Embodiment as a Strategy for Mathematics Education

Michael Eisenberg
University of Colorado, Dept. of Computer Science
Boulder, CO USA
duck@cs.colorado.edu

ABSTRACT
Embodiment has long been regarded as an important (if sometimes intellectually suspect) means of introducing students to mathematical concepts such as whole numbers, fractions, geometric forms, and so forth. Indeed, the venerable educational tradition of creating "mathematical manipulatives" may be seen as an attempt to exploit students' physical intuitions as a stepping-stone toward the understanding of abstract mathematical concepts. Historically, however, this tradition of manipulative design has tended to focus on relatively elementary mathematical concepts; and there has been little systematic effort to create tangible artifacts and activities that promote an introduction to more advanced mathematical ideas typically taught at the high school, undergraduate, and graduate levels. This paper argues that a productive strategy for expanding the range of embodied design is to focus precisely on this underexplored landscape of "difficult" mathematics. We sketch some potential areas in which novel technologies could enable an expansion of "embodied advanced mathematics".

Keywords
Embodied cognition, educational technology, mathematics education.

INTRODUCTION
In mathematics education, the notion of "embodiment" as a pedagogical strategy is weighted down with a long history of philosophical debate. On the one hand, the subject matter of mathematics deals fundamentally with abstractions–numbers, functions, groups, and so forth–whose generality and importance are derived, arguably, from their immateriality. None of us has ever seen or touched a number, a straight line, a bijection, or a set; and mathematical expertise consists, in part, of being able to operate on such notions as "pure" abstractions. At the same time, our first encounters with these ideas are motivated and mediated by physical experience: small collections of discrete objects inform us about numbers, sketches and craftwork introduce us to geometric shapes, repetitive or stereotyped tasks provide us with intuitions about algorithms and functions. In other words, even if embodied activities and ideas do not represent the endpoint (or the desired endpoint) of mathematical expertise, they appear to represent an inevitable and perhaps crucial step in the development of that expertise. (Cf. the introductory section of Chao et al. [1] for a good discussion of the ways in which manipulatives are hypothesized to help students.)

In practice, this viewpoint is reflected in the use of a wide variety of "mathematical manipulatives" in classrooms worldwide. Such artifacts include (among many others): rods of varying lengths to represent whole numbers; balance beams to provided physical intuitions about multiplication; clock faces to illustrate modular arithmetic; and pegboards to introduce notions of geometric shapes. It should be stressed that these artifacts are by no means "royal roads" to surefire mathematical understanding, as researchers such as Resnick and Omanson [4] and Uttal *et al.* [7] have argued; for instance, understanding the implied connection between a manipulative and its abstract referent may itself be a difficult task, requiring explicit instruction and practice. Nonetheless, allowing for these cautionary notes, the continued prominence and use of manipulatives in mathematics classroom reflects, we believe, at least the pragmatic confidence of educators that they have positive impact (likely both intellectual and affective) on children's mathematical development.

The intent of this article is not, in any event, to debate the merits of manipulatives in general or any one manipulative in particular; for our purposes, we accept that given appropriate circumstances (which may well included explicit instruction), the mathematical metaphors realized by concrete manipulatives can be both helpful and motivating in math education. Rather, our goal here is to reflect on the potential for novel technologies to enhance the exploration of as-yet-uncharted territory for manipulatives. This is a notion pioneered by the work of Resnick and his colleagues at the MIT Media Lab in their design of "digital manipulatives" [5]; but those efforts, important as they were (and are), represent only a portion of a much more expansive investigation of manipulatives that is still waiting to be done.

In pursuing this investigation, we might well begin with an important issue (though not by any means the only important issue): namely, the choice of mathematical content represented by manipulatives. Broadly speaking, manipulative designers seem to focus almost exclusively on elementary mathematics: whole numbers, fractions, simple two-dimensional shapes, and so forth. This is not a universal rule–some popular mathematical puzzles might be cast as "manipulatives" for more advanced content. (One might, for example, plausibly describe an artifact like the Rubik's Cube puzzle as a manipulative for the subject of group theory,) Still, the standard, garden-variety examples of classroom manipulatives are virtually all drawn from elementary mathematics.

It is worthwhile to speculate on why this should be the case. Why don't we see manipulatives designed for use by high school students, or undergraduates–or, for that matter, by professional mathematicians? In part, the answer may lie in biases originating in the constructivist learning theories articulated by Piaget and others, in which the use of concrete objects is associated with an earlier stage of cognition–a stage that precedes facility with abstract concepts. For Piagetian theory, then, it is singularly appropriate to employ concrete manipulatives at an earlier stage of cognition (the "concrete operational" stage), and less appropriate (or even unnecessary) to employ such objects at the later, "formal operational", stage.

This intellectual stance–in which there is presumably less need for manipulatives as "abstraction training wheels" for older students–is reinforced by all sorts of subtle cultural biases and traditions as well. The culture of higher-level mathematics is undeniably more focused on issues such as proof and symbol manipulation than on "building intuitions" in the informal way typical of manipulative use. Indeed, there is a historical thread of distrust for physical intuitions exemplified by the "Bourbaki" school of mathematics education, which largely shunned even the use of visual aids such as diagrams. (Cf. [6]) The Bourbaki mathematicians felt that the goal of pure mathematics education should be to emphasize "rigor" (and presumably felt that visual intuition was inconstant with that goal). Admittedly, the Bourbaki school represents a somewhat extreme point of view, pedagogically speaking, in its near-total reliance on symbolic presentation; but this is an exaggerated version of a subtle tone of distrust in visual/embodied intuition that appears to run through much of the advanced mathematics curriculum.

We feel that this distrust is as misplaced for higher-level mathematics as it is for elementary mathematics. Indeed, there are plausible counterarguments to be made in favor of a renewed exploration of manipulative design for more advanced students. First, one might argue that there is an even greater call for intuition-building when the subject matter is advanced or arcane than there is for elementary mathematics. After all, a young child learning about whole numbers can find, in her everyday environment, a variety of ready-made "manipulatives" for the purpose: money, calendar numbers, floor numbers of tall buildings, and so forth. But an undergraduate attempting to learn about (say) Laplace transforms, quaternions, or complex polynomials has virtually no support in doing so from her day-to-day environment. If anything, this should suggest that there ought to be a proportionally greater effort in designing educational artifacts for those topics whose "environmental support" is invisible.

Beyond this, there is an affective or motivational component to the use of manipulatives that should spur exploration in this area. For many students, mathematics is a forbidding subject precisely because of its distance from physical intuition. How does one form embodied representations of matrices, transcendental numbers, or higher-dimensional spaces? The very difficulty of the task reflects the essential dividing ground between what we tend to think of as "elementary" and "advanced" mathematics. (Cf. also Eisenberg and DiBiase [3].)

The time is now right, we believe, for a renewed exploration of embodied design aimed at topics in higher mathematics. As noted above, there is both an intellectual argument to be made–that advanced mathematics is singularly in need of intuitive examples and activities–and an affective argument to be made–that embodied activities form a motivational bridge to the study of higher math. Perhaps most provocatively, the advent of a wide variety of accessible new technologies–fabrication tools, projection devices, responsive or adaptive materials, embedded computation–collectively offer a tremendous opportunity for trying out novel designs.

The remainder of this paper is devoted to briefly sketching out some of the possible directions for creating embodied representations of ideas in higher mathematics. Individually, each of these ideas may prove infeasible or unsuccessful; but collectively, they illustrate the sorts of ideas that we believe to be most promising for the near future of mathematics education. It should also be stressed beforehand that although the mathematical topics targeted through these examples are "advanced", the artifacts themselves need not be thought of as exclusively for older students. In fact, it might well be the case that this new genre of manipulatives could prove entertaining and motivating for younger children as well, and could thus introduce younger children to ideas supposedly "beyond their age".

New-Wave Manipulatives: a Sampler

1.Embodying the Concepts Behind Fourier Transforms

One of the more difficult topics for undergraduates studying (e.g.) signal processing in electrical engineering is the Fourier transform, in which a signal represented in the time domain may be uniquely re-represented in the frequency domain. Students have little in the way of physical intuition to help understand this concept.

We might imagine, then, constructing a device that combines a computational display, including several long rectangular windows, with a sturdy "pump-handle-like" input device, as sketched in Figure 1. The essential idea behind this device would be that the top display shows a periodic signal scrolling by at a constant rate. The student then attempts to match the periodicity of the signal by pumping the handle at the side of the device; her movements are mirrored in the signal shown in the second window. Once the student has matched the basic frequency of the topmost signal (or, conceivably, once she has matched a multiple of that frequency), that frequency component of the topmost signal is displayed in the bottom window.

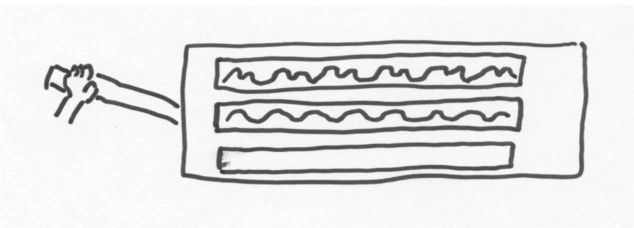

Figure 1. A rough sketch of a "Fourier series manipulative" as described in the text. A student pumps a handle at left that creates a waveform shown in the middle window of the device. That periodic component may then be subtracted from the signal in the top window to produce a reduced signal (which might be displayed in the bottom window, though that is depicted as blank in the figure).

The crucial point here is that a device such as this could allow a student to "feel" the component frequencies of a composite signal through the use of a physical input device. The basic design here might be extended so that a variety of wave-composition activities could be explored. For instance, the student might repeatedly subtract component frequencies from the topmost signal, eliminating components one by one until the original signal is flat; or she might add a sequence of "pumped" signals to create a composite signal that would be shown, as each new component is added, in the upper window.

2.Interacting with Non-Euclidean Geometry: Projection on Specialized Surfaces

For many students of geometry, the transition from the intuitions of Euclidean to non-Euclidean systems is baffling. How could it possibly be the case that (as in spherical geometry) a triangle might have three 90-degree angles; or that similar triangles must also be congruent? How could it be the case that (as in hyperbolic geometry) there may be multiple parallels drawn to a given line through a chosen point off the line?

It is feasible, we believe, to create devices through which students may interactively experiment with computer graphics on surfaces such as a sphere, concave bowl, or cube. In effect, we have begun work on such a system, developing an interactive programming system for use with a large spherical display at our university's planetarium [2]; in this system, students can write programs that move a Logo-style "turtle" about on the sphere in accordance with the rules of spherical geometry.

Exciting as this initial effort is proving to be, it really represents only one of what could be a much more extended genre of interactive systems derived from projecting computer graphics onto a standing surface. Consider, as a simple example, the possibility of suspending a large cube from the ceiling of a room, and employing six projectors to create images on each face of the cube, as sketched in Figure 2. If the projectors are controlled by a computer that coordinates the six projected images in the appropriate way, one could develop a "graphics-on-the-cube" system similar to the aforementioned spherical display.

For this device, one could experiment with "cubic geometry" (which in many respects is similar to spherical geometry, though here the surface curvature is concentrated at the eight vertices of the cube): for example, one can make an equiangular triangle with three right angles by having that triangle surround one of the vertices of the cube. Beyond writing programs to display on such a surface, one might also experiment with pointing devices (e.g., a laser-pointer) to act as a mouse-like selection tool for points on the surface; the act of pointing or drawing lines "directly" on a surface would likely prove an important element in building embodied intuitions about non-planar geometry. Finally, it should be noted that a "cubic display" is but one example of a surface display that might be explored in this fashion: for example, a set of projectors employed to create images on the surface of a pseudosphere could allow students to experiment, interactively, with the (notoriously non-intuitive) rules of hyperbolic geometry.

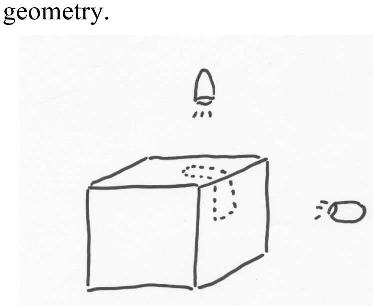

Figure 2. Displaying a curve on a cubic surface. Here, two projectors (of a presumed six) are shown projecting a continuous closed curve drawn on two faces of the cube. The projectors are controlled by a desktop computer whose job is to coordinate the separate light sources in accordance with the forms drawn on the cube's surface.

3. Handheld Objects to Represent Infinite Series

Consider the infinite series of values: 1, ½, ¼, … in which each successive value is half that of the previous one. Such series usually represent something of a hurdle to students when they are first encountered (often in connection with a calculus course): for instance, the series approaches 0 as a limiting value, and the sum of the values in the series is well-defined (with the value 2). For students, series of this type may be an early (and confusing) encounter with manipulating objects that are represented as potentially infinite symbolic expressions.

We might experiment with the creation of "manipulatives"–handheld artifacts with embedded computation–to represent objects of this sort. For example, suppose we imagine a small computational device, about the size of a pocket-watch, equipped with a display screen and a rotating dial, as sketched in Figure 3. The idea behind this device is that one can treat it as containing, in one finite space, the entirety of an infinite series of symbols that can only be viewed a bit at a time through the display screen. In the figure, we are looking at a few of the terms of the infinite series mentioned above; by turning the dial, we could "scroll through" more terms of the series for as long as we wish (in practice, of course, the user will only look at a finite number of terms, so the illusion of a "potentially infinite" series is maintained). By turning the dial backward, we could return the displayed series to its initial values at the head of the series.

The "handheld series box" of Figure 3 is thus a physical object that represents an infinite series in such a way that at any given moment, the user can look at any chosen chunk of the series. Conceivably, such artifacts could be designed so that they can be manipulated and combined in ways appropriate to the represented objects: for instance, one might add two series together, elementwise, by linking their corresponding handheld objects (we'll leave this technique unspecified for now), and use the addition process as a means of specifying the values for a third handheld object. Thus, one might add together (e.g.) two physical copies of our sample series to create the new series: 2, 1, ½, ¼, … which could itself be represented in a freshly-initialized handheld object.

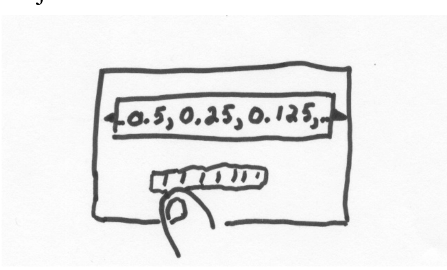

Figure 3. An "infinite series" object. The display window in the box shows elements of an infinite series. By scrolling the dial beneath, one can shift the view of the series to the right or left, revealing any arbitrary portion of the series and giving the illusion of a functionally infinite object.

The same basic design employed for representing infinite series could likewise be explored for purposes of representing other infinite (but easily computed) series of values. For example, one might create a box like that of Figure 3 to display the value of pi to arbitrary precision, employing the dial to scroll through successive values for as long as one might wish.

4. Other Possibilities

The three sample manipulatives mentioned here represent only a few illustrative possibilities along a path of "embodied mathematics" for advanced topics. Still other possibilities (for which there is insufficient space in this essay) include artifacts to represent and construct models of such notions as Markov chains and discrete finite automata. More generally, these sample ideas signal what could be a renewed interest in examining the role of embodied cognition in mathematical understanding at all levels, and in making accessible those areas of mathematics usually thought to be beyond the capabilities of all but the most dedicated students.

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