Integral Approach to the Control Volume analysis of Fluid Flow

Basic Concepts

Velocity Field

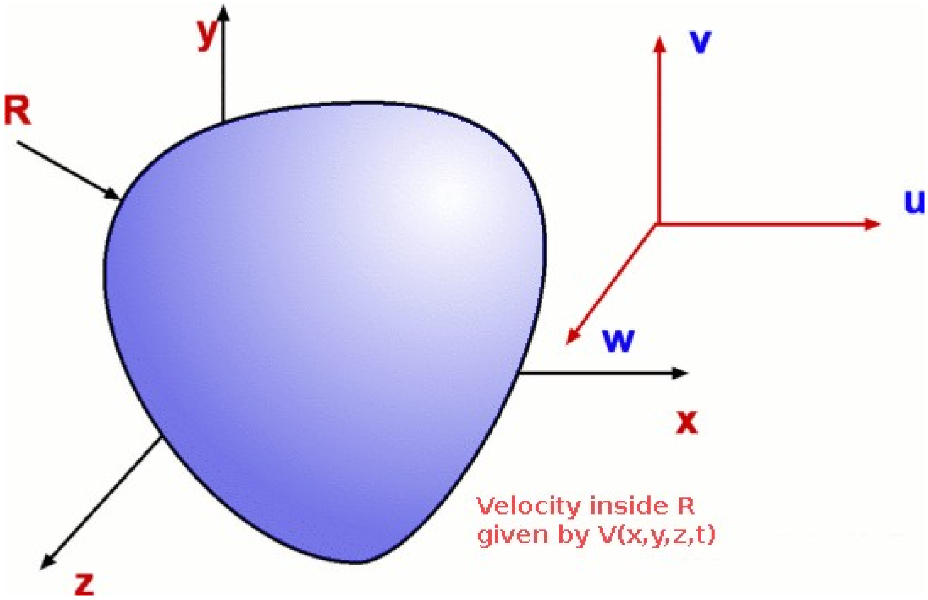

Figure 1 : Velocity Field

Velocity field implies a distribution of velocity in a given region say R (Fig.1). It is denoted in a functional form as **V(x,y,z,t)** meaning that velocity is a function of the spatial and time coordinates. It is useful to recall that we are studying fluid flow under the Continuum Hypothesis which allows us to define velocity at a point. Further velocity is a vector quantity i.e., it has a direction along with a magnitude. This is indicated by writing velocity field as

$$\vec{V} = \vec{V}(x, y, z, t) \qquad (1)$$

Velocity may have three components, one in each direction, i.e, **u,v** and **w** in **x, y** and **z** directions respectively. It is usual to write $\vec{V}$ as

$$\vec{V} = u\,\vec{i} + v\,\vec{j} + w\,\vec{k} \qquad (2)$$

It is clear that each of **u,v** and **w** can be functions of **x,y,z** and **t**. Thus

$$\vec{V} = u(x, y, z, t)\,\vec{i} + v(x, y, z, t)\,\vec{j} + w(x, y, z, t)\,\vec{k} \qquad (3)$$

Each of the other variables involved in a fluid flow can also be given a field representation. We have temperature field, **T(x,y,z,t)**, pressure field, **P(x,y,z,t)**, density field $\rho(x, y, z, t)$, etc.

Steady and Unsteady Flows

We have noted previously that velocity, pressure and other properties of fluid flow can be functions of time (apart from being functions of space). If a flow is such that the properties at every point in the flow do not depend upon time, it is called a steady flow.

Mathematically speaking for steady flows,

$$\frac{\partial X}{\partial t}=0 \tag{4}$$

where **X** is any property like pressure, velocity or density. Thus,

$$X=X(x, y, z) \tag{5}$$

Unsteady or non-steady flow is one where the properties do depend on time.

It is needless to say that any start up process is unsteady. Many examples can be given from everyday life - water flow out of a tap which has just been opened. This flow is unsteady to start with, but with time does become steady.

Some flows, though unsteady, become steady under certain frames of reference. These are called pseudosteady flows. On the other hand a flow such as the wake behind a bluff body is always unsteady.

Unsteady flows are undoubtedly difficult to calculate while with steady flows, we have one degree less complexity.

One, Two and Three Dimensional Flows

Term one, two or three dimensional flow refers to the number of space coordinated required to describe a flow. It appears that any physical flow is generally three-dimensional. But these are difficult to calculate and call for as much simplification as possible. This is achieved by ignoring changes to flow in any of the directions, thus reducing the complexity. It may be possible to reduce a three-dimensional problem to a two-dimensional one, even an one-dimensional one at times.

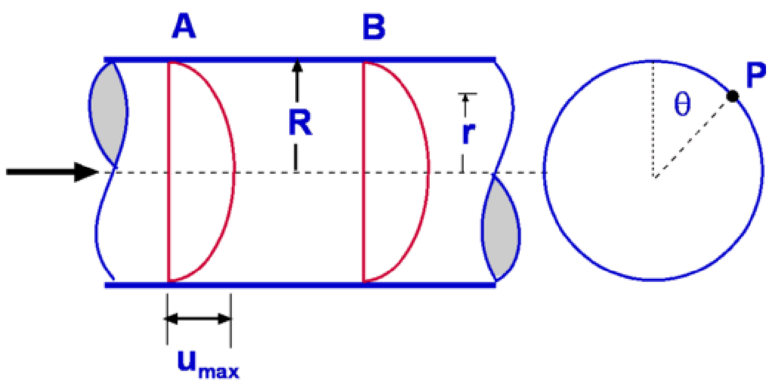

Figure 2 : Example of one-dimensional flow

Consider flow through a circular pipe. This flow is complex at the position where the flow enters the pipe. But as we proceed downstream the flow simplifies considerably and attains the state of a fully developed flow. A characteristic of this flow is that the velocity becomes invariant in the flow direction as shown in Fig 2. Velocity for this flow is given by

$$u=u_{max}\left(1-\left(\frac{r}{R}\right)^2\right) \tag{6}$$

It is readily seen that velocity at any location depends just on the radial distance **r** from the centreline and is independent of distance, **x** or of the angular position θ . This represents a typical **one-dimensional flow.**

Now consider a flow through a diverging duct as shown in Fig Velocity at any location depends not only upon the radial distance **r** but also on the **x**-distance. This is therefore a **two-dimensional flow**.

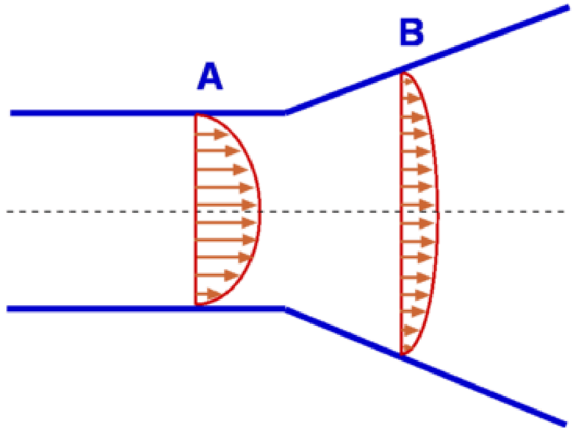

Figure 3: Example of a two-dimensional flow

Concept of a uniform flow is very handy in analysing fluid flows. A uniform flow is one where the velocity and other properties are constant independent of directions. we usually assume a uniform flow at the entrance to a pipe, far away from a aerofoil or a motor car as shown in Fig. 4.

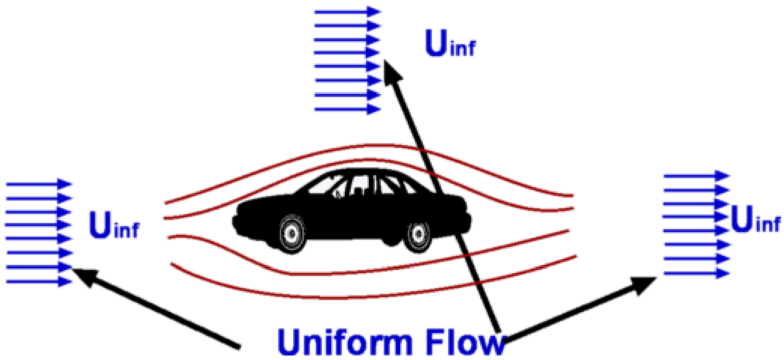

Figure 4 : Uniform Flow

Flow Description, Streamline, Pathline, Streakline and Timeline

Streamline, pathline, streakline and timeline form convenient tools to describe a flow and visualise it. They are defined below.

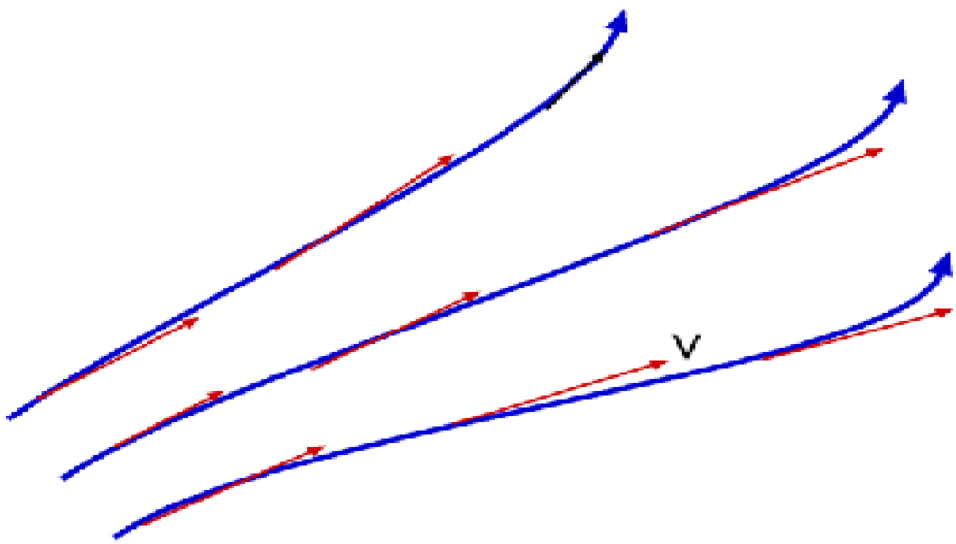

Figure 5 : Streamlines

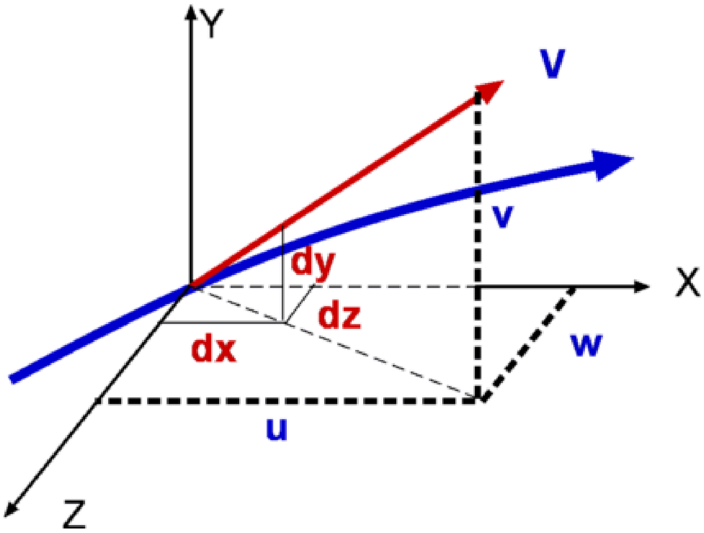

Figure 6: Streamline definition

A streamline is one that drawn is tangential to the velocity vector at every point in the flow at a given instant and forms a powerful tool in understanding flows. This definition leads to the equation for streamlines.

$$\frac{du}{u} = \frac{dv}{v} = \frac{dw}{w} \tag{7}$$

where **u,v,** and **w** are the velocity components in **x, y** and **z** directions respectively as sketched.

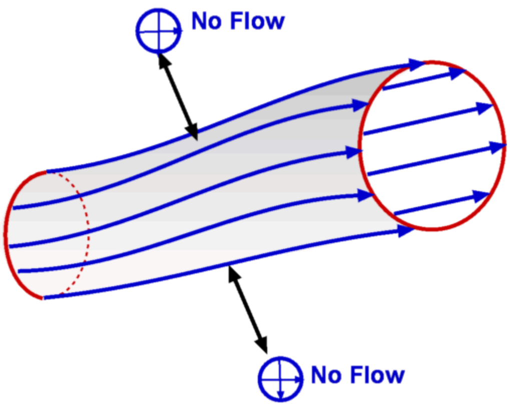

Figure 7 : Streamtube

Hidden in the definition of streamline is the fact that there cannot be a flow across it; i.e. there is no flow normal to it. Sometimes, as shown in Fig.7 we pull out a bundle of streamlines from inside of a general flow for analysis. Such a bundle is called stream tube and is very useful in analysing flows. If one aligns a coordinate along the stream tube then the flow through it is one-dimensional.

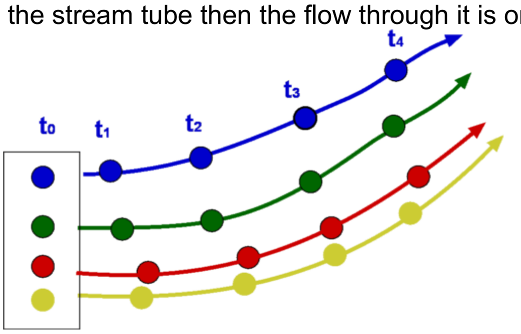

Figure 8: Pathlines

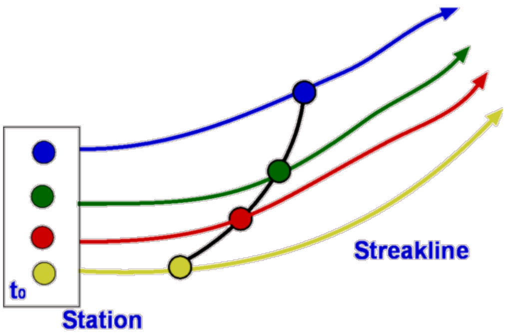

Figure 9: Streaklines

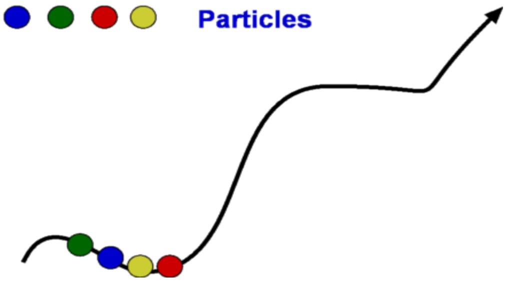

Figure 10: Timeline

Pathline is the line traced by a given particle. This is generated by injecting a dye into the fluid and following its path by photography or other means (Fig.8). Streakline concentrates on fluid particles that have gone through a fixed station or point. At some instant of time the position of all these particles are marked and a line is drawn through them. Such a line is called a streakline (Fig.9). Timeline is generated by drawing a line through adjacent particles in flow at any instant of time. Fig.10 shows a typical timeline.

In a steady flow the streamline, pathline and streakline all coincide. In an unsteady flow they can be different. Streamlines are easily generated mathematically while pathline and streaklines are obtained through experiments.

Eulerian and Lagrangian approaches

Eulerian and Lagrangian approaches seem to be the two methods to study fluid motion. The Eulerian approach concentrates on fluid properties at a point **X(x,y,z,t)**. Thus it is a field approach. In the Lagrangian approach one identifies a particle or a group of particles and follows them with time. This is bound to be a cumbersome method. But there may be situations where it is unavoidable. One such is the two phase flow involving particles.

System and Control Volume

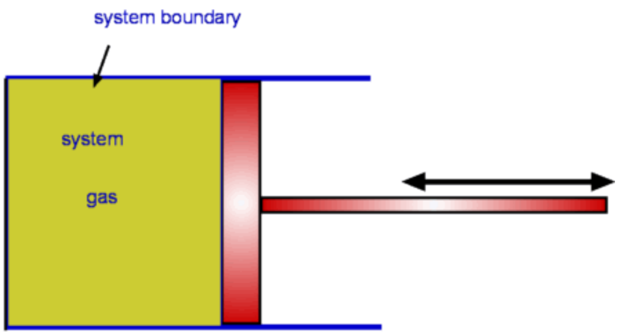

Figure 12 : Piston cylinder arrangement

Terms system and control volume are very familiar to the one who has studied thermodynamics. The word system refers to a fixed mass with a boundary. However, with time the boundary of the system may change, but the mass remains the same. The usual example given is that of a piston-cylinder arrangement as shown in Fig.12. Consider a gas filled in the cylinder which is closed by a piston at the right hand end. Let us define gas as our system. If the piston is now operated by pushing or pulling it the gas compresses or expands. The boundary of our system moves. But the mass does not move out of the boundary since by definition system is a fixed mass. The definition does not prevent work or energy crossing the boundary.

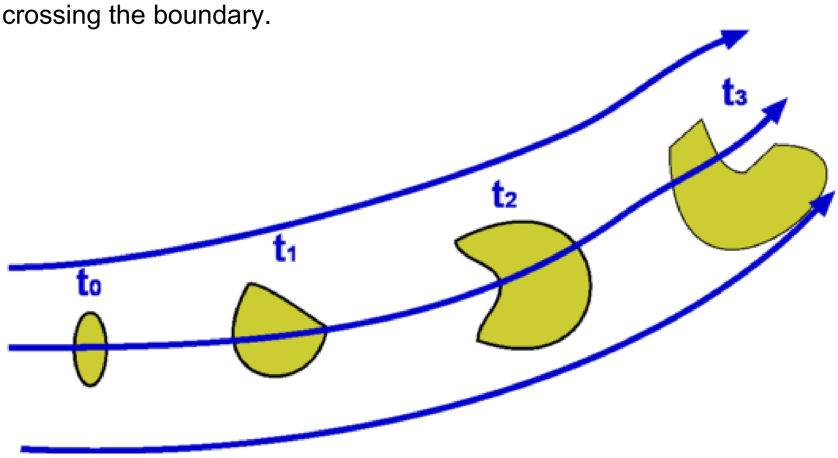

Figure 13 : System Approach

It is easy to analyse the system in the example of piston-cylinder arrangement that we have considered before. But the question is - Are all systems as simple as this? The answer is obviously a "no". In fluid dynamics we consider systems which are far more complicated. Take the flow about an aeroplane for example. If we define a system in such a flow and try to analyse it we find that it undergoes many changes as ilustrated in Fig. 1 The boundary changes rapidly and undergoes unmanageable distortions. The system approach is almost ruled out. The other examples are flow through turbomachinery, flow in hydraulic systems and many such.

The other method we have is the Control Volume approach. Here we do not focus our attention on a fixed mass of fluid. Instead we establish a "window" for observation in the flow. This is what we call the control volume shown in Fig. 14. As against the system, a control volume has a fixed boundary. Mass, momentum and energy are allowed to cross the boundary. We perform a balance of mass, momentum and energy that flow across the boundary and deduce the changes that could take place to properties of flow within the control volume. The shape of the control volume does not change normally. It is easy to visualise that this is a convenient approach. in fact, it is the one that is commonly used in fluid dynamics.

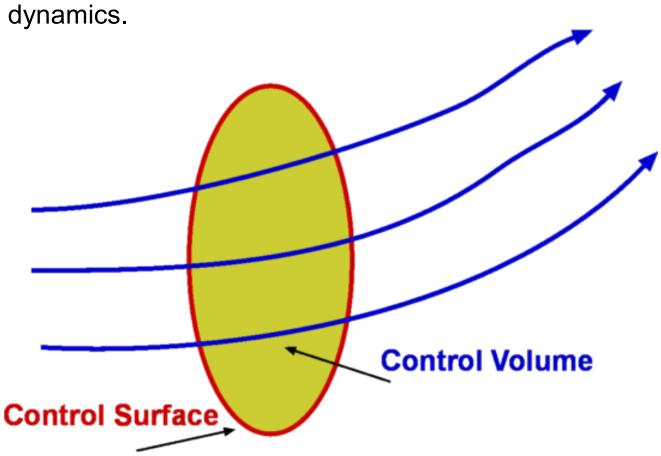

Figure 14 : Control Volume

We will consider a fixed control volume most of the time. But it is possible to have control volumes that change their boundary, those that deform etc. Obviously, these lead to more complicated equations. Examples of such control volumes are given in Fig.15.

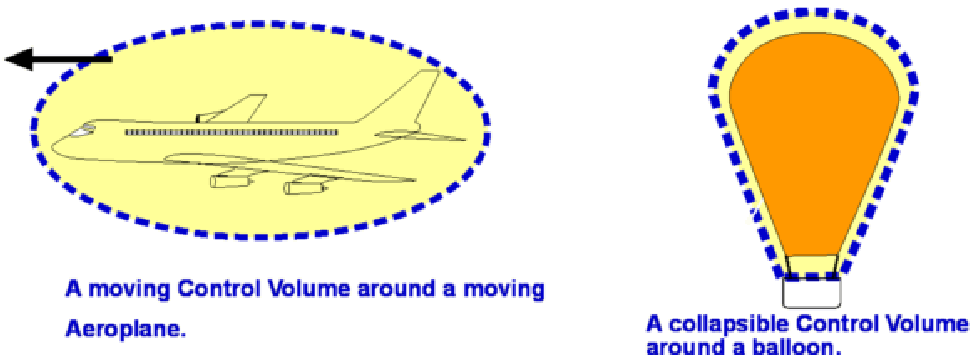

Figure 15 : Moving and Collapsible Control Volumes

The boundary of the control volume is referred to as control surface.

From the above discussion it is clear that the system and control volume approaches are akin to Lagrangian and Euler approaches.

Differential and Integral Approach

Differential approach aims to calculate flow at every point in a given flow field in the form **X(x,y,z,t)**. When we determine the flow about an aerofoil using this approach we try to obtain the needed properties like ρ, P, T everywhere within the region **R** surrounding the aerofoil as shown in Fig.16. From the detailed knowledge of the flow field we deduce features such as drag and lift. Aerofoil flow is complicated and we will have to solve the differential equations of motion. Obviously this is a costly approach calling for methods in computational fluid dynamics to be used.

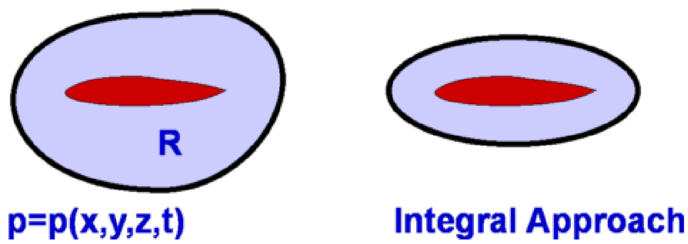

Figure 16: Differential and Integral approaches to calculate flow about an Aerofoil.

Not every flow is as complicated as an aerofoil flow. In addition there is not necessary always to get a detailed information of the flow. One may establish a big control volume to encompass the region **R** and calculate the overall features like drag and lift by studying what happens at the boundary of the control volume, i.e., at the control surface.This procedure is called the Integral approach. Both these approaches are important to us. First we discuss the integral approach. When we are more familiar with the methods to analyse the flow we take up the differential approach.

Integral Equations

Basic Laws for Fluid Flow

What laws do govern a fluid flow? Surprisingly, these are the well known conservation principles and only a handful of them. They are the same ones as will calculate problems in solid mechanics. These laws could be introduced as follows. Consider a system and its surroundings. By definition everything outside of a system is the surrounding. The system is subject to a few laws.

Conservation of Mass

Consider a system of a fixed mass, **m,** as shown in Fig 17. We know that this mass does not change and is conserved. This leads to the law of conservation of mass , namely,

$$m_{system} = constant$$

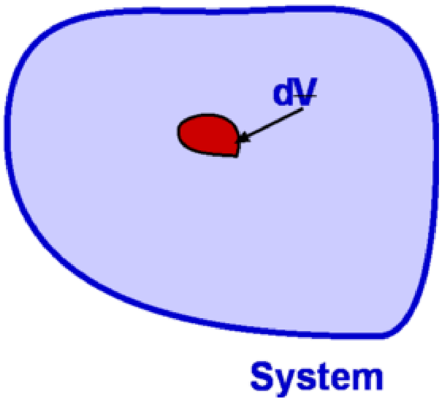

Figure 17: A System

$$\left(\frac{dm}{dt}\right)_{system}=0 \qquad \textbf{(8)}$$

with $m_{system}=\int_{system} dm=\int_{system} \rho \, d\forall$

Newton's Second Law of Motion

Newton's second law is the next one to be imposed upon fluid motion. It is known that the rate of change of momentum is proportional to the applied force. If **F** is the force upon a system,

$$F=\frac{dM}{dt} \qquad (9)$$

where **M** is the linear momentum. Further,

$$M=\int_{system} V \, dm=\int_{system} V \rho \, d\forall \qquad (10)$$

It is to be realised that momentum **M** and velocity **V** are vectors and each of a component in each of the coordinate directions. Accordingly, Eq. 10 represents three equations.

The form of this equation holds good for angular momentum. If a torque **T** acts upon the system. We have,

$$T=\frac{dH}{dt} \qquad (11)$$

$$H=\int_{system} (r\times V)\, dm=\int_{system} (r\times V)\rho \, d\forall$$

which again is a vector equation. Torque **T** can be due to body forces and/or surface forces. In addition there can also be torque directly introduced into the system such as that through a mechanical shaft connected to the system.

Conservation of Energy

The first law of thermodynamics which is a statement of the conservation of energy principle states,

$$dQ-dW=dE \qquad (12)$$

i.e., $$\frac{dQ}{dt}-\frac{dW}{dt}=\frac{dE}{dt}$$

where **dQ** is the heat added to the system, **dW** is the work done by the system and **dE** is the consequent change in energy of the system.

In addition, we have

$$E_{system} = \int_{system} e\, dm = \int_{system} e\, \rho\, d\forall \qquad (13)$$

Energy, **e** is a sum of internal energy, **u**, kinetic energy and potential energy. Thus

$$e = u + \frac{V^2}{2} + gz \qquad (14)$$

Second Law of Thermodynamics

While the first law of thermodynamics states that energy is conserved, the second law establishes a direction in which a process can take place. If **dS** is the change in entropy and **dQ** is the heat added and **T** the temperature,

$$ds \geq \frac{dQ}{T} \qquad (15)$$

In addition to the above relations we may need an equation of state $P = P(\rho, T)$,

Reynolds Transport Theorem

You may have already seen the dilemma we are in. First of all we favoured a control volume approach because it is easier and very relevant to study motion of fluids. Then we enunciated the basic laws that a fluid motion has to obey and hence lead to the equation of motion. But these are all valid for a system. The question is "How are we going to connect the basic laws for a system with a control volume approach for fluids?". This question has been foreseen by many already. The result is what is called the Reynolds Transport Theorem.

The derivation of the Reynolds Transport Theorem may seem too involved. But when the basis of the theorem is understood, it is indeed easy to follow its derivation. We shall start with a system and the rate at at which an extensive property **N** changes in it. This we try to express in terms of a corresponding intensive property η associated with the control volume, which to start with coincides with the system. To make the concept clear it seems beneficial to consider first an one-dimensional flow to derive the equation. As a second step we extend them to a general flow.

Derivation of the theorem for one-dimensional flow

Consider a stream tube in an one-dimensional flow as sketched. we remind ourselves that the flow takes place entirely through the stream tube and there is no flow across it, i.e., in a direction normal to it. Let is consider a system **S** in the flow. Let us prescribe a control volume CV coincident with it at time t_0 (Fig.18). We recall that the system is an entity of fixed mass and is allowed to move and deform. On the other hand a control volume has fixed a boundary, which we denote as CS. In this analysis we keep it stationary. After the lapse of time Δt i.e., at time $t_o + \Delta t$ we find that the control volume remains at the same position, **I+II** while the system has moved to occupy the position **II + III**. We see that during the time interval mass contained in region **I** has entered the control volume and that in **III** has left the control volume.

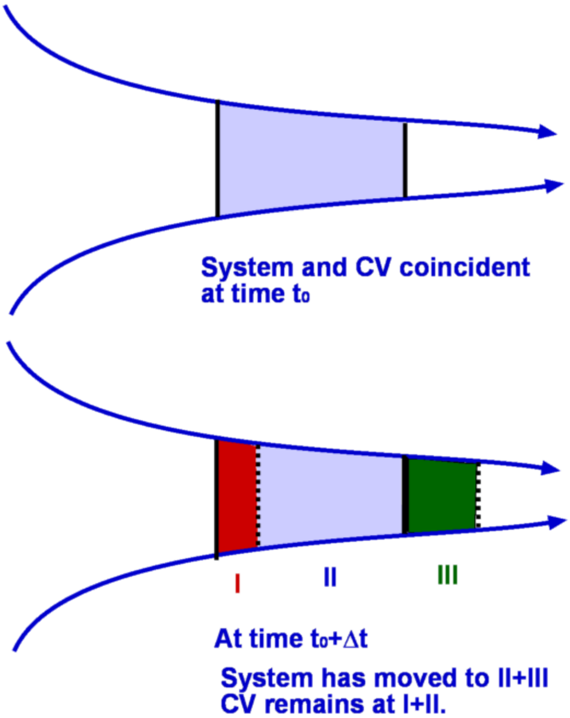

Figure 18: Control Volume and system for an one-dimensional flow

Consider an extensive property **N** associated with the control volume. By definition we have,

$$\frac{dN_s}{dt} = \lim_{\Delta t \to 0} \left(\frac{N_s(t_o + \Delta t) - N_s(t_o)}{\Delta t} \right) \tag{16}$$

where subscript **s** denotes a system.

Further we have at $t_o + \Delta t$

$$N_s(t_o + \Delta t) = (N_{II} + N_{III}) \quad at \quad t = t_o + \Delta t$$

$$N_s = (N_{CV} - N_I + N_{III}) \quad at \quad t = t_o + \Delta t \tag{17}$$

On substituting these into Eq.16 and noting that at $\mathbf{t_0}$ the system and the control volume coincide, i.e., $N_s(t_o) = N_{CV}(t_o)$, we have

$$\frac{dN_s}{dt} = \lim_{\Delta t \to 0} \left(\frac{[N_{CV} - N_I + N_{III}](t_o + \Delta t) - N_{CV}(t_o)}{\Delta t} \right) \tag{18}$$

By readjusting the terms we have,

$$\frac{dN_s}{dt} = \lim_{\Delta t \to 0} \left(\frac{N_{CV}(t_o + \Delta t) - N_{CV}(t_o)}{\Delta t} \right) + \lim_{\Delta t \to 0} \left(\frac{N_{III}(t_o + \Delta t)}{\Delta t} \right) - \lim_{\Delta t \to 0} \left(\frac{N_I(t_o + \Delta t)}{\Delta t} \right)$$

(19)

We can now take up each of the three limits on the RHS of the above equation.

The first limit gives,

$$\lim_{\Delta t \to 0}\left(\frac{N_{CV}(t_o+\Delta t)-N_{CV}(t_o)}{\Delta t}\right)=\frac{\partial N_{CV}}{\partial t}=\frac{\partial}{\partial t}\int_{CV}\eta\rho\, d\forall \quad (20)$$

recalling that **N** is an extensive property and η is the corresponding intensive property such that $N=\eta m$ where **m** is the mass given by ρ times volume, i.e, $\rho\times\forall$.

The second limit, which gives the rate of change of **N** within **III** could be written as

$$\frac{\lim_{\Delta t\to 0} N_{III}(t_o+\Delta t)}{\Delta t}=\lim_{\Delta t\to 0}\left[\frac{\eta\rho\forall}{\Delta t}\right](t_o+\Delta t) \quad (21)$$

The right hand side simply the rate at which **N** is going out of the control volume though the boundary, i.e., the control surface at right and is equal to

$$(\eta\rho A V)_{out} \quad (22)$$

where **A** is the area of cross section of **III**, **V** is the velocity normal to the area.

Similarly we have for **I**, i.e., the rate at which **N** enters the control volume through the boundary or control surface at left,

$$(\eta\rho A V)_{\text{in}} \quad (23)$$

Upon substituting Eqns. 20, 22 and 23 into Eqn. 19, we have

$$\frac{dN_s}{dt} = \frac{\partial}{\partial t}\int_{CV}\eta\rho\, d\forall + (\eta\rho A V)_{out} - (\eta\rho A V)_{\text{in}} \quad (24)$$

Eqn. 24 is the Reynolds Transport equation for the control volume considered. Each of the terms in the equation tells something significant. Putting the equation is words we have,

Rate of change of property N within the system =	Rate of change of property N within the control volume + Rate of outflow of property N through the control surface - Rate of inflow of property through the control surface.

The above result can be generalised to any control volume of any shape, but fixed in space. Let us now consider such a general control volume as shown in Fig 19. For such a control volume it is difficult to define an inlet boundary and an outlet boundary. It is best to consider the net flow of property **N** into the control volume. Accordingly, the above verbal equation is written as

Rate of change of property N within the system =	Rate of change of property N within the control volume + Net rate of change of property N through the control surface

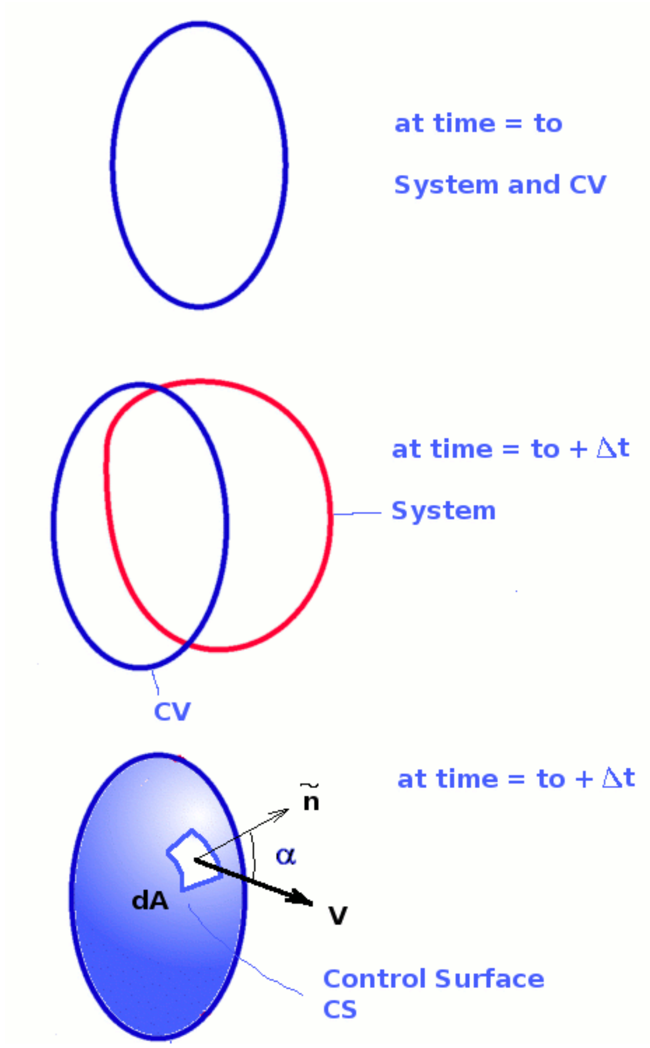

Figure 19: General Control Volume and System

Whether the flow at any small segment of control surface is an inflow or an outflow is decided by the direction of the velocity vector and that of the area vector at that segment. Consider a small area $d\vec{A}$ at the control surface (Fig.19). Let the velocity acting upon it be $\vec{V}$. The rate at which property **N** escapes or enters the control volume through $d\vec{A}$ depends upon the velocity component normal to $d\vec{A}$, i.e, $\vec{V}\,d\vec{A}$. In fact the rate of flow of **N** through $d\vec{A}$ is given by

$$\eta\,\rho\,\vec{V}\,d\vec{A} \tag{25}$$

Integrating this for the entire control surface gives the net rate of flow of **N** into the control volume. I.e,

$$\int_{CS}\eta\,\rho\,\vec{V}\,d\vec{A} \tag{26}$$

Consequently we can write the Reynolds Transport theorem for a general control volume as

$$\frac{dN_s}{dt} = \frac{\partial}{\partial t}\int_{CV}\eta\,\rho\,d\forall + \int_{CS}\eta\,\rho\,\vec{V}\,d\vec{A} \tag{27}$$

Abstract as it seems, Eqn. 27 simplifies when we consider concrete control volumes and many times becomes self-evident. This will become clear as we consider many applications of the Reynolds Transport theorem.

Conservation of Mass

First we apply the Reynolds Transport theorem, Eq.27 to derive an equation for conservation of mass. We note that in the equation, **N** is the extensive property of interest which now is mass **m**. The corresponding intensive property is

$$\eta = \frac{N}{m} = \frac{m}{m} = 1 \qquad (28)$$

Accordingly we substitute for **m** and η in Equ 27. We have

$$\frac{dm}{dt} = \frac{\partial}{\partial t}\int_{CV} \rho \, d\tilde{V} + \int_{CS} \rho \vec{V} \, d\vec{A} \qquad (29)$$

By definition that a system is an entity of fixed mass, the left hand side of the above equation is zero, thus giving the equation for conservation of mass as

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\tilde{V} + \int_{CS} \rho \vec{V} \, d\vec{A} = 0 \qquad (30)$$

which expresses that the rate of accumulation of mass within a control volume is equal to the net rate of flow of mass into the control volume. This equation is also called the Continuity Equation.

Steady Flow

For a steady flow the time derivative in the equation vanishes. As a result,

$$\int_{CS} \rho \vec{V} \, d\vec{A} = 0 \qquad (31)$$

In addition if the flow is incompressible, ρ = constant and we have

$$\int_{CS} \vec{V} \, d\vec{A} = 0 \qquad (32)$$

Incompressible Flow

The equation simplifies further when we consider an incompressible flow where density ρ is a constant. Consequently we have,

$$\rho \frac{\partial}{\partial t}\int_{CV} d\tilde{V} + \rho \int_{CS} \vec{V} \, d\vec{A} = 0 \qquad (33)$$

Dividing by density, ρ ,

$$\frac{\partial}{\partial t}\int_{CV} d\tilde{V} + \int_{CS} \vec{V} \, d\vec{A} = 0 \qquad (34)$$

The first term is the rate of change of volume within a control volume, which for a fixed control volume is zero by definition. This gives a simple form of the equation for the conservation of mass for the control volume as

$$\int_{CS} \vec{V} \, d\vec{A} = 0 \qquad (35)$$

Thus for an incompressible flow the continuity equation is the same irrespective of whether the flow is steady or unsteady.

Term V.dA

The term appears in almost all the equations for a control volume analysis - mass, momentum and energy. We need to understand it and its sign convention properly. Consider any part of a control surface and let the area be **dA**. Let the velocity vector acting on it be $\vec{V}$. We are interested in the velocity normal to the area that is convecting the mass, momentum or energy.This is given by $\vec{V}\vec{n}d\vec{A}$. This term is given in terms of scalars as

$$V\,d\,A\cos(\alpha) \qquad (36)$$

where as shown α is the angle between area vector and velocity vector.

A negative $\vec{V}d\vec{A}$ suggests an inflow into the control volume while a positive $\vec{V}d\vec{A}$ is an outflow from the control volume.

Application to an one-dimensional control volume

Consider an one-dimensional stream tube flow as shown in Fig.20. Let us mark a control volume bound by surface **1** and surface **2**. We know that there is any inflow/outflow of mass only through these two surfaces. The remaining surface **S** being made up of streamlines does not allow any mass flow through it.

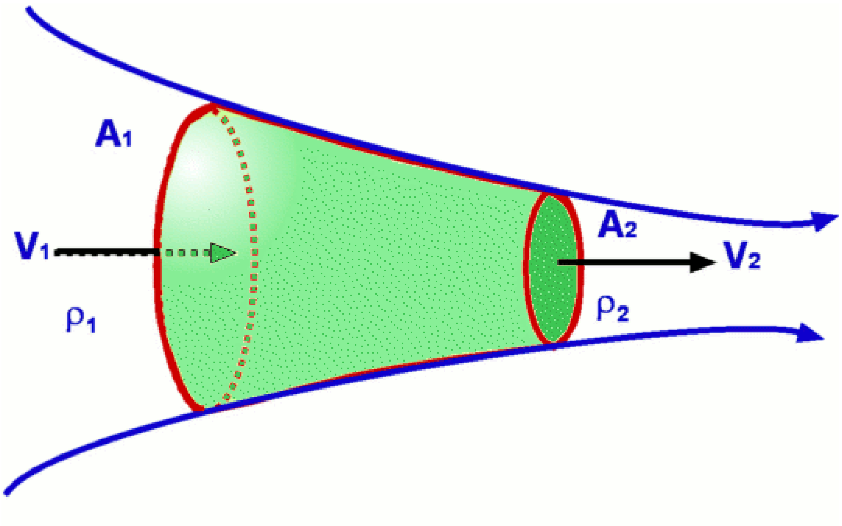

Figure 20 : Control Volume for an one-dimensional steady flow

We assume that a uniform flow prevails at surfaces **1** and **2**, V_1 and V_2 being the velocities. If the areas of cross section are A_1 and A_2, an application of the continuity Equ 31 gives

$$\int_{CS}\rho\vec{V}d\vec{A}=\int_{1}\rho\vec{V}d\vec{A}+\int_{2}\rho\vec{V}d\vec{A}=0 \qquad (37)$$

simplifying to

$$\rho_1 V_1 A_1=\rho_2 V_2 A_2 \qquad (38)$$

Momentum Equation

Let us now derive the momentum equation resulting from the Reynolds Transport theorem, Equ 27. Now we have $N=\vec{M}$ where $\vec{M}$ is the momentum. Note that momentum is a vector quantity and that it has a component in every coordinate direction. Thus,

$$N=M \quad \text{and} \quad \eta=\frac{\vec{M}}{m}=\vec{V} \qquad (39)$$

Consider the left hand side of Eqn 27. We have $\frac{d\vec{M}}{dt}$ which is proportional to the applied force as per Newton's Second Law of motion. Thus,

$$\frac{d\vec{M}}{dt}=\vec{F} \tag{40}$$

Where $\vec{F}$ is again a vector. It is necessary to include both body forces, $\vec{F}_B$ and surface forces, $\vec{F}_S$. Thus,

$$\frac{d\vec{M}}{dt}=\vec{F}_B+\vec{F}_S \tag{41}$$

Now we substitute for η in the right hand side of Eqn.27 giving,

$$\vec{F}_B+\vec{F}_S=\frac{\partial}{\partial t}\int_{CV}\vec{V}\,\rho\,d\not{V} \quad + \quad \int_{CS}\vec{V}\,\rho\,\vec{V}\,d\vec{A} \tag{42}$$

Writing this as three equations, one for each coordinate direction we have,

$$\begin{aligned}
\vec{F}_{Bx}+\vec{F}_{Sx}&=\frac{\partial}{\partial t}\int_{CV}u\,\rho\,d\not{V} \quad + \quad \int_{CS}u\,\rho\,\vec{V}\,d\vec{A}\\
\vec{F}_{By}+\vec{F}_{Sy}&=\frac{\partial}{\partial t}\int_{CV}v\,\rho\,d\not{V} \quad + \quad \int_{CS}v\,\rho\,\vec{V}\,d\vec{A}\\
\vec{F}_{Bz}+\vec{F}_{Sz}&=\frac{\partial}{\partial t}\int_{CV}w\,\rho\,d\not{V} \quad + \quad \int_{CS}w\,\rho\,\vec{V}\,d\vec{A}
\end{aligned} \tag{43}$$

The term $u\,\rho\,\vec{V}\,d\vec{A}$ represents the **u** momentum that is convected in/out by the surface $d\vec{A}$ in a direction normal to it. In fact momentum in other direction can also be convected out from the same area. These are given by $v\,\rho\,\vec{V}\,d\vec{A}$ and $w\,\rho\,\vec{V}\,d\vec{A}$.
As stated before the term $\rho\,\vec{V}\,d\vec{A}$ is replaced by $\rho\,V\,d\,A\cos(\alpha)$.

The equation thus derived finds immense application in fluid dynamic calculations such as force at the bending of a pipe, thrust developed at the foundation of a rocket nozzle, drag about an immersed body etc. We consider some of these later.

Bernoulli Equation

The momentum equation we have just derived allows us to develop the Bernoulli Equation after Bernoulli (1738). This equation basically connects pressure at any point in flow with velocity. It is one of the widely used equations in fluid dynamics to calculate pressure with the knowledge of velocity. We derive the equation for a stream tube and consider its generalisation , its applicability and limitations later.

Since we are interested in the fluid behaviour at a point consider a differential stream tube within a flow and a small control volume within it as shown in Fig.21. Since we are considering a stream tube, any flow takes place only along it and through the ends of it. The flow is therefore one dimensional in nature and takes places in a direction **s** along the stream tube. Accordingly, we denote the velocity by $\mathbf{V_s}$**.** There is no flow across the tube. Let the length of the stream tube be **ds**.

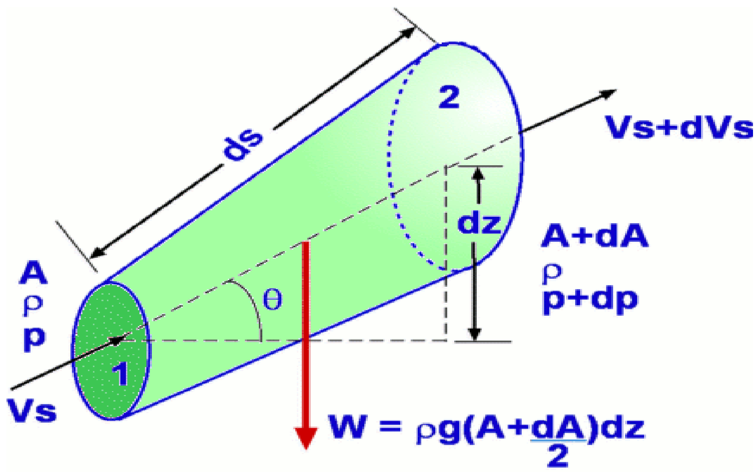

Figure 21: Differential Control Volume for an one-dimensional steady flow

Since it is a small stream tube any property changes only slightly along it. If the area, velocity, density and pressure at the left hand end i.e., the inlet end, (1) be A, V_s, ρ and **P.** Let us treat the flow as incompressible (this restriction can be removed later). We assume that at the outlet end (2) the corresponding properties to be $A+dA, V_s+dV_s, \rho+d\rho$ and $P+dP$.

Let us now apply the momentum equation to the differential control volume we have considered. In any derivations or while solving problems involving fluid flows it helps to list out the assumptions made. Accordingly we start with a listing of the assumptions.

Assumptions

1. A stream tube with no cross flow considered.
2. The flow is steady $\frac{\partial}{\partial t}=0$
3. Fluid is incompressible (ρ =constant).

It is necessary to note that any application of the momentum equation should be preceded by the Continuity Equation. We cannot obtain complete information about the flow by applying momentum equation alone.

Application of Continuity Equation

Equ 30 gives

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\forall \quad + \quad \int_{CS}\rho \vec{V}_s d\vec{A} \quad = \quad 0$$

The first term in the equation cancels out because of the steady flow assumption (2). Since all the flow takes place through (1) and (2) only the remaining term reduces to

$$-\rho V_s A+\rho(V_s+dV_s)(A+dA)=0$$

giving

$$\rho(V_s+dV_s)(A+dA) \quad = \quad \rho V_s A \quad = \quad \dot{m} \qquad (44)$$

where $\dot{m}$ is the mass flow rate through the control volume.

Application of Momentum Equation

From Eqn. 43 we have the momentum Equation-

$$F_{Bs}+F_{Ss}=\frac{\partial}{\partial t}\int_{CV} V_s \rho \, d\forall \quad + \quad \int_{CS} V_s \rho \vec{V}_s d\vec{A} \tag{45}$$

Since the flow is steady, the first term on the RHS drops out. We need to evaluate the body forces F_{Bs} and surface forces F_{Ss} acting on the control volume.

Body Force.

The only body force acting is the weight of the fluid within the control volume. We need to consider the component of this in the **s** direction. Accordingly,

$$F_{Bs}=-dW\sin(\theta)$$

$$F_{Bs}=-(m)g\sin(\theta)$$

$$F_{Bs}=-(\rho \, d\forall)g\sin(\theta)$$

$$F_{Bs}=-\left(\rho ds(A+\frac{dA}{2})\right)g\sin(\theta)$$

$$F_{Bs}=-\rho g\left(A+\frac{dA}{2}\right)dz$$

$$F_{Bs}=-\rho g A\,dz \tag{46}$$

Surface Forces

The surface force is due to pressure acting upon the boundaries of the control surface. There are three terms that contribute - end (1), end (2) and the bounding surface of the stream tube. Force on each of these is given by the product of pressure and area. For the bounding surface we take this force to be the product of an average pressure, $\frac{1}{2}(P+(P+dP))=P+\frac{dP}{2}$ multiplied by the effective area, **dA**. Thus we have for the surface forces,

$$F_{Ss}=PA-(P+dP)(A+dA)+(P+\frac{dP}{2})dA$$

$$F_{Ss}=PA-PA-PdA-AdP-dPdA+PdA+\frac{dP}{2}dA$$

$$F_{Ss}=-A\,dP \tag{47}$$

Terms on the Right Hand Side

we have on the **RHS** of Eqn 45.,

$$-V_s\rho A V_s+(V_s+dV_s)\rho(A+dA)(V_s+dV_s)$$

substituting for **V**$_s$ ρ **A V**$_s$ from Eqn. 44 , we have RHS =

$$-V_s \rho A V_s + (V_s + dV_s) \rho A V_s = \rho A V_s dV_s \tag{48}$$

Now collecting terms for the LHS and RHS we have,

$$\rho A V_s dV_s = -\rho g A\, dz - A\, dP$$

i.e.,

$$V_s dV_s + g\,dz + \frac{dP}{\rho} = 0$$

i.e.,

$$d\left(\frac{V_S^2}{2}\right) + g\,dz + \frac{dP}{\rho} = 0 \tag{49}$$

The above equation is readily integrated for an incompressible flow (ρ =constant). As a result we have,

$$\frac{P}{\rho} + \frac{V_s^2}{2} + g z = constant \tag{50}$$

Equation 50 is called the Bernoulli Equation. Note that it connects pressure (**P**), elevation (**z**) and velocity (**V**$_s$). Once it is understood that the equation is valid along a streamline (i.e., within a stream tube) we can drop the subscript **s** for velocity giving,

$$\frac{P}{\rho} + \frac{V^2}{2} + g z = constant \tag{51}$$

It may be pointed out that the equation is valid for steady flows only in absence of any friction such as the one due to viscosity. Further the flow is to be incompressible. We will derive the Bernoulli equation again but based on energy considerations.

Application to moving Control Volumes

The Continuity and the Momentum Equations we have derived can be extended to cases where the control volume is not fixed in space. One such case is when the control volume is moving with a constant velocity, say an aeroplane or a ship moving at a constant speed. Note that the equations we have derived assume that the speeds are all referred to the control volume. So it becomes a simple matter to consider a control volume moving at a constant speed, **V**$_{cv}$. Define

$$V_{rel} = V - V_{CV} \tag{52}$$

which now is the speed relative to the control volume. The equation for Reynolds Transport theorem, Eqn 27 gets altered as

$$\frac{dN_s}{dt} = \frac{\partial}{\partial t}\int_{CV} \eta \rho\, d\forall + \int_{CS} \eta \rho \vec{V}_{rel} d\vec{A} \tag{53}$$

Equation for Angular Momentum

Many of the flow devices and machinery involve rotating components. Examples are Centrifugal pumps, Turbines and Compressors. The analysis of such systems is facilitated by the Reynolds Transport theorem written for angular momentum. we have from Eqn.41,

$$T = \frac{dH}{dt}$$

where,

$$H=\int_{system}(r\times V)\,dm=\int_{system}(r\times V)\,\rho\,d\forall$$

It becomes necessary now to calculate the angular momentum about some point, say **O**. Then we have,

$$N=H_o \quad \text{and} \quad \eta=\frac{dH_o}{dm}=(r\times V) \qquad (54)$$

Substitution into the equation for Reynolds theorem (Eqn 27) gives,

$$\frac{dH_{os}}{dt}=\frac{\partial}{\partial t}\int_{CV}(r\times V)\,\rho\,d\forall+\int_{CS}(r\times V)\,\rho\,\vec{V}\,d\vec{A} \qquad (55)$$

Further, the LHS of the above equation is the sum of all the moments about the point **O**, ie., $\Sigma(r\times F)$. Accordingly we have,

$$\Sigma(r\times F)=\frac{\partial}{\partial t}\int_{CV}(r\times V)\,\rho\,d\forall+\int_{CS}(r\times V)\,\rho\,\vec{V}\,d\vec{A} \qquad (56)$$

Deformable Control Volumes and Control Volumes with non-inertial acceleration

It is possible to extend our analysis to the general cases of deformable control volumes and those that undergo acceleration. But these are not necessary in a first course in fluid mechanics. However, there are many textbooks that do cover such advanced topics.

Energy Equation

We now apply the Reynolds Transport theorem (Eqn 27) to derive an equation for energy conservation in a control volume. Now we have,

$$N=E \quad \text{and} \quad \eta=\frac{E}{m}=e \qquad (57)$$

On the LHS we have $\frac{dE}{dt}$, which from the First Law of Thermodynamics is

$$\frac{dE}{dt}=\dot{Q}-\dot{W} \qquad (58)$$

Where $\dot{Q}$ is the rate at which heat is added to the system and $\dot{W}$ is the rate at which work is done on/by the system.

Substituting in Eqn 27 we have

$$\dot{Q}-\dot{W}=\frac{\partial}{\partial t}\int_{CV}e\,\rho\,d\forall+\int_{CS}e\,\rho\,\vec{V}\,d\vec{A} \qquad (59)$$

In the above equation, **e** should include all forms of energy - internal, potential, kinetic and others. The *others* category will include nuclear, electromagnetic and other sources of energy. But for simple fluid flows these are not important. Fields such as Magneto Hydrodynamics and Relativistic Fluid Dynamics will involve these forms of energy too. We have then

$$e=u+gz+\frac{1}{2}V^2 \qquad (60)$$

Concerning work, we have different kinds - shaft work, **W**$_s$, work done by pressure, **W**$_p$ and work due to shear forces on the control surface. Shaft work includes any work that is directly added to the system by means of a pump, piston etc. Work done by pressure is calculated as

$$d\dot{W}_P = -P\,dA\,V_n = P\,dA\,V \tag{61}$$

where **dA** ia an elemental area over the control surface, the velocity **V**$_n$ is into the control volume (hence gets a negative sign). This equation is integrated over the control surface to obtain the total work due to pressure. Thus,

$$\dot{W}_P = \int_{CS} P\,dA.V \tag{62}$$

Work due to shear forces is small and is usually neglected. Heat added $\dot{Q}$ becomes important only occasionally in problems involving heat transfer. Upon substituting for various terms we have,

$$\dot{Q} - \dot{W}_s - \dot{W}_P = \frac{\partial}{\partial t}\int_{CV}(u + gz + \frac{1}{2}V^2)\rho\,d\forall + \int_{CS}(u + gz + \frac{1}{2}V^2)\rho\,\vec{V}\,d\vec{A}$$

i.e.,

$$\dot{Q} - \dot{W}_s - \int_{CS} P\,dA.V = \frac{\partial}{\partial t}\int_{CV}(u + gz + \frac{1}{2}V^2)\rho\,d\forall + \int_{CS}(u + gz + \frac{1}{2}V^2)\rho\,\vec{V}\,d\vec{A}$$

$$\dot{Q} - \dot{W}_s = \frac{\partial}{\partial t}\int_{CV}(u + gz + \frac{1}{2}V^2)\rho\,d\forall + \int_{CS}(u + \frac{P}{\rho} + gz + \frac{1}{2}V^2)\rho\,\vec{V}\,d\vec{A}$$

$$\dot{Q} - \dot{W}_s = \frac{\partial}{\partial t}\int_{CV}(u + gz + \frac{1}{2}V^2)\rho\,d\forall + \int_{CS}(h + gz + \frac{1}{2}V^2)\rho\,\vec{V}\,d\vec{A} \tag{63}$$

where **h** is specific enthalpy given by $u + \frac{P}{\rho}$. Equation 63 is the general form of the Energy Equation for a control volume.

Energy equation for a one-dimensional control volume

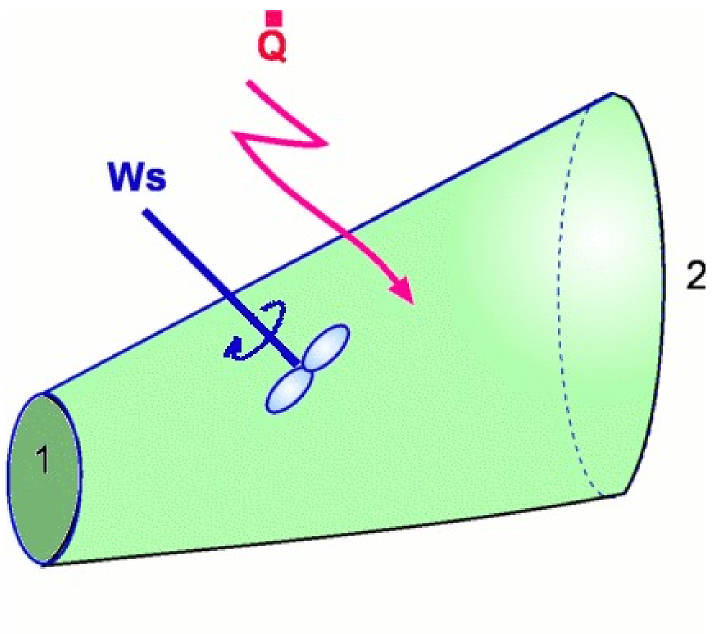

Figure 22 : Control Volume for a one-dimensional steady flow

Consider the one-dimensional control volume that we have analysed before and shown in Fig 22. If we interpret the velocity, density, pressure and other variables to be uniform across the ends or that they are the averaged values we have for a steady flow

$$\dot{Q}-\dot{W}_s=-(\rho A V)_1\left(h+gz+\frac{1}{2}V^2\right)_1+(\rho A V)_2\left(h+gz+\frac{1}{2}V^2\right)_2 \tag{64}$$

Note that for continuity, $(\rho A V)_1=(\rho A V)_2=\dot{m}$

$$\dot{Q}-\dot{W}_s=-\dot{m}\left(h+gz+\frac{1}{2}V^2\right)_1+\dot{m}\left(h+gz+\frac{1}{2}V^2\right)_2 \tag{65}$$

On division by $\dot{m}$ and denoting $\frac{\dot{Q}}{\dot{m}}$ by **q** and $\frac{\dot{W}}{\dot{m}}$ by $\mathbf{w_s}$ we have after rearrangement of terms,

$$\left(h+gz+\frac{1}{2}V^2\right)_1=\left(h+gz+\frac{1}{2}V^2\right)_2-q+w_s \tag{66}$$

Note that the term $\left(h+gz+\frac{1}{2}V^2\right)$ is equal to the Total enthalpy denoted by $\mathbf{H_0}$. Accordingly the Eqn 66 becomes

$$H_{O1}=H_{O2}-q+w_s \tag{67}$$

That is to say that the total enthalpy of a control volume is conserved unless heat or work is added to / taken out of the control volume.

Low Speed Application

In Low Speed application, especially in civil engineering, it is usual to express energy as a Head, with each of the terms in Eqn 66 having the units of a Length, **m**. This is done by dividing the equation throughout by **g**. Thus,

$$\left(\frac{u}{g}+\frac{P}{\rho g}+z+\frac{V^2}{2g}\right)_1=\left(\frac{u}{g}+\frac{P}{\rho g}+z+\frac{V^2}{2g}\right)_2-\frac{q}{g}+\frac{w_s}{g}$$

or

$$\left(\frac{u}{g}+\frac{P}{\gamma}+z+\frac{V^2}{2g}\right)_1=\left(\frac{u}{g}+\frac{P}{\gamma}+z+\frac{V^2}{2g}\right)_2-h_q+h_s \tag{68}$$

The term $\frac{P}{\gamma}$ is called the Pressure Head and $\frac{V^2}{2g}$ the velocity Head. Terms $\mathbf{h_q}$ and $\mathbf{h_s}$ represent the heat added and shaft work converted to "head" units.

If we consider a simple pipe flow without the shaft work then the equation becomes

$$\left(\frac{P}{\gamma}+z+\frac{V^2}{2g}\right)_1=\left(\frac{P}{\gamma}+z+\frac{V^2}{2g}\right)_2+\frac{u_2-u_1-q}{g} \tag{69}$$

The terms within the parenthesis is what is called the Total Head or Available Head. Clearly with the flow some available head is lost because of friction and heat transfer. It is a common practice to use the above equation in the following form,

$$\left(\frac{P}{\gamma}+z+\frac{V^2}{2g}\right)_1=\left(\frac{P}{\gamma}+z+\frac{V^2}{2g}\right)_2-h_{friction}-h_{pump}+h_{turbine} \tag{70}$$

The losses that take place between "inlet" i.e., (1) and "outlet" i.e., (2)2 are obtained through measurements and correlations.

Relationship between Energy Equation and Bernoulli Equation

An examination of Eqns 70. and 51 brings out the connection between the Energy equation and the Bernoulli equation. It is clear that two equations become one when losses that occur between (1) and (2) are ignored.

Bernoulli Equation can be used only when we are considering a frictionless flow along a streamline. Further it is required that the flow be incompressible without any addition of heat or shaft work.

Bernoulli Equation for Aerodynamic Flow

In aerodynamics one deals with considerably higher speeds than in flows of interest to civil engineers. An aeroplane flies at speeds of the order of 500 kmph and more, while river flows or household pipe flows may involve 10 kmph or so. Consequently, the kinetic energy in aerodynamic flows is very large when compared to the potential energy. Accordingly, it is usual to neglect potential energy for such flows. The Bernoulli Equation as a consequence becomes,

$$P+\frac{1}{2}\rho V^2=constant \tag{71}$$

Stagnation Pressure

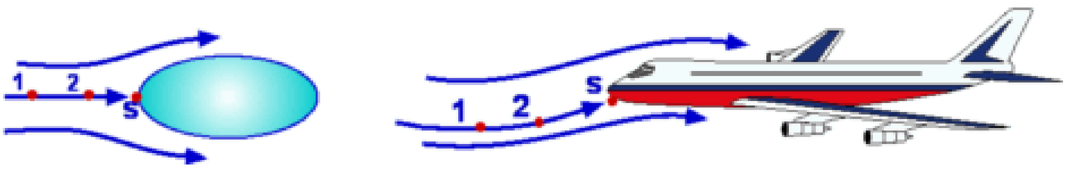

Figure 23 : Stagnation Point on (a) Simple Body and (b) a complicated Body

Consider the application of the above form of Bernoulli equation for the flow about a body such as an aeroplane as shown in Fig 23 Let **1-s** be a streamline that passes through the stagnation point of the flow, i.e., the point where the flow is brought to rest or where the velocity is zero. Assuming constant density and no losses then we can apply the Bernoulli equation along **1-s** we have,

$$P_1+\frac{1}{2}\rho V_1^2=P_2+\frac{1}{2}\rho V_2^2=P_3+\frac{1}{2}\rho V_3^2= \quad ... \quad =P_S+\frac{1}{2}\rho V_S^2 \tag{72}$$

where $\mathbf{P_s}$ and $\mathbf{V_s}$ are the pressure and velocity at the point **s**. It is known that $\mathbf{V_s= 0}$. Therefore,

$$P_1+\frac{1}{2}\rho V_1^2=P_2+\frac{1}{2}\rho V_2^2=P_3+\frac{1}{2}\rho V_3^2= \quad ... \quad =P_S \tag{73}$$

$\mathbf{P_s}$ is referred to as Stagnation Pressure. Obviously it is the maximum pressure experienced by the fluid. It becomes a very convenient constant for the Bernoulli Equation for aerodynamics flows. It is the pressure experienced by the fluid when it is brought to rest. it is as if the kinetic energy of the flowing fluid is converted into pressure as a consequence of the fluid being brought to rest.

The term "**P**" is the pressure seen by the moving fluid and is referred to as Static Pressure.

Energy Grade Line

Terms Energy Grade Line and Hydraulic Grade Line are frequently used by hydraulic engineers. Let us express each of the terms of the Bernoulli equation as a head. We have seen that in absence of work and heat transfer,

$$z_1 + \frac{P_1}{\gamma} + \frac{V_1^2}{2g} = z_2 + \frac{P_2}{\gamma} + \frac{V_2^2}{2g} = \quad \ldots \quad = H \tag{74}$$

where term **H** is not to be mistaken for enthalpy and is to be taken as the total head. If the above equation is graphically represented we see that the total energy value being constant becomes a horizontal line as shown in Fig 24 and is called the Energy Grade Line. One other line that is defined is the Hydraulic Grade Line, which is the Energy Grade Line take away the velocity head (i.e., $\mathbf{V^2/2g}$).

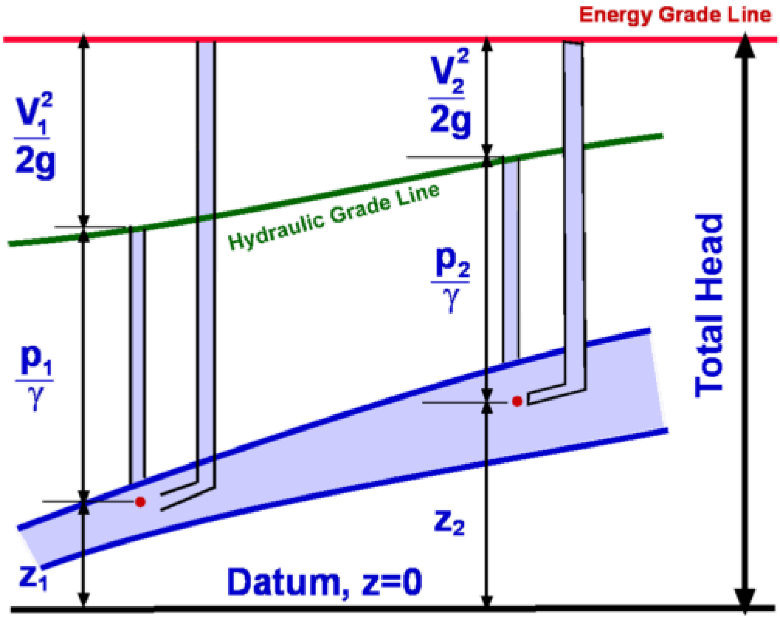

Figure 24: Energy Grade Line (EGL) and Hydraulic Grade Line (HGL) for an one-dimensional flow.

If the losses are taken into account the EGL will drop accordingly. Any work extraction along the path as with a turbine, will be seen as a sudden drop in the EGL. Any work addition will be reflected as a sharp rise. HGL follows similar trends.

Kinetic Energy Correction Factor

We have assumed in the derivation of Bernoulli equation that the velocity at the end sections (1) and (2) is uniform. But in a practical situation this may not be the case and the velocity can very across the cross section. A remedy is to use a correction factor for the kinetic energy term in the equation. If $\bar{V}$ is the average velocity at an end section then we can write for energy,

$$\int_A \frac{V^2}{2} \rho V \, dA = \alpha \dot{m} \frac{\bar{V}^2}{2} \tag{75}$$

After simplification we find that

$$\alpha = \frac{1}{A} \int_A \left(\frac{u}{\bar{V}} \right)^3 dA \tag{76}$$

Consequently, Eqn 70 is written as

$$\left(\frac{P}{\gamma} + z + \alpha_1 \frac{V^2}{2g} \right)_1 = \left(\frac{P}{\gamma} + z + \alpha_2 \frac{V^2}{2g} \right)_2 - h_{friction} - h_{pump} + h_{turbine} \tag{77}$$

where α is the Kinetic Energy Factor. Its value for a fully developed laminar pipe flow is around **2**, whereas for a turbulent pipe flow it is between **1.04** to **1.11.** It is usual to take it is **1** for a turbulent flow. It should not be neglected for a laminar flow.

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