1 Holomorphic functions

1.1 The complex derivative

The basic objects of complex analysis are the holomorphic functions. These are functions that posses a complex derivative. As we will see this is quite a strong requirement and will allow us to make far reaching statements about this type of functions. To properly understand the concept of a complex derivative, let us recall first the concept of derivative in $\mathbb{R}^n$.

Definition 1.1. Let U be an open set in $\mathbb{R}^n$ and $A : U \to \mathbb{R}^m$ a function. Given $x \in U$ we say that A is *(totally) differentiable* at x iff there exists an $m \times n$-matrix A' such that,

$$A(x + \xi) = A(x) + A'\xi + \mathrm{o}(\|\xi\|)$$

for $\xi \in \mathbb{R}^n$ sufficiently small. Then, A' is called the *derivative* of A at x.

Recall that the matrix elements of A' are the partial derivatives

$$A'_{ij} = \frac{\partial A_i}{\partial x_j}.$$

Going from the real to the complex numbers, we can simply use the decomposition $z = x+\mathrm{i}y$ of a complex number z into a pair of real numbers (x, y) to define a concept of derivative. Thus, let U be an open set in $\mathbb{C}$ and consider a function $f : U \to \mathbb{C}$. We view U as an open set in $\mathbb{R}^2$ with coordinates (x, y) and $f = u + \mathrm{i}v$ as a function with values in $\mathbb{R}^2$ with coordinates (u, v). The total derivative of f, if it exists, is then a 2×2-matrix f' given by

$$f' = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

So far we have only recited concepts from real analysis and not made use of the fact that the complex numbers do not merely form a 2-dimensional real vector space, but a field. Indeed, this implies that there are special 2×2-matrices, namely those that correspond to multiplication by a complex number. As is easy to see, multiplication by $a+\mathrm{i}b$ corresponds to the matrix,

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

The crucial step that leads us from real to complex analysis is now the additional requirement that the derivative f' take this form. It is then more useful to think of f' as the complex number $a + \mathrm{i}b$, rather than this 2×2-matrix.

Definition 1.2. Let U be an open set in $\mathbb{C}$ and $f : U \to \mathbb{C}$ a function. Given $z \in U$ we say that f is *complex differentiable* at z iff there exists $f'(z) \in \mathbb{C}$ such that,

$$f(z + \zeta) = f(z) + f'(z)\zeta + \mathrm{o}(|\zeta|)$$

for $\zeta \in \mathbb{C}$ sufficiently small. Then, $f'(z)$ is called the *complex derivative* of f at z. f is called *holomorphic* at z iff f is complex differentiable in an open neighborhood of z.

Proposition 1.3. *Let U be an open set in $\mathbb{C}$ and $f : U \to \mathbb{C}$ a function. f is complex differentiable at $z \in U$ iff f is totally differentiable at z and its partial derivatives at z satisfy the* Cauchy-Riemann equations*,*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If $U \subseteq \mathbb{C}$ is open we say that $f : U \to \mathbb{C}$ is holomorphic on U if it is holomorphic at all $z \in U$. We denote the space of functions that are holomorphic on U by $\mathcal{O}(U)$. In the following, non-empty connected open subsets of the complex plane will be of particular importance. We will refer to such open sets as *regions*. Since any non-empty open set in the complex plane is a disjoint union of regions it is sufficient to consider the spaces of holomorphic functions of the type $\mathcal{O}(D)$, where $D \subseteq \mathbb{C}$ is a region. The elements of $\mathcal{O}(\mathbb{C})$ are called *entire* functions.

Exercise 1. Let U be an open set in $\mathbb{C}$ and $f : U \to \mathbb{C}$ a function. Given $z \in U$ we say that f is *complex conjugate differentiable* at z iff there exists $f_{\overline{z}}(z) \in \mathbb{C}$ such that,

$$f(z + \zeta) = f(z) + f_{\overline{z}}(z)\overline{\zeta} + \mathrm{o}(|\zeta|)$$

for $\zeta \in \mathbb{C}$ sufficiently small. Then, $f_{\overline{z}}(z)$ is called the *complex conjugate derivative* of f at z. f is called *anti-holomorphic* at z iff f is complex conjugate differentiable in an open neighborhood of z.

1. Show that the total derivative of f as a real 2×2-matrix takes the form
$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad \text{for} \quad a, b \in \mathbb{R},$$
where f is complex conjugate differentiable.

2. Deduce the corresponding modified Cauchy-Riemann equations.

3. Show that a function is anti-holomorphic iff it is the complex conjugate of a holomorphic function.

1.2 Elementary Properties of Holomorphic functions

Proposition 1.4. *Let $D \subseteq \mathbb{C}$ be a region and $f \in \mathcal{O}(D)$. Then, f is constant iff $f'(z) = 0$ for all $z \in D$.*

Proof. If f is constant it follows immediately that $f' = 0$. Conversely, suppose that $f' = 0$. Then, viewing f as a function from an open set D in $\mathbb{R}^2$ to $\mathbb{R}^2$ we know that its total derivative is zero. By results of real analysis it follows that f is constant along any path in D. But since D is connected it is also path connected and f must be constant on D. □

Proposition 1.5. *Let $D \subseteq \mathbb{C}$ be a region.*

1. *If $f \in \mathcal{O}(D)$ and $\lambda \in \mathbb{C}$, then $\lambda f \in \mathcal{O}(D)$ and $(\lambda f)'(z) = \lambda f'(z)$.*
2. *If $f, g \in \mathcal{O}(D)$, then $f + g \in \mathcal{O}(D)$ and $(f + g)'(z) = f'(z) + g'(z)$.*
3. *If $f, g \in \mathcal{O}(D)$, then $fg \in \mathcal{O}(D)$ with $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.*
4. *If $f, g \in \mathcal{O}(D)$ and $g(z) \neq 0$ for all $z \in D$, then $f/g \in \mathcal{O}(D)$ and*

$$(f/g)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}.$$

Proof. The proofs are completely analogous to those for real functions on open subsets of the real line with the ordinary real differential. Alternatively, 1.-3. follow from statements in real analysis by viewing $\mathbb{C}$ as $\mathbb{R}^2$. □

Note that items 1.-3. imply that $\mathcal{O}(D)$ is an *algebra* over the complex numbers.

Proposition 1.6. *Let $D_1, D_2 \subseteq \mathbb{C}$ be regions. Let $f \in \mathcal{O}(D_1)$ such that $f(D_1) \subseteq D_2$ and let $g \in \mathcal{O}(D_2)$. Then $g \circ f \in \mathcal{O}(D_1)$ and moreover the* chain rule *applies,*

$$(g \circ f)'(z) = g'(f(z))f'(z) \quad \forall z \in D_1.$$

Proof. This is again a result of real analysis, obtained by viewing $\mathbb{C}$ as $\mathbb{R}^2$. (Note that g' and f' are then 2×2-matrices whose multiplication translates to multiplication of complex numbers here.) □

Proposition 1.7. *Let $D_1, D_2 \subseteq \mathbb{C}$ be regions. Let $f : D_1 \to \mathbb{C}$ such that $f(D_1) \subseteq D_2$. Let $g \in \mathcal{O}(D_2)$ be such that $g \circ f(z) = z$ for all $z \in D_1$. Let*

$z \in D_1$. Suppose that $g'(f(z)) \neq 0$ and that g' is continuous at $f(z)$. Then, f is complex differentiable at z and

$$f'(z) = \frac{1}{g'(f(z))}.$$

Proof. Again, this is a statement imported from real analysis on $\mathbb{R}^2$. (There, the condition $g'(f(z)) \neq 0$ is the condition that the determinant of the 2×2-matrix $g'(f(z))$ does not vanish.) $\square$

A few elementary examples together with the properties of holomorphic functions we have identified so far already allow us to generate considerable families of holomorphic functions.

Example 1.8. The following are elementary entire functions.

- The *constant functions*: They have vanishing complex derivative.
- The *identity function*: $f(z) = z$ has complex derivative $f'(z) = 1$.

Example 1.9. We define the complex exponential function $\exp : \mathbb{C} \to \mathbb{C}$ as follows. For all $x, y \in \mathbb{R}$ define

$$\exp(x + \mathrm{i}y) := \exp(x)\left(\cos(y) + \mathrm{i}\sin(y)\right),$$

where exp, cos and sin are the functions known from real analysis.

Exercise 2. Using results from real analysis about the real analytic functions exp, cos and sin show that the complex exponential function $f(z) = \exp(z)$ is entire and that $f'(z) = \exp(z)$ for all $z \in \mathbb{C}$.

Example 1.10. The following are (classes of) holomorphic functions produced from the elementary entire functions of Example 1.8 by addition, multiplication, division or composition.

- *Polynomials*: Any polynomial $p(z) = \sum_n \lambda_n z^n$, where $\lambda_n \in \mathbb{C}$, is entire with $p'(z) = \sum_{n \neq 0} \lambda_n n z^{n-1}$.
- *Rational functions*: Let $p(z)$ and $q(z)$ be polynomials with $q \neq 0$ and suppose that p and q have no common zeros. Let $D = \mathbb{C} \setminus N$, where N is the set of zeros of q. Then, $f(z) = p(z)/q(z) \in \mathcal{O}(D)$.
- *Hyperbolic functions*: The following are entire functions,

$$\cosh(z) := \frac{\exp(z) + \exp(-z)}{2} \quad \text{and} \quad \sinh(z) := \frac{\exp(z) - \exp(-z)}{2}.$$

- *Trigonometric functions*: The following are entire functions,

$$\cos(z) := \frac{\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)}{2} \quad \text{and} \quad \sin(z) := \frac{\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)}{2\mathrm{i}}.$$

- *The logarithm*: Since $\exp(z + 2\pi\mathrm{i}) = \exp(z)$ we have to restrict the domain of exp in order to find a unique inverse. It is customary to make the following choice: Consider the region $D_2 := \mathbb{R} + \mathrm{i}(-\pi, \pi)$. Then exp is a bijective function $D_2 \to D_1$, where $D_1 = \mathbb{C} \setminus \mathbb{R}_0^-$. We define log as the unique function such that $\exp(\log(z)) = z$ for all $z \in D_1$ and such that the image of log lies in $D_2 \subseteq \mathbb{C}$. Then, $\log \in \mathcal{O}(D_1)$ and $\log'(z) = 1/z$ for all $z \in D_1$. This version of the logarithm is also called the *principal branch*.

Exercise 3. Suppose f is a holomorphic function on a region $D \subseteq \mathbb{C}$. Suppose that the real or the imaginary part of f is constant. Show that f must be constant on D.

Exercise 4. At which points in the complex plane are the following functions complex differentiable and at which points are they holomorphic?

1. $f(x + \mathrm{i}y) = x^4 y^5 + \mathrm{i}xy^3$
2. $f(x + \mathrm{i}y) = \sin^2(x + y) + \mathrm{i}\cos^2(x + y)$

Exercise 5. Define another version ("branch") of the logarithm function that is holomorphic in the region $D = \mathbb{C} \setminus \mathbb{R}_0^+$.

Exercise 6. Define $\tan z := \frac{sinz}{\cos z}$. Where is this function defined and where is it holomorphic?

Exercise 7. Define a function $z \mapsto \sqrt{z}$ on $\mathbb{C}$ or on a subset of $\mathbb{C}$. Is this function holomorphic and if yes, where? Comment on possible choices in the construction.

1.3 Conformal mappings

Recall that we have the standard Euclidean *scalar product* on the complex plane, by viewing $\mathbb{C}$ as a two-dimensional real vector space. That is, we have

$$\langle z, z' \rangle := aa' + bb' = \Re(\bar{z}z'),$$

where $z = a + \mathrm{i}b$ and $z' = a' + \mathrm{i}b'$. Recall also that $|z| = \sqrt{\langle z, z \rangle}$. In geometric terms we have,

$$\langle z, z' \rangle = |z||z'| \cos\theta,$$

where θ is the angle between z and z', viewed as vectors in the complex plane.

We shall now be interested in mappings $A : \mathbb{C} \to \mathbb{C}$ that preserve angles between intersecting curves. First, we consider $\mathbb{R}$-linear mappings. Then, for A to be angle-preserving clearly is equivalent to the identity,

$$|z||z'|\langle A(z), A(z')\rangle = |A(z)||A(z')|\langle z, z'\rangle \quad \forall z, z' \in \mathbb{C}.$$

(We also require of course that A not be zero.)

Lemma 1.11. *Let $A : \mathbb{C} \to \mathbb{C}$ be an $\mathbb{R}$-linear mapping. Then, A preserves angles iff*

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad or \quad A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$$

where $a, b \in \mathbb{R}$ and a and b are not both equal to zero.

Proof. **Exercise.** □

More generally, to make sense of the concept of angle-preservation for a map $f : D \to \mathbb{C}$, where D is a region, it is necessary that f possesses a continuous total differential. Then, f preserves angles iff its total differential f' preserves angles at every point of D.

Proposition 1.12. *Let $D \subseteq \mathbb{C}$ be a region and $f : D \to \mathbb{C}$ a function possessing a continuous total differential in D. Then, f is angle-preserving iff f is holomorphic in D or anti-holomorphic in D and its derivative never vanishes.*

Proof. **Exercise.** □

A *conformal* mapping is a mapping that preserves both angles and orientation. Recall that a linear map is orientation preserving iff its determinant is positive. More generally, a mapping is orientation preserving iff its total derivative has positive determinant everywhere.

Proposition 1.13. *Let $D \subseteq \mathbb{C}$ be a region and $f : D \to \mathbb{C}$ a function possessing a continuous total differential in D. Then, f is conformal iff f is holomorphic in D and its derivative never vanishes.*

Proof. **Exercise.** □

1.4 Power series and analytic functions

With each sequence $\{c_n\}_{n\in\mathbb{N}}$ of complex numbers and each point $z_0 \in \mathbb{C}$ we can associate a power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

around z_0. Recall the following result from real analysis.

Lemma 1.14. *The* radius of convergence r *of the power series is given by*

$$\frac{1}{r} = \limsup_{n\to\infty} |c_n|^{1/n}.$$

That is, the power series converges absolutely in the open disk $B_r(z_0)$ to a complex function $f : B_r(z_0) \to \mathbb{C}$. For any $0 < \rho < r$ the convergence is uniform in the open disk $B_\rho(z_0)$. It diverges for z outside of the closed disk $\overline{B_r(z_0)}$.

Proof. **Exercise.** □

Definition 1.15. Let $D \subseteq \mathbb{C}$ be a region and $f : D \to \mathbb{C}$. We say that f is *analytic* in D iff for every point $z \in D$ and any $r > 0$ such that $B_r(z) \subseteq D$ the function f can be expressed as a power series around z_0 with radius of convergence greater or equal to r.

Theorem 1.16. *Let $D \subseteq \mathbb{C}$ be a region. Suppose that f is analytic in D. Then $f \in \mathcal{O}(D)$ and f' is also analytic in D. Moreover, if*

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n,$$

converges in $B_r(z_0)$, then

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z - z_0)^{n-1},$$

converges in $B_r(z_0)$.

Proof. (Adapted from Rudin.) Fix $z_0 \in D$ and $r > 0$ such that $B_r(z_0) \subseteq D$. Suppose $f(z)$ is given by the power series given above and covering in $B_r(z_0)$. Denote the second power series above by $g(z)$. It is then enough to show that $g(z)$ converges in $B_r(z_0)$ and that $g(z)$ is the complex derivative of f for all $z \in B_r(z_0)$.

Firstly, it is clear by Lemma 1.14 that $g(z)$ has the same radius of convergence as $f(z)$. In particular, $g(z)$ converges in $B_r(z_0)$. Fix $z \in B_r(z_0)$ and define $\xi := z - z_0$. Then, set ρ arbitrarily such that $|\xi| < \rho < r$. Let $\zeta \in B_s(0) \setminus \{0\}$ where $s := \rho - |\xi|$ and set

$$h(\zeta) := \frac{f(z+\zeta) - f(z)}{\zeta} - g(z).$$

We have to show that $h(\zeta) \to 0$ when $|\zeta| \to 0$. $h(\zeta)$ can be written as

$$h(\zeta) = \sum_{n=0}^{\infty} c_n a_n(\zeta),$$

where

$$a_n(\zeta) := \frac{(\xi + \zeta)^n - \xi^n}{\zeta} - n\xi^{n-1}.$$

Note that $a_0(\zeta) = 0$ and $a_1(\zeta) = 0$. By explicit computation we find for $n \geq 2$,

$$a_n(\zeta) = \zeta \sum_{k=1}^{n-1} k\xi^{k-1}(\xi + \zeta)^{n-k-1}.$$

Now, $|\xi| < \rho$ and $|\xi + \zeta| < \rho$ so that we get the estimate,

$$|a_n(\zeta)| < |\zeta| \frac{1}{2} n(n-1)\rho^{n-2}.$$

This implies

$$|h(\zeta)| < |\zeta| \frac{1}{2} \sum_{n=2}^{\infty} |c_n| n(n-1)\rho^{n-2}.$$

However, since $\rho < r$, the sum converges by Lemma 1.14 showing that there is a constant M such that

$$|h(\zeta)| < |\zeta| M.$$

This completes the proof. □

The first remarkable result of complex analysis is that the converse of this theorem is also valid: Every holomorphic function is analytic. However, in order to show this we will have to introduce the integral calculus in the complex plane.

Exercise 8. Let $a, b, c, d \in \mathbb{R}$ such that $ad - bc \neq 0$. Show that $f(z) := \frac{az+b}{cz+d}$ is analytic in $D := \mathbb{C} \setminus \{-\frac{d}{c}\}$ according to Definition 1.15.