the Euclidean $\mathbb{R}$ define $x \sim y$ if $x - y \in \mathbb{Z}$. Show that $\mathbb{R}/\sim$ is homeomorphic to S^1. The space $\mathbb{R}/\sim$ is also described as "$\mathbb{R}$ quotient by the action of the group $\mathbb{Z}$".

12.4. On the Euclidean $\mathbb{R}^2$, define $(x_1, y_1) \sim (x_2, y_2)$ if $(x_1 - x_2, y_1 - y_2) \in \mathbb{Z} \times \mathbb{Z}$. Show that $\mathbb{R}^2/\sim$ is homeomorphic to T^2.

12.5. Show that the following spaces are homeomorphic (one of them is the Klein bottle).



12.6. Describe the space that is the sphere S^2 quotient by its equator S^1.

12.7. If X is connected then $X/\sim$ is connected.

12.8. The one-point compactification of the open Mobius band (the Mobius band without the boundary circle) is the projective space $\mathbb{RP}^2$.

12.9. Show that the projective space $\mathbb{RP}^n$ is a Hausdorff space.

12.10. * Show that identifying antipodal boundary points of D^n is equivalent to identifying antipodal points of S^n. In other words, the projective space $\mathbb{RP}^n$ is homeomorphic to $S^n/x \sim -x$.

12.11. In order for the quotient space $X/\sim$ to be a Hausdorff space, a necessary condition is that each equivalence class $[x]$ must be a closed subset of X. Is this condition sufficient?

Guide for further reading

The book by Kelley [**Kel55**] has been a classics and a standard reference although it was published in 1955. Its presentation is rather abstract. The book contains no figure!

Munkres' book [**Mun00**] is presently a standard textbook. The treatment there is somewhat more modern than that in Kelley's book, with many examples, figures and exercises. It also has a section on Algebraic Topology.

Hocking and Young's book [**HY61**] contains many deep and difficult results. This book together with Kelley's and Munkres' books contain many topics not discussed in our lectures.

For General Topology as a service to Analysis, [**KF75**] is an excellent textbook. [**Cai94**] and [**VINK08**] are other good books on General Topology.

A more recent textbook by Roseman [**Ros99**] works mostly in $\mathbb{R}^n$ and is more down-to-earth. The new textbook [**AF08**] contains many interesting applications of Topology.

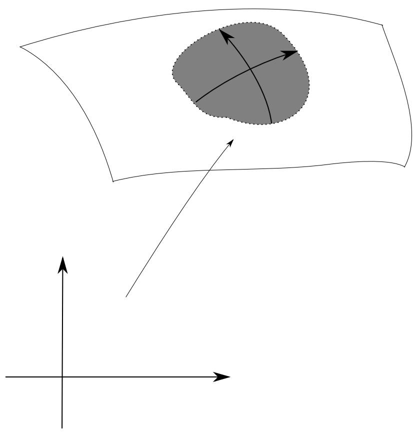
If you want to have some ideas about current research in General Topology you can visit the website of Topology Atlas [**Atl**], or you can browse the journal Topology and Its Applications, available on the web.

Algebraic Topology

13. Structures on topological spaces

Topological manifold. If we only stay around our small familiar neighborhood then we might not be able to recognize that surface of the Earth is curved, and to us it is indistinguishable from a plane. When we begin to travel farther and higher, we can realize that the surface of the Earth is a sphere, not a plane. In mathematical language, a sphere and a plane are locally same although globally different. Briefly, *a manifold is a space that is locally Euclidean*.

Definition. A *topological manifold* (đa tạp tôpô) of dimension n is a topological space each point of which has a neighborhood homeomorphic to the Euclidean space $\mathbb{R}^n$.



Remark. In this chapter we assume $\mathbb{R}^n$ has the Euclidean topology unless we mention otherwise.

An equivalent definition of manifold is:

Proposition. *A manifold of dimension n is a space such that each point has a neighborhood homeomorphic to an open subset of* $\mathbb{R}^n$.

We can think of a manifold as a space which could be covered by a collection of open subsets each of which homeomorphic to $\mathbb{R}^n$.

Remark. By Invariance of dimension 5.34, $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic unless $m = n$, therefore a manifold has a unique dimension.

Example. Any open subspace of $\mathbb{R}^n$ is a manifold of dimension n.

Example. If $f : \mathbb{R} \to \mathbb{R}$ is continuous then the graph of f is a one-dimensional manifold. More generally, let $f : D \to \mathbb{R}$ be a continuous function where $D \subset \mathbb{R}^n$ is an open set, then the graph of f, the set $\{(x, f(x)) \mid x \in D\}$ as a subspace of $\mathbb{R}^{n+1}$ is an n-dimensional manifold.

Example. The sphere S^n is an n-dimensional manifold. One way to show this is to cover S^n with two neighborhoods $S^n \setminus \{(0, 0, \ldots, 0, 1)\}$ and $S^n \setminus \{(0, 0, \ldots, 0, -1)\}$. Each of these neighborhoods is homeomorphic to $\mathbb{R}^n$ via stereographic projections. Another way is covering S^n by hemispheres $\{(x_1, x_2, \ldots, x_{n+1}) \in S^n \mid x_i > 0\}$ and $\{(x_1, x_2, \ldots, x_{n+1}) \in S^n \mid x_i < 0\}$, $1 \le i \le n+1$.

Example. The torus is a two-dimensional manifold. Let us consider the torus as the quotient space of the square $[0,1]^2$ by identifying opposite edges. Each point has a neighborhood homeomorphic to an open disk, as can be seen easily in the following figure, though explicit description would be time consuming. We can

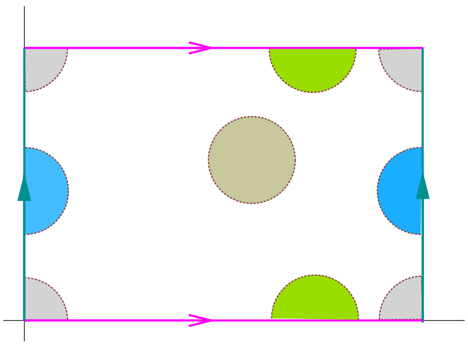


FIGURE 13.1. The sets with same colors are glued to form a neighborhood of a point on the torus. Each such neighborhood is homeomorphic to an open ball.

also view the torus as a surface in $\mathbb{R}^3$, given by the equation $(\sqrt{x^2+y^2} - a)^2 + z^2 = b^2$. As such it can be covered by the open subsets of $\mathbb{R}^3$ corresponding to $z > 0$, $z < 0$, $x^2 + y^2 < a^2$, $x^2 + y^2 > a^2$.

Remark. The interval $[0,1]$ is not a manifold, it is a "manifold with boundary". We will not give a precise definition of manifold with boundary here.

A two-dimensional manifold is often called a *surface*.

Simplicial complex. For an integer $n \ge 0$, an n-dimensional *simplex* (đơn hình) is a subspace of a Euclidean space $\mathbb{R}^m$, $m \ge n$, which is the convex linear combination of $(n+1)$ points $v_0, v_1, \ldots, v_n$ where $v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0$ are n linearly independent vectors (it can be checked that this condition does not depend on the order of the points). As a set it is given by $\{t_0v_0 + t_1v_1 + \cdots + t_nv_n \mid t_0, t_1, \ldots, t_n \in [0,1],\ t_0 + t_1 + \cdots + t_n = 1\}$. The points $v_0, v_1, \ldots, v_n$ are called the *vertices* of the simplex.

Example. A 0-dimensional simplex is just a point. A 1-dimensional simplex is a straight segment in $\mathbb{R}^m$, $m \geq 1$. A 2-dimensional simplex is a triangle in $\mathbb{R}^m$, $m \geq 2$. A 3-dimensional simplex is a tetrahedron in $\mathbb{R}^m$, $m \geq 3$.

In particular, the *standard n-dimensional simplex* (đơn hình chuẩn) Δ_n is the convex linear combination of the $(n+1)$ vectors $(1,0,0,\dots)$, $(0,1,0,0,\dots)$, $\dots$ in the standard linear basis of $\mathbb{R}^{n+1}$. Thus

$$\Delta_n = \{(t_0, t_1, \dots, t_n) \mid t_0, t_1, \dots, t_n \in [0,1],\ t_0 + t_1 + \cdots + t_n = 1\}.$$

The convex linear combinations of any subset of the set of vertices of a simplex is called a *face* of the simplex.

Example. For a 2-dimensional simplex (a triangle) its faces are the vertices, the edges, and the triangle itself.

An n-dimensional *simplicial complex* (phức đơn hình) in $\mathbb{R}^m$ is a finite collection S of simplexes of dimensions at most n in $\mathbb{R}^m$ such that:

(a) any face of an element of S is an element of S,
(b) the intersection of any two elements of S is a common face,
(c) at least one element of S is of dimension n.

The union of all elements of S is called its *underlying space*, denoted by $|S|$. Such a space is called a *polyhedron* (đa diện).

Example. A 1-dimensional simplicial complex is a graph.

Triangulation. A *triangulation* (phép phân chia tam giác) of a topological space X is a homeomorphism from the underlying space of a simplicial complex to X, the space X is then said to be *triangulated*.

For example, a triangulation of a surface is an expression of the surface as a union of finitely many triangles, with a requirement that two triangles are either disjoint, or have one common edge, or have one common vertex.

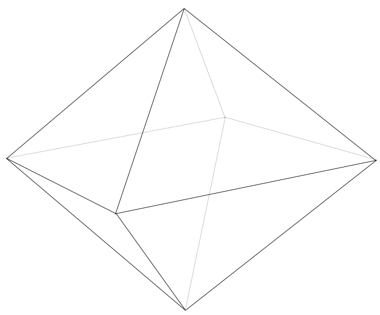


FIGURE 13.2. A triangulation of the 2-dimensional sphere.

It is known that any two or three dimensional manifold can be triangulated, and that there exists a 4-dimensional manifold with no triangulation. The situations in higher dimensions are still being studied.

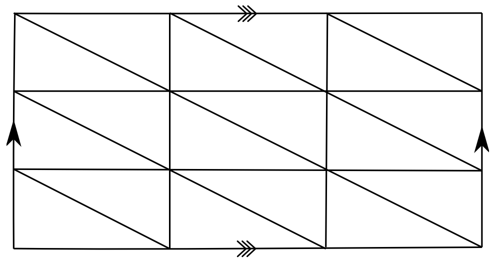


FIGURE 13.3. A triangulation of the torus.

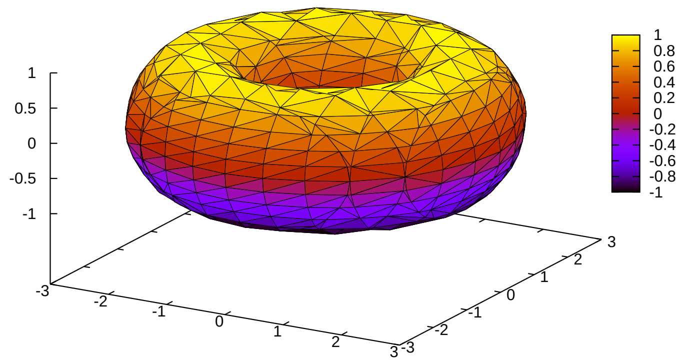


FIGURE 13.4. Another triangulation of the torus.

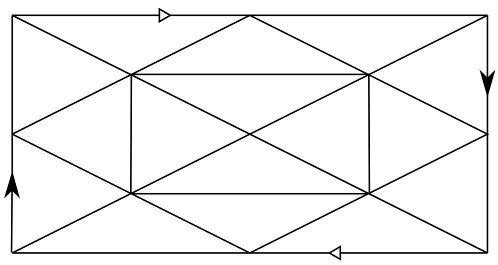


FIGURE 13.5. A triangulation of the projective plane.

A simplicial complex is specified by a finite set of points. If a space can be triangulated then we can study that space combinatorially, using constructions and computations in finitely many steps.

Cell complex.

Definition. A 0-dimensional *cell* (ô) is a point. For $n \geq 1$ an n-dimensional cell is an open ball in the Euclidean space $\mathbb{R}^n$.

By an n-dimensional *disk* we mean a closed ball in the Euclidean space $\mathbb{R}^n$. In particular the unit disk centered at the origin $B'(0,1)$ is denoted by D^n. Thus the boundary ∂D^n is the sphere S^{n-1} and the interior $\text{int}(D^n)$ is an n-cell.

Definition. Let X be a topological space and let. By *attaching a cell to a space* X we mean taking a continuous function $f : \partial D^n \to X$ then forming the quotient space $(X \sqcup D^n)/(x \sim f(x),\ x \in \partial D^n)$. Intuitively, we glue a disk to the space by gluing each point on the boundary of the disk with a point on the space. We can attach many cells simultaneously in the same way.

A (finite) *cell complex* (phức ô) or *CW-complex* X is a topological space built as follows:

(a) X^0 is a finite discrete space.
(b) For each $1 \le i \le n \in \mathbb{Z}^+$, X^i is obtained from X^{i-1} by attaching finitely many i-cells.
(c) $X = X^n$.

Briefly, a cell complex is a topological space which can be built by attaching cells. The subspace X^n is called the n-dimensional *skeleton* (khung) of X.

Example. A topological circle has a cell complex structure as a triangle with three 0-cells and three 1-cells. There is another cell complex structure with only one 0-cell and one 1-cell.

Example. The 2-dimensional sphere has a cell complex structure as a tetrahedron with four 0-cells, six 1-cells, and four 2-cells. There is another cell complex structure with only one 0-cell and one 2-cell.

Example. The torus has a cell complex structure with one 0-cells, two 1-cells, and one 2-cells.

It's intutively easy to be convinced that a simplicial complex gives a cell complex:

Proposition. *Any polyhedron is a cell complex.*

PROOF. Let X be a simplicial complex. Let X^i be the union of all simplexes of X of dimensions at most i. Then X^{i+1} is the union of X^i with finitely many $(i+1)$-dimensional simplexes. Let Δ^{i+1} be such an $(i+1)$-dimensional simplex. The faces of Δ^{i+1} are simplexes of X, so the union of those faces, which is the boundary of Δ^{i+1}, belongs to X^i. There is a homeomorphism from an $(i+1)$-disk to Δ^{i+1}, bringing the boundary of the disk to the boundary of Δ^{i+1}. Thus including Δ^{i+1} means attaching an $(i+1)$-cell to X^i. □

The example of the torus indicates that cell complexes may require less cells than simplicial complexes. On the other hand we loose the combinatorial setting, because we need to specify the attaching maps.

It is known that any compact manifold of dimension different from 4 has a cell complex structure. Whether that is true or not in dimension 4 is not known yet [**Hat01**, p. 529].

Problems.

13.1. Show that if two spaces are homeomorphic and one space is an n-dimensional manifold then the other is also an n-dimensional manifold.

13.2. Show that $\mathbb{RP}^n$ is an n-dimensional topological manifold.

13.3. Draw a cell complex structure on the torus with two holes.

13.4. Find a cell complex structure on $\mathbb{RP}^n$.

Further readings

Bernard Riemann proposed the idea of manifold in his Habilitation dissertation. A translation of this article is available in [**Spi99**].

Two conditions are often added to the definition of a manifold: it is Hausdorff, and it has a countable basis. The first condition is useful for doing Analysis on manifolds, and the second condition guarantees the existence of Partition of Unity.

Theorem 13.5 (Partition of Unity). *Let U be an open cover of a manifold M. Then there is a collection F of continuous real functions $f : M \to [0,1]$ such that*

(a) *For each $f \in F$, there is $V \in U$ such that $supp(f) \subset V$.*
(b) *For each $x \in M$ there is a neighborhood of x such that there are only finitely many $f \in F$ which is non-zero on that neighborhood.*
(c) *For each $x \in M$,*
$$\sum_{f \in F} f(x) = 1$$

A Partition of Unity allows us to extend some local properties to global ones, by "patching" neighborhoods. It is needed for such important results as the existence of a Riemannian metric on a manifold in Differential Geometry, the definition of integration on manifold in Theory of Differential Forms. It is also used in the proof of the Riesz Representation Theorem in Measure Theory [**Rud86**].

13.6. Check that $\mathbb{R}^n$ has a countable basis.

13.7. Any subset of $\mathbb{R}^n$ is Hausdorff and has a countable basis.

With the above additional assumptions we can show:

13.8. A manifold is locally compact.

13.9. A manifold is a regular space.

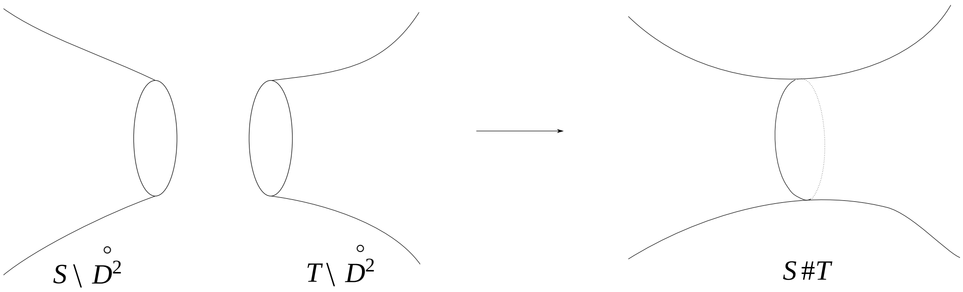
By the Urysohn Metrizability Theorem 11.12 we have:

13.10. Any manifold is metrizable.

14. Classification of compact surfaces

In this section by a *surface* (mặt) we mean a two-dimensional manifold (without boundary).

Connected sum. Let S and T be two surfaces. From each surface deletes an open disk, then glue the two boundary circles. The resulting surface is called the *connected sum* (tổng liên thông) of the two surfaces, denoted by $S\#T$.



It is known that the connected sum does not depend on the choices of the disks.

Example. If S is any surface then $S\#S^2 = S$.

Classification.

Theorem (Classification of compact surfaces). *A connected compact surface is homeomorphic to either the sphere, or a connected sum of tori, or a connected sum of projective planes.*

We denote by T_g the connected sum of g tori, and by M_g the connected sum of g projective planes. The number g is called the *genus* (giống) of the surface.

The sphere and the surfaces T_g are *orientable* (định hướng được) surface , while the surfaces M_g are *non-orientable* (không định hướng được) surfaces. We will not give a precise definition of orientability here.

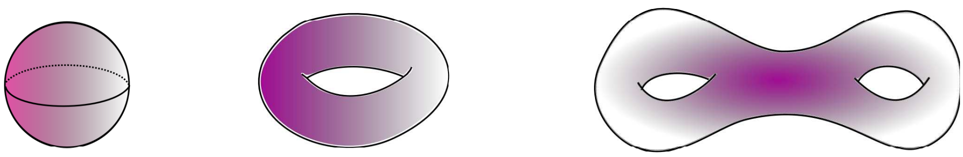


FIGURE 14.1. Orientable surfaces: $S^2, T_1, T_2, \ldots$

Notice that at this stage we have not yet been able to prove that those surfaces are distinct.

The Classification theorem is a direct consequence of the following:

Theorem 14.1. *A connected compact surface is homeomorphic to the space obtained by identifying the edges of a polygon in one of the following ways:*

(a) aa^{-1},
(b) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$,
(c) $c_1^2c_2^2\cdots c_g^2$.

Proof of 14.1. Let S be a triangulated surface. Cut S along the triangles. Label the edges by alphabet characters and mark the orientations of each edge. In this way each edge will appear twice on two different triangles.

Take one triangle. Pick a second triangle which has one common edge with the first one, then glue the two along the common edge following the orientation of the edge. Continue this gluing process in such a way that at every step the resulting polygon is planar. This is possible if at each stage the gluing is done in such a way that there is one edge of the polygon such that the entire polygon is on one of its side. The last polygon P is called a *fundamental polygon* of the surface.

The boundary of the fundamental polygon consists of labeled and oriented edges. Choose one edge as the initial one then follow the edge of the polygon in a predetermined direction. This way we associate each polygon with a word w.

We consider two words equivalent if they give rise to homeomorphic surfaces.

In the reverse direction, the surface can be reconstructed from an associated word. We consider two words equivalent if they give rise to homeomorphic surfaces. In order to find all possible surfaces we will find all possible associated words up to equivalence.

Theorem 14.1 is a direct consequence of the following:

Proposition 14.2. *An associated word to a connected compact without boundary surface is equivalent to a word of the forms:*

(a) aa^{-1},
(b) $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\cdots a_gb_ga_g^{-1}b_g^{-1}$,
(c) $c_1^2c_2^2\cdots c_g^2$.

We will prove 14.2 through a series of lemmas.

Let w be the word of a fundamental polygon.

Lemma 14.3. *A pair of the form aa^{-1} in w can be deleted, meaning that this action will give an equivalent word corresponding to a homeomorphic surface.*

PROOF. If w is not aa^{-1} then it can be reduced as illustrated in the figure. □

Lemma 14.4. *The word w is equivalent to a word whose all of the vertices of the associated polygon is identified to a single point on the associated surface (w is "reduced").*

PROOF. When we do the following operation, the number of P vertices is decreased.

When there is only one P vertex left, we arrive at the situation in Lemma 14.3.

□

Lemma 14.5. *A word of the form $-a-a-$ is equivalent to a word of the form $-aa-$.*

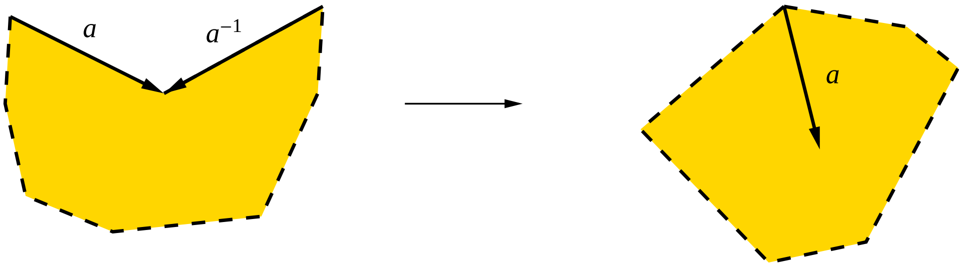


FIGURE 14.2. Lemma 14.3.

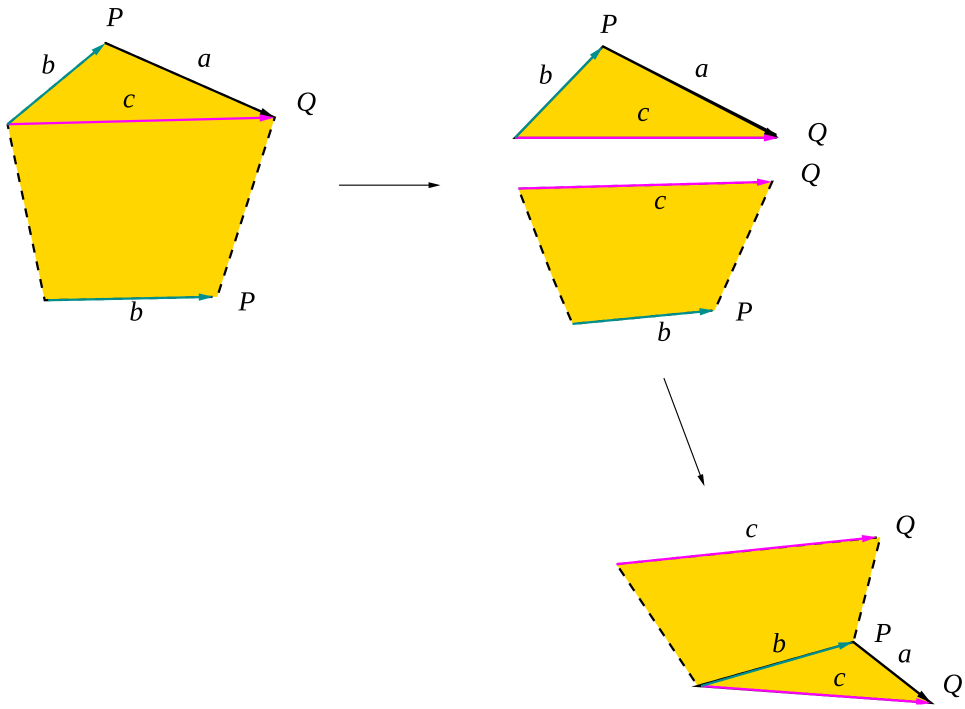


FIGURE 14.3. Lemma 14.4.

Lemma 14.6. *Suppose that w is reduced. Assume that w has the form $-a\alpha a^{-1}-$ where α is a non-empty word. Then there is a letter b such that b is in α but the other b or b^{-1} is not.*

PROOF. If all letters in α appear in pairs then the vertices in the part of the polygon associated to α are identified only with themselves, and are not identified with a vertex outside of that part. This contradicts the assumption that w is reduced. $\square$

Lemma 14.7. *A word of the form $-a-b-a^{-1}-b^{-1}-$ is equivalent to a word of the form $-aba^{-1}b^{-1}-$.*

Lemma 14.8. *A word of the form $-aba^{-1}b^{-1}-cc-$ is equivalent to a word of the form $-a^2-b^2-c^2-$.*

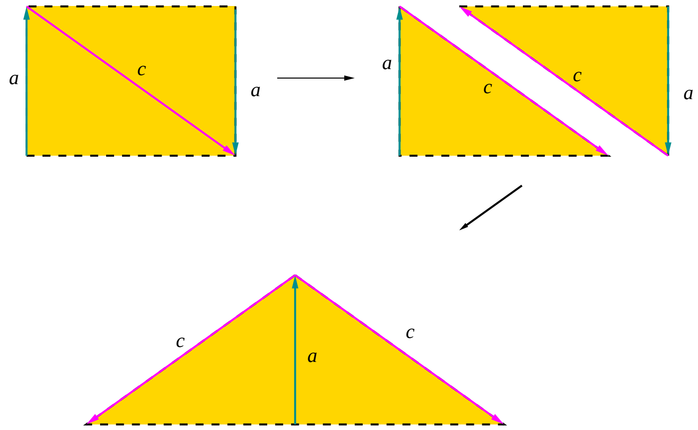


FIGURE 14.4. Lemma 14.5.

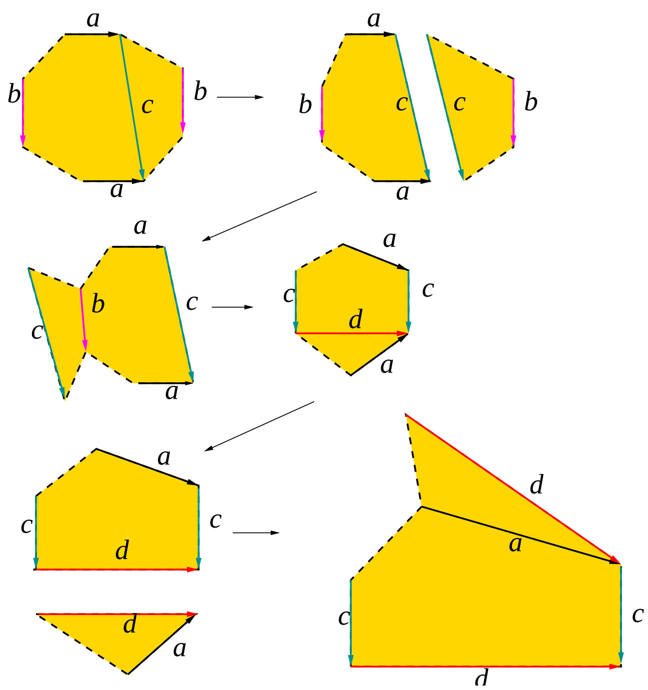


FIGURE 14.5. Lemma 14.7.

PROOF. Do the operation in the figure, after that we are in a situation where we can apply Lemma 14.5 three times.

□

PROOF OF 14.2. The proof follows the following steps.
1. Bring w to the reduced form by using 14.4 finitely many times.
2. If w has the form $-aa^{-1}-$ then go to 2.1, if not go to 3.
2.1. If w has the form aa^{-1} then stop, if not go to 2.2.

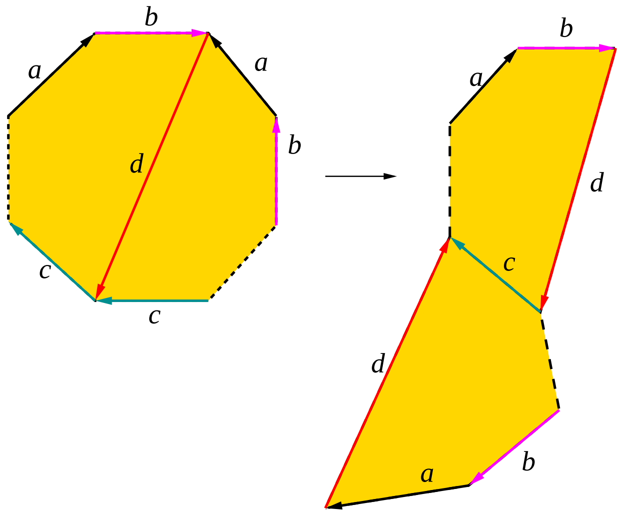


FIGURE 14.6. Lemma 14.8.

2.2. w has the form $aa^{-1}\alpha$ where $\alpha \neq \emptyset$. Repeatedly apply 14.3 finitely many times, deleting pairs of the form aa^{-1} in w until no such pair is left or w has the form aa^{-1}. If no such pair is left go to 3.

3. w does not have the form $-aa^{-1}-$. Repeatedly apply 14.5 finitely many times until w no longer has the form $-a\alpha a-$ where $\alpha \neq \emptyset$. Note that if we apply 14.5 then some pairs of of the form $-a\alpha a-$ with $\alpha \neq \emptyset$ could become a pair of the form $-a - a^{-1}-$, but a pair of the form $-aa-$ will not be changed. Therefore 14.5 could be used finitely many times until there is no pair $-a\alpha a-$ with $\alpha = \emptyset$ left.

Also it is crucial from the proof of 14.5 that this step will not undo the steps before it.

4. If there is no pair of the form $-a\alpha a^{-1}$ where $\alpha \neq \emptyset$, then stop: w has the form $a_1^2 a_2^2 \cdots a_g^2$.

5. w has the form $-a\alpha a^{-1}$ where $\alpha \neq \emptyset$. By 14.6 w must has the form $-a - b - a^{-1} - b^{-1}-$, since after Step 3 there could be no $-b - a^{-1} - b-$.

6. Repeatedly apply 14.7 finitely many times until w no longer has the form $-a\alpha b\beta a^{-1}\gamma b^{-1}-$ where at least one of α, β, or γ is non-empty.

7. If w is not of the form $-aa-$ then stop: w has the form $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$.

8. w has the form $-aba^{-1}b^{-1} - cc-$. Use 14.8 finitely many times to transform w to the form $a_1^2 a_2^2 \cdots a_g^2$. $\square$

Euler Characteristics. The *Euler Characteristics* (đặc trưng Euler) $\chi(S)$ of a triangulated surface S is the number V of vertices minus the number E of edges plus the number F of triangles (faces):

$$\chi(S) = V - E + F.$$

Theorem 14.9. *The Euler Characteristics with respect to two triangulations of the same surface are equal.*

By this theorem the Euler Characteristics of a surface is defined and does not depend on the choice of triangulation. Since the Euler Characteristics does not change under homeomorphisms, just like the number of connected components, it is said to be a *topological invariant* (bất biến tôpô). If two surfaces have different Euler Characteristics, then they are not homeomorphic.

Example. By Theorem 14.9 we have $\chi(S^2) = 2$. A consequence is the famous formula of Leonhard Euler: For any convex polyhedron, $V - E + F = 2$.

Example. From any triangulation of the torus, we get $\chi(T^2) = 0$. For the projective plane, $\chi(\mathbb{R}P^2) = 1$. As a consequence, the sphere, the torus, and the projective plane are not homeomorphic to each other: they are different surfaces.

Problems.

14.10. Show that $T^2 \# \mathbb{R}P^2 = K \# \mathbb{R}P^2$, where K is the Klein bottle.

14.11. Show that gluing two Mobius bands along their boundaries gives the Klein bottle. In other words, $\mathbb{R}P^2 \# \mathbb{R}P^2 = K$. [13]

14.12 (Surfaces are homogeneous). A space is *homogeneous* (đồng nhất) if given two points there exists a homeomorphism from the space to itself bringing one point to the other point.

(a) Show that the sphere S^2 is homogeneous.
(b) Show that the torus T^2 is homogeneous.

It is known that any manifold is homogeneous, see 30.1.

14.13. (a) Show that $T_g \# T_h = T_{g+h}$.
(b) Show that $M_g \# M_h = M_{g+h}$.
(c) What is $M_g \# T_h$?

14.14. Show that $\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2$.

14.15. Compute the Euler Characteristics of all connected compact without boundary surfaces.

Deduce that the orientable surfaces S^2 and T_g, for different g, are distinct, meaning not homeomorphic to each other. Similarly the non-orientable surfaces M_g are all distinct.

14.16. From 14.1, describe a cell complex structure on any given compact without boundary surface.

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[13] There is a humorous poem:
A mathematician named Klein
Thought the Mobius band was divine
Said he, "If you glue
The edges of two,
You'll get a weird bottle like mine."

15. Homotopy

Homotopy of maps. Let X and Y be topological spaces and $f, g : X \to Y$. We say that f and g are *homotopic* (đồng luân) if there is a continuous map

$$\begin{aligned} F : X \times [0,1] &\to Y \\ (x,t) &\mapsto F(x,t) \end{aligned}$$

such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for all $x \in X$. The map F is called a *homotopy* (phép đồng luân) from f to g.

We can think of t as a time parameter and F as a continuous process in time that starts with f and ends with g. To suggest this view $F(x,t)$ is often written as $F_t(x)$.

Proposition. *Being homotopic is an equivalence relation on the set of continuous maps between two given topological spaces.*

Homotopic spaces.

Definition. Two topological spaces X and Y are *homotopic* if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ is homotopic to Id_X and $f \circ g$ is homotopic to Id_Y. Each of the maps f and g is called a *homotopy equivalence*.

Immediately we have:

Proposition. *Homeomorphic spaces are homotopic.*

So being homotopic is a weaker notion than being homeomorphic.

We can check that homotopy between spaces is a relation with these properties: reflective, symmetric, and transitive.

A space which is homotopic to a space containing only one point is called a *contractible space* (thắt được).

Example. Any ball in a normed space is contractible.

Let X be a space, and let A be a subspace of X. We say that A is a *retract* (rút) of X if there is a continuous map $r : X \to A$ such that $r|_A = \mathrm{id}_A$, called a retraction (phép rút) from X to A. In other words A is a retract of X if the identity map id_A can be extended to X.

A *deformation retraction* (phép rút biến dạng) from X to A is a homotopy F on X that starts with id_X, ends with a retraction from X to A, and fixes A throughout, i.e., $F_0 = \mathrm{id}_X$, $F_1(X) = A$, and $F_t|_A = \mathrm{id}_A$, $\forall t \in [0,1]$. If there is such a deformation retraction we say that A is a *deformation retract* (rút biến dạng) of X.

In such a deformation retraction each point $x \in X \setminus A$ "moves" along the path $F_t(x)$ to a point in A, while every point of A is fixed.

Example. A normed space minus a point has a deformation retraction to a sphere. Indeed a normed space minus the origin has a deformation retraction $F_t(x) = (1-t)x + t\frac{x}{||x||}$ to the unit sphere at the origin.

Example. An annulus $S^1 \times [0,1]$ has a deformation retract to one of its circle boundary $S^1 \times \{0\}$.

Proposition. *If a space X has a deformation retraction to a subspace A then X is homotopic to A.*

PROOF. Suppose that F_t is a deformation retraction from X to A. Consider $F_1 : X \to A$ and the inclusion map $g : A \to X$, $g(x) = x$. Then id_X is homotopic to $g \circ F_1$ via F_t, while $F_1 \circ g = \mathrm{id}_A$. □

Example. The letter A is homotopic to the letter O, as subspaces of the Euclidean plane.

Example. The circle, the annulus, and the Mobius band are homotopic each other but are not homeomorphic to each other.

Example. A subset A of $\mathbb{R}^n$ is called *star-shaped* if there is a point $x_0 \in A$ such that for any $x \in A$ the straight segment from x to x_0 is contained in A. Since A has a deformation retraction to x_0, it is contractible.

Homotopy of paths. Recall that a *path* (đường đi) in a space X is a continuous map α from the Euclidean interval $[0,1]$ to X. The point $\alpha(0)$ is called the initial end point, and $\alpha(1)$ is called the final end point. In this section for simplicity of presentation we assume the domain of a path is the Euclidean interval $[0,1]$ instead of any Euclidean closed interval as before.

Definition. Let α and β be two paths from a to b in X. A *path-homotopy* (phép đồng luân đường) from α to β is a continuous map $F : [0,1] \times [0,1] \to X$, $F(s,t) = F_t(s)$, such that $F_0 = \alpha$, $F_1 = \beta$, and for each t the path F_t goes from a to b, i.e. $F_t(0) = a$, $F_t(1) = b$.

If there is a path-homotopy from α to β we say that α is *path-homotopic* (đồng luân đường) to β.

Remark. A homotopy of path is a homotopy of maps defined on $[0,1]$, *with the further requirement that the homotopy fixes the initial point and the terminal point*. To emphasize this we have used the word path-homotopy, but some sources (e.g. **[Hat01**, p. 25]) simply use the term homotopy, taking this further requirement implicitly.

Example. In a normed space any two paths α and β with the same initial points and end points are homotopic, via the homotopy $(1-t)\alpha + t\beta$. This is also true for any convex subset of a normed space.

Proposition. *Homotopic relation on the set of all paths from a to b is an equivalence relation.*

PROOF. If α is path-homotopic to β via a path-homotopy F then we can easily find a homotopy from β to α, for instance $G_t = F_{1-t}$.

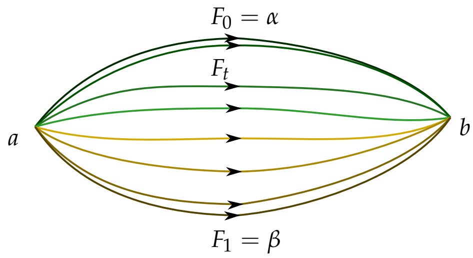


FIGURE 15.1. We can think of a path-homotopy from α to β as a way to continuously brings α to β, similar to a motion picture, keeping the endpoints fixed.

We check that if α is homotopic to β via a homotopy F and β is homotopic to γ via a homotopy G then α is homotopic to γ. Let

$$H_t = \begin{cases} F_{2t}, & 0 \le t \le \frac{1}{2} \\ G_{2t-1}, & \frac{1}{2} \le t \le 1. \end{cases}$$

Note that continuity of a map is not the same as continuity with respect to each variable (see 9.3). To see the continuity of H it is better to write it as

$$H(s,t) = \begin{cases} F(s,2t), & 0 \le s \le 1,\ 0 \le t \le \frac{1}{2} \\ G(s,2t-1), & 0 \le s \le 1,\ \frac{1}{2} \le t \le 1 \end{cases}$$

then use 4.4. So H is a homotopy from α to γ. □

A *loop* (vòng) or a *closed path* (đường đi đóng) *based* at a point $a \in X$ is a path whose initial point and end point are a. In other words it is a continuous map $\alpha : [0,1] \to X$ such that $\alpha(0) = \alpha(1) = a$. The *constant loop* at a is the loop $\alpha(t) = a$ for all $t \in [0,1]$.

A space is said to be *simply connected* (đơn liên) if it is path-connected and any loop is path-homotopic to a constant loop.

Example. As in a previous example, any convex subset of a normed space is simply connected.

Problems.

15.1. Show that the Mobius band has a deformation retract to a circle.

15.2. Show that the homotopy type of the Euclidean plane with a point removed does not depend on the choice of the point.

15.3. Show that contractible spaces are path-connected.

15.4. Show that deformation retract to a point ⇒ contractible.

15.5. Let X be a topological space, and Y be a subspace of X.

(a) Show that if Y is a retract of X then any map from Y to a topological space Z can be extended to X.

(b) Show that a subset consisting of two points cannot be a retract of $\mathbb{R}^2$.

Note: This shows that the Tiestze extension theorem cannot be automatically generalized to maps to general topological spaces.

15.6. Show that if B is contractible then $A \times B$ is homotopic to A.

15.7. Classify the alphabetical characters according to homotopy types, that is, which of the characters are homotopic to each other as subspaces of the Euclidean plane? Do the same for the Vietnamese alphabetical characters. Note that the result depends on the font you use.

Further readings

One of the most celebrated achievements in Topology is the resolution of the Poincaré conjecture:

Theorem (Poincaré conjecture). *A compact manifold that is homotopic to the sphere is homeomorphic to the sphere.*

The proof of this statement is the result of a cumulative effort of many mathematicians, including Stephen Smale (for dimension ≥ 5, early 1960s), Michael Freedman (for dimension 4, early 1980s), and Grigory Perelman (for dimension 3, early 2000s).

16. The fundamental group

Let α be a path from a to b. Then the *inverse path* of α is defined to be the path $\alpha^{-1}(t) = \alpha(1-t)$ from b to a.

Let α be a path from a to b, and β be a path from b to c, then the *composition* (hợp) of α with β is defined to be the path

$$\gamma(t) = \begin{cases} \alpha(2t), & 0 \le t \le \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \le t \le 1. \end{cases}$$

The path γ is often denoted as $\alpha \cdot \beta$. By 4.4, $\alpha \cdot \beta$ is continuous.

Lemma 16.1. *If α is path-homotopic to α_1 and β is path-homotopic to β_1 then $\alpha \cdot \beta$ is path-homotopic to $\alpha_1 \cdot \beta_1$.*

PROOF. Let F be the first homotopy and G be the second homotopy. Consider $H_t = F_t \cdot G_t$. □

Lemma 16.2. *If α is a path from a to b then $\alpha \cdot \alpha^{-1}$ is path-homotopic to the constant loop at a.*

PROOF. Our homotopy from $\alpha \cdot \alpha^{-1}$ to the constant loop at a can be described as follows. At a fixed t, the loop F_t starts at time 0 at a, goes along α but at twice the speed of α, until time $\frac{1}{2} - \frac{t}{2}$, stays there until time $\frac{1}{2} + \frac{t}{2}$, then catches the inverse path α^{-1} at twice its speed to come back to a.

More precisely,

$$F_t(s) = \begin{cases} \alpha(2s), & 0 \le s \le \frac{1}{2} - \frac{t}{2} \\ \alpha(\frac{1}{2} - \frac{t}{2}), & \frac{1}{2} - \frac{t}{2} \le s \le \frac{1}{2} + \frac{t}{2} \\ \alpha^{-1}(2s), & \frac{1}{2} + \frac{t}{2} \le s \le 1. \end{cases}$$

□

Lemma 16.3 (reparametrization). *If $\varphi : [0,1] \to [0,1]$ is a continuous map such that $\varphi(0) = 0$ and $\varphi(1) = 1$ then for any path α the path $\alpha \circ \varphi$ (a "reparametrization" of α) is path-homotopic to α.*

Roughly speaking, a reparametrization does not change the homotopy class.

PROOF. Let $F_t = (1-t)\varphi + t\mathrm{Id}_{[0,1]}$. Then $\alpha \cdot F_t$ gives a path-homotopy from $\alpha \circ \varphi$ to α. □

The fundamental group. Consider the set of loops based at a point x_0 under the path-homotopy relation, denoted by $\pi_1(X, x_0)$. On this set we define a multiplication operation $[\alpha] \cdot [\beta] = [\alpha \cdot \beta]$. By 16.1 this operation is well-defined.

Theorem 16.4. *The set of all path-homotopy classes of loops of X based at a point x_0 is a group under the above operation.*

This group is called *the fundamental group* (nhóm cơ bản) of X at x_0, denoted by $\pi_1(X, x_0)$. The point x_0 is called the *base point*.

PROOF. Let's write 1 as the constant loop at x_0. By Lemma 16.3 $1 \cdot \alpha$ is path-homotopic to α, thus $[1] \cdot [\alpha] = [\alpha]$. So $[1]$ is the identity in $\pi_1(X, x_0)$.

We define $[\alpha]^{-1} = [\alpha^{-1}]$. It is easy to check that this is well-defined. By 16.2 $[\alpha]^{-1}$ is indeed the inverse element of $[\alpha]$.

Since $(\alpha \cdot \beta) \cdot \gamma$ is a reparametrization of $\alpha \cdot (\beta \cdot \gamma)$, we have associativity: $([\alpha] \cdot [\beta]) \cdot [\gamma] = [\alpha] \cdot ([\beta] \cdot [\gamma])$. □

Example. If X is a convex subset of a normed space then $\pi_1(X, x_0)$ is trivial.

A simply connected space is precisely as path-connected space with trivial fundamental group.

The dependence of the fundamental group on the base point is explained in the following proposition.

Proposition 16.5 (dependence on base point). *If there is a path from x_0 to x_1 then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.*

PROOF. Let α be a path from x_0 to x_1. Consider the map

$$\begin{array}{rccc} h_\alpha: & \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ & [\gamma] & \mapsto & [\alpha \cdot \gamma \cdot \alpha^{-1}] \end{array}$$

Using 16.1 we can check that this is a well-defined map, a group homomorphism with an inverse homomorphism:

$$\begin{array}{rccc} h_\alpha^{-1}: & \pi_1(X, x_1) & \to & \pi_1(X, x_0) \\ & [\gamma] & \mapsto & [\alpha^{-1} \cdot \gamma \cdot \alpha] \end{array}$$

□

For a path-connected space the fundamental group is the same up to group isomorphism for any choice of the base point. Therefore if X is a path-connected space we often drop the base point in the notation, and write $\pi_1(X)$.

Induced homomorphisms on fundamental groups. Let X and Y be topological spaces, and $f : X \to Y$. Then f induces the following map

$$\begin{array}{rcl} f_* : \pi_1(X, x_0) & \to & \pi_1(Y, f(x_0)) \\ [\gamma] & \mapsto & [f \circ \gamma] \end{array}$$

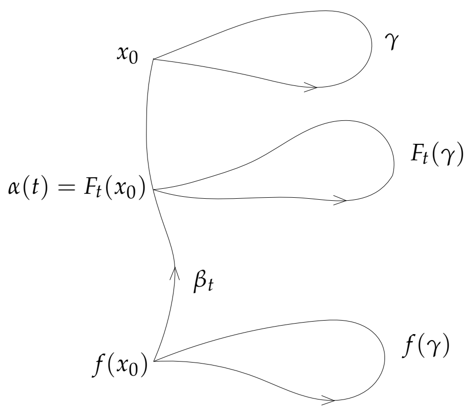
It can be checked that this is a well-defined map. Furthermore $f_*([\gamma_1] \cdot [\gamma_2]) = f_*([\gamma_1 \cdot \gamma_2]) = [f \circ (\gamma_1 \cdot \gamma_2)]$. It can be checked directly that $f \circ (\gamma_1 \cdot \gamma_2) = (f \circ \gamma_1) \cdot (f \circ \gamma_2)$, thus $f_*([\gamma_1] \cdot [\gamma_2]) = f_*([\gamma_1]) \cdot f_*([\gamma_2])$, therefore f_* is a homomorphism.

Proposition ($(g \circ f)_* = g_* \circ f_*$). *If $f : X \to Y$ and $g : Y \to Z$ then $(g \circ f)_* = g_* \circ f_* : \pi_1(X, x_0) \to \pi_1(Z, g(f(x_0)))$.*

PROOF. $(g \circ f)_*([\gamma]) = [(g \circ f) \circ \gamma] = [g \circ (f \circ \gamma)] = g_*([f \circ \gamma]) = g_*(f_*([\gamma]))$. □

Lemma. *If $f : X \to X$ is homotopic to the identity then $f_* : \pi_1(X, x_0) \to \pi_1(X, f(x_0))$ is an isomorphism.*

PROOF. We will show that $f_*([\gamma]) = [f \circ \gamma] = [\alpha \cdot \gamma \cdot \alpha^{-1}]$, where α is a path from $f(x_0)$ to x_0. In other words $f_* = h_\alpha$, where h_α is the map used in the proof of 16.5, which was shown there to be an isomorphism.



From the assumption there is a homotopy F from f to id_X. Then $F_t(x_0)$, $0 \le t \le 1$ is a continuous path from $f(x_0)$ to x_0. Denote this path by α. For each fixed $0 \le t \le 1$, let β_t be the path that goes along α from $\alpha(0) = f(x_0)$ to $\alpha(t)$, namely $\beta_t(s) = \alpha(st)$, $0 \le s \le 1$. Let $G_t = \beta_t \cdot F_t(\gamma) \cdot \beta_t^{-1}$. That G is continuous can be checked by writing down the formula for G explicitly. Then G is a path-homotopy from $f(\gamma)$ to $\alpha \cdot \gamma \cdot \alpha^{-1}$. □

Theorem. *If $f : X \to Y$ is a homotopy equivalence then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.*

PROOF. Since f is a homotopy equivalence there is $g : Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y. By the above lemma, the composition

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g(f(x_0)))$$

is an isomorphism, which implies that g_* is surjective. Similarly the composition

$$\pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g(f(x_0))) \xrightarrow{f_*} \pi_1(Y, f(g(f(x_0))))$$

is an isomorphism, which implies that g_* is injective. Since g_* is bijective from the first composition we see that f_* is bijective. □

Corollary (homotopy invariance). *If two path-connected spaces are homotopic, then their fundamental groups are isomorphic.*

We say that for path-connected spaces, the fundamental group is a *homotopy invariant*.

Problems.

16.6. If X_0 is a path-connected component of X and $x_0 \in X_0$ then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X_0, x_0)$.

16.7. Show that a topological space is simply-connected if and only if it is path-connected and its fundamental group is trivial.

16.8. Let X and Y be topological spaces, $f : X \to Y$, $f(x_0) = y_0$. Show that the induced map

$$\begin{aligned} f_* : \pi_1(X, x_0) &\to \pi_1(Y, y_0) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned}$$

is a well-defined.

16.9. Suppose that $f : X \to Y$ is a homeomorphism. Show that the induces homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.

17. The fundamental group of the circle

Theorem ($\pi_1(S^1) \cong \mathbb{Z}$). *The fundamental group of the circle is infinite cyclic.*

Let γ_n be the loop $(\cos(n2\pi t), \sin(n2\pi t))$, $0 \le t \le 1$, the loop on the circle S^1 based at the point $(1,0)$ that goes n times around the circle at uniform speed in the counter-clockwise direction if $n > 0$ and in the clockwise direction if $n < 0$. Consider the map

$$\begin{aligned} \Phi : \mathbb{Z} &\to \pi_1(S^1, (1,0)) \\ n &\mapsto [\gamma_n]. \end{aligned}$$

This map associates each integer n with the path-homotopy class of γ_n. We will show that Φ is a group isomorphism, where $\mathbb{Z}$ has the usual additive structure. This implies that the fundamental group of the circle is generated by a loop that goes once around the circle in the counter-clockwise direction, and the homotopy class of a loop in the circle corresponds to an integer representing the "number of times" that loop goes around the circle, with the counter-clockwise direction being the positive direction.

PROOF. First we check that Φ is a group homomorphism. This means γ_{m+n} is path-homotopic to $\gamma_m \cdot \gamma_n$. This is true because the two paths are reparametrizations of each other. This is not difficult, the details can be given as follows.

Let $p : \mathbb{R} \to S^1$, $p(t) = (\cos(2\pi t), \sin(2\pi t))$, a map that wraps the line around the circle countably infinitely many times in the counter-clockwise direction. This is related to the usual parametrization of the circle by angle. Then γ_n is the path $p(nt), 0 \le t \le 1$. Let

$$\begin{aligned} \widetilde{\gamma_{m+n}} : [0,1] &\to [0, m+n] \\ t &\mapsto (m+n)t, \end{aligned}$$

then $\gamma_{m+n} = p \circ \widetilde{\gamma_{m+n}}$. Let

$$\begin{aligned} \widetilde{\gamma_m \cdot \gamma_n} : [0,1] &\to [0, m+n] \\ t &\mapsto \begin{cases} m2t, & 0 \le t \le \frac{1}{2}, \\ n(2t-1)+m, & \frac{1}{2} \le t \le 1, \end{cases} \end{aligned}$$

then $\gamma_m \cdot \gamma_n = p \circ \widetilde{\gamma_m \cdot \gamma_n}$. Let $\varphi = \widetilde{\gamma_{m+n}}^{-1} \circ \widetilde{\gamma_m \cdot \gamma_n}$ (here $\widetilde{\gamma_{m+n}}^{-1}$ denotes the inverse map), then $\widetilde{\gamma_{m+n}} \circ \varphi = \widetilde{\gamma_m \cdot \gamma_n}$:

$$\begin{tikzcd} {[0,1]} \arrow[r, "\widetilde{\gamma_{m+n}}"] & {[0,m+n]} \\ {[0,1]} \arrow[u, "\varphi"] \arrow[ur, "\widetilde{\gamma_m \cdot \gamma_n}"'] & \end{tikzcd}$$

This implies $p \circ (\widetilde{\gamma_{m+n}} \circ \varphi) = (p \circ \widetilde{\gamma_{m+n}}) \circ \varphi = \gamma_{m+n} \circ \varphi = \gamma_m \cdot \gamma_n = p \circ (\widetilde{\gamma_m \cdot \gamma_n})$. Thus $\gamma_m \cdot \gamma_n$ is a reparametrization of γ_{m+n}.

Now we prove that Φ is surjective. This means every loop γ on the circle based at $(1,0)$ is path-homotopic to a loop γ_n. Our is based on the fact that there is a path $\tilde{\gamma}$ on $\mathbb{R}$ starting at 0 such that $\gamma = p \circ \tilde{\gamma}$. This is an important result in its own right and will be proved separately below at 17.1. Then $\tilde{\gamma}(1)$ is an integer n. Since $\mathbb{R}$ is simply-connected, $\tilde{\gamma}$ is path-homotopic to the path $\widetilde{\gamma_n}(t) = nt$, $0 \leq t \leq 1$, namely through a path-homotopy such as $F(s,t) = (1-s)\tilde{\gamma}(t) + s\widetilde{\gamma_n}(t)$, $0 \leq s \leq 1$. Then $\gamma = p \circ \tilde{\gamma}$ is path-homotopic to γ_n through the path-homotopy $G = p \circ F$.

Finally we check that Φ is injective. This is reduced to showing that if γ_m is path-homotopic to γ_n then $m = n$. Our proof is based on another important result below, 17.2, which claims that if γ_m is path-homotopic to γ_n then $\widetilde{\gamma_m}$ is path-homotopic to $\widetilde{\gamma_n}$. This implies the terminal point m of $\widetilde{\gamma_m}$ must be the same as the terminal point n of $\widetilde{\gamma_n}$. $\square$

Covering spaces. The map $p : \mathbb{R} \to S^1$, $p(t) = (\cos(2\pi t), \sin(2\pi t))$ is called the *covering map* associated with of the *covering space* $\mathbb{R}$ of S^1. For a path $\gamma : [0,1] \to S^1$, a path $\tilde{\gamma} : [0,1] \to \mathbb{R}$ such that $p \circ \tilde{\gamma} = \gamma$ is called a *lift* of γ.

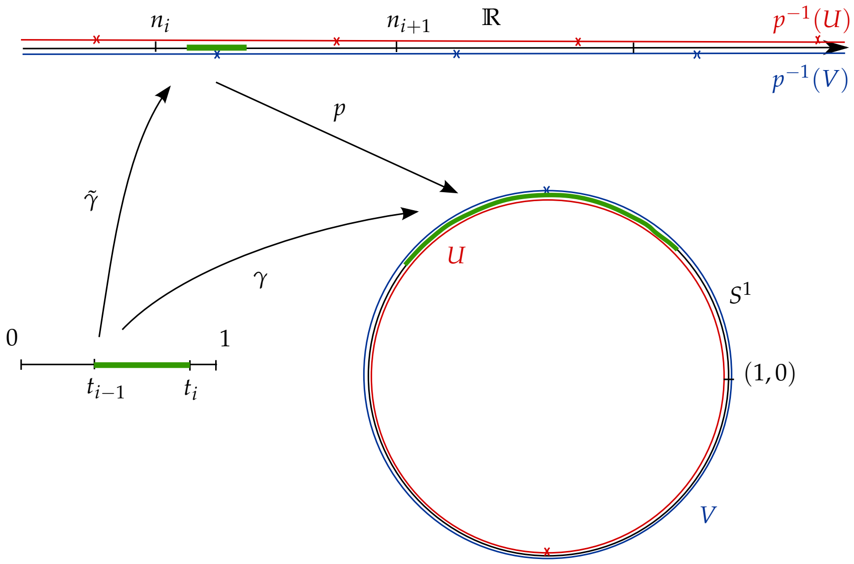
$$\begin{tikzcd}
& \mathbb{R} \arrow[d, "p"] \\
{[0,1]} \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & S^1
\end{tikzcd}$$

Lemma 17.1 (existence of lift). *Every path in S^1 has a lift to $\mathbb{R}$. Furthermore if the initial point of the lift is specified then the lift is unique.*

PROOF. Let us write $S^1 = U \cup V$ with $U = S^1 \setminus \{(0,-1)\}$ and $V = S^1 \setminus \{(0,1)\}$. Then $p^{-1}(U) = \bigcup_{n \in \mathbb{Z}} (n - \frac{1}{4}, n + \frac{3}{4})$ and $p^{-1}(V) = \bigcup_{n \in \mathbb{Z}} (n + \frac{1}{4}, n + \frac{5}{4})$. The key observation is that the preimage $p^{-1}(U)$ consists of infinitely many disjoint open subsets of $\mathbb{R}$, each of which is homemorphic to U via p, i.e. $p : (n - \frac{1}{4}, n + \frac{3}{4}) \to U$ is a homeomorphism, in particular the inverse map exists and is continuous. The same thing happens with respect to V.

Let $\gamma : [0,1] \to S^1$, $\gamma(0) = (1,0)$. We can divide $[0,1]$ into sub-intervals with endpoints $0 = t_0 < t_1 < \cdots < t_n = 1$ such that on each sub-interval $[t_{i-1}, t_i]$, $1 \leq i \leq n$, the path γ is either contained in U or in V. This is guaranteed by the existence of a Lebesgue number 8.3 with respect to the open cover $\gamma^{-1}(U) \cup \gamma^{-1}(V)$ of $[0,1]$.

Suppose a lift $\tilde{\gamma}(0)$ is chosen (which is an integer). Suppose that $\tilde{\gamma}$ has been constructed on $[0, t_{i-1}]$ for a certain $1 \leq i \leq n$. If $\gamma([t_{i-1}, t_i]) \subset U$ then there is a unique $n_i \in \mathbb{Z}$ such that $\tilde{\gamma}(t_{i-1}) \in (n_i - \frac{1}{4}, n_i + \frac{3}{4})$. There is only one way to continuously extend $\tilde{\gamma}$ to $[t_{i-1}, t_i]$, that is by defining $\tilde{\gamma} = p|_{(n_i - \frac{1}{4}, n_i + \frac{3}{4})}^{-1} \circ \gamma$. Similarly, if $\gamma([t_{i-1}, t_i]) \subset V$ there is $n_i \in \mathbb{Z}$ such that $\tilde{\gamma}(t_{i-1}) \in (n_i + \frac{1}{4}, n_i + \frac{5}{4})$,



then on $[t_{i-1}, t_i]$ we define $\tilde{\gamma} = p|^{-1}_{(n_i+\frac{1}{4}, n_i+\frac{5}{4})} \circ \gamma$. In either case we obtain a unique continuous lift $\tilde{\gamma}$ on $[0, t_i]$. In this way $\tilde{\gamma}$ is extended continuously to $[0, 1]$. $\square$

Examining the above proof we can see that the key property of the covering space $p : \mathbb{R} \to S^1$ is the following: each point on the circle has an open neighborhood U such that the preimage $p^{-1}(U)$ is the disjoint union of open subsets of $\mathbb{R}$, each of which is homeomorphic to U via p. This is the defining property of general covering spaces. This is also the property used in the next proof.

Lemma 17.2 (homotopy of lifts). *Lifts of path-homotopic paths with same initial points are path-homotopic.*

PROOF. The proof is similar to the above proof of 17.1. Let $F : [0, 1] \times [0, 1] \to S^1$ be a path-homotopy from the path F_0 to the path F_1. If the two lifts $\widetilde{F_0}$ and $\widetilde{F_1}$ have same initial points then that initial point is the lift of the point $F((0, 0))$.

As we noted earlier, the circle has an open cover O such that each $U \in O$ we have $p^{-1}(U)$ is the disjoint union of open subsets of $\mathbb{R}$, each of which is homeomorphic to U via p. The collection $F^{-1}(O)$ is an open cover of the square $[0, 1] \times [0, 1]$. By the existence of Lebesgue's number, there is a partition of $[0, 1] \times [0, 1]$ into sub-rectangles such that each sub-rectangle is contained in an element of $F^{-1}(O)$. More concisely, we can divide $[0, 1]$ into sub-intervals with endpoints $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for each $1 \le i, j \le n$ there is $U \in O$ such that $F([t_{i-1}, t_i] \times [t_{j-1}, t_j]) \subset U$.

We already have $\tilde{F}((0, 0))$. Suppose that $\tilde{F}((t_{i-1}, t_{j-1}))$, $1 \le i, j \le n$ is already defined. Suppose that $F((t_{i-1}, t_{j-1})) \in F([t_{i-1}, t_i] \times [t_{j-1}, t_j]) \subset U$ for some $U \in O$. We can write $p^{-1}(U) = \cup_{k \in K} U_k$ with $U_k \cap U_l = \emptyset$ if $k \neq l$, and each U_k is an open subset of $\mathbb{R}$ such that $p|_{U_k} : U_k \to U$ is a homeomorphism. Suppose that the known lift of the point $F((t_{i-1}, t_{j-1}))$ is in U_k for some $k \in K$. Then we define $\tilde{F}$ on the sub-rectangle $[t_{i-1}, t_i] \times [t_{j-1}, t_j]$ to be $p|^{-1}_{U_k} \circ F$.

We need to check $\tilde{F}$ is continuous on the extended domain. Since we extend one sub-rectangle at a time in this way, the intersection of the previous domain of $\tilde{F}$ and the sub-rectangle $[t_{i-1}, t_i] \times [t_{j-1}, t_j]$ is connected. That implies $\tilde{F}$ must bring the entire common domain to a unique U_k for some $k \in K$, therefore on this common domain $\tilde{F}$ is $p|_{U_k}^{-1} \circ F$, agreeing with the new definition.

Thus we obtained a continuous lift $\tilde{F}$ of F. Since the initial point is given, by uniqueness of lifts of paths in 17.1, the restriction of $\tilde{F}$ to $[0,1] \times \{0\}$ is $\widetilde{F_0}$ while the restriction of $\tilde{F}$ to $[0,1] \times \{1\}$ is $\widetilde{F_1}$. Thus $\tilde{F}$ is a path-homotopy from $\widetilde{F_0}$ to $\widetilde{F_1}$. $\square$

Applications.

Corollary. *The circle is not contractible.*

Corollary. *The plane minus a point is not simply connected.*

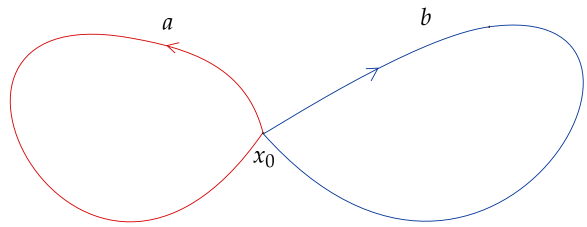
Problems.

17.3. Find the fundamental groups of the Mobius band and the cylinder.

18. Van Kampen theorem

Van Kampen theorem is about giving the fundamental group of a union of subspaces from the fundamental groups of the subspaces.

Example. Two circles with one common point (the figure 8) is called a wedge product $S^1 \vee S^1$. Let x_0 be the common point, let a be a loop starting at x_0 going once around the first circle and let b the a loop starting at x_0 going once around the second circle. Then a and b generate the fundamental groups of the two circles with based points at x_0. Intuitively we can see that $\pi_1(S^1 \vee S^1, x_0)$ consists of path-homotopy classes of loops like a, ab, bba, $aabab^{-1}a^{-1}a^{-1}, \ldots$ This is a group called the free group generated by a and b, denoted by $\langle a, b\rangle$.



Free group. Let S be a set. Let S^{-1} be a set having a bijection with S. Corresponding to each element $x \in S$ is an element in S^{-1} denoted by x^{-1}. A *word with letters in S* is a finite sequence of elements in S or S^{-1}. The sequence with no element is called the *empty word*. In a word, if two elements x and x^{-1} are consecutive then they can be cancelled, i.e. they can be replaced by the empty word. Given two words we form a new word by juxtaposition (đặt kề): $(s_1s_2\cdots s_n)\cdot(s_1's_2'\cdots s_m') = s_1s_2\cdots s_ns_1's_2'\cdots s_m'$. With this operation the set of all words with letters in S becomes a group. The identity element 1 is the empty word. The inverse element of a word $s_1s_2\cdots s_n$ is the word $s_n^{-1}s_{n-1}^{-1}\cdots s_1^{-1}$. This group is called the *free group generated by the set S*, denoted by $\langle S\rangle$.

Example. The free group $\langle\{a\}\rangle$ generated by the set $\{a\}$ is often written as $\langle a\rangle$. As a set $\langle a\rangle$ can be written as $\{a^n \mid n \in \mathbb{Z}\}$. The product is given by $a^m \cdot a^n = a^{m+n}$. The identity is a^0. Thus as a group $\langle a\rangle$ is an infinite cyclic group, isomorphic to $(\mathbb{Z}, +)$.

Let G be a set and let R be a set of words with letters in G, i.e. a finite subset of the free group $\langle G\rangle$. Let N be the smallest normal subgroup of $\langle G\rangle$ containing R. The quotient group $\langle G\rangle/N$ is written $\langle G \mid R\rangle$. Elements of G are called *generators* of this group and elements of R are called *relators* of this group. We can think of $\langle G \mid R\rangle$ as consisting of words in G subjected to the *relations* $r = 1$ for all $r \in R$.

Example. $\langle a \mid a^2\rangle = \{a^0, a\} \cong \mathbb{Z}_2$.

Free product of groups. Let G and H be groups. Form the set of all words with letters in G or H. In such a word, two consecutive elements from the same group can be reduced by the group operation. For example $ba^2ab^3b^{-5}a^4 = ba^3b^{-2}a^4$. In particular if x and x^{-1} are next to each other then they will be cancelled. So the identities of G and H can be reduced. For example $abb^{-1}c = a1c = ac$.

As with free group, given two words we form a new word by juxtaposition. For example $(a^2b^3a^{-1}) \cdot (a^3ba) = a^2b^3a^{-1}a^3ba = a^2b^3a^2ba$. This is a group operation, with the identity element 1 being the empty word, the inverse of a word $s_1s_2\cdots s_n$ is the word $s_n^{-1}s_{n-1}^{-1}\cdots s_1^{-1}$. This group is called the *free product* of G with H.

Proposition. *If*

$$G = \langle g_1, g_2, \dots, g_{m_1} \mid r_1, r_2, \dots, r_{n_1} \rangle$$

and

$$H = \langle h_1, h_2, \dots, h_{m_2} \mid s_1, s_2, \dots, s_{n_2} \rangle$$

then

$$G * H = \langle g_1, g_2, \dots, g_{m_1}, h_1, h_2, \dots, h_{m_2} \mid r_1, r_2, \dots, r_{n_1}, s_1, s_2, \dots, s_{n_2} \rangle .$$

Example ($G * H \neq G \times H$). We have

$$\langle g \rangle * \langle h \rangle = \langle g, h \rangle = \{g^{m_1}h^{n_1}g^{m_2}h^{n_2}\cdots g^{m_k}h^{n_k} \mid m_1, n_1, \dots, m_k, n_k \in \mathbb{Z},\ k \in \mathbb{Z}^+\}.$$

Compare that to $\langle g \rangle \times \langle h \rangle = \{(g^m, h^n) \mid m, n \in \mathbb{Z}\}$ with component-wise multiplication. This group can be identified with $\langle g, h \mid gh = hg \rangle = \{g^m h^n \mid m, n \in \mathbb{Z}\}$. Thus $\mathbb{Z} * \mathbb{Z} \neq \mathbb{Z} \times \mathbb{Z}$.

For more details on free group and free product, see textbooks on Algebra such as [**Gal10**] or [**Hun74**].

Van Kampen theorem.

Theorem (Van Kampen theorem). *Suppose that $X = U \cup V$ with U, V open, path-connected, $U \cap V$ is path-connected, and $x_0 \in U \cap V$. Let $i_U : U \cap V \hookrightarrow U$ and $i_V : U \cap V \hookrightarrow V$ be inclusion maps. Then*

$$\pi_1(U \cup V, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{\langle (i_U)_*(\alpha)(i_V)_*(\alpha^{-1}) \mid \alpha \in \pi_1(U \cap V, x_0) \rangle}.$$

Corollary. *If $X = U \cup V$ with U, V open, path-connected, $U \cap V$ is simply connected, and $x_0 \in U \cap V$, then $\pi_1(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0)$.*

Example. Consider the sphere S^n, $n \geq 2$. Let $A = S^n \setminus \{(0, 0, \dots, 0, 1)\}$ and $B = S^n \setminus \{(0, 0, \dots, 0, -1)\}$. Then A and B are contractible. By Van Kampen theorem, $\pi_1(S^2) \cong \pi_1(A) * \pi_1(B) = 1$.

Thus we obtain:

Corollary.

$$\pi_1(S^n) \cong \begin{cases} \mathbb{Z}, & n = 1 \\ 1, & n > 1. \end{cases}$$

Corollary. *The spheres of dimensions greater than one are simply connected.*

Example. Consider the wedge of two circles. Let U be the union of the first circle with an open arc on the second circle containing the common point. Similarly let V be the union of the second circle with an open arc on the first circle containing the common point. Clearly U and V have deformation retractions to the first and the second circles respectively, while $U \cap V$ has a deformation retraction to the common point. Applying the Van Kampen theorem we get

$$\pi_1(S^1 \vee S^1) \cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z}.$$

The fundamental group of a cell complex. A simple application of the Van Kampen give us:

Theorem 18.1. *Let X be a topological space and consider the space $X \sqcup_f D^n$ obtained by attaching an n-dimensional cell to X via the map $f : \partial D^n = S^{n-1} \to X$. Let the base point $x_0 \in f(\partial D^n)$. Then*

$$\pi_1(X \sqcup_f D^n, x_0) \cong \begin{cases} \pi_1(X, x_0)/[f(\partial D^n)], & n = 2, \\ \pi_1(X, x_0) & n > 2. \end{cases}$$

Intuitively, gluing a 2-disk destroys the boundary circle of the disk homotopically, but gluing disks of dimensions greater than 2 does not affect the fundamental group.

PROOF. In view of 16.6, we can assume X is path-connected, otherwise we can focus on the path-connected component containing x_0. Let $Y = X \sqcup_f D^n$. Let $U = X \sqcup_f \{x \in D^n \mid \|x\| > \frac{1}{2}\} \subset Y$. Let $V = \{x \in D^n \mid \|x\| < 1\} \subset Y$. Then $U \cap V = \{x \in D^n \mid \frac{1}{2} < \|x\| < 1\} \subset Y$. Let $y_0 \in U \cap V$. We apply Van Kampen theorem to the pair (U, V).

Consider the case $n = 2$. Let γ be a loop starting at y_0 going once around the annulus $U \cap V$. Then $[\gamma]$ is a generator of $\pi_1(U \cap V, y_0)$. In V the loop γ is homotopically trivial since V has a deformation retraction to y_0. Thus $\pi_1(Y, y_0) \cong \pi_1(U, y_0)/([\gamma] = 1)$. Since there is a path from x_0 to y_0, we have $\pi_1(Y, y_0) \cong \pi_1(Y, x_0)$. Since U has a deformation retraction to X we have $\pi_1(U, y_0) \cong \pi_1(X, x_0)$. Under this deformation retraction, the image of γ becomes $f(\partial D^2)$. Therefore $\pi_1(Y, x_0) \cong \pi_1(X, x_0)/([f(\partial D^2)] = 1)$.

When $n = 2$ the space $U \cap V$ is contractible, therefore $\pi_1(Y, y_0) \cong \pi_1(U, y_0) \cong \pi_1(X, x_0)$. $\square$

This result shows that the fundamental group only gives information about the two-dimensional skeleton of a cell complex, it does not give information on cells of dimensions greater than 2.

The fundamental groups of surfaces. By the classification theorem, any compact without boundary surface is obtained by identifying the edges of a polygon following a word as in 14.1. As such it has a cell complex structure with a two-dimensional disk glued to the boundary of the polygon under the equivalence relation, which is a wedge of circles. An application of 18.1 gives us:

Theorem. *The fundamental group of a connected compact surface S is isomorphic to one of the following groups:*

(a) *trivial group, if $S = S^2$,*

(b) $\left\langle a_1, b_1 a_2, b_2, \dots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \right\rangle$, *if S is the orientable surface of genus g,*

(c) $\left\langle c_1, c_2, \dots, c_g \mid c_1^2 c_2^2 \cdots c_g^2 \right\rangle$, *if S is the unorientable surface of genus g.*

Problems.

18.2. Find the fundamental groups of the following spaces:

(a) The Mobius band.

(b) The cylinder.

(c) A wedge of finitely many circles.

(d) $S^1 \vee S^2$.

(e) $S^2 \vee S^3$.

(f) The plane minus finitely many points.

(g) The Euclidean space $\mathbb{R}^3$ minus finitely many points.

18.3. Give a rigorous definition of the wedge product of two spaces. For example, what really is $S^1 \vee S^1$?

18.4. Is the fundamental group of the Klein bottle abelian?

18.5. Show that the fundamental groups of the one-hole torus and the two-holes torus are not isomorphic. Therefore the two surfaces are different.

18.6. Find a space whose fundamental group is isomorphic to $\mathbb{Z}_3$.

18.7. Find a space whose fundamental group is isomorphic to $\mathbb{Z} * \mathbb{Z}_5$.

19. Simplicial homology

Oriented simplex. Consider the relation on the collection of ordered sets of vertices of a simplex whereas two order sets of vertices are related if they differ by a even permutation. This is an equivalence relation. Each of the two equivalence classes is called an *orientation* of the simplex. If we choose an orientation, then the simplex is said to be *oriented*.

Example. A 1-dimensional simplex in $\mathbb{R}^n$ is a straight segment connecting two points. Choosing one point as the first point and the other point as the second gives an orientation to this simplex. Intuitively, this is the same as to give a direction to the straight segment.

Chain. Let X be a simplicial complex in a Euclidean space. For each integer n, let $S_n(X)$ be the free abelian group generated by all n-dimensional oriented simplexes in X modulo the relation that if σ and σ' are the same simplex with opposite orientations, then $\sigma = -\sigma'$. Each element of $S_n(X)$, called an n-dimensional *chain* (xích), is a finite sum of integer multiples of n-dimensional oriented simplexes, i.e. of the form $\sum_{i=1}^m n_i\sigma_i$ where σ_i is an n-dimensional oriented simplex of X and $n_i \in \mathbb{Z}$. If n is less than 0 or bigger than the dimension of X then $S_n(X)$ is assigned to be the trivial group 0.

Boundary. Let σ be an n-dimensional oriented simplex, i.e., a convex hull of $(n+1)$ ordered points $v_0, v_1, \dots, v_n$ where $v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$ are n linearly independent vectors. Denote such a convex hull by $[v_0, v_1, \dots, v_n]$. Define the *boundary* of σ to be the following $(n-1)$-dimensional chain, the alternating sum of the $(n-1)$-dimensional faces of σ:

$$\partial_n\sigma = \sum_{i=0}^{n}(-1)^i[v_0, v_1, \dots, v_{i-1}, \widehat{v_i}, v_{i+1}, \dots, v_n],$$

where the notation $\widehat{v_i}$ is traditionally used to indicate that this point is dropped.

This map is extended linearly to become an operator from $S_n(X)$ to $S_{n-1}(X)$, namely

$$\partial_n\left(\sum_{i=1}^m n_i\sigma_i\right) = \sum_{i=1}^m n_i\partial_n\sigma_i.$$

Remark. When $n = 0$, we assign $\partial_0 = 0$. This is consistent with the convention that $S_{-1}(X) = 0$. Similarly if n is bigger than the dimension of X then $\partial_n = 0$.

Example. An oriented 1-dimensional simplex in $\mathbb{R}^n$ is a straight segment v_0v_1. Its boundary of this simplex is the 0-dimensional chain $v_1 - v_0$.

Example. An oriented 2-dimensional simplex in $\mathbb{R}^n$ is a triangle with three vertices v_0, v_1, v_2 in this order. The boundary of this simplex is the 1-dimensional chain $v_2v_3 - v_1v_3 + v_1v_2 = v_2v_3 + v_3v_1 + v_1v_2$.

Example. From the previous two examples, if $[v_0, v_1, v_3]$ is a an oriented 2-simplex, then

$$\partial_1(\partial_2([v_0, v_1, v_3])) = \partial_1(v_2v_3 + v_3v_1 + v_1v_2) = (v_3 - v_2) + (v_1 - v_3) + (v_2 - v_1) = 0.$$

This example illustrates that intuitively "a boundary has empty boundary":

Proposition 19.1 (boundary of boundary is zero). *$\partial_{n-1} \circ \partial_n = 0$ for all $n \geq 2$.*

PROOF. Let $\sigma = [v_0, v_1, \ldots, v_n]$, an oriented n-simplex. As defined,

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_n].$$

Then

$$\begin{aligned}
\partial_{n-1}\partial_n(\sigma) &= \sum_{i=0}^{n} (-1)^i \partial_{n-1}([v_0, v_1, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_n]) \\
&= \sum_{i=0}^{n} (-1)^i (\sum_{j=0}^{i-1} (-1)^j [v_0, \ldots, \widehat{v_j}, \ldots, \widehat{v_i}, \ldots, v_n] + \\
&\quad + \sum_{j=i}^{n-1} (-1)^j [v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_{j+1}}, \ldots, v_n]) \\
&= \sum_{0 \leq j < i \leq n} (-1)^{i+j} [v_0, \ldots, \widehat{v_j}, \ldots, \widehat{v_i}, \ldots, v_n] + \\
&\quad + \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} [v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_{j+1}}, \ldots, v_n] \\
&= \sum_{0 \leq j < i \leq n} (-1)^{i+j} [v_0, \ldots, \widehat{v_j}, \ldots, \widehat{v_i}, \ldots, v_n] + \\
&\quad + \sum_{0 \leq i < k \leq n,\ (k=j+1)} (-1)^{i+k-1} [v_0, \ldots, \widehat{v_i}, \ldots, \widehat{v_k}, \ldots, v_n] \\
&= \sum_{0 \leq j < i \leq n} (-1)^{i+j} [v_0, \ldots, \widehat{v_j}, \ldots, \widehat{v_i}, \ldots, v_n] + \\
&\quad + \sum_{0 \leq j < i \leq n} -(-1)^{i+j} [v_0, \ldots, \widehat{v_j}, \ldots, \widehat{v_i}, \ldots, v_n] \\
&= 0.
\end{aligned}$$

□

The above result can be interpreted as

$$\mathrm{Im}(\partial_{n+1}) \subset \ker(\partial_n),\ \forall n \geq 0.$$

In general, a sequence of groups and homomorphisms

$$\cdots \xrightarrow{\partial_{n+2}} S_{n+1} \xrightarrow{\partial_{n+1}} S_n \xrightarrow{\partial_n} S_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_1} S_0$$

satisfying $\mathrm{Im}(\partial_{n+1}) \subset \ker(\partial_n)$, $\forall n \geq 0$ is called a *chain complex* (phức xích). If furthermore $\mathrm{Im}(\partial_{n+1}) = \ker(\partial_n)$, $\forall n \geq 0$ then the chain complex is called *exact* (khớp).

Notice that if the group S_n are abelian then $\operatorname{Im}(\partial_{n+1})$ is a normal subgroup of $\ker(\partial_n)$.

Definition. The n-dimensional simplicial *homology group* (nhóm đồng điều) of a simplicial complex X is defined to be the quotient group

$$H_n(X) = \frac{\ker(\partial_n)}{\operatorname{Im}(\partial_{n+1})}.$$

For more on simplicial homology one can read **[Mun84]**. In **[Hat01]** Hatcher used a modified notion called Δ-complex, different from simplicial complex.

20. Singular homology

A *singular simplex* (đơn hình suy biến, kì dị) is a continuous map from a standard simplex to a topological space. More precisely, an n-dimensional singular simplex in a topological space X is a continuous map $\sigma : \Delta_n \to X$.

Let $S_n(X)$ be the free abelian group generated by all n-dimensional singular simplexes in X. As a set

$$S_n(X) = \{\sum_{i=1}^{m} n_i\sigma_i \mid \sigma_i : \Delta_n \to X,\ m \in \mathbb{Z}^+,\ k \in \mathbb{Z}\}.$$

Each element of $S_n(X)$ is a finite sum of integer multiples of n-dimensional singular simplexes, called a *singular n-chain* .

Boundary. Let σ be an n-dimensional singular simplex in X, i.e., a map

$$\begin{aligned} \sigma : \Delta_n &\to X \\ (t_0, t_1, \ldots, t_n) &\mapsto \sigma(t_0, t_1, \ldots, t_n). \end{aligned}$$

For $0 \leq i \leq n$ define the *ith face* of σ to be the $(n-1)$-singular simplex

$$\begin{aligned} \delta_i\sigma : \Delta_{n-1} &\to X \\ (t_0, t_1, \ldots, t_{n-1}) &\mapsto \sigma(t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}). \end{aligned}$$

This map is extended linearly to become an operator from $S_n(X)$ to $S_{n-1}(X)$, namely

$$\delta_i\left(\sum_{j=1}^{m} n_j\sigma_j\right) = \sum_{j=1}^{m} n_j\delta_i\sigma_j.$$

Define the *boundary* of σ to be the singular $(n-1)$-chain $\partial_n\sigma = \sum_{i=0}^{n}(-1)^i\delta_i\sigma$. More generally, ∂_n is defined on $S_n(X)$ by $\partial_n = \sum_{i=0}^{n}(-1)^i\delta_i$. Clearly ∂_n is a group homomorphism from $S_n(X)$ to $S_{n-1}(X)$.

Example. A 1-dimensional singular simplex in $\mathbb{R}^n$ is a continuous map $\sigma(t_0, t_1)$ with $t_0, t_1 \in [0, 1]$ and $t_0 + t_1 = 1$. Its image is a curve between the points $A = \sigma(1, 0)$ and $B = \sigma(0, 1)$. Its boundary is $-A + B$.

A 2-dimensional singular simplex in $\mathbb{R}^n$ is a continuous map $\sigma(t_0, t_1, t_2)$ with $t_0, t_1, t_2 \in [0, 1]$ and $t_0 + t_1 + t_2 = 1$. Its image is a "curved triangle" between the points $A = \sigma(1, 0, 0)$, $B = \sigma(0, 1, 0)$, and $C = (0, 0, 1)$. Intuitively, the image of the face δ_0 is the "curved edge" BC, the image of δ_1 is CA, and the image of δ_2 is AB. The boundary is $\delta_0 - \delta_1 + \delta_2$.

Similar to the case of simplicial complex 19.1, we have

Proposition. $\partial_{n-1} \circ \partial_n = 0, \forall n \geq 2.$

So like the case of simplicial complex we make the following definition:

Definition. The n-dimensional singular *homology group* (nhóm đồng điều) of a topological space X is defined to be the quotient group

$$H_n(X) = \frac{\ker(\partial_n)}{\operatorname{Im}(\partial_{n+1})}.$$

Induced homomorphism. Suppose X and Y are topological spaces and $f : X \to Y$ is continuous. Then f induces the following group homomorphism:

$$\begin{aligned} f_\# : S_n(X) &\to S_n(Y) \\ \sigma &\mapsto f \circ \sigma. \end{aligned}$$

It can be checked straightforwardly that

Proposition. $\partial(f \circ \sigma) = f \circ (\partial\sigma)$.

A consequence of this result is that $f_\#$ induces a group homomorphism $f_* : H_n(X) \to H_n(Y)$.

Proposition. $(g \circ f)_* = g_* \circ f_*$.

Theorem. *If $f : X \to Y$ is either a homeomorphism or a homotopy equivalence then $f_* : H_n(X) \to H_n(Y)$ is an isomorphism.*

Thus the homology groups are not only topological invariants, they are *homotopy invariants* of topological spaces.

Mayer-Vietoris sequence.

Theorem. *Let X be a topological space. Suppose $U, V \subset X$ and $int(U) \cup int(V) = X$. Then the following chain complex, called the Mayer-Vietoris sequence, is exact:*

$$\begin{aligned} \cdots \to \quad & H_n(U \cap V) \xrightarrow{\phi_*} H_n(U) \oplus H_n(V) \xrightarrow{\psi_*} H_n(U \cup V) \xrightarrow{\Delta} H_{n-1}(U \cap V) \to \cdots \\ \cdots \quad & \to H_0(U \cup V) \to 0. \end{aligned}$$

The Mayer-Vietoris sequence allows us to study the homology of a space from the homologies of subspaces, in a similar manner to the Van Kampen theorem.

Using the Mayer-Vietoris sequence we get:

Theorem.

$$H_n(S^m) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, m \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 20.1. *For $n \geq 2$ there cannot be any retraction from the disk D^n to its boundary S^{n-1}.*

PROOF. Suppose there is a retraction $r : D^n \to S^{n-1}$. Let $i : S^{n-1} \hookrightarrow D^n$ be the inclusion map. From the diagram $S^{n-1} \xrightarrow{i} D^n \xrightarrow{r} S^{n-1}$ we have $r \circ i = \mathrm{id}_{S^{n-1}}$, therefore on the $(n-1)$-dimensional homology groups $(r \circ i)_* = \mathrm{id}_{H_{n-1}(S^{n-1})}$ is non-trivial. On the other hand $(r \circ i)_* = r_* \circ i_*$, where $r_* : H_{n-1}(D^n) \to H_{n-1}(S^{n-1})$ is trivial for $n > 1$, since D^n is contractible. This is a contradiction. $\square$

A proof of this result in differentiable setting using Differential Topology is presented in 28.1.

The important Brouwer fixed point theorem follows from that simple result:

Theorem 20.2 (Brouwer fixed point theorem). *A continuous map from the disk D^n to itself has a fixed point.*

PROOF. Suppose that $f : D^n \to D^n$ does not have a fixed point, i.e. $f(x) \neq x$ for all $x \in D^n$. The straight line from $f(x)$ to x will intersect the boundary ∂D^n at a point $g(x)$. Then $g : D^n \to \partial D^n$ is a retraction. That is impossible. □

For more on singular homology, one can read **[Vic94]**.

Problems.

20.3. If X is a path-connected space then $H_0(X) \cong \mathbb{Z}$. If X has k path-connected components then $H_0(X) \cong \mathbb{Z}^k$.

20.4. Show that if X has two connected components A and B then $H_i(X) \cong H_i(A) \oplus H_i(B)$ for all $i \geq 0$.

20.5. Show that if $A \cap B$ is contractible then $H_i(A \cup B) \cong H_i(A) \oplus H_i(B)$ for $i \geq 1$. Is this true if $i = 0$?

20.6. Compute the homology groups of $S^2 \times [0,1]$.

20.7. Compute the fundamental group and the homology groups of the Euclidean space $\mathbb{R}^3$ minus a straight line.

20.8. Compute the fundamental group and the homology groups of the Euclidean space $\mathbb{R}^3$ minus two intersecting straight lines.

20.9. Compute the fundamental group and the homology groups of $\mathbb{R}^3 \setminus S^1$.

21. Cellular homology

Degrees of maps on spheres. A continuous map $f : S^n \to S^n$ induces a homomorphism $f_* : H_n(S^n) \to H_n(S^n)$. We know $H_n(S^n) \cong \mathbb{Z}$, so there is a generator a such that $H_n(S^n) = \langle a \rangle$. Then $f_*(a) = ma$ for a certain integer m, called the *topological degree* of f, denoted by $\deg f$.

Example. If f is the identity map then $\deg f = 1$. If f is the constant map then $\deg f = 0$.

Relative homology groups. Let A be a subspace of X. Viewing each singular simplex in A as a singular simplex in X, we have a natural inclusion $S_n(A) \hookrightarrow S_n(X)$. In this way $S_n(A)$ is a normal subgroup of $S_n(X)$. The boundary map ∂_n induces a homomorphism $\partial_n : S_n(X)/S_n(A) \to S_{n-1}(X)/S_{n-1}(A)$, giving a chain complex

$$\cdots \to S_n(X)/S_n(A) \xrightarrow{\partial_n} S_{n-1}(X)/S_{n-1}(A) \xrightarrow{\partial_{n-1}} S_{n-2}(X)/S_{n-2}(A) \to \cdots$$

The homology groups of this chain complex is called the *relative homology groups* of the pair (X, A), denoted by $H_n(X, A)$.

If $f : X \to Y$ is continuous and $f(A) \subset B$ then as before it induces a homomorphism $f_* : H_n(X, A) \to H_n(Y, B)$.

Homology of a cell complex. Let X be a cellular complex. Recall that X^n denote the n-dimensional skeleton of X. Suppose that X^n is obtained from X^{n-1} by attaching the n-dimensional disks $D_1^n, D_2^n, \ldots, D_k^n$. Let $e_1^n, e_2^n, \ldots, e_k^n$ be the corresponding cells. Then

$$H_n(X^n, X^{n-1}) \cong \langle e_1^n, e_2^n, \ldots, e_k^n \rangle = \{\sum_{i=1}^{k} m_i e_i^n \mid m_i \in \mathbb{Z}\}.$$

Consider following sequence

$$C(X) = \cdots \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{d_{n-1}} \cdots$$
$$\cdots \xrightarrow{d_2} H_1(X^1, X^0) \xrightarrow{d_1} H_0(X).$$

Here the map d_n is given by

$$d_n(e_i^n) = \sum_j d_{i,j} e_j^{n-1},$$

where the sum is taken over all $(n-1)$-dimensional cells and the integer number $d_{i,j}$ is given as the degree of the map on spheres:

$$S_i^{n-1} = \partial D_i^n \to X^{n-1} \to X^{n-1}/X^{n-2} = S_1^{n-1} \vee S_2^{n-1} \vee \cdots \to S_j^{n-1}.$$

Theorem. *The sequence $C(X)$ is a chain complex and its homology coincides with the homology of X.*

Homology groups of surfaces. As an application we get:

Theorem. *The fundamental group of a connected compact orientable surface S of genus $g \geq 0$ is*

$$H_n(S) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 2 \\ \mathbb{Z}^g & \text{if } n = 1. \end{cases}$$

For more on cellular homology one can read [**Hat01**, p. 137].

Problems.

21.1. Using cellular homology compute the homology groups of the following spaces:

(a) The Klein bottle.

(b) $S^1 \vee S^1$.

(c) $S^1 \vee S^2$.

(d) $S^2 \vee S^3$.

Guide for further reading

The book [**Vas01**] gives a modern overview of many aspects of both Algebraic and Differential Topology, aims at undergraduate students. Although it often only sketch proofs, it introduces the general ideas very well.

The book of Munkres [**Mun00**], also aims at undergraduate students, has a part on Algebraic Topology, but stops before homology.

For homology the book of Hatcher [**Hat01**] is very popular, but it aims at graduate students, and sometimes one needs to read other sources too.

Recently Algebraic Topology has begun to be applied to science and engineering. One can read the book [**EH10**].

Differential Topology

22. Smooth manifolds

In this chapter we always assume that $\mathbb{R}^n$ has the Euclidean topology.

Roughly, a smooth manifold is a space that is locally diffeomorphic to $\mathbb{R}^m$. This allows us to bring the differential and integral calculus from $\mathbb{R}^m$ to manifolds.

Smooth maps on $\mathbb{R}^n$. Recall that for a function f from a subset D of $\mathbb{R}^k$ to $\mathbb{R}^l$ we say that f is *smooth* (or infinitely differentiable) at an *interior point* x of D if all partial derivatives of all orders of f exist at x.

If x is a boundary point of D, then f is said to be *smooth* at x if f can be extended to be a function which is smooth at every point in an open neighborhood in $\mathbb{R}^k$ of x. Precisely, f is smooth at x if there is an open set $U \subset \mathbb{R}^k$ containing x, and function $F : U \to \mathbb{R}^l$ such that F is smooth at every point of U and $F|_{U \cap D} = f$.

If f is smooth at every point of D then we say that f is smooth on D, in other words $f \in C^\infty(D)$.

Let $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$. Then $f : X \to Y$ is a *diffeomorphism* if it is bijective and both f and f^{-1} are smooth. If there is a diffeomorphism from X to Y then we say that they are *diffeomorphic*.

Example. Any open ball $B(x, r)$ in $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.

Smooth manifolds.

Definition. A subspace $M \subset \mathbb{R}^k$ is a *smooth manifold* of dimension $m \in \mathbb{Z}^+$ if every point in M has a neighborhood in M which is diffeomorphic to $\mathbb{R}^m$.

Recall that by Invariance of dimension 5.34, $\mathbb{R}^m$ cannot be homeomorphic to $\mathbb{R}^n$ if $m \neq n$, therefore a manifold has a unique dimension.

Remark. A diffeomorphism is a homeomorphism, therefore a smooth manifold is a topological manifold. In this chapter unless stated otherwise manifolds mean smooth manifolds.

The following is a simple but convenient observation:

Proposition. *A subspace $M \subset \mathbb{R}^k$ is a smooth manifold of dimension m if every point in M has an open neighborhood in M which is diffeomorphic to an open subset of $\mathbb{R}^m$.*

Although this proposition seems to be less intuitive than our original definition, it is technically more convenient to use, therefore from now on we will usually take it as the definition.

PROOF. Suppose that (U, ϕ) is a local coordinate on M where U is a neighborhood of x in M and $\phi : U \to \mathbb{R}^m$ is a diffeomorphism. There is an open subset U' of M such that $x \in U' \subset U$. Since ϕ is a homeomorphism, $\phi(U')$ is an open neighborhood of $\phi(x)$. There is a ball $B(\phi(x), r) \subset \phi(U')$. Let $U'' = \phi^{-1}(B(\phi(x), r))$. Then U'' is open in U', so is open in M. Furthermore $\phi|_{U''} : U'' \to B(\phi(x), r)$ is a diffeomorphism.

We have just shown that any point in the manifold has an open neighborhood diffeomorphic to an open ball in $\mathbb{R}^m$. For the reverse direction, we recall that any open ball in $\mathbb{R}^m$ is diffeomorphic to $\mathbb{R}^m$. □

By this result, each point x in a manifold has an open neighborhood U in M and a diffeomorphism $\varphi : U \to V$ where V is an open subset of $\mathbb{R}^m$. The pair (U, φ) is called a *local coordinate* at x. The pair (V, φ^{-1}) is called a *local parametrization* at x.

Example. Any open subset of $\mathbb{R}^m$ is a smooth manifold of dimension m.

Example. The graph of a smooth function $y = f(x)$ for $x \in (a, b)$ (a smooth curve) is a 1-dimensional smooth manifold.

More generally:

Proposition. *The graph of a smooth function $f : D \to \mathbb{R}^l$, where $D \subset \mathbb{R}^k$ is an open set, is a smooth manifold of dimension k.*

PROOF. Let $G = \{(x, f(x)) \mid x \in D\} \subset \mathbb{R}^{k+l}$ be the graph of f. The map $x \mapsto (x, f(x))$ from D to G is smooth. Its inverse is the projection $(x, y) \mapsto x$. This projection is the restriction of the projection given by the same formula from $\mathbb{R}^{k+l}$ to $\mathbb{R}^k$, which is a smooth map. Therefore D is diffeomorphic to G. □

Example (The circle). Let $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. It is covered by four neighborhoods which are half circles, each corresponds to points $(x, y) \in S^1$ such that $x > 0$, $x < 0$, $y > 0$ and $y < 0$. Each of these neighborhoods is diffeomorphic to $(-1, 1)$. For example consider the projection from $U = \{(x, y) \in S^1 \mid x > 0\} \to (-1, 1)$ given by $(x, y) \mapsto y$. The map $(x, y) \mapsto y$ is smooth on $\mathbb{R}^2$, so it is smooth on U. The inverse map $y \mapsto (\sqrt{1 - y^2}, y)$ is smooth on $(-1, 1)$. Therefore the projection is a diffeomorphism.

Remark. By convention, a manifold of dimension 0 is a discrete subspace of a Euclidean space.

Remark. We are discussing smooth manifolds embedded in Euclidean spaces. There is a notion of abstract smooth manifold, but we do not discuss it now.

Problems.

22.1. From our definition, a smooth function f defined on $D \subset \mathbb{R}^k$ does not necessarily have partial derivatives defined at boundary points of D. However, show that if D is the closure of an open subspace of $\mathbb{R}^k$ then the partial derivatives of f are defined and are continuous on D. For example, $f : [a,b] \to \mathbb{R}$ is smooth if and only if f has right-derivative at a and left-derivative at b, or equivalently, f is smooth on an open interval (c,d) containing $[a,b]$.

22.2. If X and Y are diffeomorphic and X is an m-dimensional manifold then so is Y.

22.3. The sphere $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$ is a smooth manifold of dimension n, covered by the hemispheres.

There is a another way to see that S^n as a manifold, by using two stereographic projections, one from the North Pole and one from the South Pole.

22.4. Show that the hyperboloid $x^2 + y^2 - z^2 = 1$ is a manifold. Is the surface $x^2 + y^2 - z^2 = 0$ a manifold?

22.5. The torus can be obtained by rotating around the z axis a circle on the xOz plane not intersecting the z axis. Show that the torus is a smooth manifold.

22.6. Consider the union of the curve $y = \sin\frac{1}{x}$, $x > 0$ and the segment $\{(0,y) | -1 \leq y \leq 1\}$ (the Topologist's sine curve, see Section 5.1). Is it a manifold?

22.7. Consider the union of the curve $y = x^3 \sin\frac{1}{x}$, $x \neq 0$ and the point $(0,0)$. Is it a smooth manifold?

22.8. Is the trace of the path $\gamma(t) = (\frac{1}{2}\sin(2t), \cos(t))$, $t \in (0, 2\pi)$ (the figure 8) a smooth manifold?

22.9. A simple closed regular path is a map $\gamma : [a,b] \to \mathbb{R}^m$ such that γ is injective on $[a,b)$, γ is smooth, $\gamma^{(k)}(a) = \gamma^{(k)}(b)$ for all integer $k \geq 0$, and $\gamma'(t) \neq 0$ for all $t \in [a,b]$. Show that the trace of a simple closed regular path is a smooth 1-dimensional manifold.

22.10. The trace of the path$((2 + \cos(1.5t))\cos t, (2 + \cos(1.5t))\sin t, \sin(1.5t))$, $0 \leq t \leq 4\pi$ is often called the trefoil knot. Draw it (using computer). Show that the trefoil knot is a smooth 1-dimensional manifold (in fact it is diffeomeorphic to the circle S^1, but this is more difficult).

22.11. Show that any open subset of a manifold is a manifold.

22.12. Show that a connected manifold is also path-connected.

22.13. Show that any diffeomorphism from S^{n-1} onto S^{n-1} can be extended to a diffeomorphism from $D^n = B'(0,1)$ onto D^n.

22.14. Show that our definition of smooth manifold coincides with the definitions in [**Spi65**, p. 109] and [**Mun91**, p. 196].

23. Tangent spaces and derivatives

Derivatives of maps on $\mathbb{R}^n$. We summarize here several results about derivatives of functions defined on open sets in $\mathbb{R}^n$. See for instance **[Spi65]** or **[Lan97]** for more details.

Let U be an open set in $\mathbb{R}^k$ and V be an open set in $\mathbb{R}^l$. Let $f : U \to V$ be smooth. We define the derivative of f at $x \in U$ to be the linear map df_x such that

$$\begin{aligned} df_x : \mathbb{R}^k &\to \mathbb{R}^l \\ h &\mapsto df_x(h) = \lim_{h\to 0} \frac{f(x+th) - f(x)}{t}. \end{aligned}$$

Thus $df_x(h)$ is the directional derivative of f at x in the direction of h.

The derivative df_x is a linear approximation of f at x.

Because we assumed that all the first order partial derivatives of f exist and are continuous, the derivative of f exists. In the canonical coordinate system of $\mathbb{R}^n$ the derivative map df_x is represented by an $l \times k$-matrix $Jf_x = [\frac{\partial f_i}{\partial x_j}(x)]_{1\le i\le l,\, 1\le j\le k}$, called the Jacobian of f at x, thus $df_x(h) = Jf_x \cdot h$.

Theorem (The chain rule). *Let U, V, W be open subsets of $\mathbb{R}^k$, $\mathbb{R}^l$, $\mathbb{R}^p$ respectively, let $f : U \to V$ and $g : V \to W$ be smooth maps, and let $y = f(x)$. Then*

$$d(g\circ f)_x = dg_y \circ df_x.$$

In other words, the following commutative diagram

$$\begin{tikzcd} & V \arrow[dr, "g"] & \\ U \arrow[ur, "f"] \arrow[rr, "g\circ f"'] & & W \end{tikzcd}$$

induces the commutative diagram

$$\begin{tikzcd} & TV_y \arrow[dr, "dg_y"] & \\ TU_x \arrow[ur, "df_x"] \arrow[rr, "d(g\circ f)_x"'] & & TW_{g(y)} \end{tikzcd}$$

Proposition. *Let U and V be open subsets of $\mathbb{R}^k$ and $\mathbb{R}^l$ respectively. If $f : U \to V$ is a diffeomorphism then the derivative df_x is a linear isomorphism, and $k = l$.*

Remark. As a corollary, $\mathbb{R}^k$ and $\mathbb{R}^l$ are not diffeomorphic if $k \neq l$.

Tangent spaces of manifolds.

Example. To motivate the definition of tangent spaces of manifolds we recall the notion of tangent spaces of surfaces. Consider a parametrized surface in $\mathbb{R}^3$ given by $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$. Consider a point $\varphi(u_0,v_0)$ on the surface. Near to (u_0,v_0) if we fix $v = v_0$ and only allow u to change then we get a parametrized path $\varphi(u,v_0)$ passing through $\varphi(u_0,v_0)$. The velocity vector of the curve $\varphi(u,v_0)$

is a "tangent vector" to the curve at the point $\varphi(u_0, v_0)$, and is given by the partial derivative with respect to u, that is, $\frac{\partial \varphi}{\partial u}(u_0, v_0)$. Similarly we have another "tangent vector" $\frac{\partial \varphi}{\partial v}(u_0, v_0)$. Then the "tangent space" of the surface at $\varphi(u_0, v_0)$ is the plane spanned the above two tangent vectors (under some further conditions for this notion to be well-defined).

We can think of a manifold as a multi-dimensional surface. Therefore our definition of tangent space of manifold is a natural generalization.

Definition. Let M be an m-dimensional manifold in $\mathbb{R}^k$. Let $x \in M$ and let $\varphi : U \to M$, where U is an open set in $\mathbb{R}^m$, be a parametrization of a neighborhood of x. Assume that $x = \varphi(u_0)$ where $u \in U$. We define the *tangent space* of M at x, denoted by TM_x, to be the vector space in $\mathbb{R}^k$ spanned by the vectors $\frac{\partial \varphi}{\partial u_i}(u_0)$, $1 \le i \le m$.

Since $\frac{\partial \varphi}{\partial u_i}(u_0) = d\varphi(u_0)(e_i)$, we can see that $TM_x = d\varphi_u(TU_u) = d\varphi_u(\mathbb{R}^m)$.

Example. Consider a surface $z = f(x, y)$. Then the tangent plane at $(x, y, f(x, y))$ consists of the linear combinations of the vectors $(1, 0, f_x(x, y))$ and $(0, 1, f_y(x, y))$.

Example. Consider the circle S^1. Let $(x(t), y(t))$ be any path on S^1. The tangent space of S^1 at (x, y) is spanned by the velocity vector $(x'(t), y'(t))$ if this vector is not 0. Since $x(t)^2 + y(t)^2 = 1$, differentiating both sides with respect to t we get $x(t)x'(t) + y(t)y'(t) = 0$, or in other words $(x'(t), y'(t))$ is perpendicular to $(x(t), y(t))$. Thus the tangent space is perpendicular to the radius.

Proposition. *The tangent space does not depend on the choice of parametrization.*

PROOF. Consider the following diagram, where U, U' are open, φ and φ' are parametrizations of open neighborhood of $x \in M$.

$$\begin{tikzcd} & M & \\ U \arrow[ur, "\varphi"] \arrow[rr, "\varphi'^{-1}\circ\varphi"'] & & U' \arrow[ul, "\varphi'"'] \end{tikzcd}$$

Notice that the map $\varphi'^{-1} \circ \varphi$ is to be understood as follows. We have that $\varphi(U) \cap \varphi'(U')$ is a neighborhood of $x \in M$. Restricting to $\varphi^{-1}(\varphi(U) \cap \varphi'(U'))$, the map $\varphi'^{-1} \circ \varphi$ is well-defined, and is a diffeomorphism. The above diagram gives us, with any $v \in \mathbb{R}^m$:

$$d\varphi_u(v) = d\varphi'_{\varphi'^{-1}\circ\varphi(u)}\big(d(\varphi'^{-1} \circ \varphi)_u(v)\big).$$

Thus any tangent vector with respect to the parametrization φ is also a tangent vector with respect to the parametrization φ'. We conclude that the tangent space does not depend on the choice of parametrization. $\square$

Proposition (Tangent space has same dimension as manifold). *If M is an m-dimensional manifold then the tangent space TM_x is an m-dimensional linear space.*

PROOF. Since a parametrization φ is a diffeomorphism, there is a smooth map F from an open set in $\mathbb{R}^k$ to $\mathbb{R}^m$ such that $F \circ \varphi = \mathrm{Id}$. So $dF_{\varphi(0)} \circ d\varphi_0 = \mathrm{Id}_{\mathbb{R}^m}$. This implies that the dimension of the image of $d\varphi_0$ is m. $\square$

Derivatives of maps on manifolds. Let $M \subset \mathbb{R}^k$ and $N \subset \mathbb{R}^l$ be manifolds of dimensions m and n respectively. Let $f : M \to N$ be smooth. Let $x \in M$. There is a neighborhood W of x in $\mathbb{R}^k$ and a smooth extension F of f to W. The derivative of f is defined to be the restriction of the derivative of F. Precisely:

Definition. The *derivative* of f at x is defined to be the linear map

$$\begin{array}{rcl} df_x : TM_x & \to & TN_{f(x)} \\ h & \mapsto & df_x(h) = dF_x(h). \end{array}$$

Observe that $df_x = dF_x|_{TM_x}$.

We need to show that the derivative is well-defined.

Proposition. *$df_x(h) \in TN_{f(x)}$ and does not depend on the choice of F.*

PROOF. We have a commutative diagram

$$\begin{tikzcd} W \arrow[r, "F"] & N \\ U \arrow[u, "\varphi"] \arrow[ur, "f\circ\varphi"] \arrow[r, "\psi^{-1}\circ f\circ\varphi"'] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Let us explain this diagram. Assume that $\varphi(u) = x$, $\psi(v) = f(x)$, $h = d\varphi_u(w)$. Take a parametrization $\psi(V)$ of a neighborhood of $f(x)$. Then $f^{-1}(\psi(V))$ is an open neighborhood of x in M. From the definition we can find an open set W in $\mathbb{R}^k$ such that $W \cap M$ is an open neighborhood of x in M parametrized by $\varphi(U)$, and f has an extension to a function F defined on W which is smooth.

The diagram induces that if $w \in \mathbb{R}^m$ then $df_x(d\varphi_u(w)) = dF_x\big(d\varphi_u(w)\big) = d\psi_v\big(d(\psi^{-1} \circ f \circ \varphi)_u(w)\big)$. From this identity we get the desired conclusion. $\square$

Thus, although as noted in the previous section a smooth map defined on a general subset of $\mathbb{R}^k$ may not have derivatives, on a manifold the derivative can be defined, in a natural manner as the restriction of the derivative of the extension map to the tangent space of the manifold.

Proposition (The chain rule). *If $f : M \to N$ and $g : N \to P$ are smooth functions between manifolds, then*

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

PROOF. There is an open neighborhood V of y in $\mathbb{R}^l$ and a smooth extension G of g to V. There is an open neighborhood U of x in $\mathbb{R}^k$ such that $U \subset F^{-1}(V)$ and there is a smooth extension F of f U. Then $d(g \circ f)_x = d(G \circ F)_x|_{TM_x} = (dG_y \circ dF_x)|_{TM_x} = dG_y|_{TN_y} \circ dF_x|_{TM_x} = dg_y \circ df_x$. $\square$

Definition. If M and N are two smooth manifolds in $\mathbb{R}^k$ and $M \subset N$ then we say that M is a *submanifold* of N.

Proposition. *If $f : M \to N$ is a diffeomorphism then $df_x : TM_x \to TN_{f(x)}$ is a linear isomorphism. In particular the dimensions of the two manifolds are same.*

PROOF. Let $m = \dim M$ and $n = \dim N$. Since $df_x \circ df^{-1}_{f(x)} = \mathrm{Id}_{TN_{f(x)}}$ and $df^{-1}_{f(x)} \circ df_x = \mathrm{Id}_{TM_x}$ we deduce, via the rank of df_x that $m \geq n$. Doing the same with $df^{-1}_{f(x)}$ we get $m \leq n$, hence $m = n$. From that df_x must be a linear isomorphism. $\square$

Problems.

23.1. Calculate the tangent spaces of S^n.

23.2. Calculate the tangent spaces of the hyperboloid $x^2 + y^2 - z^2 = a$, $a > 0$.

23.3. Show that if $\mathrm{Id} : M \to M$ is the identify map then $d(\mathrm{Id})_x$ is $\mathrm{Id} : TM_x \to TM_x$.

23.4. Show that if M is a submanifold of N then TM_x is a subspace of TN_x.

23.5. In general, a curve on a manifold M is a smooth map c from an open interval of $\mathbb{R}$ to M. The derivative of this curve is a linear map $\frac{dc}{dt}(t_0) : \mathbb{R} \to TM_{c(t_0)}$, represented by the vector $c'(t_0) \in \mathbb{R}^k$, this vector is called the velocity vector of the curve at $t = t_0$.

Show that any vector in TM_x is the velocity vector of a curve in M.

23.6. Show that if M and N are manifolds and $M \subset N$ then $TM_x \subset TN_x$.

23.7 (Cartesian products of manifolds). If $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$ are manifolds then $X \times Y \subset \mathbb{R}^{k+l}$ is also a manifold. Furthermore $T(X \times Y)_{(x,y)} = TX_x \times TY_y$.

23.8. (a) Calculate the derivative of the map $f : (0, 2\pi) \to S^1$, $f(t) = (\cos t, \sin t)$.

(b) Calculate the derivative of the map $f : S^1 \to \mathbb{R}$, $f(x, y) = e^y$.

24. Regular values

Let $f : M \to N$ be smooth. A point in M is called a *regular point* (điểm thường, điểm chính qui) of f if the derivative of f at that point is surjective. Otherwise the point is called a *critical point* (điểm dừng, điểm tới hạn) of f.

A point in N is called a *critical value* of f if it is the value of f at a critical point. Otherwise the point is called a *regular value* of f.

Thus y is a critical value of f if and only if $f^{-1}(y)$ contains a critical point. In particular, if $f^{-1}(y) = \emptyset$ then y is considered a regular value.

Example. If $f : M \to N$ where $\dim(M) < \dim(N)$ then every $x \in M$ is a critical point and every $y \in N$ is a critical value of f.

Example. Let U be an open set in $\mathbb{R}^n$ and let $f : U \to \mathbb{R}$ be smooth. Then $x \in U$ is a critical point of f if and only if $\nabla f(x) = 0$.

The Inverse function theorem and the Implicit function theorem. First we state the Inverse function theorem in Multivariables Calculus.

Theorem 24.1 (Inverse function theorem). *Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be smooth. If df_x is bijective then f is locally a diffeomorphism.*

More concisely, if $\det(Jf_x) \neq 0$ then there is an open neighborhood U of x and an open neighborhood V of $f(x)$ such that $f|_U \colon U \to V$ is a diffeomorphism.

Remark 24.2. For a proof, see for instance **[Spi65]**. Usually the result is stated for continuously differentiable function (i.e. C^1), but the result for smooth functions follows, since the Jacobian matrix of the inverse map is the inverse matrix of the Jacobian of the original map, and the entries of an inverse matrix can be obtained from the entries of the original matrix via smooth operations, namely $A^{-1} = \frac{1}{\det A} A^*$, where $A^*_{i,j} = (-1)^{i+j} \det(A^{j,i})$, and $A^{j,i}$ is obtained from A by omitting the ith row and jth column.

Theorem 24.3 (Implicit function theorem). *Suppose that $f : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is smooth and $f(x) = y$. If df_x is onto then locally at x the level set $f^{-1}(y)$ is a graph of dimension m.*

More concisely, suppose that $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth and the matrix $[D^{m+j} f_i(x_0, y_0)]$, $1 \leq i, j \leq n$ is non-singular, then there is a neighborhood $U \times V$ of (x_0, y_0) such that for each $x \in U$ there is a unique $g(x) \in V$ satisfying $f(x, g(x)) = 0$. The function g is smooth.

The Implicit function theorem is obtained by setting $F(x, y) = (x, f(x, y))$ and applying the Inverse function theorem to F.

Theorem 24.4 (Inverse function theorem for manifolds). *Let M and N be two manifolds of the same dimensions, and let $f : M \to N$ be smooth. If x is a regular point of f then there is a neighborhood in M of x on which f is a diffeomorphism onto its image.*

PROOF. Consider

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ U \arrow[u, "\varphi"] \arrow[r, "\psi^{-1}\circ f\circ\varphi"] & V \arrow[u, "\psi"'] \end{tikzcd}$$

Since df_x is surjective, it is bijective. Then $d(\psi^{-1}\circ f\circ\varphi)_u = d\psi^{-1}_{f(x)}\circ df_x\circ d\varphi_u$ is an isomorphism. The Inverse Function Theorem can be applied to $\psi^{-1}\circ f\circ\varphi$, giving that it is a local diffeomorphism at u, so f is a local diffeomorphism at x. $\square$

Preimage of a regular value.

Proposition 24.5. *If* $\dim(M) = \dim(N)$ *and y is a regular value of f then $f^{-1}(y)$ is a discrete set. In other words, $f^{-1}(y)$ is a zero dimensional manifold. Furthermore if M is compact then $f^{-1}(y)$ is a finite set.*

PROOF. If $x \in f^{-1}(y)$ then there is a neighborhood of x on which f is a bijection. That neighborhood contains no other point in $f^{-1}(y)$. Thus $f^{-1}(y)$ is a discrete set.

If M is compact then the set $f^{-1}(y)$ is compact. If it the set is infinite then it has a limit point x_0. Because of the continuity of f, we have $f(x_0) = y$. That contradicts the fact that $f^{-1}(y)$ is discrete. $\square$

The following theorem is the Implicit function theorem for manifolds.

Theorem 24.6 (Preimage of a regular value is a manifold). *If y is a regular value of $f : M \to N$ then $f^{-1}(y)$ is a manifold of dimension* $\dim(M) - \dim(N)$.

PROOF. Let $m = \dim(M)$ and $n = \dim(N)$. The case $m = n$ is already considered in 24.5. Now we assume $m > n$. Let $x_0 \in f^{-1}(y_0)$. Consider the diagram

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ O \arrow[u, "\varphi"] \arrow[r, "g"'] & W \arrow[u, "\psi"'] \end{tikzcd}$$

where $g = \psi^{-1}\circ f\circ\varphi$ and $\psi(w_0) = y_0$.

Since df_{x_0} is onto, $dg_{\varphi^{-1}(x_0)}$ is also onto. If needed we can change g, O and φ by permuting variables appropriately such that the matrix $[D^j g_i(\varphi^{-1}(x_0))]$, $1 \le i \le n$, $m-n+1 \le j \le m$ is non-singular. Denote $\varphi^{-1}(x_0)) = (u_0, v_0) \in \mathbb{R}^{m-n}\times\mathbb{R}^n$. By the Implicit Function Theorem applied to g there is an open neighborhood U of u_0 in $\mathbb{R}^{m-n}$ and an open neighborhood V of v_0 in $\mathbb{R}^n$ such that $U\times V$ is contained in O and on $U\times V$ we have $g(u,v) = w_0$ if and only if $v = h(u)$ for a certain smooth function $h : U \to V$. In other words, on $U\times V$ the equation $g(u,v) = w_0$ determines a graph $(u, h(u))$.

Now we have $\varphi(U \times V) \cap f^{-1}(y_0) = \{\varphi(u, h(u)) \mid u \in U\}$. Let $\tilde{\varphi}(u) = \varphi(u, h(u))$ then $\tilde{\varphi}$ is a diffeomorphism from U onto $\varphi(U \times V) \cap f^{-1}(y_0)$, a neighborhood of x_0 in $f^{-1}(y_0)$. $\square$

Example. To be able to follow the proof more easily the reader can try to work it out for an example, such as the case where M is the graph of the function $z = x^2 + y^2$, and f is the height function $f((x, y, z)) = z$ defined on M.

Example. The n-sphere S^n is a subset of $\mathbb{R}^{n+1}$ determined by the implicit equation $\sum_{i=1}^{n+1} x_i^2 = 1$. Since 1 is a regular value of the function $f(x_1, x_2, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} x_i^2$ we conclude that S^n is a manifold of dimension n.

Lie groups. The set $M_n(\mathbb{R})$ of $n \times n$ matrices over $\mathbb{R}$ can be identified with the Euclidean manifold $\mathbb{R}^{n^2}$.

Consider the map $\det : M_n(\mathbb{R}) \to \mathbb{R}$. Let $A = [a_{i,j}] \in M_n(\mathbb{R})$. Since $\det(A) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)} = \sum_j (-1)^{i+j} a_{i,j} \det(A^{i,j})$, we can see that det is a smooth function.

Let us find the critical points of det. A critical point is a matrix $A = [a_{i,j}]$ at which $\frac{\partial \det}{\partial a_{i,j}}(A) = (-1)^{i+j} \det(A^{i,j}) = 0$ for all i, j. In particular, $\det(A) = 0$. So 0 is the only critical value of det.

Therefore $\mathrm{SL}_n(\mathbb{R}) = \det^{-1}(1)$ is a manifold of dimension $n^2 - 1$.

Furthermore we note that the group multiplication in $\mathrm{SL}_n(\mathbb{R})$ is a smooth map from $\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$ to $\mathrm{SL}_n(\mathbb{R})$. The inverse operation is a smooth map from $\mathrm{SL}_n(\mathbb{R})$ to itself. We then say that $\mathrm{SL}_n(\mathbb{R})$ is a Lie group.

Definition. A *Lie group* is a smooth manifold which is also a group, for which the group operations are compatible with the smooth structure, namely the group multiplication and inversion are smooth.

Let $O(n)$ be the group of orthogonal $n \times n$ matrices, the group of linear transformation of $\mathbb{R}^n$ that preserves distances.

Proposition. *The orthogonal group $O(n)$ is a Lie group.*

PROOF. Let $S(n)$ be the set of symmetric $n \times n$ matrices. This is clearly a manifold of dimension $\frac{n^2+n}{2}$.

Consider the smooth map $f : M(n) \to S(n)$, $f(A) = AA^t$. We have $O(n) = f^{-1}(I)$. We will show that I is a regular value of f.

We compute the derivative of f at $A \in f^{-1}(I)$:

$$df_A(B) = \lim_{t \to 0} \frac{f(A + tB) - f(A)}{t} = BA^t + AB^t.$$

We note that the tangents spaces of $M(n)$ and $S(n)$ are themselves. To check whether df_A is onto for $A \in O(n)$, we need to check that given $C \in S(n)$ there is a $B \in M(n)$ such that $C = BA^t + AB^t$. We can write $C = \frac{1}{2}C + \frac{1}{2}C$, and the equation $\frac{1}{2}C = BA^t$ will give a solution $B = \frac{1}{2}CA$, which is indeed a solution to the original equation. $\square$

Problems.

24.7. Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - y^2$. Show that if $a \neq 0$ then $f^{-1}(a)$ is a 1-dimensional manifold, but $f^{-1}(0)$ is not. Show that if a and b are both positive or both negative then $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic.

24.8. Let $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y) = x^2 + y^2 - z^2$. Show that if $a \neq 0$ then $f^{-1}(a)$ is a 2-dimensional manifold, but $f^{-1}(0)$ is not. Show that if a and b are both positive or both negative then $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic.

24.9. Show that the equation $x^5 + y^4 + z^3 = 1$ determine a manifold in $\mathbb{R}^3$.

24.10. Is the intersection of the two surfaces $z = x^2 + y^2$ and $z = 1 - x^2 - y$ a manifold?

24.11. Show that the height function $(x, y, z) \mapsto z$ on the sphere S^2 has exactly two critical points.

24.12. Show that if f achieves local extremum at x then x is a critical point of f.

24.13. Show that a smooth function on a compact manifold must have at least two critical points.

24.14. Let $\dim(M) = \dim(N)$, M be compact and S be the set of all regular values of $f : M \to N$. For $y \in S$, let $|f^{-1}(y)|$ be the number of elements of $f^{-1}(y)$. Show that the map

$$\begin{array}{ccc} S & \to & \mathbb{N} \\ y & \mapsto & |f^{-1}(y)|. \end{array}$$

is locally constant. In other words, each regular value has a neighborhood where the number of preimages of regular values is constant.

24.15. Let M be a compact manifold and let $f : M \to \mathbb{R}$ be smooth. Show that the set of regular values of f is open.

24.16. Use regular value to show that the torus T^2 is a manifold.

24.17. Find the regular values of the function $f(x, y, z) = [4x^2(1 - x^2) - y^2]^2 + z^2 - \frac{1}{4}$ (and draw a corresponding level set).

24.18. Find a counter-example to show that 24.5 is not correct if regular value is replaced by critical value.

24.19. $\checkmark$ If $f : M \to N$ is smooth, y is a regular of f, and $x \in f^{-1}(y)$, then $\ker df_x = Tf^{-1}(y)_x$.

24.20. Show that S^1 is a Lie group.

24.21. Show that the set of all invertible $n \times n$-matrices $\mathrm{GL}(n; \mathbb{R})$ is a Lie group and find its dimension.

24.22. In this problem we find the tangent spaces of $\mathrm{SL}_n(\mathbb{R})$.

(a) Check that the derivative of the determinant map $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is represented by a gradient vector whose (i, j)-entry is $(-1)^{i+j} \det(A^{i,j})$.
(b) Determine the tangent space of $\mathrm{SL}_n(\mathbb{R})$ at $A \in \mathrm{SL}_n(\mathbb{R})$.
(c) Show that the tangent space of $\mathrm{SL}_n(\mathbb{R})$ at the identity matrix is the set of all $n \times n$ matrices with zero traces.

25. Critical points and the Morse lemma

Partial derivatives. Let $f : M \to \mathbb{R}$. Let U be an open neighborhood in M parametrized by φ. For each $x = \varphi(u)$ we define the *first partial derivatives*:

$$\left(\frac{\partial}{\partial x_i} f\right)(x) = \frac{\partial}{\partial u_i}(f \circ \varphi)(u).$$

In other words, $\left(\frac{\partial}{\partial x_i} f\right)(\varphi(u)) = \frac{\partial}{\partial u_i}(f \circ \varphi)(u)$. Of course this definition depends on local coordinates.

If f is defined on $\mathbb{R}^m$ then this is the usual partial derivative.

To understand $\left(\frac{\partial}{\partial x_i} f\right)(x)$ better, we can think that the parametrization φ brings the coordinate system of $\mathbb{R}^m$ to the neighborhood U, then $\left(\frac{\partial}{\partial x_i} f\right)(x)$ is the rate of change of $f(x)$ when the variable x changes along the path in U which is the composition of the standard path te_i along the ith axis of $\mathbb{R}^m$ with φ.

We can write

$$\begin{aligned}\left(\frac{\partial}{\partial x_i} f\right)(x) &= \frac{\partial}{\partial u_i}(f \circ \varphi)(u) = d(f \circ \varphi)(u)(e_i) \\ &= (df(x) \circ d\varphi(u))(e_i) = df(x)(d\varphi(u)(e_i)).\end{aligned}$$

Thus $\left(\frac{\partial}{\partial x_i} f\right)(x)$ is the value of the derivative map $df(x)$ at the image of the unit vector e_i of $\mathbb{R}^m$.

Gradient vector. The tangent space TM_x inherits the Euclidean inner product from the ambient space $\mathbb{R}^k$. In this inner product space the linear map $df_x : TM_x \to \mathbb{R}$ is represented by a vector in TM_x which we called the *gradient vector* $\nabla f(x)$. This vector is determined by the property $\langle \nabla f(x), v\rangle = df_x(v)$ for any $v \in TM_x$. Notice that the gradient vector $\nabla f(x)$ is defined on the manifold, *not depending on local coordinates*.

In a local parametrization the vectors $d\varphi(u)(e_i) = \frac{\partial \varphi}{\partial u_i}(u)$, $1 \le i \le m$ constitutes a vector basis for TM_x. In this basis the coordinates of $\nabla f(x)$ are

$$\langle \nabla f(x), d\varphi(u)(e_i)\rangle = df_x(d\varphi(u)(e_i)) = \frac{\partial f}{\partial x_i}(x).$$

In other words, in that basis we have the familiar formula $\nabla f = \langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_m}\rangle$. This formula depends on local coordinates. It implies that $\nabla f : M \to \mathbb{R}^m$ is a smooth function.

We have several simple observations:

Proposition. *A point is a critical point if and only if the gradient vector at that point is zero.*

Proposition. *At a local extremum point the gradient vector must be zero.*

Second derivatives. Since $\frac{\partial}{\partial x_i} f$ is a smooth function on U, we can take its partial derivatives. Thus we define the *second partial derivatives*:

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j} f\right)(x).$$

In other words,

$$\begin{aligned}
\frac{\partial^2 f}{\partial x_i \partial x_j}(\varphi(u)) &= \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j} f\right)(\varphi(u)) = \frac{\partial}{\partial u_i}\left(\frac{\partial f}{\partial x_j} \circ \varphi\right)(u) \\
&= \frac{\partial}{\partial u_i}\left(\frac{\partial}{\partial u_j}(f \circ \varphi)\right)(u) = \frac{\partial^2}{\partial u_i \partial u_j}(f \circ \varphi)(u).
\end{aligned}$$

Non-degenerate critical points. Consider the Hessian matrix of second partial derivatives:

$$Hf(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right)_{1 \le i,j \le m}.$$

If this matrix is non-degenerate, then we say that x is a *non-degenerate critical point* of f.

Lemma 25.1. *The non-degeneracy of a critical point does not depend on choices of local coordinates.*

PROOF. We can see that the problem is reduced to the case of functions on $\mathbb{R}^m$. If $f : \mathbb{R}^m \to \mathbb{R}$ and φ is a change of variables (i.e. a diffeomorphism) of $\mathbb{R}^m$ then we have

$$\frac{\partial}{\partial u_i}(f \circ \varphi)(u) = \sum_k \frac{\partial f}{\partial x_k}(x) \cdot \frac{\partial \varphi_k}{\partial u_i}(u).$$

Then

$$\begin{aligned}
\frac{\partial^2}{\partial u_j \partial u_i}(f \circ \varphi)(u) &= \sum_k \left[\left(\sum_l \frac{\partial^2 f}{\partial x_l \partial x_k}(x) \cdot \frac{\partial \varphi_l}{\partial u_j}(u)\right) \cdot \frac{\partial \varphi_k}{\partial u_i}(u) + \frac{\partial f}{\partial x_k}(x) \cdot \frac{\partial^2 \varphi}{\partial u_j \partial u_i}(u)\right] \\
&= \sum_{k,l} \frac{\partial^2 f}{\partial x_l \partial x_k}(x) \cdot \frac{\partial \varphi_l}{\partial u_j}(u) \cdot \frac{\partial \varphi_k}{\partial u_i}(u).
\end{aligned}$$

In other words: $H(f \circ \varphi)(u) = J\varphi(u)^t [Hf(\varphi(u))] J\varphi(u)$. This formula immediately gives us the conclusion. □

Morse lemma.

Theorem (Morse's lemma). *Suppose that $f : M \to \mathbb{R}$ is smooth and p is a non-degenerate critical point of f. There is a local coordinate φ in a neighborhood of p such that $\varphi(p) = 0$ and in that neighborhood*

$$f(x) = f(p) - \varphi(x)_1^2 - \varphi(x)_2^2 - \cdots - \varphi(x)_k^2 + \varphi(x)_{k+1}^2 + \varphi(x)_{k+2}^2 + \cdots + \varphi(x)_m^2.$$

In other words, in a neighborhood of 0,

$$(f\circ\varphi^{-1})(u) = (f\circ\varphi^{-1})(0) - u_1^2 - u_2^2 - \cdots - u_k^2 + u_{k+1}^2 + u_{k+2}^2 + \cdots + u_m^2.$$

If we abuse notations by using local coordinates and write x_i for $u_i = \varphi(x)_i$ then

$$f(x) = f(p) - x_1^2 - x_2^2 - \cdots - x_k^2 + x_{k+1}^2 + x_{k+2}^2 + \cdots + x_m^2.$$

The number k does not depend on the choice of such local coordinates and is called the index of the non-degenerate critical point p.

Example. Non-degenerate critical points of index 0 are local minima, and the ones with maximum indexes are local maxima.

PROOF. Since we only need to prove the formula for $f\circ\varphi^{-1}$, we only need to work in $\mathbb{R}^m$.

First, we write

$$\begin{aligned} f(x) &= f(0) + \int_0^1 \frac{d}{dt} f(tx)\,dt \\ &= f(0) + \sum_{i=1}^m \int_0^1 \left(\frac{\partial f}{\partial x_i}(tx)\right) x_i\,dt \\ &= f(0) + \sum_{i=1}^m x_i \int_0^1 \left(\frac{\partial f}{\partial x_i}(tx)\right) dt. \end{aligned}$$

A result of Analysis (see for example [**Lan97**, p. 276]) tells us that the functions $g_i(x) = \int_0^1 \left(\frac{\partial}{\partial x_i} f(tx)\right) dt$ are smooths. Notice that $g_i(0) = \frac{\partial f}{\partial x_i}(0) = 0$. Furthermore

$$\frac{\partial g_i}{\partial x_j}(x) = \int_0^1 \frac{\partial^2 f}{\partial x_j \partial x_i}(tx) t\,dt,$$

therefore $\frac{\partial g_i}{\partial x_j}(0) = \frac{1}{2}\frac{\partial^2 f}{\partial x_j \partial x_i}(0)$.

Apply this construction once again to g_i we obtain smooth functions $g_{i,j}$ such that $g_{i,j}(0) = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(0)$ and

$$f(x) = f(0) + \sum_{i,j=1}^m x_i x_j g_{i,j}(x).$$

Set $h_{i,j} = (g_{i,j} + g_{j,i})/2$ then $h_{i,j} = h_{j,i}$, $h_{i,j}(0) = \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}(0)$, and

$$f(x) = f(0) + \sum_{i,j=1}^m x_i x_j h_{i,j}(x).$$

The rest of the proof is a simple completing the square. Since the matrix $\big(h_{i,j}(0)\big)$ is non-degenerate by a permutation of variables if necessary, we can assume that $h_{1,1}(0) \neq 0$. Then there is a neighborhood of 0 such that $h_{1,1}(x)$ does not change its sign. In that neighborhood, if $h_{1,1}(0) > 0$ then

$$
\begin{aligned}
f(x) &= f(0)+ \quad h_{1,1}(x)x_1^2 + \sum_{1<j}(h_{1,j}(x)+h_{j,1}(x))x_1x_j + \sum_{1<i,j} h_{i,j}(x)x_ix_j \\
&= f(0)+ \quad h_{1,1}(x)x_1^2 + 2\sum_{1<j} h_{1,j}(x)x_1x_j + \sum_{1<i,j} h_{i,j}(x)x_ix_j \\
&= f(0)+ \quad \left(\sqrt{h_{1,1}(x)}x_1\right)^2 + 2\sqrt{h_{1,1}(x)}x_1 \frac{\sum_{1<j} h_{1,j}(x)x_j}{\sqrt{h_{1,1}(x)}} + \sum_{1<i,j} h_{i,j}(x)x_ix_j \\
&= f(0)+ \quad \left[\sqrt{h_{1,1}(x)}x_1 + \sum_{1<j}\frac{h_{1,j}(x)}{\sqrt{h_{1,1}(x)}}x_j\right]^2 \\
&\qquad - \left(\sum_{1<j}\frac{h_{1,j}(x)}{\sqrt{h_{1,1}(x)}}x_j\right)^2 + \sum_{1<i,j} h_{i,j}(x)x_ix_j.
\end{aligned}
$$

Similarly, if $h_{1,1}(0) < 0$ then

$$
\begin{aligned}
f(x) &= f(0)- \quad \left[\sqrt{-h_{1,1}(x)}x_1 - \sum_{1<j}\frac{h_{1,j}(x)}{\sqrt{-h_{1,1}(x)}}x_j\right]^2 \\
&\qquad + \left(\sum_{1<j}\frac{h_{1,j}(x)}{\sqrt{-h_{1,1}(x)}}x_j\right)^2 + \sum_{1<i,j} h_{i,j}(x)x_ix_j.
\end{aligned}
$$

Combining both cases, we define the new variables:

$$
\begin{aligned}
v_1 &= \sqrt{|h_{1,1}(x)|}x_1 + \mathrm{sign}(h_{1,1}(0))\sum_{1<j}\frac{h_{1,j}(x)}{\sqrt{|h_{1,1}(x)|}}x_j, \\
v_i &= x_i,\ i>1.
\end{aligned}
$$

Since

$$\frac{\partial v_1}{\partial x_1}(0) = \sqrt{|h_{1,1}(0)|} \neq 0$$

the Jacobian matrix $\left(\frac{\partial v_i}{\partial x_j}(0)\right)$ is non-singular. By the Inverse function theorem, there is a neighborhood of 0 where the correspondence $x \mapsto v$ is a diffeomorphism, that is, a change of variables. With the new variables we have

$$f(v) = f(0) + \mathrm{sign}(h_{1,1}(0))v_1^2 + \sum_{1<i,j} h'_{i,j}(v)v_iv_j.$$

By a direct calculation, we can check that in these variables

$$Hf(0) = \begin{pmatrix} \mathrm{sign}(h_{1,1}(0)) & 0 \\ 0 & (2h'_{i,j}(0))_{1<i,j\leq m} \end{pmatrix}.$$

Using 25.1 we conclude that the matrix $(h'_{i,j}(0))_{1<i,j\leq m}$ must be non-singular. Thus the induction process can be carried out. Finally we can permute the variables such that in the final form of f the negative signs are in front.

□

Problems.

25.2. Show that the gradient vector is always normal to level surfaces.

25.3. Give a generalization of the method of Lagrange multipliers to manifolds.

25.4. For the specific case of $f(x) = \sum_{1 \le i,j \le m} a_{i,j} x_i x_j$ where $a_{i,j}$ are real numbers, to prove the Morse's lemma we can use a diagonalization of a quadratic form or a symmetric matrix, considered in Linear Algebra. The change of variables corresponds to using a new vector basis consisting of eigenvectors of the matrix.

25.5. Recover the classification of critical points from Calculus.

26. Flows

Vector fields.

Definition. A smooth tangent *vector field* on a manifold $M \subset \mathbb{R}^k$ is a smooth map $V : M \to \mathbb{R}^k$ such that $V(x) \in TM_x$ for each $x \in M$.

Example. If $f : M \to \mathbb{R}$ is smooth then the gradient ∇f is a smooth vector field on M.

An *integral curve* at a point $x \in M$ with respect to the vector field V is a smooth path $\gamma : (a,b) \to M$ such that $0 \in (a,b)$, $\gamma(0) = x$, and $\gamma'(t) = V(\gamma(t))$ for all $t \in (a,b)$. It is a path going through x and at every moment taking the vectors of the given vector field as velocity vectors. In picture, an integral curve is tangent to the vector field. Integral curves are also called *solution curves*, *trajectories*, or *flow lines*.

In a local coordinate around x, a vector field on that neighborhood corresponds to a vector field on $\mathbb{R}^m$, and an integral curve in that neighborhood corresponds to an integral curve on $\mathbb{R}^m$. Thus, by using local coordinate, we can consider a local integral curve as a solution to the differential equation $\gamma'(t) = V(\gamma(t))$ in $\mathbb{R}^m$ subjected to the initial condition $\gamma(0) = x$.

Flows. For each $x \in M$, let $\phi(t,x)$, or $\phi_t(x)$, be an integral curve at x, with t belongs to an interval $J(x)$. We have a map

$$\begin{aligned} \phi : D = \{(t,x) \mid x \in M,\ t \in J(x)\} \subset \mathbb{R} \times M &\to M \\ (t,x) &\mapsto \phi_t(x), \end{aligned}$$

with the properties $\phi_0(x) = x$, and $\frac{d}{dt}(\phi)(t,x) = V(\phi(t,x))$. This map ϕ is called a *flow* (dòng) generated by the vector field V.

Theorem. *For each smooth vector field there exists a unique smooth flow, in the sense that any two integral curves at the same point must agree on the intersection of their domains. The domain of this flow can be taken to be an open set.*

This theorem is just an interpretation of the theorem in Differential Equations on the existence, uniqueness, and dependence on initial conditions of solutions to differential equations, see for example **[HS74]**, **[Lan97]**.

Theorem (Group law). *Any flow satisfies*

$$\phi_{t+s}(x) = \phi_t(\phi_s(x)).$$

PROOF. Define $\gamma(t) = \phi_{t+s}(x)$. Then $\gamma(0) = \phi_s(x)$, and $\gamma'(t) = \frac{d}{dt}(\phi)(t+s,x) = V(\phi(t+s,x)) = V(\gamma(t))$. Thus $\gamma(t)$ is an integral curve at $\phi_s(x)$. But $\phi_t(\phi_s(x))$ is another integral curve at $\phi_s(x)$. By uniqueness of integral curves, $\gamma(t)$ must agree with $\phi_t(\phi_s(x))$ on their common domains. $\square$

When every integral curve can be extended without bound in both directions, in other words, for all x the map $\phi_t(x)$ is defined for all $t \in \mathbb{R}$, we say that the flow is *complete*.

Theorem. *On a compact manifold any flow is complete.*

PROOF. Although generally each integral curve has its own domain, first we will show that for compact manifolds all integral curves can have same domains. Since the domain D of the flow can be taken to be an open subset of $\mathbb{R} \times M$, each $x \in M$ has an open neighborhood U_x and a corresponding interval $(-\epsilon_x, \epsilon_x)$ such that $(-\epsilon_x, \epsilon_x) \times U_x$ is contained in D. The collection $\{U_x \mid x \in M\}$ is an open cover of M therefore there is a finite subcover. That implies there is a positive real number ϵ such that for every $x \in M$ the integral curve $\phi_t(x)$ is defined on $(-\epsilon, \epsilon)$.

Now $\phi_t(x)$ can be extended by intervals of length $\epsilon/2$ to be defined on $\mathbb{R}$. For example, if $t > 0$ then there is $n \in \mathbb{N}$ such that $n\frac{\epsilon}{2} \le t < (n+1)\frac{\epsilon}{2}$, then define

$$\phi_t(x) = \phi_{t-n\frac{\epsilon}{2}}\left(\phi_{n\frac{\epsilon}{2}}(x)\right),$$

where $\phi_{n\frac{\epsilon}{2}}(x) = \phi_{\frac{\epsilon}{2}}\left(\phi_{(n-1)\frac{\epsilon}{2}}(x)\right)$ if $n \ge 1$. ☐

Theorem. *If the map $\phi_t : M \to M$ is defined then it is a diffeomorphism.*

For example, if the flow is complete then ϕ_t is defined for all $t \in \mathbb{R}$, we can think of ϕ_t as moving every point along integral curves for an amount of time t.

PROOF. Since the flow ϕ is smooth the map ϕ_t is smooth. Its inverse map ϕ_{-t} is also smooth. ☐

Theorem. *Let M be a compact smooth manifold and $f : M \to \mathbb{R}$ be smooth. If the interval $[a, b]$ only contains regular values of f then the level sets $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic.*

PROOF. The idea of the proof is to construct a diffeomorphism from $f^{-1}(a)$ to $f^{-1}(b)$ by pushing along the flow lines of the gradient vector field of f. However since $\nabla f(x)$ can be zero outside of $f^{-1}([a, b])$ we need a modification to ∇f.

Suppose that $a < b$. By 24.15 there are intervals $[a, b] \subset (c, d) \subset [c, d] \subset (h, k)$ such that (h, k) contains only regular values of f. Thus on $f^{-1}((h, k))$ the vector $\nabla f(x)$ never vanish.

By 26.4 there is a smooth function ψ that is 1 on $f^{-1}([c, d])$ and is 0 outside $f^{-1}((h, k))$. Let $F = \psi \frac{\nabla f}{||\nabla f||^2}$, then F is a well-defined smooth vector field on M. Notice that F is basically a rescale of ∇f.

Let ϕ be the flow generated by F. We have:

$$\frac{d}{dt} f(\phi_t(x)) = df_{\phi_t(x)}\left(\frac{d}{dt}\phi_t(x)\right) = \langle \nabla f(\phi_t(x)), F(\phi_t(x))\rangle = \psi(\phi_t(x)).$$

Fix $x \in f^{-1}(a)$. Since $\phi_t(x)$ is continuous with respect to t and $\phi_0(x) = x$, there is an $\epsilon > 0$ such that $\phi_t(x) \in f^{-1}((c, d))$ for $t \in [0, \epsilon)$. Let ϵ_0 be the supremum (or

∞) of the set of such ϵ. Then for $t \in [0, \epsilon_0)$ we have $\phi_t(x) \in f^{-1}((c,d))$, and so

$$\frac{d}{dt} f(\phi_t(x)) = \psi(\phi_t(x)) = 1.$$

This means the flow line is going at constant speed 1. We get $f(\phi_t(x)) = t + a$ for $t \in [0, \epsilon_0)$. If $\epsilon_0 \leq b - a$ then by continuity $f(\phi_{\epsilon_0}(x)) = \epsilon_0 + a \leq b < d$. This implies there is $\epsilon' > \epsilon_0$ such that $f(\phi_t(x)) < d$ for $t \in [\epsilon_0, \epsilon')$, a contradiction. Thus $\epsilon_0 > b - a$. We now observe that $f(\phi_{b-a}(x)) = b$. Thus ϕ_{b-a} maps $f^{-1}(a)$ to $f^{-1}(b)$, so it is the desired diffeomorphism. ☐

Theorem 26.1 (Homogeneity of manifolds). *On a connected manifold there is a self diffeomorphism that brings any given point to any given point.*

PROOF. First we can locally bring any point to a given point without outside disturbance. That translates to a problem on $\mathbb{R}^n$: we will show that for any $c \in B(0,1)$ there is a diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ such that $h|_{\mathbb{R}^n \setminus B(0,1)} = 0$ and $h(0) = c$.

By 26.2 there is a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f|_{B'(0,||c||)} = 1$ and $f|_{\mathbb{R}^n \setminus B(0,1)} = 0$. Consider the vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, $F(x) = f(x)c$. This is a smooth vector field with compact support. The flow generated by this vector field is a smooth map $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{aligned} \phi_0(x) &= x, \\ \frac{d}{dt}\phi_t(x) &= F(\phi_t(x)). \end{aligned}$$

☐

Problems.

26.2. √ The following is a common smooth function:

$$f(x) = \begin{cases} e^{-1/x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0. \end{cases}$$

(a) Show that $f(x)$ is smooth.

(b) Let $a < b$ and let $g(x) = f(x-a)f(b-x)$. Then g is smooth, $g(x)$ is positive on (a,b) and is zero everywhere else.

(c) Let
$$h(x) = \frac{\int_{-\infty}^{x} g(x)\,dx}{\int_{-\infty}^{\infty} g(x)\,dx}.$$
Then $h(x)$ is smooth, $h(x) = 0$ if $x \leq a$, $0 < h(x) < 1$ if $a < x < b$, and $h(x) = 1$ if $x \geq b$.

(d) The function
$$k(x) = \frac{f(x-a)}{f(x-a) + f(b-x)}$$
also has the above properties of $h(x)$.

(e) In $\mathbb{R}^n$, construct a smooth function whose value is 0 outside of the ball of radius b, 1 inside the ball of radius a, where $0 < a < b$, and between 0 and 1 in between the two balls.

26.3 (Smooth Urysohn lemma). Let $A \subset U \subset \mathbb{R}^n$ where A is compact and U is open. We will show that there exists a smooth function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $0 \leq \varphi(x) \leq 1$, $\varphi|_A = 1$, $\varphi|_{\mathbb{R}^n \setminus U} = 0$.

26.4 (Smooth Urysohn lemma for manifolds). Let M be a smooth manifold, $A \subset U \subset M$ where A is compact and U is open in M. Show that there is a smooth function $\varphi : M \to \mathbb{R}$ such that $0 \leq \varphi(x) \leq 1$, $\varphi|_A = 1$, $\varphi|_{M \setminus U} = 0$.

27. Manifolds with boundaries

The closed half-space $\mathbb{H}^m = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid x_m \geq 0\} \subset \mathbb{R}^m$ whose topological boundary is $\partial\mathbb{H}^m = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid x_m = 0\}$ is our model for a manifold with boundary.

Definition. A subspace M of $\mathbb{R}^k$ is called a *manifold with boundary* of dimension m if each point in M has a neighborhood diffeomorphic to either $\mathbb{R}^m$ or $\mathbb{H}^m$, where in the second case the point is sent to $\partial\mathbb{H}^m$. The set of all points of the first type is called the *interior* of M. The set of all points of the second type is called the *boundary* of M, denoted by ∂M.

A point belongs to either the interior or the boundary, not both, because of the following:

Lemma. *$\mathbb{H}^m$ is not diffeomorphic to $\mathbb{R}^m$.*

PROOF. Suppose $f : \mathbb{R}^m \to \mathbb{H}^m$ is a diffeomorphism. For any $x \in \mathbb{R}^m$, df_x is non-singular, therefore by the Inverse function theorem f is a diffeomorphism from an open ball containing x onto an open ball containing $f(x)$. Thus $f(x)$ must be an interior point (in topological sense) of $\mathbb{H}^m$. This implies that f cannot be onto $\mathbb{H}^m$, a contradiction.

Alternatively we can use Invariance of dimension 5.34. $\square$

Remark. The boundary of a manifold is generally not the same as its topological boundary.

Remark. On convention, when we talk about a manifold we still mean a manifold as earlier defined, that is, with no boundary. A manifold with boundary can have empty boundary, in which case it is a manifold.

Proposition. *The interior of an m-manifold with boundary is an m-manifold without boundary. The boundary of an m-manifold with boundary is an $(m-1)$-manifold without boundary.*

PROOF. The part about the interior is clear. Let us consider the part about the boundary.

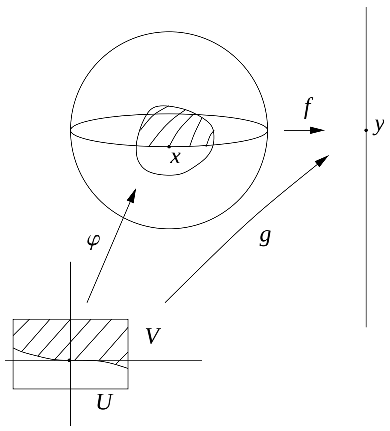
Let M be an m-manifold and let $x \in \partial M$. Let φ be a diffeomorphism from a neighborhood U of x in M to $\mathbb{H}^m$. We can check that if $y \in U$ then $\varphi(y) \in \partial\mathbb{H}^m$ if and only if $y \in \partial M$. Thus the restriction $\varphi|_{U \cap \partial M}$ is a diffeomorphism from a neighborhood of x in ∂M to $\partial\mathbb{H}^m$, which is diffeomorphic to $\mathbb{R}^{m-1}$. $\square$

The *tangent space of a manifold with boundary* M is defined as follows. It x is an interior point of M then TM_x is defined as before. If x is a boundary point then there is a parametrization $\varphi : \mathbb{H}^m \to M$, where $\varphi(0) = x$. Notice that by continuity φ has well-defined partial derivatives at 0. This implies that the derivative $d\varphi_0 : \mathbb{R}^m \to \mathbb{R}^k$ is well-defined. Then TM_x is still defined as $d\varphi_0(\mathbb{R}^m)$. The Chain rule still holds. The notion of critical point is defined exactly as for manifolds.

Theorem 27.1. *Let M be an m-dimensional manifold without boundary. Let $f : M \to \mathbb{R}$ be smooth and let y be a regular value of f. Then the set $f^{-1}([y,\infty))$ is an m-dimensional manifold with boundary $f^{-1}(y)$.*

PROOF. Let $N = f^{-1}([y,\infty))$. Since $f^{-1}((y,\infty))$ is an open subspace of M, it is an m-manifold without boundary.

The crucial case is when $x \in f^{-1}(y)$. Let φ be a parametrization of a neighborhood of x in M, with $\varphi(0) = x$. Let $g = f \circ \varphi$. As in the proof of 24.6, by the Implicit function theorem, there is an open ball U in $\mathbb{R}^{m-1}$ containing 0 and an open interval V in $\mathbb{R}$ containing 0 such that in $U \times V$ the set $g^{-1}(y)$ is a graph $\{(u,h(u)) \mid u \in U\}$ where h is smooth.



Since $(U \times V) \setminus g^{-1}(y)$ consists of two connected components, exactly one of the two is mapped via g to (y,∞), otherwise x will be a local extremum point of f, and so $df_x = 0$, violating the assumption. In order to be definitive, let us assume that $W = \{(u,v) \mid v \geq h(u)\}$ is mapped by g to $[y,\infty)$. Then $\varphi(W) = \varphi(U \times V) \cap f^{-1}([y,\infty))$ is a neighborhood of x in N parametrized by $\varphi|_W$. On the other hand W is diffeomorphic to an open neighborhood of 0 in $\mathbb{H}^m$. To show this, consider the map $\psi(u,v) = (u, v - h(u))$ on $U \times V$. Then ψ is a smooth bijection on open subspaces of $\mathbb{R}^m$, whose Jacobian is non-singular, therefore is a diffeomorphism. The restriction $\psi|_W$ is a diffeomorphism to $\psi(U \times V) \cap \mathbb{H}^m$. Thus x is a boundary point of N. $\square$

Example. Let f be the height function on S^2 and let y be a regular value. Then the set $f^{-1}((-\infty,y])$ is a disk with the circle $f^{-1}(y)$ as the boundary.

Example. If y is a regular value of the height function on D^2 then $f^{-1}(y)$ is a 1-dimensional manifold with boundary on ∂D^2.

Example. The closed disk D^n is an n-manifold with boundary.

Theorem 27.2. *Let M be an m-dimensional manifold with boundary, let N be an n-manifold with or without boundary. Let $f : M \to N$ be smooth. Suppose that $y \in N$ is a regular value of both f and $f|_{\partial M}$. Then $f^{-1}(y)$ is an $(m-n)$-manifold with boundary $\partial M \cap f^{-1}(y)$.*

PROOF. That $f^{-1}(y) \setminus \partial M$ is an $(m-n)$-manifold without boundary is already proved in 24.6.

We consider the crucial case of $x \in \partial M \cap f^{-1}(y)$.

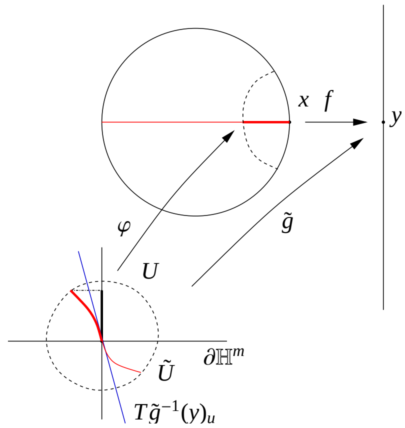


FIGURE 27.1.

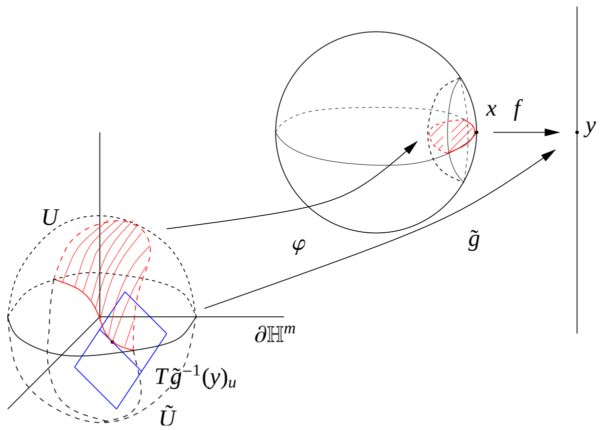


FIGURE 27.2.

The map g can be extended to $\tilde{g}$ defined on an open neighborhood $\tilde{U}$ of 0 in $\mathbb{R}^m$. As before, $\tilde{g}^{-1}(y)$ is a graph of a function of $(m-n)$ variables so it is an $(m-n)$-manifold without boundary.

Let $p : \tilde{g}^{-1}(y) \to \mathbb{R}$ be the projection to the last coordinate (the height function). We have $g^{-1}(y) = p^{-1}([0,\infty))$ therefore if we can show that 0 is a regular value of p then the desired result follows from 27.1 applied to $\tilde{g}^{-1}(y)$ and p. For that we need to show that the tangent space $T\tilde{g}^{-1}(y)_u$ at $u \in p^{-1}(0)$ is not contained in $\partial\mathbb{H}^m$. Note that since $u \in p^{-1}(0)$ we have $u \in \partial\mathbb{H}^m$.

Since $\tilde{g}$ is regular at u, the null space of $d\tilde{g}_u$ on $T\tilde{U}_u = \mathbb{R}^m$ is exactly $T\tilde{g}^{-1}(y)_u$, of dimension $m-n$. On the other hand, $\tilde{g}|_{\partial\mathbb{H}^m}$ is regular at u, which implies that the null space of $d\tilde{g}_u$ restricted to $T(\partial\mathbb{H}^m)_u = \partial\mathbb{H}^m$ has dimension $(m-1)-n$. Thus $T\tilde{g}^{-1}(y)_u$ is not contained in $\partial\mathbb{H}^m$. $\square$

Problems.

27.3. Check that $\mathbb{R}^m$ cannot be diffeomorphic to $\mathbb{H}^m$.

27.4. Show that the subspace $\{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid x_m > 0\}$ is diffeomorphic to $\mathbb{R}^m$.

27.5. A simple regular path is a map $\gamma : [a, b] \to \mathbb{R}^m$ such that γ is injective, smooth, and $\gamma^{(k)}(t) \neq 0$ for all $t \in [a, b]$. Show that the trace of a simple closed regular path is a smooth 1-dimensional manifold with boundary.

27.6. Suppose that M is an n-manifold without boundary. Show that $M \times [0, 1]$ is an $(n+1)$-manifold with boundary. Show that the boundary of $M \times [0, 1]$ consists of two connected components, each of which is diffeomeorphic to M.

27.7. Let M be a compact smooth manifold and $f : M \to \mathbb{R}$ be smooth. Show that if the interval $[a, b]$ only contains regular values of f then the sublevel sets $f^{-1}((-\infty, a])$ and $f^{-1}((-\infty, b])$ are diffeomorphic.

28. Sard theorem

Sard theorem. We use the following result from Analysis:

Theorem (Sard Theorem). *The set of critical values of a smooth map from $\mathbb{R}^m$ to $\mathbb{R}^n$ is of Lebesgue measure zero.*

For a proof see for instance **[Mil97]**. Sard theorem also holds for smooth functions from $\mathbb{H}^m$ to $\mathbb{R}^n$. This is left as a problem.

Since a set of measure zero must have empty interior, we have:

Corollary. *The set of regular values of a smooth map from $\mathbb{R}^m$ to $\mathbb{R}^n$ is dense in $\mathbb{R}^n$.*

An application of Sard theorem for manifolds is the following:

Theorem. *If M and N are two manifolds with boundary and $f : M \to N$ is smooth then the set of all regular values of f is dense in N. In particular f has a regular value in N.*

PROOF. Consider any open subset V of N parametrized by $\psi : V' \to V$. Then $f^{-1}(V)$ is an open submanifold of M. We only need to prove that $f|_{f^{-1}(V)}$ has a regular value in V. Let C be the set of all critical points of $f|_{f^{-1}(V)}$.

We can cover $f^{-1}(V)$ (or any manifold) by a *countable* collection I of parametrized open neighborhoods. This is possible because a Euclidean space has a countable topological basis (see 2.9).

For each $U \in I$ we have a commutative diagram:

$$\begin{tikzcd} U \arrow[r, "f"] & V \\ U' \arrow[u, "\varphi_U"] \arrow[r, "g_U"] & V' \arrow[u, "\psi"'] \end{tikzcd}$$

where U' is an open subset of $\mathbb{H}^m$ and V' is an open subset of $\mathbb{H}^n$. From this diagram, x is a critical point of f in U if and only if $\varphi_U^{-1}(x)$ is a critical point of g_U. Thus the set of critical points of g_U is $\varphi_U^{-1}(C \cap U)$.

Now we write

$$f(C) = \bigcup_{U \in I} f(C \cap U) = \bigcup_{U \in I} \psi(g_U(\varphi^{-1}(C \cap U))) = \psi\left(\bigcup_{U \in I} g_U(\varphi^{-1}(C \cap U))\right).$$

By Sard Theorem the set $g_U(\varphi_U^{-1}(C \cap U))$ is of measure zero. This implies that the set $D = \bigcup_{U \in I} g_U(\varphi_U^{-1}(C \cap U))$ is of measure zero, since a countable union of sets of measure zero is a set of measure zero. As a consequence D must have empty topological interior.

Since ψ is a homeomorphism, $\psi(D) = f(C)$ must also have empty topological interior. Thus $f(C) \subsetneq V$, so there must be a regular value of f in V. $\square$

If $N \subset M$ and $f : M \to N$ such that $f|_N = \mathrm{id}_N$ then f is called a *retraction* from M to N and N is a *retract* of M.

Lemma 28.1. *Let M be a compact manifold with boundary. There is no smooth map $f : M \to \partial M$ such that $f|_{\partial M} = id_{\partial M}$. In other words there is no smooth retraction from M to its boundary.*

PROOF. Suppose that there is such a map f. Let y be a regular value of f. Since $f|_{\partial M}$ is the identity map, y is also a regular value of $f|_{\partial M}$. By Theorem 27.2 the inverse image $f^{-1}(y)$ is a 1-manifold with boundary $f^{-1}(y) \cap \partial M = \{y\}$. But a 1-manifold cannot have boundary consisting of exactly one point. This result is contained in the classification of compact one-dimensional manifolds. □

Theorem 28.2 (Classification of compact one-dimensional manifolds). *A smooth compact connected one-dimensional manifold is diffeomorphic to either a circle, in which case it has no boundary, or an arc, in which case its boundary consists of two points.*

See **[Mil97]** for a proof.

Brouwer fixed point theorem.

Theorem 28.3 (Smooth Brouwer fixed point theorem). *A smooth map from the disk D^n to itself has a fixed point.*

This is a repeat of the proof for the continuous case using Algebraic Topology in 20.2.

PROOF. Suppose that f does not have a fixed point, i.e. $f(x) \neq x$ for all $x \in D^n$. The straight line from $f(x)$ to x will intersect the boundary ∂D^n at a point $g(x)$. Then $g : D^n \to \partial D^n$ is a smooth function which is the identity on ∂D^n. That is impossible, by 28.1. □

Actually the Brouwer fixed point theorem holds true for continuous maps. A proof can start by approximating a continuous function by smooth ones then use the smooth version of the theorem, see for instance **[Mil97]**.

Problems.

28.4. Show that Sard theorem also holds for smooth functions from $\mathbb{H}^m$ to $\mathbb{R}^n$.

28.5. Show that a smooth loop on S^2 (i.e. a smooth map from S^1 to S^2) cannot cover S^2. Similarly, there is no smooth surjective maps from $\mathbb{R}$ to $\mathbb{R}^n$ with $n > 1$. In other words, there is no smooth space filling curves, in contrast to the continuous case (compare 5.35).

28.6. Prove the Brouwer fixed point theorem for $[0, 1]$ directly.

28.7. Check that the function g in the proof of 28.3 is smooth.

28.8. Is the Brouwer fixed point theorem correct for open balls?

28.9. Is the Brouwer fixed point theorem correct for spheres?

28.10. Is the Brouwer fixed point theorem correct for tori?

28.11. Show that the Brouwer fixed point theorem is correct for any space homeomorphic to a disk.

28.12. Let A be an $n \times n$ matrix whose entries are all nonnegative real numbers. We will derive the Frobenius theorem which says that A must have a real nonnegative eigenvalue.

(a) Suppose that A is not singular. Check that the map $v \mapsto \frac{Av}{||Av||}$ brings $Q = \{(x_1, x_2, \dots, x_n) \in S^{n-1} \mid x_i \geq 0, 1 \leq i \leq n\}$ to itself.

(b) Prove that Q is homeomorphic to the closed ball D^{n-1}.

(c) Use the continuous Brouwer fixed point theorem to prove that A has a real nonnegative eigenvalue.

29. Orientation

Orientation on vector spaces. On a finite dimensional real vector space, two vector bases are said to determine the same orientation of the space if the change of bases matrix has positive determinant. Being of the same orientation is an equivalence relation on the set of all bases. With this equivalence relation the set of all bases is divided into two equivalence classes. If we choose one of the two classes as the prefered one, then we say the vector space is *oriented* and the chosen equivalence class is called the *orientation* (or the positive orientation).

Thus any finite dimensional real vector space is *orientable* (i.e. can be oriented) with two possible orientations.

Example. The standard positive orientation of $\mathbb{R}^n$ is represented by the basis

$$\{e_1 = (1,0,\dots,0), e_2 = (0,1,0,\dots,0), \dots, e_n = (0,\dots,0,1)\}.$$

Unless stated otherwise, $\mathbb{R}^n$ is always oriented this way.

Let T be an isomorphism from an oriented finite dimensional real vector space V to an oriented finite dimensional real vector space W. Then T brings a basis of V to a basis of W. There are only two possibilities. Either T brings a positive basis of V to a positive basis of W, or T brings a positive basis of V to a negative basis of W. In the first case we say that T is *orientation-preserving*, and in the second case we say that T is *orientation-reversing*.

Orientation on manifolds. Roughly, a manifold is oriented if at each point an orientation for the tangent space is chosen and this orientation should be smoothly depended on the point.

Definition. A manifold M is said to be *oriented* if at each point x an orientation for the tangent space TM_x is chosen and at each point there exists a local coordinate (U, φ) such that for each x in U the derivative $d\varphi_x : TM_x \to \mathbb{R}^m$ is orientation-preserving.

Thus in this local coordinate the orientation of TM_x is given by the basis $\{\frac{\partial\varphi}{\partial x_1}(x), \frac{\partial\varphi}{\partial x_2}(x), \dots, \frac{\partial\varphi}{\partial x_m}(x)\}$ where $\frac{\partial\varphi}{\partial x_1}(x) = d\varphi^{-1}_{\varphi(x)}(e_i)$. Roughly, the local coordinate brings the orientation of $\mathbb{R}^m$ to the manifold.

If a manifold is oriented then the set of orientations of its tangent spaces is called an *orientation* of the manifold and the the manifold is said to be *orientable*.

Another approach to orientation of manifold is to orient each parametrized neighborhood then require that the orientations on overlapping neighborhood agree. Concisely, suppose that $\varphi : U \to M$ is a parametrization of a neighborhood in M. At each point, the orientation on TM_x is given by the image of the standard basis of $\mathbb{R}^n$ via $d\varphi_u$, i.e. it is given by the basis $\{\frac{\partial\varphi}{\partial u_1}(u), \frac{\partial\varphi}{\partial u_2}(u), \dots, \frac{\partial\varphi}{\partial u_n}(u)\}$ where $\frac{\partial\varphi}{\partial u_i}(u) = d\varphi_u(e_i)$. Suppose that $\psi : V \to M$ parametrizes an overlapping neighborhood. Since $d\psi_v = d(\psi \circ \varphi^{-1})_v \circ d\varphi_u$, the consistency requirement is that the map $d(\psi \circ \varphi^{-1})_v$ must be orientation preserving on $\mathbb{R}^m$. In other words, we can say that the change of coordinates must be orientation preserving.

Example. If a manifold is parametrized by one parametrization, that is, it is covered by one local coordinate, then it is orientable, since we can take the unique parametrization to bring an orientation of $\mathbb{R}^m$ to the entire manifold. In particular, any open subset of $\mathbb{R}^k$ is an orientable manifold.

Example. The graph of a smooth function $f : D \to \mathbb{R}^l$, where $D \subset \mathbb{R}^k$ is an open set, is an orientable manifold, since this graph can be parametrized by a single parametrization, namely $x \mapsto (x, f(x))$.

Proposition. *If $f : \mathbb{R}^k \to \mathbb{R}$ is smooth and a is a regular value of f then $f^{-1}(a)$ is an orientable manifold.*

PROOF. Let $M = f^{-1}(a)$. If $x \in M$ then $\ker df_x = TM_x$, so the gradient vector $\nabla f(x)$ is perpendicular to TM_x. In other words the gradient vector is always perpendicular to the level set. In particular, $\nabla f(x)$ does not belong to TM_x. Choose the orientation on TM_x represented by a basis $b(x) = \{b_1(x), \ldots, b_{k-1}(x)\}$ such that the ordered set $\{b_1(x), \ldots, b_{k-1}(x), \nabla f(x)\}$ is a positive basis in the standard orientation of $\mathbb{R}^k$. That means $\det(b_1(x), \ldots, b_{k-1}(x), \nabla f(x)) > 0$.

We check that this orientation is smoothly depended on the point. Let $\varphi : \mathbb{R}^{k-1} \to U \subset M$ be a local parametrization of a neighborhood U of x, with $\varphi(0) = x$. We can assume that basis $\left\{\frac{\partial \varphi}{\partial u_1}(0), \frac{\partial \varphi}{\partial u_2}(0), \ldots, \frac{\partial \varphi}{\partial u_{k-1}}(0)\right\}$ is in the same orientation as $b(x)$, if that is not the case we can interchange two variables of φ. We can check that $\left\{\frac{\partial \varphi}{\partial u_1}(u), \frac{\partial \varphi}{\partial u_2}(u), \ldots, \frac{\partial \varphi}{\partial u_{k-1}}(u)\right\}$ is in the same orientation as $b(\varphi(u))$ for all $u \in \mathbb{R}^{k-1}$. Indeed, consider $\det\left(\frac{\partial \varphi}{\partial u_1}(u), \frac{\partial \varphi}{\partial u_2}(u), \ldots, \frac{\partial \varphi}{\partial u_{k-1}}(u), \nabla f(\varphi(u))\right)$. This is a continuous real function on $u \in \mathbb{R}^{k-1}$ whose value at 0 is positive, therefore its value is always positive. $\square$

Example. The sphere is orientable.

Example. The torus is orientable.

Proposition. *A connected orientable manifold has exactly two orientations.*

PROOF. Suppose the manifold M is orientable. There is an orientation o on M. Then $-o$ is a different orientation on M. Suppose that o_1 is an orientation on M, we show that o_1 is either o or $-o$.

If two orientations agrees at a point they must agree locally around that point. Indeed, from the definition there is a neighborhood V of x and a local coordinates $\varphi : V \to \mathbb{R}^m$ that brings the orientation o_1 to the standard orientation of $\mathbb{R}^m$, and a local coordinates $\psi : V \to \mathbb{R}^k$ that brings the orientation o to the standard orientation of $\mathbb{R}^m$. Assuming $\varphi(x) = \psi(x) = 0$, then $\det J(\psi^{-1} \circ \varphi)$ is smooth on $\mathbb{R}^m$ and is positive at 0, therefore it is always positive. That imples o_1 and o agree on V.

Let U be the set of all points x in M such that the orientation of TM_x with respect to o_1 is the same with the orientation of TM_x with respect to o. Then U is open in M. Similarly the complement $M \setminus U$ is also open. Since M is connected, either $U = M$ or $U = \emptyset$. $\square$

Orientable surfaces. A two dimensional manifold in $\mathbb{R}^3$ is called a *surface*. A surface is *two-sided* if there is a smooth way to choose a unit normal vector $N(p)$ at each point $p \in S$. That is, there is a smooth map $N : S \to \mathbb{R}^3$ such that at each $p \in S$ the vector $N(p)$ has length 1 and is perpendicular to TS_p.

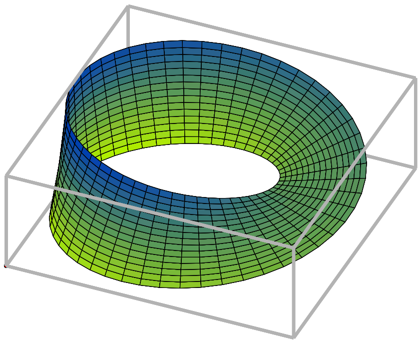


FIGURE 29.1. The Mobius band is not orientable and is not two-sided.

Proposition. *A surface is orientable if and only if it is two-sided.*

PROOF. If the surface S is orientable then its tangent spaces could be oriented smoothly. That means at each point $p \in S$ there is a local parametrization $r(u, v)$ such that $\{r_u(u, v), r_v(u, v)\}$ gives the orientation of TS_p. Then the unit normal vector $\dfrac{r_u(u,v) \times r_v(u,v)}{||r_u(u,v) \times r_v(u,v)||}$ is defined smoothly on the surface.

Conversely, if there is a smooth unit normal vector N on the surface then we orient each tangent plane TS_p by a basis $\{v_1, v_2\}$ such that $\{v_1, v_2, N(p)\}$ is in the same orientation as the standard orientation of $\mathbb{R}^3$. For each point p take a local parametrization $r : \mathbb{R}^2 \to S$, $r(0, 0) = p$, such that $\{r_u(0, 0), r_v(0, 0)\}$ is in the orientation of TS_p (take any local parametrization, if it gives the opposite orientation at p then just switch the variables). Since $\langle r_u(u, v) \times r_v(u, v), N(r(u, v))\rangle$ is smooth, its sign does not change, and since the sign at $(0, 0)$ is positive, the sign is always positive. Thus $\{r_u(u, v), r_v(u, v)\}$ is in the orientation of $TS_{r(u,v)}$. That means the orientation is smooth. $\square$

Orientation on the boundary of an oriented manifold. Suppose that M is a manifold with boundary and the interior of M is oriented. We orient the boundary of M as follows. Suppose that under an orientation-preserving parametrization φ the point $\varphi(x)$ is on the boundary ∂M of M. Then the orientation $\{b_2, b_3, \ldots, b_n\}$ of $\partial\mathbb{H}^n$ such that the ordered set $\{-e_n, b_2, b_3, \ldots, b_n\}$ is a positive basis of $\mathbb{R}^n$ will induce the positive orientation for $T\partial M_{\varphi(x)}$ through $d\varphi(x)$. This is called *the outer normal first orientation of the boundary*.

Problems.

29.1. Show that two diffeomorphic manifolds are are either both orientable or both unorientable.

29.2. Suppose that $f : M \to N$ is a diffeomorphism of connected oriented manifolds with boundary. Show that if there is an x such that $df_x : TM_x \to TN_{f(x)}$ is orientation-preserving then f is orientation-preserving.

29.3. Let $f : \mathbb{R}^k \to \mathbb{R}^l$ be smooth and let a be a regular value of f. Show that $f^{-1}(a)$ is an orientable manifold.

29.4. Consider the map $-\mathrm{id} : S^n \to S^n$ with $x \mapsto -x$. Show that $-\mathrm{id}$ is orientation-preserving if and only if n is odd.

30. Topological degrees of maps

Let M and N be boundaryless, oriented manifolds of the same dimensions m. Further suppose that M is compact.

Let $f : M \to N$ be smooth. Suppose that x is a regular point of f. Then df_x is an isomorphism from TM_x to $TN_{f(x)}$. Let $\operatorname{sign}(df_x) = 1$ if df_x preserves orientations, and $\operatorname{sign}(df_x) = -1$ otherwise.

For any regular value y of f, let

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \operatorname{sign}(df_x).$$

Notice that the set $f^{-1}(y)$ is finite because M is compact (see 24.5).

This number $\deg(f, y)$ is called the *Brouwer degree* (bậc Brouwer) [14] or *topological degree* of the map f with respect to the regular value y.

From the Inverse Function Theorem 24.4, each regular value y has a neighborhood V and each preimage x of y has a neighborhood U_x on which f is a diffeomorphism onto V, either preserving or reversing orientation. Therefore we can interpret that $\deg(f, y)$ *counts the algebraic number of times the function f covers the value y*.

Example. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Then $\deg(f, 1) = 0$. This could be explained geometrically from the graph of f, as f covers the value 1 twice in opposite directions at $x = -1$ and $x = 1$.

Example. Consider $f(x) = x^3 - x$ with the regular value 0. From the graph of f we see that f covers the value 0 three times in positive direction at $x = -1$ and $x = 1$ and negative direction at $x = 0$, therefore we see right away that $\deg(f, 0) = 1$.

On the other hand, if we consider the regular value -1 then f covers this value only once in positive direction, thus $\deg(f, 1) = 1$.

Homotopy invariance. In this section we will show that the Brouwer degree does not depend on the choice of regular values and is invariant under smooth homotopy.

Lemma. *Let M be the boundary of a compact oriented manifold X, oriented as the boundary of X. If $f : M \to N$ extends to a smooth map $F : X \to N$ then $\deg(f, y) = 0$ for every regular value y.*

PROOF. (a) Assume that y is a regular value of F. Then $F^{-1}(y)$ is a 1-dimensional manifold of dimension 1 whose boundary is $F^{-1}(y) \cap M = f^{-1}(y)$, by Theorem 27.1.

By the Classification of one-dimensional manifolds, $F^{-1}(y)$ is the disjoint union of arcs and circles. Let A be a component that intersects M. Then A is an arc with boundary $\{a, b\} \subset M$.

[14] L. E. J. Brouwer (1881–1966) is a Dutch mathematician. He had many important contributions in the early development of topology, and founded Intuitionism.

We will show that $\operatorname{sign}(\det(df_a)) = -\operatorname{sign}(\det(df_b))$. Taking sum over all arc components of $F^{-1}(y)$ would give us $\deg(f, y) = 0$.

An orientation on A. Let $x \in A$. Recall that TA_x is the kernel of $dF_x : TX_x \to TN_y$. We will choose the orientation on TA_x such that this orientation together with the pull-back of the orientation of TN_y via dF_x is the orientation of X. Let $(v_2, v_3, \ldots, v_{n+1})$ be a positive basis for TN_y. Let $v_1 \in TA_x$ such that $\{v_1, dF_x^{-1}(v_2), \ldots, dF_x^{-1}(v_{n+1})\}$ is a positive basis for TX_x. Then v_1 determine the positive orientation on TA_x.

At $x = a$ or at $x = b$ we have $df_x = dF_x|_{TM_x}$. Therefore df_x is orientation-preserving on TM_x oriented by the basis $\{df_x^{-1}(v_2), \ldots, df_x^{-1}(v_{n+1})\}$.

We claim that exactly one of the two above orientations of TM_x at $x = a$ or $x = b$ is opposite to the orientation of TM_x as the boundary of X. This would show that $\operatorname{sign}(\det(df_a)) = -\operatorname{sign}(\det(df_b))$.

Observe that if at a the orientation of TA_a is pointing outward with respect to X then b the orientation of TA_b is pointing inward, and vice versa. Indeed, since A is a smooth arc it is parametrized by a smooth map $\gamma(t)$ such that $\gamma(0) = a$ and $\gamma(1) = b$. If we assume that the orientation of $TA_{\gamma(t)}$ is given by $\gamma'(t)$ then it is clear that at a the orientation is inward and at b it is outward.

(b) Suppose now that y is not a regular value of F. There is a neighborhood of y in the set of all regular values of f such that $\deg(f, z)$ does not change in this neighborhood. Let z be a regular value of F in this neighborhood, then $\deg(f, z) = \deg(F, z) = 0$ by (a), and $\deg(f, z) = \deg(f, y)$. Thus $\deg(f, y) = 0$. ☐

Lemma. *If f is smoothly homotopic to g then $\deg(f, y) = \deg(g, y)$ for any common regular value y.*

PROOF. Let $I = [0, 1]$ and $X = M \times I$. Since f be homotopic to g there is a smooth map $F : X \to N$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$.

The boundary of X is $(M \times \{0\}) \sqcup (M \times \{1\})$. Then F is an extension of the pair f, g from ∂X to X, thus $\deg(F|_{\partial X}, y) = 0$ by the above lemma.

Note that one of the two orientations of $M \times \{0\}$ or $M \times \{1\}$ as the boundary of X is opposite to the orientation of M (this is essentially for the same reason as in the proof of the above lemma). Therefore $\deg(F|_{\partial X}, y) = \pm(\deg(f, y) - \deg(g, y)) = 0$, so $\deg(f, y) = \deg(g, y)$. ☐

Lemma 30.1 (Homogeneity of manifold). *Let N be a connected boundaryless manifold and let y and z be points of N. Then there is a self diffeomorphism $h : N \to N$ that is smoothly isotopic to the identity and carries y to z.*

We do not present a proof for this lemma. The reader can find a proof in [**Mil97**, p. 22].

Theorem 30.2. *Let M and N be boundaryless, oriented manifolds of the same dimensions. Further suppose that M is compact and N is connected. The Brouwer degree of a map from M to N does not depend on the choice of regular values and is invariant under smooth homotopy.*

Therefore from now on we will write $\deg(f)$ instead of $\deg(f,y)$.

PROOF. We have already shown that degree is invariant under homotopy.

Let y and z be two regular values for $f : M \to N$. Choose a diffeomorphism h from N to N that is isotopic to the identity and carries y to z.

Note that h preserves orientation. Indeed, there is a smooth isotopy $F : N \times [0,1] \to N$ such that $F_0 = h$ and $F_1 = \mathrm{id}$. Let $x \in N$, and let $\varphi : \mathbb{R}^m \to N$ be an orientation-preserving parametrization of a neighborhood of x with $\varphi(0) = x$. Since $dF_t(x) \circ d\varphi_0 : \mathbb{R}^m \times \mathbb{R}$ is smooth with respect to t, the sign of $dF_t(x)$ does not change with t.

As a consequence, $\deg(f,y) = \deg(h \circ f, h(y))$.

Finally since $h \circ f$ is homotopic to $id \circ f$, we have $\deg(h \circ f, h(y)) = \deg(\mathrm{id} \circ f, h(y)) = \deg(f, h(y)) = \deg(f,z)$. □

Example. Let M be a compact, oriented and boundaryless manifold. Then the degree of the identity map on M is 1. On the other hand the degree of a constant map on M is 0. Therefore the identity map is not homotopic to a constant map.

Example 30.3 (Proof of the Brouwer fixed point theorem via the Brouwer degree). We can prove that D^{n+1} cannot retract to its boundary (this is 28.1 for the case of D^{n+1}) as follows. Suppose that there is such a retraction, a smooth map $f : D^{n+1} \to S^n$ that is the identity on S^n. Define $F : [0,1] \times S^n$ by $F(t,x) = f(tx)$. Then F is a smooth homotopy from a constant map to the identity map on the sphere. But these two maps have different degrees.

Theorem (The fundamental theorem of Algebra). *Any non-constant polynomial with real coefficients has at least one complex root.*

PROOF. Let $p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n$, with $a_i \in \mathbb{R}$, $1 \le i \le n$. Suppose that p has no root, that is, $p(z) \neq 0$ for all $z \in \mathbb{C}$. As a consequence, $a_n \neq 0$.

For $t \in [0,1]$, let

$$q_t(z) = (1-t)^n z^n + a_1(1-t)^{n-1} t z^{n-1} + \cdots + a_{n-1}(1-t)t^{n-1} z + a_n t^n.$$

Then $q_t(z)$ is continuous with respect to the pair (t,z). Notice that if $t \neq 0$ then $q_t(z) = t^n p((1-t)t^{-1} z)$, and $q_0(z) = z^n$ while $q_1(z) = a_n$.

If we restrict z to the set $\{z \in \mathbb{C} \mid |z| = 1\} = S^1$ then $q_t(z)$ has no roots, so $\frac{q_t(z)}{|q_t(z)|}$ is a continuous homotopy of maps from S^1 to itself, starting with the polynomial z^n and ending with the constant polynomial $\frac{a_n}{|a_n|}$. But these two polynomials have different degrees, a contradiction. □

Example. Let $v : S^1 \to \mathbb{R}^2$, $v((x,y)) = (-y,x)$, then it is a nonzero (not zero anywhere) tangent vector field on S^1.

Similarly we can find a nonzero tangent vector field on S^n with odd n.

Theorem 30.4 (The Hairy Ball Theorem). *If n is even then every smooth tangent vector field on S^n has a zero.*

PROOF. Suppose that v is a nonzero tangent smooth vector field on S^n. Let $w(x) = \frac{v(x)}{||v(x)||}$, then w is a unit smooth tangent vector field on S^n.

Notice that $w(x)$ is perpendicular to x. On the plane spanned by x and $w(x)$ we can easily rotate vector x to vector $-x$. Precisely, let $F_t(x) = \cos(t) \cdot x + \sin(t) \cdot w(x)$ with $0 \leq t \leq \pi$, then F is a homotopy on S^n from x to $-x$. But the degrees of these two maps are different, see 30.14. □

Problems.

30.5. Find the topological degree of a polynomial on $\mathbb{R}$. Notice that although the domain $\mathbb{R}$ is not compact, the topological degree is well-defined for polynomial.

30.6. Let $f : S^1 \to S^1$, $f(z) = z^n$, where $n \in \mathbb{Z}$. We can also consider f as a vector-valued function $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (f_1(x, y), f_2(x, y))$. Then $f = f_1 + if_2$.

(a) Recalling the notion of complex derivative and the Cauchy-Riemann condition, check that $\det(Jf_z) = |f'(z)|^2$.
(b) Check that all values of f are regular.
(c) Check that $\deg(f, y) = n$ for all $y \in S^1$.

30.7. Show that $\deg(f, y)$ is locally constant on the subspace of all regular values of f.

30.8. What happens if we drop the condition that N is connected in Theorem 30.2? Where do we use this condition?

30.9. Let M and N be oriented boundaryless manifolds, M is compact and N is connected. Let $f : M \to N$. Show that if $\deg(f) \neq 0$ then f is onto, i.e. the equation $f(x) = y$ always has a solution.

30.10. Let $r_i : S^n \to S^n$ be the reflection map

$$r_i((x_1, x_2, \ldots, x_i, \ldots, x_{n+1})) = (x_1, x_2, \ldots, -x_i, \ldots, x_{n+1}).$$

Compute $\deg(r_i)$.

30.11. Let $f : S^n \to S^n$ be the map that interchanges two coordinates:

$$f((x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots, x_{n+1})) = (x_1, x_2, \ldots, x_j, \ldots, x_i, \ldots, x_{n+1}).$$

Compute $\deg(f)$.

30.12. Suppose that M, N, P are compact, oriented, connected, boundaryless m-manifolds. Let $M \xrightarrow{f} N \xrightarrow{g} P$. Then $\deg(g \circ f) = \deg(f) \deg(g)$.

30.13. Let M be a compact connected smooth manifold. Let $f : M \to M$ be smooth.

(a) Show that if f is bijective then $\deg f = \pm 1$.
(b) Let $f^2 = f \circ f$. Show that $\deg(f^2) \geq 0$.

30.14. Let $r : S^n \to S^n$ be the antipodal map

$$r((x_1, x_2, \ldots, x_{n+1})) = (-x_1, -x_2, \ldots, -x_{n+1}).$$

Compute $\deg(r)$.

30.15. Let $f : S^4 \to S^4$, $f((x_1, x_2, x_3, x_4, x_5)) = (x_2, x_4, -x_1, x_5, -x_3)$. Find $\deg(f)$.

30.16. Find a map from S^2 to itself of any given degree.

30.17. If $f, g : S^n \to S^n$ be smooth such that $f(x) \neq -g(x)$ for all $x \in S^n$ then f is smoothly homotopic to g.

30.18. Let $f : M \to S^n$ be smooth. Show that if $\dim(M) < n$ then f is homotopic to a constant map.

30.19 (Brouwer fixed point theorem for the sphere). Let $f : S^n \to S^n$ be smooth. If $\deg(f) \neq (-1)^{n+1}$ then f has a fixed point.

30.20. Show that any map of from S^n to S^n of odd degree carries a certain pair of antipodal points to a pair of antipodal point.

Guide for further reading

We have closely followed John Milnor's masterpiece [**Mil97**]. Another excellent text is [**GP74**]. There are not many textbooks such as these two books, presenting differential topology to undergraduate students.

The book [**Hir76**] is a technical reference for some advanced topics. The book [**DFN85**] is a masterful presentation of modern topology and geometry, with some enlightening explanations, but it sometimes requires knowledge of many topics. The book [**Bre93**] is rather similar in aim, but is more like a traditional textbook.

An excellent textbook for differential geometry of surfaces is [**dC76**].

Suggestions for some problems

1.11: There is an infinitely countable subset of B.

1.12: Use 1.11.

1.19: $\bigcup_{n=1}^{\infty}[n, n+1] = [1, \infty)$.

1.23: Proof by contradiction.

4.8: Consider a metric space with only two points.

4.20: Compare the subinterval $[1, 2\pi)$ and its image via φ.

5.27: Let A be countable and $x \in \mathbb{R}^2 \setminus A$. There is a line passing through x that does not intersect A (by an argument involving cardinalities of sets).

5.28: Delete a point from $\mathbb{R}$. Use 4.12 and 5.3.

6.10: Consider the set of all irrational numbers.

7.9: Suppose that there are two points x and y that could not be separated by open sets. Consider the directed set whose elements are pairs (U_x, V_y) of open neighborhoods of x and y, under set inclusion. Take a net n such that $n(U_x, V_y)$ is a point in $U_x \cap V_y$.

7.10: (b) Let C be a countable subset of $[0, \Omega)$. The set $\bigcup_{c \in C}[0, c)$ is countable while the set $[0, \Omega)$ is uncountable.

8.9: Use Lebesgue's number.

8.12: See the proof of 8.1.

8.14: Use 8.12.

8.15: Use 8.14.

9.5: Look at their bases.

9.11: Only need to show that the projection of an element of the basis is open.

9.15: Use 9.2 to prove that the inclusion map is continuous.

9.16: Use 9.15.

9.17: Let (x_i) and (y_i) be in $\prod_{i \in I} X_i$. Let $\gamma_i(t)$ be a continuous path from x_i to y_i. Let $\gamma(t) = (\gamma_i(t))$.

9.18: (b) Use 9.15. (c) Fix a point $x \in \prod_{i \in I} X_i$. Use (b) to show that the set A_x of points that differs from x at at most finitely many coordinates is connected. Furthermore A_x is dense in $\prod_{i \in I} X_i$.

9.19: Use 9.15. It is enough to prove for the case an open cover of $X \times Y$ by open sets of the form a product of an open set in X with an open set in Y. For each "slice" $\{x\} \times Y$ there is finite subcover $\{U_{x,i} \times V_{x,i} \mid 1 \leq i \leq n_x\}$. Take $U_x = \bigcap_{i=1}^{n_x} U_{x,i}$. The collection $\{U_x \mid x \in X\}$ covers X so there is a subcover $\{U_{x_j} \mid 1 \leq j \leq n\}$. The collection $\{U_{x_j,i} \times V_{x_j,i} \mid 1 \leq i \leq n_{x_j}, 1 \leq j \leq n\}$ is a finite subcover of $X \times Y$.

10.4: Use10.3.

10.12: Use 8.12

10.13: Use 10.12.

10.15: ($\Leftarrow$) Use 8.15 and 6.2.

10.18: ($\Leftarrow$) Use 8.15 and the Urysohn lemma 11.1.

11.3: ($\Leftarrow$) Use 8.15 and the Urysohn lemma 11.1.

11.7: Use 11.6.

11.9: See 9.1 and 9.4.

11.11: Any real function on L is continuous. The cardinality of the set of such function is c^c. A real function on $\mathbb{H}$ is continuous if and only if it is continuous on the dense subset of points with rational coordinates, so the cardinality of the set of such functions is at most $c^{\aleph_0}$. Since $c^c > c^{\aleph_0}$, the space $\mathbb{H}$ cannot be normal, by Tiestze Extension Theorem.

12.10: The idea is easy to be visualized in the cases $n = 1$ and $n = 2$. Let $S^+ = \{x = (x_1, x_2, \ldots, x_{n+1}) \in S^n \mid x_1 \geq 0\}$, the upper hemisphere. Let $S^0 = \{x = (x_1, x_2, \ldots, x_{n+1}) \in S^n \mid x_1 = 0\}$, the equator. Let $f : S^n \to S^+$ be given by $f(x) = x$ if $x \in S^+$ and $f(x) = -x$ otherwise. Then the following diagram is commutative:

$$\begin{tikzcd}
S^n \arrow[r, "f"] \arrow[d] \arrow[dr, "p\circ f"] & S^+ \arrow[d, "p"] \\
S^n/x \sim -x \arrow[r, "\tilde{f}"] & S^+/x \sim -x,\ x \in S^0
\end{tikzcd}$$

Then it is not difficult to show that $S^+/x \sim -x,\ x \in S^0$ is homeomorphic to $\mathbb{R}\mathrm{P}^n = D^n/x \sim -x,\ x \in \partial D^n$.

13.6: The set of all balls with rational radius whose center has rational coordinates forms a basis for the Euclidean topology of $\mathbb{R}^n$.

13.9: By 10.13.

14.14: Deleting an open disk is the same as deleting the interior of a triangle.

18.6: See [**Hat01**, p. 52].

20.5: Use Mayer-Vietoris sequence.

20.8: First take a deformation retraction to a sphere.

20.9: Show that $\mathbb{R}^3 \setminus S^1$ is homotopic to Y which is a closed ball minus a circle inside. Show that $Y = S^1 \vee S^2$, [**Hat01**, p. 46]. Or write Y as a union of two halves, each of which is a closed ball minus a straight line, and use the Van Kampen theorem.

22.5: The torus is given by the equation $(\sqrt{x^2 + y^2} - b)^2 + z^2 = a^2$ where $0 < a < b$.

22.6: Consider a neighborhood of a point on the y-axis. Can it be homeomorphic to an open neighborhood in $\mathbb{R}$?

22.12: See 5.7.

22.13: See 4.16 and 26.2.

22.14: Use the Implicit function theorem.

24.14: Each $x \in f^{-1}(y)$ has a neighborhood U_x on which f is a diffeomorphism. Let $V = [\bigcap_{x \in f^{-1}(y)} f(U_x)] \setminus f(M \setminus \bigcup_{x \in f^{-1}(y)} U_x)$. Consider $V \cap S$.

24.22: Use Problem 24.19.

26.2: Show that $f^{(n)}(x) = e^{-1/x} P_n(1/x)$ where $P_n(x)$ is a polynomial.

26.3: Cover A by finitely many balls $B_i \subset U$. For each i there is a smooth function φ_i which is positive in B_i and is zero outside of B_i.

30.19: If f does not have a fixed point then f will be homotopic to the reflection map.

30.17: Note that $f(x)$ and $g(x)$ will not be antipodal points. Use the homotopy

$$F_t(x) = \frac{(1-t)f(x) + tg(x)}{||(1-t)f(x) + tg(x)||}.$$

30.18: Using Sard Theorem show that f cannot be onto.

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You may say I'm a dreamer

But I'm not the only one

I hope someday you'll join us ...

John Lennon, Imagine.