Multilayer Reflectivity

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January 5, 2014

1 Introduction

The purpose of this document is to present an *ab initio* derivation of the reflectivity for a plane electromagnetic wave reflecting off a flat multilayer surface (see Figure 1). Each layer is assumed to be isotropic and described by a complex dielectric constant. The final result will be in the form of a recursion relation first derived by Parratt[1].

2 Conventions

In this document, the phase convention $e^{-i\omega t}$ will be used. The Henke[2] tables[1] assume the opposite convention. For this reason, the sign of β that appears in the supporting documentation for the Henke data is opposite of what one would use from the formulas derived here. In particular, the index of refraction n will become $n = 1 - \delta + i\beta$, and *not* $n = 1 - \delta - i\beta$ as is stated in the supporting documentation for the tables.

The SI form of the Maxwell equations is used in this work. In SI units, the Maxwell equations assume a simpler form than if Gaussian units were employed. The SI forms of the macroscopic Maxwell equations are

$$\nabla \cdot \vec{B} = 0, \tag{1}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{2}$$

$$\nabla \cdot \vec{D} = \rho, \tag{3}$$

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}, \tag{4}$$
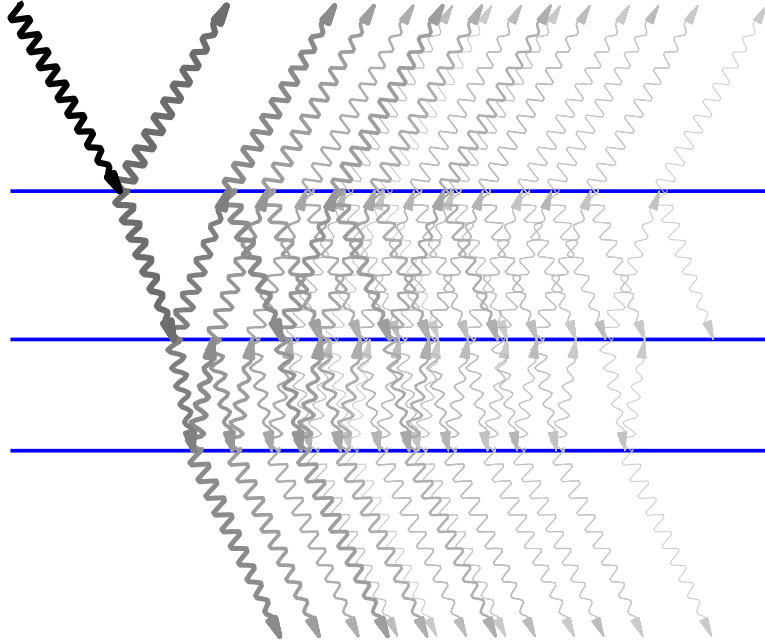
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[1] http://henke.lbl.gov/optical_constants/

Figure 1: This figure shows the possible paths of a photon within a multilayer that is composed of two dielectric slabs sandwiched between the vacuum at the top and a bulk substrate at the bottom. When the photon encounters the boundary between two layers, it either reflects or refracts. The shade and thickness of a photon line is used to denote the relative probability of the line.

where $\vec{j}$ and ρ are the macroscopic (or free) current and charge densities, respectively.

3 Plane Waves

In dealing with harmonic plane waves, it often simpler to use a complex representation for the fields:

$$\vec{E}(\vec{x}, t) = \vec{E}(\vec{k}, \omega) e^{i\vec{k}\cdot\vec{x} - i\omega t} \tag{5}$$

$$\vec{H}(\vec{x}, t) = \vec{H}(\vec{k}, \omega) e^{i\vec{k}\cdot\vec{x} - i\omega t} \tag{6}$$

$$\vec{B}(\vec{x}, t) = \vec{B}(\vec{k}, \omega) e^{i\vec{k}\cdot\vec{x} - i\omega t} \tag{7}$$

$$\vec{D}(\vec{x}, t) = \vec{D}(\vec{k}, \omega) e^{i\vec{k}\cdot\vec{x} - i\omega t} \tag{8}$$

The physical fields are obtained by taking the real parts of the corresponding complex fields. The amplitudes such as $\vec{E}(\vec{k},\omega)$ are in general complex valued where the phase of $\vec{E}(\vec{k},\omega)$ corresponds to a phase shift of the wave.

If the medium is a lossy one, then $\vec{k}$ will also be a complex (bi-)vector. The real part of $\vec{k}$ corresponds to the direction of constant phase of the wave, whereas the imaginary part will give the direction of constant amplitude. In general these directions will be different.

It is important to note that a complex representation such as the above is permissible only because the Maxwell equations are linear in the fields with real coefficients. Then both the real and imaginary parts of the fields independently satisfy the Maxwell equations with no mixing of the real and imaginary components. When dealing with expressions that are non-linear in the fields, one must first revert back to the real fields before evaluating the expression in order to avoid the undesired mixing.

In the absence of sources, the Maxwell equations can be written in terms of the complex field amplitudes as:

$$\vec{k} \times \vec{E}(\vec{k},\omega) = \omega \vec{B}(\vec{k},\omega) \tag{9}$$

$$\vec{k} \times \vec{H}(\vec{k},\omega) = -\omega \vec{D}(\vec{k},\omega) \tag{10}$$

$$\vec{k} \cdot \vec{D}(\vec{k},\omega) = 0 \tag{11}$$

$$\vec{k} \cdot \vec{B}(\vec{k},\omega) = 0. \tag{12}$$

For a linear homogeneous isotropic medium, we will assume that the macroscopic fields $(\vec{D}, \vec{H})$ are related to the microscopic ones $(\vec{E}, \vec{B})$ via the constitutive relations

$$\vec{D}(\vec{k},\omega) = \varepsilon(\omega)\vec{E}(\vec{k},\omega) \tag{13}$$

$$\vec{H}(\vec{k},\omega) = \mu^{-1}(\omega)\vec{B}(\vec{k},\omega), \tag{14}$$

where, in general, the dielectric constant $\varepsilon(\omega)$ and the permeability $\mu(\omega)$ are complex and frequency dependent.

For clarity, in the following unless otherwise indicated, the explicit functional dependence upon $\vec{k}$ and ω will be dropped. Hence, $\vec{E}(\vec{k},\omega)$ will be written simply as $\vec{E}$, and $\varepsilon(\omega)$ will be written as ε. With this understanding, equations (9)-(12) may be written

$$\vec{k} \times \vec{E} = \mu\omega\vec{H} \tag{15}$$

$$\vec{k} \times \vec{H} = -\omega\varepsilon\vec{E} \tag{16}$$

$$\vec{k} \cdot \vec{E} = 0 \tag{17}$$

$$\vec{k} \cdot \vec{H} = 0. \tag{18}$$

Taking the cross product of the first of these with $\vec{k}$ and substituting in the second results in

$$\vec{k} \cdot \vec{k} = k^2 = \omega^2 \mu \varepsilon, \tag{19}$$

which is a dispersion relation between $\vec{k}$ and ω.

In a medium where $\vec{k}$ is real, such as the vacuum, equations (15-18) imply that the vectors $\vec{k}$, $\vec{E}$, and $\vec{H}$ are mutually orthogonal. This claim cannot be made if the dielectric medium is complex.

The real index of refraction n_r of a medium is defined as the ratio of the speed of light in the vacuum to the speed of constant phase speed in the medium, i.e., $n_r = (c/\omega) \operatorname{Re}(k)$. The complex index of refraction n is defined by

$$n = \frac{ck}{\omega}. \tag{20}$$

4 The Poynting Vector

The flux in an electromagnetic wave is given by the time-averaged magnitude of the Poynting vector $\vec{S}(\vec{x}, t)$, which is proportional to the cross product of the $\vec{E}(\vec{x}, t)$ and $\vec{H}(\vec{x}, t)$ fields. In performing this cross product the real valued fields must be used. Naively using the complex fields would result in an undesirable mixing of the real and imaginary parts. To see this, write

$$\begin{aligned}
\vec{E}(\vec{x}, t) &= \vec{E}(\vec{x})e^{-i\omega t} \\
&= (\vec{E}_r + i\vec{E}_i)e^{-i\omega t} \\
&= (\vec{E}_r \cos \omega t + \vec{E}_i \sin \omega t) + i(\vec{E}_i \cos \omega t - \vec{E}_r \sin \omega t)
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\vec{H}(\vec{x}, t) &= (\vec{H}_r + i\vec{H}_i)e^{-i\omega t} \\
&= (\vec{H}_r \cos \omega t + \vec{H}_i \sin \omega t) + i(\vec{H}_i \cos \omega t - \vec{H}_r \sin \omega t)
\end{aligned} \tag{22}$$

Then the time-averaged Poynting vector is

$$\begin{aligned}
\langle \vec{S}(\vec{x}, t) \rangle &= \langle \operatorname{Re}[\vec{E}(\vec{x}, t)] \times \operatorname{Re}[\vec{H}(\vec{x}, t)] \rangle \\
&= \vec{E}_r \times \vec{H}_r \langle \cos^2 \omega t \rangle + \vec{E}_i \times \vec{H}_i \langle \sin^2 \omega t \rangle \\
&\quad + (\vec{E}_r \times \vec{H}_i + \vec{E}_i \times \vec{H}_r) \langle \cos \omega t \sin \omega t \rangle \\
&= \frac{1}{2}(\vec{E}_r \times \vec{H}_r + \vec{E}_i \times \vec{H}_i) \\
&= \frac{1}{2} \operatorname{Re}[\vec{E}(\vec{x}) \times \vec{H}^*(\vec{x})] = \frac{1}{2} \operatorname{Re}[\vec{E}^*(\vec{x}) \times \vec{H}(\vec{x})],
\end{aligned} \tag{23}$$

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which is *not* the same as using the real part of $\vec{E}(\vec{x}) \times \vec{H}(\vec{x})$.

For plane waves of the form

$$\vec{E}(\vec{x}) = \vec{E} e^{i\vec{k}\cdot(\vec{x}-\vec{x}_0)}, \tag{24}$$

we can write

$$\vec{E}^*(\vec{x}) \times \vec{H}(\vec{x}) = \vec{E}^* \times \vec{H} e^{-2\,\mathrm{Im}\,\vec{k}\cdot(\vec{x}-\vec{x}_0)} \tag{25}$$

$$= \frac{1}{\mu\omega} \vec{E}^* \times (\vec{k} \times \vec{E}) e^{-2\,\mathrm{Im}\,\vec{k}\cdot(\vec{x}-\vec{x}_0)} \tag{26}$$

$$= \frac{1}{\mu\omega} \big(\vec{k}|\vec{E}|^2 - \vec{E}(\vec{k}\cdot\vec{E}^*)\big) e^{-2\,\mathrm{Im}\,\vec{k}\cdot(\vec{x}-\vec{x}_0)}, \tag{27}$$

where where $\vec{E}$ is the field amplitude[2] at x_0, and $|\vec{E}|^2 = \vec{E}\cdot\vec{E}^*$. For a medium such as the vacuum where $\vec{k}$ is real, (17) implies that $\vec{k}\cdot\vec{E}^* = 0$. But when $\vec{k}$ is complex, $\vec{k}\cdot\vec{E} = 0$ does not mean that $\vec{k}\cdot\vec{E}^*$ is also zero. After a bit of algebra and making use of (17), we find

$$\langle\vec{S}(\vec{x},t)\rangle = e^{-2\,\mathrm{Im}\,\vec{k}\cdot(\vec{x}-\vec{x}_0)}\langle\vec{S}(\vec{x}_0,t)\rangle, \tag{28}$$

where

$$\langle\vec{S}(\vec{x}_0,t)\rangle = \frac{1}{2\mu\omega}\left[|\vec{E}|^2\,\mathrm{Re}\,\vec{k} - 2\,\mathrm{Im}\,\vec{k} \times (\mathrm{Re}\,\vec{E} \times \mathrm{Im}\,\vec{E})\right]. \tag{29}$$

When $\vec{k}$ is complex, this equation shows that the imaginary part of $\vec{k}$ leads to exponential damping of the time-averaged Poynting vector. With the addition of the second term in the brackets above, we see that energy is transported in a direction that differs from the direction of the wave, which is given by $\mathrm{Re}\,\vec{k}$. This term is zero if $\vec{k}$ is real, or if the electric field is linearly polarized since for linear polarization, the real and imaginary parts of the electric field lie in the same direction. As will be seen, for so-called TM fields, the electric field in a lossy medium is elliptically polarized in the plane of incidence causing this term to introduce a component to the flux that is parallel to the multilayer surface and in the plane of incidence. In this paper, we are primarily interested in the component of the Poynting vector that is normal to the dielectric surfaces. Hence this term will drop out of the calculation.

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[2]In a lossy medium where $\mathrm{Im}\,\vec{k}$, the field amplitude decrease along the direction of $\mathrm{Im}\,\vec{k}$.

5 Multilayer Geometry

We assume that the multilayer is composed of N parallel layers with thicknesses d_j, $j = 1, \ldots N$ sandwiched between a semi-infinite vacuum on one side and a semi-infinite substrate on the other. The vacuum will be denoted as the $j = 0$ layer, and the substrate by $j = N+1$. A coordinate system will be adopted such that the z axis is perpendicular to the interface planes and increases from the vacuum to the substrate. In particular, the z coordinate of the plane forming the interface between the j and $j+1$ layers is given by z_j, where

$$z_j = z_{j-1} + d_j. \tag{30}$$

For later purposes, it is convenient to extend the set of d_j values to include the symbols $d_0 = 0$ and $d_{N+1} = 0$.

As figure 1 indicates, each dielectric layer will consist of a superposition of reflected and refracted electromagnetic waves. The exception is in the substrate, which contains only refracted waves. The net electric field in the jth layer can be written as

$$\vec{E}^{(j)}(\vec{x}, t) = \vec{E}^{(j)}(\vec{k}^{(j)}, \omega)e^{i(\vec{k}^{(j)} \cdot \vec{x} - \omega t)} + \vec{E}'^{(j)}(\vec{k}'^{(j)}, \omega)e^{i(\vec{k}'^{(j)} \cdot \vec{x} - \omega t)}, \tag{31}$$

where $\vec{k}^{(j)}$ is the wave vector of the forward traveling wave and $\vec{k}'^{(j)}$ is that of the reflected wave. A similar equation holds for the $\vec{H}$ field.

6 Boundary Conditions

The boundary conditions of the fields may be found by applying Stokes's theorem to the Maxwell equations involving the curl, and the divergence theorem to those involving the divergence. These operations are trivial and will not be given here. It follows that the component of $\vec{E}(\vec{x}, t)$ parallel to the surface is preserved across the boundary. For a dielectric surface with finite conductivity, there can be no surface current and it follows that the parallel component of $\vec{H}(\vec{x}, t)$ is also preserved. Only for an ideal conductor with infinite conductivity does one have to worry about a surface current. If $\hat{z}$ is the surface normal, the boundary conditions on $\vec{E}$ and $\vec{H}$ at the interface between the j and $j+1$ layers at the point $\vec{x}_j = (x, y, z_j)$ is given by

$$\begin{aligned}\hat{z} \times \vec{E}^{(j+1)}(\vec{x}_j, t) &= \hat{z} \times \vec{E}^{(j)}(\vec{x}_j, t)) \\ \hat{z} \times \vec{H}^{(j+1)}(\vec{x}_j, t) &= \hat{z} \times \vec{H}^{(j)}(\vec{x}_j, t)).\end{aligned} \tag{32}$$

Let $\vec{E}_{||} = \hat{z} \times \vec{E}$ denote the components of the electric field parallel to the interface. Then using (31) in the boundary condition for $\vec{E}$ produces

$$\vec{E}_{||}^{(j)} e^{i\vec{k}^{(j)}\cdot\vec{x}_j} + \vec{E}_{||}'^{(j)} e^{i\vec{k}'^{(j)}\cdot\vec{x}_j} = \vec{E}_{||}^{(j+1)} e^{i\vec{k}^{(j+1)}\cdot\vec{x}_j} + \vec{E}_{||}'^{(j+1)} e^{i\vec{k}'^{(j+1)}\cdot\vec{x}_j}. \quad (33)$$

We can separate the vector $\vec{x}_j$ into a components parallel and perpendicular to the $z = z_j$ plane by writing $\vec{x}_j = \vec{x}_{||} + z_j\hat{z}$, whereby the previous equation becomes

$$\vec{E}_{||}^{(j)} e^{ik_z^{(j)}z_j} e^{i\vec{k}^{(j)}\cdot\vec{x}_{||}} + \vec{E}_{||}'^{(j)} e^{ik_z'^{(j)}z_j} e^{i\vec{k}'^{(j)}\cdot\vec{x}_{||}}$$
$$= \vec{E}_{||}^{(j+1)} e^{ik_z^{(j+1)}z_j} e^{i\vec{k}^{(j+1)}\cdot\vec{x}_{||}} + \vec{E}_{||}'^{(j+1)} e^{ik_z'^{(j+1)}z_j} e^{i\vec{k}'^{(j+1)}\cdot\vec{x}_{||}}. \quad (34)$$

The only way for this boundary condition to hold at all points $\vec{x}_{||}$ in the $z = z_j$ plane is for

$$\vec{k}^{(j)} \cdot \vec{x}_{||} = \vec{k}'^{(j)} \cdot \vec{x}_{||} = \vec{k}^{(j+1)} \cdot \vec{x}_{||} = \vec{k}'^{(j+1)} \cdot \vec{x}_{||}, \quad (35)$$

or

$$\begin{aligned} k_x^{(j)} &= k_x'^{(j)} = k_x^{(j+1)} = k_x'^{(j+1)} \\ k_y^{(j)} &= k_y'^{(j)} = k_y^{(j+1)} = k_y'^{(j+1)}. \end{aligned} \quad (36)$$

From these equations it follows that reflection and refraction take place in a plane (called the plane of incidence). Moreover, since the above must hold for both real and imaginary parts of the wave vectors, and assuming that the wave vector in the vacuum $\vec{k}^{(0)}$ is real, then $k_{x,y}^{(j)}$ are also real. Hence, we are left to conclude that only $k_z^{(j)}$ can have a non-zero imaginary component. It follows from the dispersion relation given by (19) that either $k_z'^{(j)} = k_z^{(j)}$ or $k_z'^{(j)} = -k_z^{(j)}$. If, as has been assumed, that the net electric field in a layer is the sum of distinct reflected and a forward traveling waves, then consistency demands that

$$k_z'^{(j)} = -k_z^{(j)} \quad (37)$$

Using the above relations, (34) can be rewritten as

$$\vec{E}_{||}^{(j)} e^{ik_z^{(j)}z_j} + \vec{E}_{||}'^{(j)} e^{-ik_z^{(j)}z_j} = \vec{E}_{||}^{(j+1)} e^{ik_z^{(j+1)}z_j} + \vec{E}_{||}'^{(j+1)} e^{-ik_z^{(j+1)}z_j}. \quad (38)$$

Similarly,

$$\vec{H}_{||}^{(j)} e^{ik_z^{(j)}z_j} + \vec{H}_{||}'^{(j)} e^{-ik_z^{(j)}z_j} = \vec{H}_{||}^{(j+1)} e^{ik_z^{(j+1)}z_j} + \vec{H}_{||}'^{(j+1)} e^{-ik_z^{(j+1)}z_j}. \quad (39)$$

7 Reflectivity

The reflectivity, $\mathcal{R}$ of a multilayer is defined as the ratio of the flux in the reflected wave to that of the incident wave at the interface between the vacuum ($j = 0$) and the first ($j = 1$) dielectric slab. This can be computed as ratio of the magnitude of the time-averaged Poynting vector of the reflected wave to that of the incident wave. Since for the vacuum, $\vec{k}$ is real, it follows trivially from (29) that

$$\mathcal{R} = \frac{|\vec{E}'|^2}{|\vec{E}|^2} = \frac{|\vec{H}'|^2}{|\vec{H}|^2}. \tag{40}$$

That latter equality is a result of replacing $\vec{E}$ by (16), expanding the resulting vector products, and using (18).

From the boundary conditions, we found that the component of the wave vector parallel to the jth multilayer interface, $\vec{k}_{||}^{(j)}$, is real for an incident plane wave from the vacuum. Since the boundary conditions also imply that this component is the same for all the layers, the superscript (j) can be dropped and the parallel component written simply as $\vec{k}_{||}$. For simplicity in what follows, the x axis will be chosen to lie in the direction of $\vec{k}_{||}$, i.e., $\vec{k}_{||} = k_x \hat{x}$. Then using

$$\vec{k}^{(j)} = k_x \hat{x} + k_z^{(j)} \hat{z}, \tag{41}$$

equations (15) and (16) can be written in component form as

$$\omega \mu_j H_x^{(j)} = -k_z^{(j)} E_y^{(j)}, \tag{42}$$

$$\omega \mu_j H_y^{(j)} = k_x E_z^{(j)} - k_z^{(j)} E_x^{(j)}, \tag{43}$$

$$\omega \mu_j H_z^{(j)} = k_x E_y^{(j)}, \tag{44}$$

and

$$\omega \varepsilon^{(j)} E_x^{(j)} = k_z^{(j)} H_y^{(j)}, \tag{45}$$

$$\omega \varepsilon^{(j)} E_y^{(j)} = k_x H_z^{(j)} - k_z^{(j)} H_x^{(j)}, \tag{46}$$

$$\omega \varepsilon^{(j)} E_z^{(j)} = -k_x H_y^{(j)}. \tag{47}$$

The electric field $\vec{E}^{(j)}$ can always be written as a component that lies in the plane of incidence, and one normal to the plane. In the vacuum, if $\vec{E}^{(0)}$ is perpendicular to the plane of incidence, then $\vec{H}^{(0)}$ will lie in the plane of incidence, and if $\vec{E}^{(0)}$ lies in the plane of incidence, then $\vec{H}^{(0)}$ must be perpendicular to the plane. Thus we can consider two distinct cases: The

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first case, known as TE "Transverse Electric", is one where the incident electric field $\vec{E}^{(0)}$ is perpendicular to the plane of incidence. In the second case, TM or "Transverse Magnetic", the $\vec{H}^{(0)}$ field normal to the plane of incidence.

7.1 TE Case

For this case, the incident electric field from the vacuum is normal to the plane of incidence. Expressed in terms of components, it can be written as $\vec{E}^{(j)} = E^{(j)}\hat{y}$. Then the only non-zero component of the $\vec{H}^{(j)}$ field that is parallel to the interface is

$$H_x^{(j)} = -\frac{k_z^{(j)}}{\omega\mu_j}E^{(j)}. \tag{48}$$

Similarly, the reflected field satisfies

$$H_x'^{(j)} = \frac{k_z^{(j)}}{\omega\mu_j}E'^{(j)}. \tag{49}$$

Using the above expressions, it is straightforward to show that the two boundary condition equations (38) and (39) may be written as

$$E^{(j)}e^{ik_z^{(j)}z_j}(1 + a_j^{-1}R_j) = E^{(j+1)}e^{ik_z^{(j+1)}z_j}(1 + R_{j+1}) \tag{50}$$

$$f_j E^{(j)}e^{ik_z^{(j)}z_j}(1 - a_j^{-1}R_j) = f_{j+1}E^{(j+1)}e^{ik_z^{(j+1)}z_j}(1 - R_{j+1}), \tag{51}$$

where

$$f_j = k_z^{(j)}/\mu_j, \tag{52}$$

$$a_j = e^{2ik_z^{(j)}d_j}, \tag{53}$$

and

$$R_j = a_j e^{-2ik_z^{(j)}z_j}\frac{E'^{(j)}}{E^{(j)}}. \tag{54}$$

Note that in defining a_j, we have assumed that both d_0 and d_{N+1} are 0, otherwise d_j corresponds to the width of the jth layer as defined by (30).

The system of equations given by (50) and 51 may be easily solved for R_j to yield the Parratt recursion relation

$$R_j = a_j\frac{F_j + R_{j+1}}{1 + F_j R_{j+1}}, \tag{55}$$

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where

$$F_j = \frac{f_j - f_{j+1}}{f_j + f_{j+1}}. \tag{56}$$

The recurrence relation (55) may be solved for R_0 iteratively by noting that $R_{N+1} = 0$ in the substrate where there is no reflected field. The quantities F_j and a_j may be precomputed since they depend only upon the thicknesses d_j of the dielectric layers, and $k_z^{(j)}$, which may be computed from (20) via

$$\begin{aligned} k_z^{(j)} &= \sqrt{(\omega/c)^2 n_j^2 - k_x^2} \\ &= \frac{\omega}{c}\sqrt{n_j^2 - \cos^2\phi}, \end{aligned} \tag{57}$$

where ϕ is the graze angle that the incident wave makes with the multilayer surface at z_0.

Then the reflectivity is given by

$$\mathcal{R} = |R_0|^2. \tag{58}$$

7.2 TM Case

In the transverse magnetic case, the incident magnetic field is normal to the plane of incidence and can be written as $\vec{H}^j = H^{(j)}\hat{y}$. The non-zero electric field component parallel to the interface plane for the forward moving wave is

$$E_x^j = \frac{k_z^{(j)}}{\omega\varepsilon^{(j)}}H^{(j)}, \tag{59}$$

and

$$E_x'^j = -\frac{k_z^{(j)}}{\omega\varepsilon^{(j)}}H'^{(j)} \tag{60}$$

for the reflected field. Using these equations, the boundary conditions given by equations (38) and (39) become

$$f_j H^{(j)} e^{ik_z^{(j)} z_j}(1 - a_j^{-1} R_j) = f_{j+1} H^{(j+1)} e^{ik_z^{(j+1)} z_j}(1 - R_{j+1}) \tag{61}$$

$$H^{(j)} e^{ik_z^{(j)}}(1 + a_j^{-1} R_j) = H^{(j+1)} e^{ik_z^{(j+1)} z_j}(1 + R_{j+1}), \tag{62}$$

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where

$$f_j = k_z^{(j)}/\varepsilon^{(j)} = (\mu_j c^2/n_j^2)k_z^{(j)}, \tag{63}$$

$$a_j = e^{2ik_z^{(j)}(z_{j+1}-z_j)} = e^{2ik_z^{(j)}d_j}, \tag{64}$$

and

$$R_j = a_j e^{-2ik_z^{(j)}z_j}\frac{H'^{(j)}}{H^{(j)}}. \tag{65}$$

Equations (61) and (62) are formally identical to equations (50) and (51). As such, the recurrence relation given by (55) may be used here for the TM case provided that f_j is given by (63).

8 Transmission Through a Multilayer

The case when the substrate is the vacuum (or air) is also of interest, where it is meaningful to talk about the transmission probability through the multilayer. This probability, or transmittance, is defined to be the fraction of incident flux that passes into the multilayer substrate. For a substrate with a complex wave vector k we find

$$\mathcal{T} = \frac{\langle \vec{S}(z_N^+)\rangle \cdot \hat{z}}{\langle \vec{S}(z_0^-)\rangle \cdot \hat{z}} \tag{66}$$

where $\vec{S}(z_0^-)$ is the value of the Poynting vector for the forward propagating incident wave at the top of the multilayer, and $\vec{S}(z_N^+)$ is the refracted wave at the boundary of the substrate. Here the plus and minus superscripts are used to signify the particular sides of the dielectric boundary, with $z^{\pm} = z\pm\epsilon$. For clarity, the dependence upon the x and y variables has been suppressed.

For the special cases of TE and TM modes, the term involving the cross product of the real and imaginary electric fields in (29) is zero since it is in a direction that is orthogonal to $\hat{z}$. Hence we find

$$\langle \vec{S}(z_0^-)\rangle \cdot \hat{z} = \frac{|\vec{E}(z_0^-)|^2}{2\mu_0\omega}k_z^{(0)} \tag{67}$$

at the top of the multilayer (where $k_z^{(0)}$ is real), and

$$\langle \vec{S}(z_N^+)\rangle \cdot \hat{z} = \frac{|\vec{E}(z_N^+)|^2}{2\mu_{N+1}\omega}\operatorname{Re}k_z^{(N+1)} \tag{68}$$

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just inside the substrate, with the result

$$\mathcal{T} = (\mu_0/\mu_{N+1})\operatorname{Re}(k_z^{(N+1)}/k_z^{(0)})\frac{|\vec{E}(z_N^+)|^2}{|\vec{E}(z_0^-)|^2}. \tag{69}$$

In these expressions, $\vec{E}(\vec{x})$ represents the complex field amplitude at the position $\vec{x}$.

8.1 TE Transmittance

For TE mode, the electric field has no z component and is perpendicular to the plane of incidence. Hence,

$$\mathcal{T} = |T|^2 \frac{\mu_0 \operatorname{Re} k_z^{(N+1)}}{\mu_{N+1} k_z^{(0)}} \tag{70}$$

where

$$T = \frac{E(z_N^+)}{E(z_0^-)}. \tag{71}$$

The latter equation can be written in the form

$$T = \frac{E(z_N^+)}{E(z_0^-)} \tag{72}$$

$$= \left(\frac{E(z_1^-)}{E(z_0^-)}\right)\left(\frac{E(z_2^-)}{E(z_1^-)}\right)\cdots\left(\frac{E(z_N^-)}{E(z_{N-1}^-)}\right)\left(\frac{E(z_N^+)}{E(z_N^-)}\right) \tag{73}$$

$$= T_0 T_1 \cdots T_{N-1} T_N, \tag{74}$$

where

$$T_j = \frac{E^{(j+1)}}{E^{(j)}} \exp[iz_j(k_z^{(j+1)} - k_z^{(j)}) + ik_z^{(j+1)}d_{j+1}] \tag{75}$$

and (31) was used to write

$$E(z_j^-) = E^{(j)} e^{ik_z^{(j)} z_j} \tag{76}$$

for the refracted wave in the jth layer. It is straightforward to show from equations (50) and (55) that

$$T_j = e^{ik_z^{(j+1)}d_{j+1}} \frac{1 + F_j}{1 + F_j R_{j+1}}. \tag{77}$$

8.2 TM Transmittance

In TM mode, the magnetic field is perpendicular to the plane of incidence so that both $\vec{k} \cdot \vec{H} = 0$ and $\vec{k}^* \cdot \vec{H}^*$ are zero. Then it is easy to show from (16) that

$$|\vec{E}|^2 = \frac{|\vec{k}|^2}{\omega^2 |\varepsilon|^2} |\vec{H}|^2 = \frac{\mu^2 c^4}{\omega^2 |n|^4} |\vec{k}|^2 |H|^2, \tag{78}$$

where c is the speed of light in the vacuum. This allows (69) to be cast into the form

$$\mathcal{T} = \left(\frac{\mu_{N+1}/\mu_0}{|n_{N+1}|^4} \right) \frac{\operatorname{Re} k_z^{(N+1)}}{k_z^{(0)}} \frac{|\vec{k}^{(N+1)}|^2}{|\vec{k}^{(0)}|^2} |T|^2 \tag{79}$$

where

$$T = \frac{H(z_N^+)}{H(z_0^-)} \tag{80}$$

$$= T_0 T_1 \cdots T_N. \tag{81}$$

By a similar analysis as for the TE case, we find

$$T_j = \frac{H^{(j+1)}}{H^{(j)}} \exp[iz_j(k_z^{(j+1)} - k_z^{(j)}) + ik_z^{(j+1)} d_{j+1}] \tag{82}$$

$$= e^{ik_z^{(j+1)} d_{j+1}} \frac{1 + F_j}{1 + F_j R_{j+1}}, \tag{83}$$

where F_j and R_j are given in section 7.2 for the TM case.

9 Examples

In this section, we will consider a couple of examples and compare the results with those in standard textbooks as a consistency check.

9.1 A Single Dielectric Interface

For this example, $N = 0$ to indicate a single interface between the the vacuum and a bulk substrate. Then (55) is easily solved to yield

$$R_0 = F_0 = \frac{f_0 - f_1}{f_0 + f_1}, \tag{84}$$

which for the specific TE and TM cases, becomes

$$R_0^{TE} = \frac{\sin\phi - \sqrt{n^2 - \cos^2\phi}}{\sin\phi + \sqrt{n^2 - \cos^2\phi}} \tag{85}$$

$$R_0^{TM} = \frac{n^2\sin\phi - \sqrt{n^2 - \cos^2\phi}}{n^2\sin\phi + \sqrt{n^2 - \cos^2\phi}}, \tag{86}$$

where n is the (complex) index of refraction of the substrate, and ϕ is the graze angle of the incoming wave with respect to the interface. If a complex refraction angle $\sin\phi'$ is defined by

$$\sin\phi' = \sqrt{n^2 - \cos\phi}, \tag{87}$$

we can write

$$R_0^{TE} = \frac{\sin\phi - \sin\phi'}{\sin\phi + \sin\phi'} \tag{88}$$

$$R_0^{TM} = \frac{n^2\sin\phi - \sin\phi'}{n^2\sin\phi + \sin\phi'} \tag{89}$$

Using (77) to find the transmission coefficient yields

$$T_0 = 1 + F_0 = 1 + R_0. \tag{90}$$

Physically, this expression corresponds to the notion that the energy in the transmitted wave is equal to that of incident one plus what is taken away by the reflected component, which is traveling in the opposite z direction.

For TE mode, evaluating $\mathcal{R}$ and $\mathcal{T}$ is straightforward producing

$$\mathcal{R}^{TE} = |R_0^{TE}|^2 \tag{91}$$

$$= \left|\frac{\mu_1\sin\phi - \mu_0 n\sin\phi'}{\mu_1\sin\phi + \mu_0 n\sin\phi'}\right|^2 \tag{92}$$

and

$$\mathcal{T}^{TE} = \frac{\mu_0}{\mu_1}\frac{\operatorname{Re}k_z^{(1)}}{k_z^{(0)}}|1 + R_0^{TE}|^2 \tag{93}$$

$$= \frac{4\mu_0\mu_1\sin\phi\operatorname{Re}(n\sin\phi')}{|\mu_1\sin\phi + \mu_0 n\sin\phi'|^2}. \tag{94}$$

After some simple algebra, it is easy to see that $\mathcal{R} + \mathcal{T} = 1$ for TE mode.

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For TM mode, we find

$$\mathcal{R}^{TM} = |R_0^{TM}|^2 \tag{95}$$

$$= \left| \frac{\mu_0 \sin\phi - (\mu_1/n)\sin\phi'}{\mu_0 \sin\phi + (\mu_1/n)\sin\phi'} \right|^2 \tag{96}$$

and

$$\mathcal{T}^{TM} = \left(\frac{\mu_1/\mu_0}{|n|^4} \right) \frac{\operatorname{Re} k_z^{(1)}}{k_z^{(0)}} \frac{|\vec{k}^{(1)}|^2}{|\vec{k}^{(0)}|^2} |1 + R_0^{TM}|^2 \tag{97}$$

$$= \frac{4\mu_0\mu_1 \sin\phi}{|\mu_0 \sin\phi + (\mu_1/n)\sin\phi'|^2} \left[\frac{\operatorname{Re}(n\sin\phi')(\cos^2\phi + |n\sin\phi'|^2)}{|n|^4} \right]. \tag{98}$$

It is easy to see that when n is real, the bracketed quantity in the transmittance reduces to $\operatorname{Re}(\sin\phi'/n)$. But for the general case of complex n the simplification of this term requires some caution. First proceed by expanding $\operatorname{Re}(n\sin\phi')$ and $|n\sin\phi'|^2$ in terms of their complex conjugates and then simplify. This produces

$$\left[\cdot \right] = \frac{(n\sin\phi' + n^*\sin\phi'^*)(\cos^2\phi + nn^*\sin\phi'\sin\phi'^*)}{2|n|^4} \tag{99}$$

$$= \frac{n|n|^2 \sin\phi'^* + n\sin\phi'(n^2\cos^2\phi' + n^{*2}\sin^2\phi^*)}{2|n|^4}, \tag{100}$$

where (87) has been used to write

$$\cos^2\phi = n^2\cos^2\phi'. \tag{101}$$

The important thing to note here is that since $\cos^2\phi$ is a real-valued quantity, $n^2\cos^2\phi'$ may be replaced by $n^{*2}\cos^2\phi'^*$ in the bracketed expression. After some some simplification we find

$$\left[\cdot \right] = \operatorname{Re}\left(\frac{\sin\phi'}{n} \right), \tag{102}$$

and finally

$$\mathcal{T}^{TM} = \frac{4\mu_0 \sin\phi \operatorname{Re}[(\mu_1/n)\sin\phi']}{|\mu_0 \sin\phi + (\mu_1/n)\sin\phi'|^2}. \tag{103}$$

10 Summary

In this document, we considered the reflection of an electromagnetic wave in the vacuum from a flat semi-infinite multilayer dielectric substance. The multilayer was assumed to be composed of N dielectric layers, each of finite thickness, on a semi-infinite substrate. We found that the reflectivity of the multilayer to be given by

$$\mathcal{R} = |R_0|^2, \tag{104}$$

where R_0 is calculated from the Parratt recursion relation

$$R_j = a_j \frac{F_j + R_{j+1}}{1 + F_j R_{j+1}}, \tag{105}$$

starting from $j = N$ using $R_{N+1} = 0$. Here,

$$a_j = e^{2i k_z^{(j)} d_j}, \tag{106}$$

$$F_j = \frac{f_j - f_{j+1}}{f_j + f_{j+1}}, \tag{107}$$

$$f_j = \begin{cases} k_z^{(j)}/\mu_j & \text{TE} \\ (\mu_j c^2/n_j^2) k_z^{(j)} & \text{TM} \end{cases}, \tag{108}$$

and

$$k_z^{(j)} = \frac{\omega}{c} \sqrt{n_j^2 - \cos^2 \phi}. \tag{109}$$

In these equations, the thickness of the jth layer is d_j, with d_0 defined to be 0, n_j is the index of refraction of the jth layer, μ_j is the magnetic permeability of the layer, and ϕ is the grazing angle of the incident electromagnetic wave with the multilayer.

Acknowledgments

References

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