STRESS RESULTANT SYSTEM IN BEAMS: BENDING MOMENT AND SHEAR FORCE DIAGRAMS

BEAMS

Stress Resultant System as and Equivalent Force System

The Section Principle:

Consider a statically determinate beam shown in Figure 1.

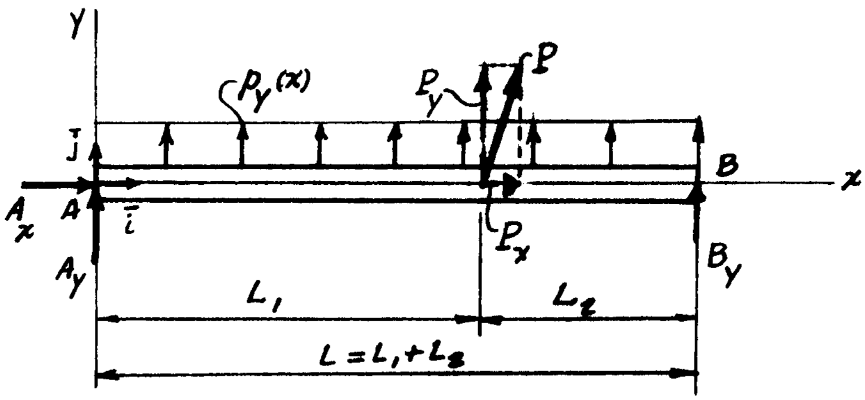
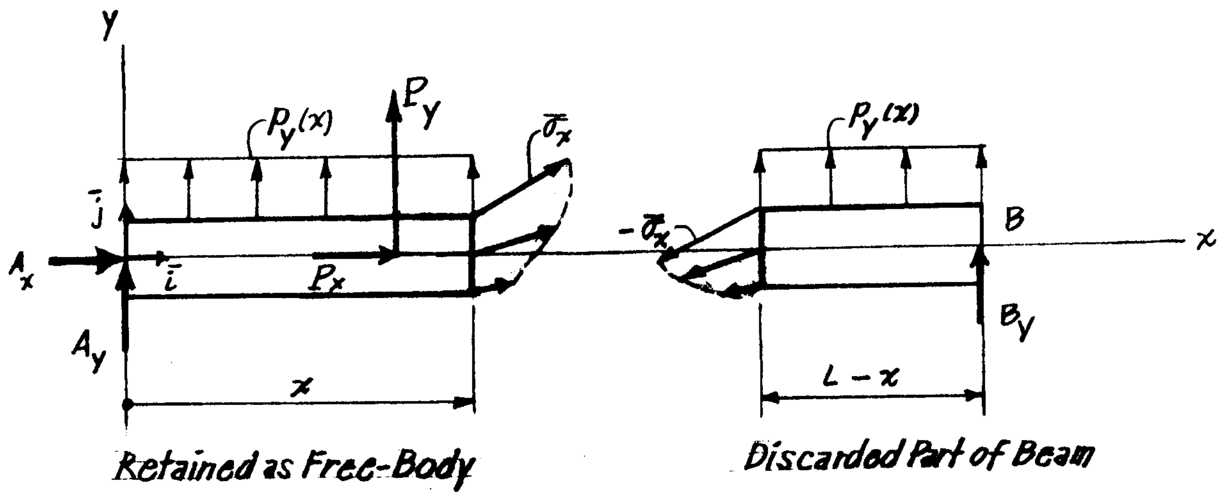


Figure 1

In order to be able to determine what are the forces acting within the material of the beam in a generic cross-section at x, a theoretical device is introduced which consists of an imaginary transverse section that is passed through the beam at x in order to remove the portion of the beam for which $x' > x$. Based on more than three hundred years of experience since the pioneering work of Galileo Galilei (1564-1642), it is imagined that a distribution of contact forces acting on the exposed cross-section of the retained portion of the beam are adequate to represent the contact action of the removed part of the beam (for which $x' > x$) on the remaining part of the beam. These contact forces, postulated to be vector-valued, are called stress vectors when they act over unit elements of the cross-section. It is hypothesised that this distribution of stress vectors is capable of maintaining the remaining part of the beam as a free-body precisely in the same mechanical state when it is an integral part of the whole beam. The very idea of using contact forces to represent the action of the removed part of the structure on the remaining part was first introduced in the studies of the strength of beams, bars and ropes by Galileo Galilei in his famous monograph, Two New Sciences (1638), the first book on strength of materials. This work resulted from Galileo's efforts to determine the strength of the components of ship structures whilst he was a consulting engineer to the Venetian shipyards. This idea was developed further by the French Jesuit Ignace Gaston Pardies (c.1636-1673) in his study of flexible suspension bridge cables in 1673, in which he passed an imaginary section through the cable, and represented the action of the removed cable on the exposed section of the remaining cable by a tangential force. This Pardies Principle was used in 1690 by the German mathematician, philosopher and engineer Gottfried Wilhelm Leibniz (1646-1716) and the Swiss mathematician Johann Bernoulli (1667-1748). Pardies Principle was systematically and explicitly used by Jacob Bernoulli (1655-1705), the older brother of Johann, and Jacob's pupil Jacob Hermann (1678-1733) in 1716, before Leonhard Euler (1707-1783), a former pupil of Johann Bernoulli, made the imaginary section principle a powerful tool of solid mechanics. The very idea of the section principle is to convert internal interacting forces acting in the section, which are internal effects and the character of which is unknown, into pseudo-external forces, which are external effects, the nature of which is considered to be well-known, and to which all the apparatus of theoretical mechanics directly applies. See Figure 2 for illustration.



Stresses Acting in Cross–Section of Beam

Figure 2

Stress Couple and Stress Resultant:

The stress vectors by their very nature are, in principle, statically indeterminate, i.e., in order to evaluate them the internal deformation effects of the beam have to be taken into consideration. Since the exposed stress vector system which is distributed over the cross-section can be treated like an external force system, it can be replaced by an equivalent force system constructed at any point in the cross-section, usually at the geometrical centroid of the cross-section – a point relative to which the first moments of the cross-sectional area vanish.

Consider a typical symmetrical cross-section under the action of stress vectors

$$\bar{\sigma}_x = \sigma_{xx}\,\bar{i} + \sigma_{xy}\bar{j}$$

which represent a planar state of stress, as shown in Figure 3.

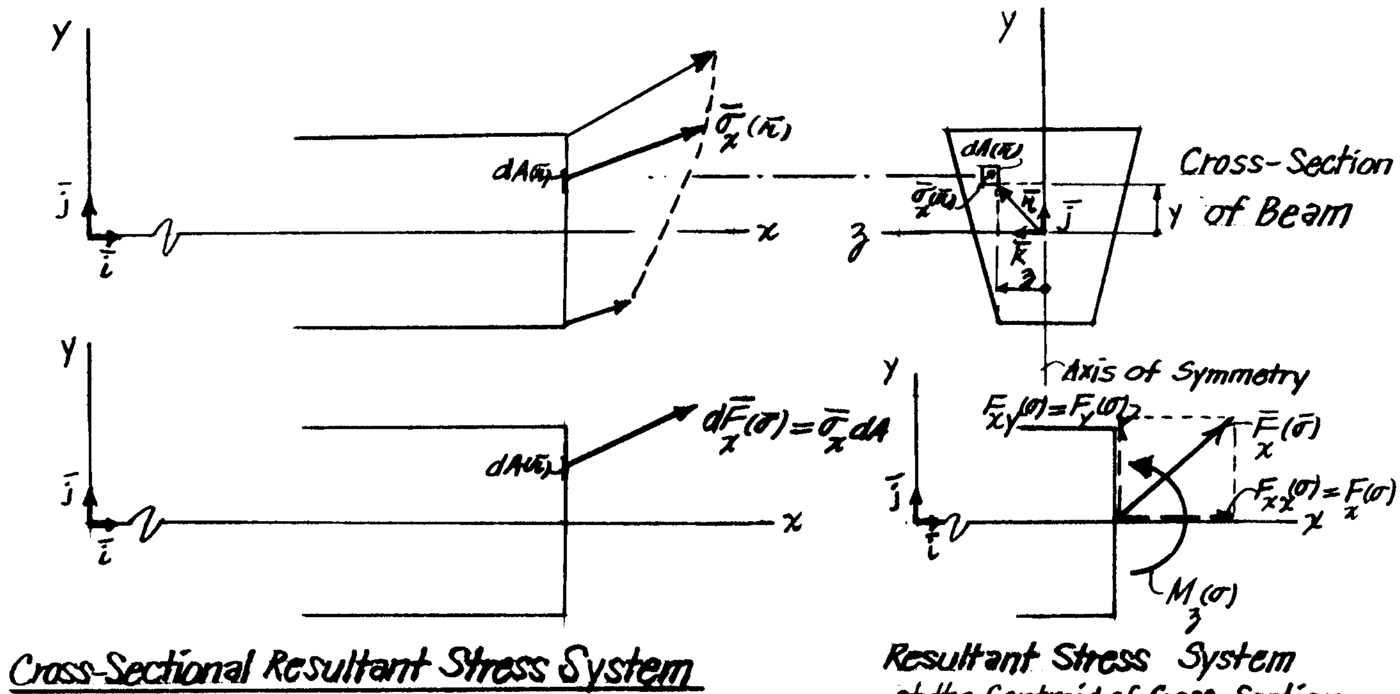


Figure 3

The stress vector $\bar{\sigma}_x$ is acting on a cross-sectional element dA located by the cross-sectional position vector

$$\bar{r} = y\,\bar{j} + z\,\bar{k}$$

relative to the centroid of the cross-section. As mentioned <u>supra</u>, the stress vector $\bar{\sigma}_x$ is considered to be the effect of the contact action of the contiguous part of the beam which has been removed by the imaginary section. Thus both $\bar{\sigma}_x$ and $\bar{r}$ have been referred to the frame of reference (x,y,z).

The stress vector component, σ_{xx}, which acts perpendicular to the cross-section, is usually called the <u>normal stress</u>. The stress vector component, σ_{xy}, which acts tangential to the cross-section, is called the shear stress. It ought to be noted that the first subscript of the stress denotes the normal to the surface on which the stress is acting, and the second subscript denotes the orientation of the stress as a force. The subscript x of the stress vector $\bar{\sigma}_x$ denotes the normal to the surface on which the stress vector $\bar{\sigma}$ is acting.

Stresses are statically indetermine internal effects in a deformable extended body, and, therefore, lie beyond the reach of rigid body mechanics. However, for externally statically - determinate beams, the equivalent force system of the stresses consisting of a Stress Resultant $\bar{F}(\sigma)$ and a Stress Couple $\bar{C}(\sigma)$ can be found by rigid body mechanics. The Stress Resultant System as an Equivalent Force System can be calculated the same way as was done for the external forces. The Equivalent Force System of stresses relative to the centroid of the cross-section consists of the Stress Resultant,

$$\bar{F}_x(\bar{\sigma}) = \int_A d\,\bar{F}_x(\bar{\sigma}) = \int_A \bar{\sigma}_x dA$$

and the stress couple,

$$\bar{C}_x(\bar{\sigma}) = \bar{M}_x(\bar{\sigma}) = \int_A \bar{r} \times d\bar{F}_x(\bar{\sigma}) = \int_A \bar{r} \times \bar{\sigma}_x dA$$

where

$$d\bar{F}_x(\bar{\sigma}) = \bar{\sigma}_x dA$$

is the differential stress-force acting over dA. See Figure 3.

Referring the stress vector to the directed base of the frame of reference gives,

$$d\bar{F}_x(\bar{\sigma}) = \bar{\sigma}_x dA = (\sigma_{xx}\,\bar{i} + \sigma_{xy}\bar{j})dA = (\sigma_{xx}dA)\bar{i} + (\sigma_{xy}dA)\bar{j}$$

and, therefore, the Stress Resultant,

$$\bar{F}_x(\bar{\sigma}) = [\int_A \sigma_{xx}dA]\bar{i} + [\int_A \sigma_{xy}dA]\bar{j} = F_{xx}(\sigma)\bar{i} + F_{xy}(\sigma)\bar{j}$$

$$= F_x(\sigma)\bar{i} + F_y(\sigma)\bar{j}$$

where $F_x(\sigma)$ is called the Normal Stress Resultant, and $F_y(\sigma)$ is called the transverse Shear Stress Resultant:

$$F_x(\sigma) = \int_A \sigma_{xx} dA$$

$$F_y(\sigma) = \int_A \sigma_{xy} dA$$

Sometimes N or T is used for $F_x(\sigma)$, and V is used for $F_y(\sigma)$. Euler used T and V.

If the Normal Stress Resultant $F_x(\sigma)$ is plotted on the affine plane $F_x(\sigma) \times X$, then the resulting graph is called the Normal, or Axial Force Diagram.

If the Shear Stress Resultant $F_y(\sigma)$ plotted on the affine plane $F_y(\sigma) \times X$, then the resulting graph is called the Shear Force Diagram.

The Stress-Couple,

$$\bar{M}_x(\bar{\sigma}) = \int_A \bar{r} \times d\bar{F}_x(\bar{\sigma}) = \int_A (y\bar{j} + z\bar{k}) \times [\sigma_{xx} dA)\bar{i} + (\sigma_{xy} dA)\bar{j}]$$

$$= \int_A y\sigma_{xx} dA\ (\bar{j} \times \bar{i}) + \int_A z\ \sigma_{xx} dA\ (\bar{k} \times \bar{i}) + \int\ y\sigma_{xy}\ (\bar{j} \times \bar{j})$$

$$+ \int_A z\ \sigma_{xy} dA\ (\bar{k} \times \bar{j})$$

$$= [-\int_A y\sigma_{xx} dA]\bar{k} + [\int_A z\sigma_{xx} dA]\bar{j} + [-\int_A z\sigma_{xy} dA]\bar{i}$$

$$= [-\int y\sigma_{xx} dA]\bar{k} = M_z(\sigma)\bar{k}$$

since for a symmetrical cross-section and planar state of stress $[\sigma_{xx}(-z) = \sigma_{xx}(z);\ \sigma_{xy}(-z) = \sigma_{xy}(z)!]$

$$\int_A z\sigma_{xx} dA = 0$$

$$\int_A z\sigma_{xy} dA = 0$$

The stress-couple component,

$$M_z(\sigma) = -\int_A y\ \sigma_{xx} dA$$

is called the Bending Moment about the z-axis.

If the Bending Moment $M_z(\sigma)$ is plotted on the affine $M_z(\sigma) \times X$

plane then the resulting graph is called the Bending Moment Diagram. All three diagrams are affine, and, therefore, non-metrical. On affine plane angles and distances are meaningless.

It must be observed that in all three diagrams the plotted stress resultants $F_x(\sigma)$, $F_y(\sigma)$ and the bending moment $M_z(\sigma)$ act on the cross-section x with positive unit normal vector $\bar{n} = \bar{i}$.

Euler's Field Equations for Beams

Consider a beam shown in Figure 4.

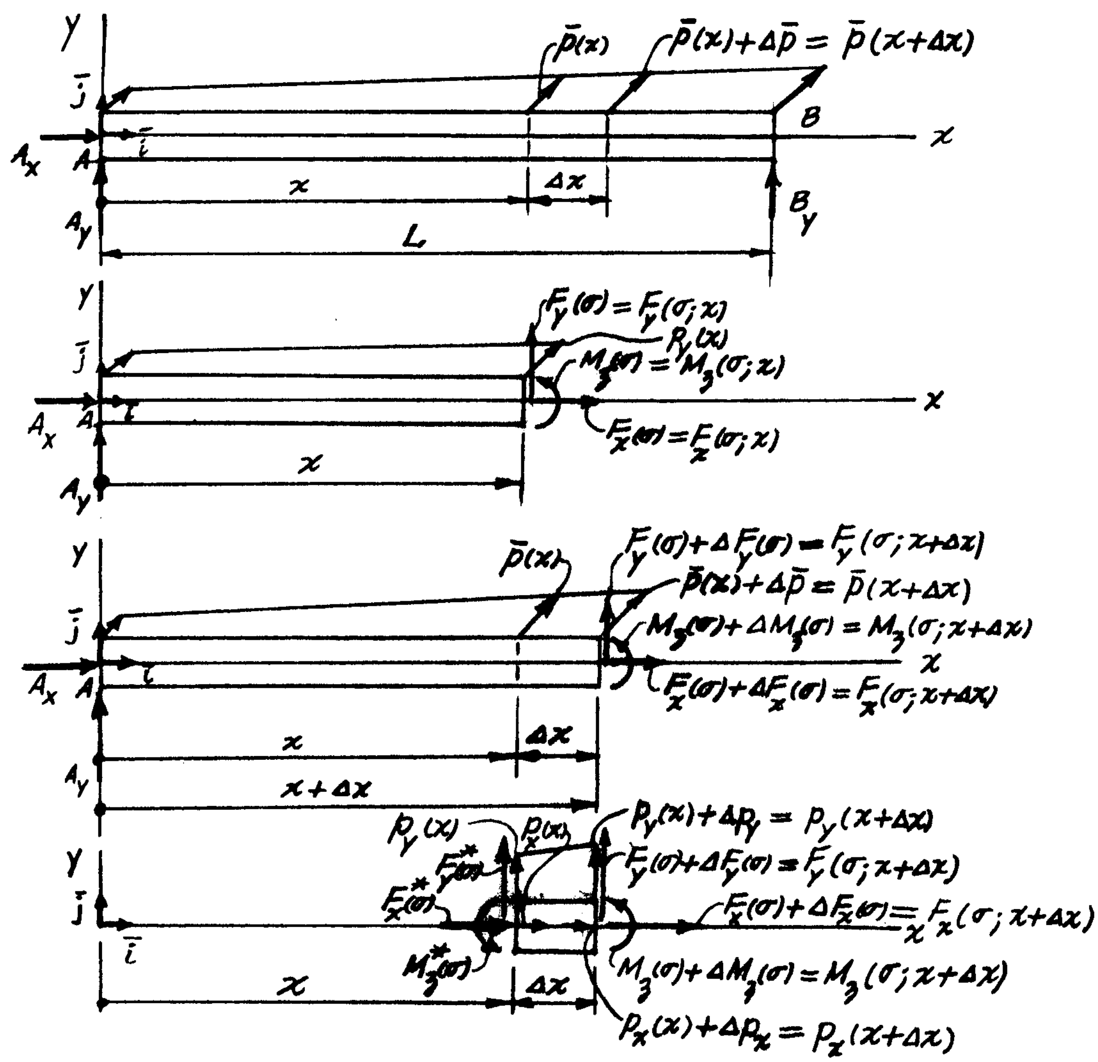


Figure 4

If the Stress Resultant System $\{F_x(\sigma), F_y(\sigma), M_z(\sigma)\}$ is constructed on the positive face of the cross-section x which has the positive unit normal vector $\bar{n}_+ = \bar{i}$, then the relationship of the Stress Resultant System $\{F_x^*(\sigma), F_y^*(\sigma), M_z^*(\sigma)\}$ acting on the negative face of the cross-section x of the finite beam element Δx which has the normal vector $\bar{n}_- = -\bar{i}$ with the Stress Resultant System $\{F_x(\sigma), F_y(\sigma), M_z(\sigma)\}$ has to be established.

If the finite element Δx of the beam is in the state of equilibrium, then the Equivalent Force System constructed at $x+\Delta x$ must vanish:

$$\bar{F}_{x+\Delta x} = \bar{0}: \; F_{x+\Delta x,x} = F_x^*(\sigma)+[F_x(\sigma)+\Delta F_x(\sigma)]+[p_x(x)+p_x(x+\Delta x)/2]\Delta x=0 \tag{1}$$

$$F_{x+\Delta x,y} = F_y^*(\sigma)+[F_y(\sigma)+\Delta F_y(\sigma)]+[p_y(x)+p_y(x+\Delta x)/2]\Delta x=0 \tag{2}$$

$$\bar{C}_{x+\Delta x} = \bar{0}: \; M_{x+\Delta x,z} = M_z^*(\sigma)-\Delta x F_y^*(\sigma)-[2p_y(x)+p_y(x+\Delta x)](\Delta x^2/6) +[M_z(\sigma)+\Delta M_z(\sigma)]=0 \tag{3}$$

Furthermore, if the requirement is imposed that the equilibrium condition must also hold in the limit $\Delta x \to 0$, then

$$\lim_{\Delta x\to 0} \Sigma F_x = F_x^*(\sigma) + F_x(\sigma) = 0 \quad \therefore \quad F_x^*(\sigma) = -F_x(\sigma) \tag{1a}$$

$$\lim_{\Delta x\to 0} \Sigma F_y(\sigma) = F_y^*(\sigma) + F_y(\sigma) = 0 \quad \therefore \quad F_y^*(\sigma) = -F_y(\sigma) \tag{2a}$$

$$\lim_{\Delta x\to 0} \Sigma M_z = M_z^*(\sigma) + M_z(\sigma) = 0 \quad \therefore \quad M_z^*(\sigma) = -M_z(\sigma) \tag{3a}$$

This Stress Resultant Principle was established by Leonhard Euler in 1774 by a different method. This result, as Euler demonstrated, shows that on both faces of a section in any beam, faces which have unit normal vectors $\bar{n} = \bar{i}$ and $-\bar{n} = -\bar{i}$, the stress resultant system is an equal and opposite vector system as shown in Figure 5:

$$\bar{F}_{-\bar{n}}(\bar{\sigma}) = -\bar{F}_{\bar{n}}(\bar{\sigma}) \quad \text{and} \quad \bar{M}_{-\bar{n}}(\bar{\sigma}) = -\bar{M}_{\bar{n}}(\bar{\sigma})$$

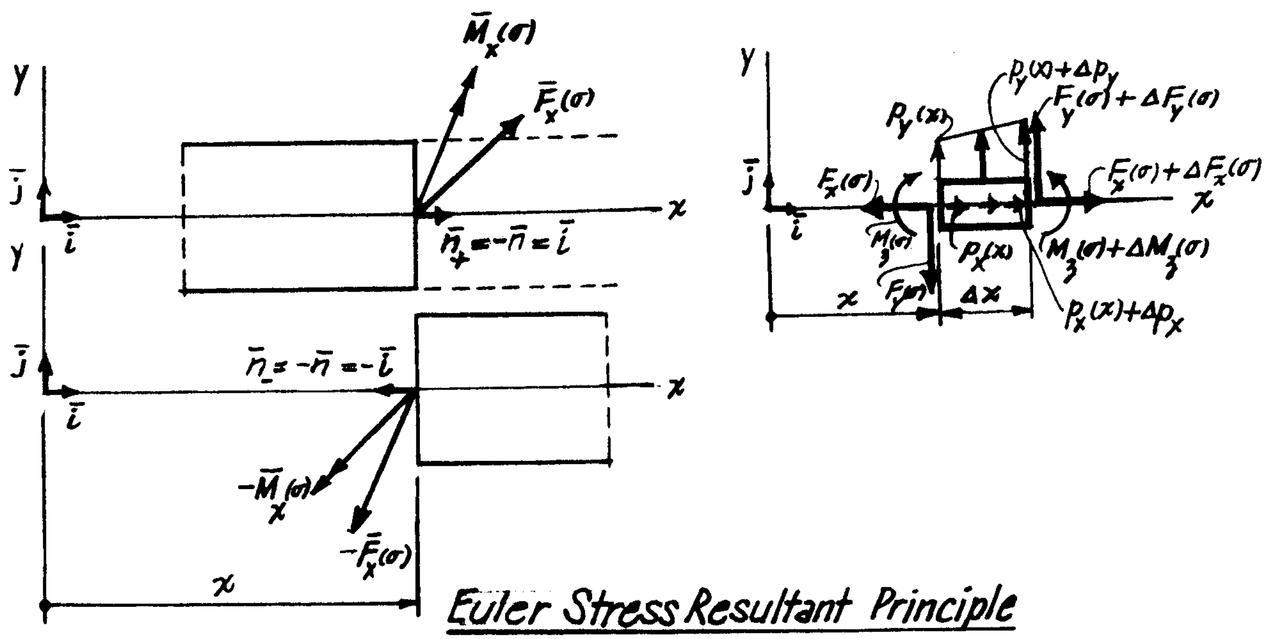


Figure 5

This idea was generalised to three-dimensional continuum by the mathematician and erstwhile hydraulic engineer, Augustin Louis Cauchy (1789-1857), in 1821 when he invented through a flash of genius the stress tensor concept which is independent of the constitutive material properties of the continuum. The remarkable fact is that Euler and Cauchy, and in this order, are the greatest and the most productive mathematicians and scientist-engineers in the history of Western Science. Cauchy's generalisation of Euler's Stress Resultant Principle for the continuum took the form:

$$\bar{\sigma}_{\bar{n}} = - \bar{\sigma}_{-\bar{n}}$$

Inspired by Euler's work on Stress Resultant in beams, Cauchy asserted that the difference between perfect fluids and solids as continua, apart from their constitution, is that the stress vector $\bar{\sigma}_n$ acts skew on any section with the normal $\bar{n}$ in solids whereas in perfect fluids it acts normal to any section. This very idea is still in use, and forms an important concept of continuum mechanics.

If the stress resultant principle expressed by (1a), (2a), (3a) are used in (1), (2) and (3), then the equilibrium equations reduce to the incremental form:

$$F_{x+\Delta x,x} = \Delta F_x(\sigma) + [p_x(x)+(\Delta p_x/2)]\ \Delta x = 0 \quad (1b)$$

$$F_{x+\Delta x,y} = \Delta F_y(\sigma) + [p_y(x)+(\Delta p_y/2)]\ \Delta x = 0 \quad (2b)$$

$$M_{x+\Delta x,z} = \Delta x\ F_y(\sigma) + [3p_y(x) + \Delta p_y](\Delta x^2/6)+\Delta M_z(\sigma)=0 \quad (3b)$$

which in the limit $\Delta x \to 0$, become the Euler Field Equilibrium Equations for Beams:

$$\lim_{\Delta x \to 0} \{[\Delta F_x(\sigma)/\Delta x] + p_x(x) + (\Delta p_x/2)\} = [dF_x(\sigma)/dx] + p_x(x)=0 \quad (1c)$$

$$\lim_{\Delta x \to 0} \{[\Delta F_y(\sigma)/\Delta x] + p_y(x) + (\Delta p_x/2)\} = [dF_y(\sigma)/dx] + p_y(x)=0 \quad (2c)$$

$$\lim_{\Delta x \to 0} \{[\Delta M_z(\sigma)/\Delta x] + F_y(\sigma) + [3p_y(x) + \Delta p_y](\Delta x/6)$$

$$= [dM_z(\sigma)/dx] + F_y(\sigma)=0 \quad (3c)$$

Euler's Field Equilibrium Equations can be expressed in the traditional form:

$$[dF_x(\sigma)/dx] = -p_x(x) \quad (1)$$

$$[dF_y(\sigma)/dx] = -p_y(x) \quad (2)$$

$$[dM_z(\sigma)/dx] = -F_y(\sigma) \quad (3)$$

These important field equations were established by Euler in 1771 as a special case of his more general dynamic field equations.

If the loading intensities $p_x(x)$ and $p_y(x)$ are smooth functions of x, then the stress resultants $F_x(\sigma)$ and $F_y(\sigma)$, and the stress couple $M_z(\sigma)$ can be obtained from the loading by integration:

From (1),

$$F_x(\sigma) = -\int p_x(x)dx + C_1$$

From (2),

$$F_y(\sigma) = -\int p_y(x)dx + C_2$$

$$M_z(\sigma) = - \int F_y(\sigma)dx + C_3$$

This integration method, whether it is analytically or graphically carried out, is often used in constructing the Normal Force, the Shear Force, and the Bending Moment Diagram.

If the Euler Field Equations for Beams given by (1), (2) and (3) are integrated between two definite limits, say x_1 and x_2, then convenient expressions for the Stress Resultant, or Stress Couple acting at x_2 can be obtained in terms of the corresponding quantity at x_1.

Normal Stress Resultant (also called Axial Force):

$$\int_1^2 [dF_x(\sigma)/dx]dx = \int_1^2 dF_x(\sigma) = - \int_1^2 p_x(x)dx$$

or

$$F_x^2(\sigma) - F_x^1(\sigma) = - \int_1^2 p_x(x)dx \qquad (1)$$

Transverse Shear Stress Resultant (also Shear Force):

$$\int_1^2[dF_y(\sigma)/dx]dx = \int_1^2 dF_y(\sigma) = - \int_1^2 p_y(x)dx$$

or

$$F_y^2(\sigma) - F_y^1(\sigma) = - \int_1^2 p_y(x)dx \qquad (2)$$

Stress Couple (Bending Moment):

$$\int_1^2 [dM_z(\sigma)/dx]dx = \int_1^2 dM_z(\sigma) = - \int_1^2 F_y(\sigma)dx$$

or

$$M_z^2(\sigma) - M_z^1(\sigma) = - \int_1^2 F_y(\sigma)dx \qquad (3)$$

These integrated results of Euler Field Equations are usually expressed in a form in which the quantity at x_2 is expressed as the difference between the same <u>prescribed</u> quantity at x_1 and another appropriate quantity integrated between limits x_1 and x_2:

$$F_x^2(\sigma) = F_x^1(\sigma) - \int_1^2 p_x(x)dx \qquad (1)$$

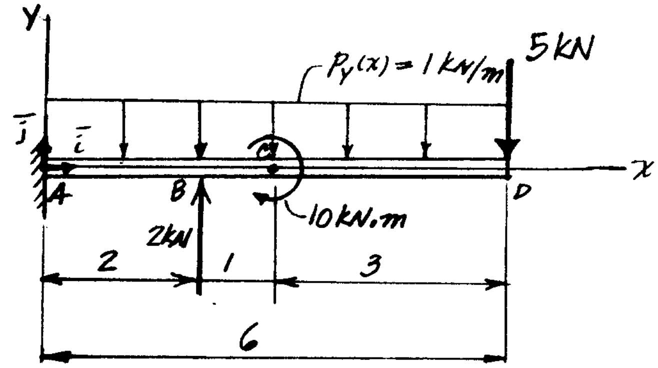
$$F_y^2(\sigma) = F_y^1(\sigma) - \int_1^2 p_y(x)dx \qquad (2)$$

$$M_z^2(\sigma) = M_z^1(\sigma) - \int_1^2 F_y(\sigma)dx \qquad (3)$$

These integrals can be carried out analytically if $p_x(x)$ and $p_y(x)$ are given as smooth function of x, and $F_x^1(\sigma)$, $F_y^1(\sigma)$, and $M_z^1(\sigma)$ are prescribed.

EXAMPLE I

Establish the Shear Force and Bending Moment Diagrams for the cantilever beam shown.



Solution:

Entire Beam as Free-Body:

Equivalent Force System at A vanishes for equilibrium:

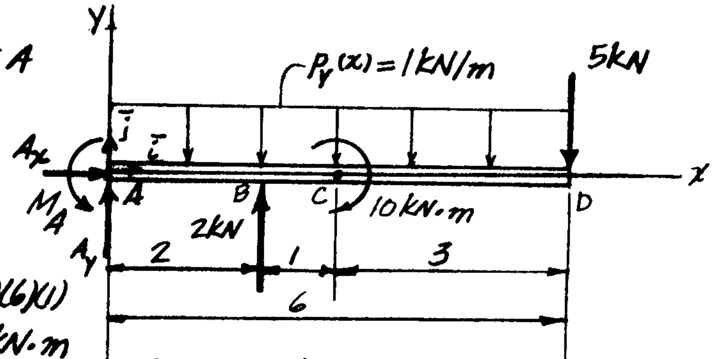
$\bar{F}_A = \bar{0}: F_{Ax} = A_x = 0$

$F_{Ay} = A_y + 2 - (6)(1) - 5 = A_y - 9 = 0$

$\therefore A_y = 9 kN$

$\bar{C}_A = \bar{M}_A = 0: M_{Az} = M_A + (2)(2) - 10 - (6)(5) - (3)(6)(1)$

$= M_A - 54 = 0 \quad \therefore M_A = 54 kN.m$



Entire Beam as Free-Body

Sectional Free-Body $0 \le x < 2$:

Equivalent Force System at x vanishes for equilibrium:

$\bar{F}_x = \bar{0}: F_{x,x} = F_x(\sigma) = 0 \quad \therefore F_x(\sigma) = 0$

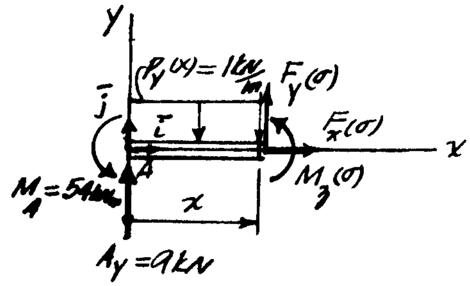
$F_{x,y} = A_y - 1x + F_y(\sigma) = 9 - 1x + F_y(\sigma) = 0$

$\therefore F_y(\sigma) = 1x - 9$

$\bar{C}_x = \bar{M}_x = \bar{0}: M_z = M_A + (1x)(\frac{x}{2}) - A_y x + M_z(\sigma)$

$= 54 + \frac{1}{2}x^2 - 9x + M_z(\sigma) = 0$

$\therefore M_z(\sigma) = -54 - 0.5x^2 + 9x$



Sectional Free-Body $0 \le x < 2$

By Integration method:

$\overset{x}{F_y}(\sigma) = 0 - \{A_y + \int_0^x p_y(x)dx\} = -9 - \int_0^x (-1)dx = -9 + 1x \quad \therefore \overset{2}{F_y}(\sigma) = -7 kN$

$\overset{x}{M_z}(\sigma) = 0 - \{M_A + \int_0^x F_y(\sigma)dx\} = -54 - \int_0^x (-9 + 1x)dx = -54 + 9x - 0.5x^2 \quad \therefore \overset{2}{M_z}(\sigma) = -38 kN.m$

Sectional Free-Body $2 \le x < 3$:

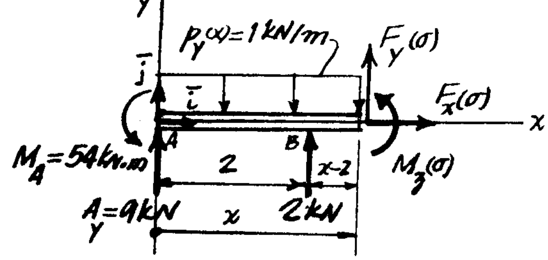
Equivalent Force System at x vanishes for equilibrium:

$\bar{F}_x = \bar{0}: F_{x,x} = F_x(\sigma) = 0 \quad \therefore F_x(\sigma) = 0$

$F_{x,y} = A_y + 2 - 1x + F_y(\sigma) = 9 + 2 - 1x + F_y(\sigma)$

$= 11 - 1x + F_y(\sigma) = 0$

$\therefore F_y(\sigma) = -11 + 1x$



Sectional Free-Body $2 \le x < 3$

$\bar{C}_x = \bar{M}_x = \bar{0}: M_{xz} = M_A - A_y x + (1x)(\frac{x}{2}) - 2(x-2) + M_z(\sigma) = 54 - 9x + 0.5x^2 - 2x + 4 + M_z(\sigma)$

$= 58 - 11x + 0.5x^2 + M_z(\sigma) = 0$

$\therefore M_z(\sigma) = -58 + 11x - 0.5x^2$

(Example I-2)

By the Integration Method:

$$F_y^x(\sigma) = F_y^2(\sigma) - \left\{2 + \int_2^x p_y(x)\,dx\right\} = -7 - \left\{2 + \int_2^x -1\,dx\right\} = -9 + [1x]_2^x$$

$$= -11 + 1x \qquad \therefore F_y^3(\sigma) = -7 \text{ kN}$$

This expression is actually valid for the interval $2 < x < 6$ since all the functions in the interval are continuous within that interval.

$$M_z^x(\sigma) = M_z^2(\sigma) - \left\{\int_2^x F_y(\sigma)\,dx\right\} = -38 - \left\{\int_2^x [-11 + 1x]\,dx\right\} = -38 - [-11x + 0.5x^2]_2^x$$

$$= -58 + 11x - 0.5x^2 \qquad \therefore M_z^3(\sigma) = -29.5 \text{ kN·m}$$

<u>Sectional Free-Body $3 \le x < 6$:</u>

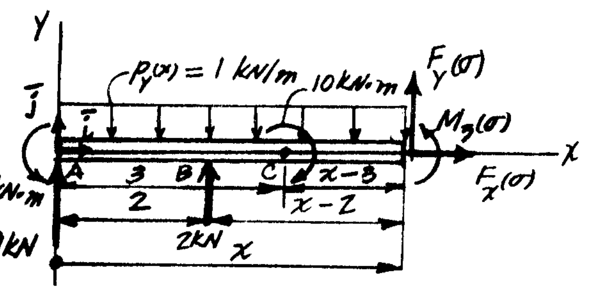
Equivalent Force System at x vanishes for equilibrium:

$\bar{F}_x = \bar{0}$: $F_{x,x} = F_x(\sigma) = 0 \quad \therefore F_x(\sigma) = 0$

$F_{x,y} = A_y + 2 - 1x + F_y(\sigma)$

$= 9 + 2 - 1x + F_y(\sigma) = 11 - 1x + F_y(\sigma) = 0$

$\therefore F_y(\sigma) = -11 + 1x$



<u>Sectional Free-Body, $3 \le x < 6$</u>

$\bar{C}_x = \bar{M}_x = \bar{0}$: $M_{x,z} = M_A + (1x)(\frac{x}{2}) - A_y x - 2(x-2) - 10 + M_z(\sigma) = 54 + 0.5x^2 - 9x - 2x + 4 - 10 + M_z(\sigma)$

$= 48 + 0.5x^2 - 11x + M_z(\sigma) = 0$

$\therefore M_z(\sigma) = -48 + 11x - 0.5x^2$

By the Integration Method:

$F_y^x(\sigma) = -11 + 1x$ (This expression is the same as for $2 < x < 3$! See note supra!)

$$M_z^x(\sigma) = M_z^3(\sigma) - \left\{-10 + \int_3^x [-11 + 1x]\,dx\right\} = -29.5 + 10 - [-11x + 0.5x^2]_3^x$$

$$= -48 - 0.5x^2 + 11x$$

<u>Shear Force Diagram</u>

<u>Bending Moment Diagram</u>