THE MANIN-MUMFORD CONJECTURE: A BRIEF SURVEY

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Introduction. This is a survey paper on the Manin-Mumford conjecture for number fields with some emphasis on effectivity. It is based on the author's lecture at the Arizona Winter School on Arithmetical Algebraic Geometry (March 1999). We discuss some of the history of this conjecture (and of related conjectures) and some recent explicit results.

1. FINITENESS RESULTS

The Manin-Mumford conjecture for number fields is a deep and important finiteness question (raised independently by Manin and Mumford) regarding the intersection of a curve with the torsion subgroup of its Jacobian:

Conjecture 1.1. *Let K be a number field. Let C be a curve of genus $g \geq 2$ defined over K. We will denote by J the Jacobian of C. Fix an embedding $C \hookrightarrow J$ defined over K. Then the set $C(\overline{K}) \cap J(\overline{K})_{tors}$ is finite.*

Conjecture 1.1 was proved by Raynaud in [48]. Various other proofs and generalizations were subsequently given by Raynaud ([49]), Serre ([53]), Coleman ([19]), and Hindry ([30]) (see also the end of this section where recent developments are mentioned).

According to Lang ([35]), Manin was led to ask the above question in connection with another famous conjecture, namely the Mordell Conjecture:

Conjecture 1.2. *Let K be a number field and let C be a curve of genus $g \geq 2$ defined over K. Then $C(K)$ is finite.*

The Mordell conjecture was proved by Faltings in his landmark paper [25] (see also [26]). Different proofs were shortly afterwards given by Vojta ([57]) and Bombieri ([6]).

The function field analogue of the Mordell conjecture in characteristic 0 was first proved by Manin in [39], using the theorem of the kernel. Coleman discovered a gap in Manin's proof of the latter theorem and managed to prove a weaker version of the theorem (sufficient for the proof of Mordell's conjecture for function fields in characteristic 0; see [21]). Manin's initial version of the theorem of the kernel was later on proved by Chai ([14]), using work of Deligne and Coleman.

Long before either of the Conjectures 1.1 or 1.2 was settled, it was Serge Lang ([34]) who realized that the two statements are special cases of the following more general conjecture, which is usually called the Mordell-Lang conjecture in characteristic 0:

Conjecture 1.3. *Let X be a closed geometrically integral subvariety of a semi-abelian variety A defined over a field K of characteristic 0. Let Γ be a finitely generated subgroup of $A(\overline{K})$ and Γ' a subgroup of the divisible hull of Γ (i.e. for*

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each $x \in \Gamma'$ there exists a non-zero integer n such that $nx \in \Gamma$). If X is not a translate of a semi-abelian subvariety of A, then $X(\overline{K}) \cap \Gamma'$ is not Zariski dense in X.

It is not hard to check the following proposition:

Proposition 1.4. *Conjecture 1.3 implies Conjectures 1.1 and 1.2.*

Proof. For the Manin-Mumford conjecture, let $X = C$, $A = J$, $\Gamma = \{0\}$ and $\Gamma' = J(\overline{K})_{tors}$. X is a curve of genus at least 2, so it is not a translate of a semi-abelian subvariety of A. By the Mordell-Lang conjecture, $X(\overline{K}) \cap \Gamma'$ is not Zariski dense in X, hence it is finite.
For the Mordell conjecture, let $X = C$, $A = J$, $\Gamma = \Gamma' = J(K)$ (note that, by the Mordell-Weil theorem, Γ is finitely generated). As before, $X(\overline{K}) \cap \Gamma' = C(K)$ is finite. □

The Mordell-Lang conjecture in characteristic 0 was proved in its entirety by McQuillan ([43]), following work of Faltings ([25], [26]), Raynaud ([48], [49]), Hindry ([30]), Vojta ([58]) and Buium ([8]). One should also mention that special cases of Conjecture 1.3 were settled earlier by Tate and Lang (see [34]), Liardet ([37], [38]), Laurent ([36]) and Bogomolov ([5]).

A generalization of the Manin-Mumford conjecture was proposed by Bogomolov:

Conjecture 1.5. *Let X be a curve of genus $g \geq 2$ defined over a number field K. Fix an embedding of X into its Jacobian J. Let h_{NT} denote the Néron-Tate height on $J(\overline{K})$. Then for sufficiently small $\epsilon > 0$, the set*

$$\{P \in X(\overline{K})\colon h_{NT}(P) \leq \epsilon\}$$

is finite.

Conjecture 1.5 has been settled by Ullmo ([56]) using work of Szpiro, Ullmo and Zhang ([55]). There is also the generalized Bogomolov conjecture:

Conjecture 1.6. *Let A be a semi-abelian variety defined over K and let X be a closed geometrically integral subvariety of A which is not a translate of a semi-abelian subvariety of A by a torsion point. Let h be a canonical height on $A(\overline{K})$ (for example, if A is an abelian variety, h can be taken to be the Néron-Tate height on $A(\overline{K})$). Then for sufficiently small $\epsilon > 0$, the set*

$$\{P \in X(\overline{K})\colon h(P) \leq \epsilon\}$$

is not Zariski dense in X.

When A is an abelian variety, Conjecture 1.6 was settled by Zhang ([63]). A quantitative version (and also another proof) of the same result was given by David and Philippon ([24]). Zhang also proved Conjecture 1.6 when A is a torus ([62]). A proof of Conjecture 1.6 for almost split semi-abelian varieties (i.e. semi-abelian varieties which are isogenous to the product of an abelian variety and a torus) was recently announced by Chambert-Loir ([15]).

Poonen has recently proposed an even more general conjecture that includes both the Mordell-Lang and the generalized Bogomolov conjecture as special cases (see [46]). Let notation be as in Conjecture 1.6. Let Γ be a finitely generated subgroup of $A(\overline{K})$ and let Γ' be the divisible hull of Γ. Fix a canonical height on $A(\overline{K})$. For $\epsilon > 0$, define

$$\Gamma'_\epsilon = \{\gamma + P : \gamma \in \Gamma',\ P \in A(\overline{K}),\ h(P) \leq \epsilon\}.$$

Poonen's conjecture is the following statement:

Conjecture 1.7. *If X is not a translate of a semi-abelian subvariety of A by a point in Γ', then for sufficiently small $\epsilon > 0$, the set $X(\overline{K}) \cap \Gamma'_\epsilon$ is not Zariski dense in X.*

Poonen has proved Conjecture 1.7 for the case of almost split semi-abelian varieties ([46]).

The survey [1] by Abbès is an account of recent advances concerning the equidistribution of small points. For a detailed account of the history of the Mordell-Lang conjecture one should refer to Lang's book [34] and to Pillay's report [44], [45]. For a survey of the characteristic p situation, one should consult Voloch's article [60]. In the next section, we will only give a brief account of the situation in positive characteristic.

2. THE MORDELL-LANG CONJECTURE IN POSITIVE CHARACTERISTIC

The formulation of the Mordell-Lang conjecture given in the previous section does not carry over in the case when the field K has positive characteristic. In fact, the same is true for the Manin-Mumford and the Mordell conjecture, as illustrated by the following example:

Example 1. Let F_q be a finite field. If C is a curve over F_q embedded in its Jacobian J (over F_q), then $C(\overline{K}) \cap J(\overline{K})_{tors} = C(\overline{K})$, which is infinite.

One can also construct less trivial counterexamples. However, it turns out that the only possible counterexamples have to be of a certain type. In [2], Abramovich and Voloch indicated that a correct version of the Mordell-Lang conjecture in positive characteristic can be formulated provided that, roughly speaking, the case when the variety X admits a purely inseparable rational map to a variety defined over a finite field is excluded. Note that there is a similar condition appearing in Manin's version of the Mordell conjecture for characteristic zero function fields (where the constant field plays the role of a finite field). Also the same condition features in the version of Mordell's conjecture for function fields in positive characteristic, as proved by Grauert ([27]) and Samuel ([52]). The latter results were extended by Coleman in [17]. Voloch ([59]) was able to prove a characteristic p-analogue of the Manin-Mumford conjecture (under certain assumptions on the curve) and Buium and Voloch ([12]) have given explicit bounds for the number of torsion points on the curve (again under certain assumptions on the curve).

As mentioned above, a general analogue of the Mordell-Lang conjecture in characteristic p was proposed by Abramovich and Voloch ([2]). They proved several important cases of their conjecture including the "prime to p"-analogue of the Manin-Mumford conjecture (see Conjecture 2.1 below). Before we state the conjecture, we need some notation (we follow Pillay's account [44], [45]):
Suppose k and K are algebraically closed fields (of any characteristic) such that k is properly contained in K. Let A be a semi-abelian variety defined over K. A closed subvariety Y of A is called special if there exists a semi-abelian subvariety A_1 of A, a semi-abelian variety A_0 defined over k, a subvariety Y_0 of A_0 also defined over k and a surjective morphism $h : A_1 \longrightarrow A_0$ such that Y is a translate of $h^{-1}(Y_0)$. The Mordell-Lang conjecture in all characteristics is the following statement:

Conjecture 2.1. *Suppose that Γ is a finitely generated subgroup of $A(K)$. Let Γ' be a subgroup contained in the prime-to-p divisible hull of Γ (i.e. for each $x \in \Gamma'$*

there exists a non-zero integer n such that $(n,p)=1$ and $nx \in \Gamma$). Let X be a closed subvariety of A. Then there exists a finitely many special subvarieties $X_1,\ldots,X_n$ of A such that

$$X \cap \Gamma' \subseteq X_1 \cup \cdots \cup X_n \subseteq X.$$

For the case when the characteristic of k is 0 and A "does not descend to k", the above statement was settled by Buium in [8] using diffential algebraic geometry. Conjecture 2.1 was settled in its entirety by Hrushovski in [31], using methods of model theory applied to arithmetical algebraic geometry. Buium's techniques seem to have influenced Hrushovski's work.

We should note that Conjecture 2.1 is still open when we allow torsion points whose order is divisible by the characteristic of the field. As Abramovich and Voloch point out ([2]), there is no evidence that the conjecture should hold in this case and there is no evidence that it should fail either. Work in progress by Scanlon may provide some answers to this question.

3. Bounds for the number of torsion points

From now on, we will focus on the Manin-Mumford conjecture for number fields. We recall some terminology introduced by Coleman:

Definition. *Let C be a curve of genus $g \geq 2$ over a number field K. Fix an embedding of C into its Jacobian J defined over K. The set $T = C(\overline{K}) \cap J(\overline{K})_{tors}$ will be called a torsion packet on C with respect to the given embedding.*

There are two important questions one can ask about torsion packets:

Question 1. *Give general bounds for the cardinality of a torsion packet.*

Question 2. *Given a specific curve C and an Albanese embedding $C \longrightarrow J$, describe the asscociated torsion packet T explicitly.*

In this section we will focus on Question 1. The following conjecture has been made by Coleman ([19]):

Conjecture 3.1. *Let notation be as in the definition above. Let v be a prime of K dividing $p \in \mathbb{Q}$ such that all of the following conditions are satisfied:*

1. *$p \geq 5$.*
2. *$K/\mathbb{Q}$ is unramified at v.*
3. *C has good reduction at v.*

Then the extension $K(T)/K$ is unramified above v.

Coleman has provided considerable evidence for the validity of this conjecture in [16], [19] and [20]:

Theorem 3.2. *Conjecture 3.1 is true in any of the following cases:*

1. *$p \geq 2g+1$.*
2. *C has ordinary reduction at v.*
3. *C has superspecial reduction at v.*
4. *C is an abelian étale covering of $\mathbb{P}^1 - \{0,1,\infty\}$.*

The proof is obtained by Coleman's p-adic integration theory ([16]). The techniques are powerful enough to also give a new proof of the Manin-Mumford conjecture ([19]). One of the most important consequences of Coleman's theory is a bound for the cardinality of a torsion packet, under certain hypotheses on C and J (see [16]):

Theorem 3.3. *Suppose that, in addition to the hypotheses of Conjecture 3.1, C has ordinary reduction at v and J has potential complex multiplication. Then*

$$\#T \leq pg.$$

The following two examples illustrate that Coleman's bound is sharp and, more importantly, that the hypotheses on C and J are essential for the bound to hold.

Example 2 (Boxall and Grant, [7]). Let C be the genus-two curve with affine model

$$y^2 = x^5 + x.$$

Embed C into its Jacobian J by sending the point at infinity to 0. The Hasse-Witt matrix of C at 11 is easily seen to be invertible, so C has ordinary reduction at 11. Also J has complex multiplication induced by

$$(x, y) \mapsto (\zeta^2 x, \zeta y),$$

where ζ is a primitive 8-th root of unity in $\overline{\mathbb{Q}}$. Now the six hyperelliptic branch points of C lie in the torsion packet T (with respect to the given embedding). Moreover, using standard arguments in the arithmetic of genus two curves (see [13]), it is not hard to show that the 16 points (x, y) for which $x^4 + 4x^2 + 1 = 0$ or $x^4 - 4x^2 + 1 = 0$ have order 6 in J. Therefore, $\#T \geq 22$. On the other hand, by Coleman's bound, $\#T \leq 22$, so we are done.
Now an easy computation shows that C is superspecial at 7. By what has been said above, $\#T \geq 15$. Therefore, the hypothesis of ordinariness of C is essential for Coleman's bound to hold.

Example 3 (Coleman, [16]). The modular curve $X_1(13)$ has genus 2, ordinary reduction at 5 and 22 points in its cuspidal torsion packet (see [16] for details). Therefore, $X_1(13)$ does not have complex multiplication. This also shows that the CM hypothesis on J is essential for Coleman's bound to hold. Coleman's paper ([16]) contains a number of interesting examples, especially for curves of genus 2 and 3.

Examples 2 and 3 seem to hold the record for the maximum number of points in a torsion packet on a genus-two curve. Poonen has in fact constructed ([47]) countably many pairwise non-isomorphic genus-two curves over $\mathbb{Q}$, each with at least 22 points in the hyperelliptic torsion packet.

A remarkable (and almost unconditional) bound for the cardinality of a torsion packet was given by Buium in [11]:

Theorem 3.4. *Suppose that, in addition to the hypotheses of Conjecture 3.1, we have $p \geq 2g + 1$. Then*

$$\#T \leq g!\ p^{4g}\ 3^g\ (p(2g-2) + 6g).$$

Buium uses p-jets to prove an unramified version of the above statement. The result then follows from Theorem 3.2. Buium has in fact given explicit bounds in a number of different contexts ([9], [10], [11] and [12]).

It is worth noting at this point that a new bound for the Manin-Mumford conjecture follows from the work of Hrushovski ([32]):

Theorem 3.5. *Fix a projective embedding of J. There exist constants α and β such that*

$$\#T \leq \alpha \ (deg(C))^{\beta},$$

where $deg(C)$ is the degree of C with respect to the given projective embedding.

The constants α and β do not depend on C or on K, but on the genus of C. They also depend on a prime of good reduction for C (I thank Alexandru Buium for pointing this out to me).

4. RECENT EXPLICIT EXAMPLES

In this section we will focus on Question 2 of the previous section. We will review some recent explicit examples of torsion packets for specific curves. In the next section, we will briefly discuss some of the ideas involved in the proofs hoping that the relevant techniques might prove to be useful in different contexts as well. The examples below are listed roughly in chronological order.

Example 4 (Coleman, Kaskel and Ribet, [22]). Consider the modular curve $X_0(37)$ and its Jacobian $J_0(37)$. Let C_0 and C_∞ be the two cusps on $X_0(37)$ and consider the Albanese embedding $X_0(37) \longrightarrow J_0(37)$ by sending C_∞ to 0. By a theorem of Drinfeld and Manin, it follows that C_0 lies in the corresponding torsion packet T. In fact, one has:

Theorem 4.1.

$$T = \{C_0, C_\infty\}.$$

Example 5 (Coleman, Tamagawa and Tzermias, [23]). Consider the Fermat curve $F_N : X^N + Y^N + Z^N = 0$, where N is an integer such that $N \geq 4$. The set of cusps C_N on F_N is the set of points (X, Y, Z) (over $\overline{\mathbb{Q}}$) satisfying $XYZ = 0$. Embed F_N into J_N by using a cusp as a base-point and let T_N be the corresponding torsion packet. Rohrlich ([51]) has shown that $C_N \subseteq T_N$. In fact, one has:

Theorem 4.2.

$$T_N = C_N.$$

It should be noted that Coleman had settled some special cases of Theorem 4.2 using rigid analytic geometry ([18]). These special cases were used in the proof of Theorem 4.2. Also, Theorem 4.2 has an analogue (which is however conditional upon a weak version of Vandiver's conjecture) for the non-hyperelliptic Fermat quotients $F_{p,s} : y^p = x^s(1 - x)$, where p is a prime such that $p \geq 11$ and s is an integer such that $1 \leq s \leq p - 2$ and $s \neq 1, (p - 1)/2, p - 2$. The latter analogue is obtained by means of the work of Greenberg ([28]) and Kurihara ([33]) and the question whether it remains valid unconditionally (i.e. independently of Vandiver's conjecture) is open (see [23] for details). Shaulis ([54]) has recently computed the cuspidal torsion packets on the hyperelliptic Fermat quotient curves $(s \in \{1, (p - 1)/2, p - 2\})$.

Example 6 (Boxall and Grant, [7]). In this recent article, a general method is developed that can sometimes explicitly compute the hyperelliptic torsion packet on a genus 2 curve (i.e. the torsion packet corresponding to the embedding of the curve in its Jacobian by taking a hyperelliptic branch point as a base-point).

Example 7 (Voloch, [61]). Consider the curve $C : y^2 = x^6 + 1$. Voloch shows that:

Theorem 4.3. *The hyperelliptic torsion packet on C consists of the six hyperelliptic branch points together with the two points at infinity.*

Example 8 (Baker, [3]). Let p be a prime, with $p \geq 23$. As in Example 4, embed the modular curve $X_0(p)$ into $J_0(p)$ by sending C_∞ to 0. When $X_0(p)$ is hyperelliptic (i.e. when $p =$23, 29, 31, 37, 41, 47, 59, 71), the hyperelliptic branch points belong to the cuspidal torsion packet T_p provided that $p \neq 37$. The last condition has to be imposed since, by a result of Mazur and Swinnerton-Dyer ([42]), the Atkin-Lehner involution on $X_0(37)$ does not coincide with the hyperelliptic involution. We define a set C_p as follows:
If C is hyperelliptic and $p \neq 37$, then C_p is the set consisting of C_0, C_∞ and the hyperelliptic branch points; in all other cases, C_p is the set consisting only of C_0 and C_∞. Baker proved the Coleman-Kaskel-Ribet conjecture:

Theorem 4.4.
$$T_p = C_p.$$

Baker also obtains similar results for the curves $X_0^+(p)$. He also studies other torsion packets (besides the cuspidal one). His results also apply to more general modular curves, namely $X_0(N)$ and $X_1(N)$, for N composite. For details one should consult Baker's preprint ([3]). A different proof of Theorem 4.4 has been recently announced by Akio Tamagawa.

5. REMARKS ON RELEVANT TECHNIQUES

We will now briefly record some observations regarding the techniques employed in settling the above examples. This is by no means a comprehensive list of such techniques. However, it is our opinion that there exist similarities (broadly construed) and therefore we feel it might be useful to record some of them here.

1. Coleman's conjecture (where known to hold) gives valuable information about the primes dividing the exponent of the torsion packet T. Stated differently, rigid analytic techniques and their consequences seem to be a starting point for tackling such problems. In Examples 4 and 8 it is useful to use the Chinese Remainder Theorem to decompose a potential torsion point P as a sum of its l-primary components P_l. Coleman, Kaskel and Ribet prove ([22]) that for $l \neq 2$, 3, the image of P_l in $J_0(p)$ is in the cuspidal group, unless either $l = p$ or $5 \leq l \leq 2g$ or $X_0(p)$ does not have ordinary reduction at l and l is ramified in the Hecke algebra. In other words, one gets very precise information for "most" primes l. Ribet has shown that, under a mild hypothesis, the situation for the prime $l = 2$ is completely understood (see [50]).
Regarding Example 5, we have the following result of Coleman ([20]) : the exponent of the cuspidal torsion packet on the curve $y^p = x^s(1-x)$ is a power of p, unless the curve is hyperelliptic, in which case 2 is the only other prime that can possibly divide the exponent of T.

2. Studying the Galois representation on $J(\overline{K})_{tors}$ also gives important information on the torsion packet T. This idea goes back to Lang ([34]), who showed that the following statement implies the Manin-Mumford conjecture:

Conjecture 5.1. *(Lang) The image of $Gal(\overline{K}/K)$ in $Aut(J(\overline{K})_{tors})$ contains an open subgroup of the homothety group $\hat{\mathbb{Z}}^* \subseteq Aut(J(\overline{K})_{tors})$.*

Conjecture 5.1 (Lang's approach to the Manin-Mumford conjecture) remains open. Partial results have been obtained by Bogomolov ([4]), Serre ([53]) and Hindry ([30]).
In Example 6, Boxall and Grant give a nice application of Lang's philosophy. They show that for certain curves of genus 2, a good knowledge of the Galois groups generated by the torsion points on the Jacobian is enough to determine the hyperelliptic torsion packet.
In Example 4, Coleman, Kaskel and Ribet use, among other things, an explicit desription (due to Kaskel) of the image of Galois acting on torsion points of $J_0(37)$. They also give a general quantitative result in the spirit of Lang. They show that if for a curve C/K there exists a $\sigma \in Gal(\overline{K}/K)$ which acts on $J[M](\overline{K})$ via the homothety n (where $M, n \in \mathbb{Z}$ and $n > 1$), then $\#(C(\overline{K}) \cap J[M](\overline{K})) \leq gn^2$.
The latter result is used in Example 5 to obtain some initial bounds on the cardinality of the cuspidal torsion packet on the Fermat quotient curves. Coleman, Tamagawa and Tzermias ([23]) use the theory of complex multiplication to produce an explicit homothety in the image of the Galois representation. The work of Shaulis ([54]) on the hyperelliptic Fermat quotients is also in the spirit of Lang's philosophy. Galois arguments are also essential in Baker's work regarding Example 8 (we refer the reader to Baker's preprint [3]).

3. It is also useful to have an explicit description of the intersection of $C(\overline{K})$ with special torsion subgroups of $J(\overline{K})$. For Examples 4 and 8, one has the theorem of Mazur ([40], [41]) and Mazur and Swinnerton-Dyer ([42]) that the intersection of $X_0(p)$ with the cuspidal group consists only of the two cusps. For Example 5, an analogous result follows from the work of Greenberg ([28]), Gross and Rohrlich ([29]) and Kurihara ([33]).

4. In certain cases, working with images of the curve in question can resolve certain technical difficulties. In Example 5, it almost suffices to consider only prime exponents, by means of the evident map $F_N \longrightarrow F_M$, whenever M divides N. Also the quotients of F_p are easier to work with because of the existence of complex multiplication.
In Example 8, Baker uses the projection from $X_0(p)$ to $X_0^+(p)$ to eliminate complications at the Eisenstein primes. The latter idea simplifies the problem considerably.

5. Finally, we should mention that Buium's techniques provide an entirely different approach to these problems, as illustrated by Voloch's Example 7 which makes essential use of Buium's methods (see [61]).

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