Chapter 9

Harmonic Analysis

9.1 The Christ-Kiselev Maximal Inequality

If $f \in L^2(\mathbb{R}, dx)$, then the integral in

$$\hat{f}_R(k) \equiv (2\pi)^{1/2} \int_{-R}^{R} e^{-ikx} f(x)\, dx$$

converges for each $R < \infty$ and each k, and by general principles (see Theorem 1.7.15.TK), f_R converges in $L^2(\mathbb{R}, dk)$ to $\hat{f}$, the Fourier transform. x-ref? It is natural to ask about pointwise convergence for a.e. k. It is a deep, complicated, and celebrated result of Carleson that this is true for any $f \in L^2(\mathbb{R}, dx)$. By settling for a slightly weaker result, we will prove this here if $f \in L^p(\mathbb{R}, dx)$ for some p with $1 \leq p < 2$. Of course, the natural approach is via a maximal function, and we will use a particularly elegant version of the construction due to Christ and Kiselev. Their result will turn out to be useful in the study of eigenfunction expansions of general Schrödinger operators; see TK. x-ref?

Let $\{A_\alpha\}_{\alpha \in \mathbb{R}}$ be a family of measurable sets of a measure space $(M, d\mu)$ which obey

$$\alpha > \beta \Rightarrow A_\beta \subset A_\alpha \tag{9.1.1}$$

We will let χ_α denote the characteristic function of A_α.

For simplicity, we will suppose two regularity conditions for the family of sets; namely, that for $\beta > \alpha$, $\mu(A_\beta \backslash A_\alpha) < \infty$ and

$$\lim_{\varepsilon \downarrow 0} \mu(A_{\alpha+\varepsilon} \backslash A_\alpha) = \lim_{\varepsilon \downarrow 0} \mu(A_\alpha \backslash A_{\alpha-\varepsilon}) = 0 \tag{9.1.2}$$

and up to sets of measure zero,

$$\lim_{\alpha \downarrow -\infty} A_\alpha = \bigcap_\alpha A_\alpha = \emptyset; \quad \lim_{\alpha \to \infty} A_\alpha = \bigcup_\alpha A_\alpha = M \tag{9.1.3}$$

This implies that for any $f \in L^p(M, d\mu)$ that

$$\alpha \mapsto \|f\chi_\alpha\|_p \text{ is continuous} \tag{9.1.4}$$

The main example that will concern us is $M = \mathbb{R}$, $A_\alpha = (-\infty, \alpha)$, in which case (9.1.2) is equivalent to μ having no pure points. With a little more effort, one can accommoodate such pure points (see the argument below in Theorem 9.1.2).

Let $T : L^p(M, d\mu) \to L^q(M, d\mu)$ be a bounded map. Define the Christ-Kiselev maximal function:

$$(T^* f)(x) = \sup_\alpha |T(\chi_\alpha f)(x)|$$

Theorem 9.1.1 (Christ-Kiselev Maximal Inequality)**.** *Let $p < q < \infty$. Then T^* is bounded from $L^p(M, d\mu)$ to $L^q(M, d\mu)$. Indeed,*

$$\|T^* f\|_q \le 2^{-\beta}(1 - 2^{-\beta})^{-1} \|T\|_{L^p \to L^q} \|f\|_p \tag{9.1.5}$$

where $\beta = p^{-1} - q^{-1}$.

Remarks. 1. Since $1 \le p < q < \infty$, $0 < \beta < 1$. Since $G(\beta) \equiv 2^{-\beta}/(p - 2^{-\beta}) = \sum_{n=1}^\infty 2^{-n\beta}$ is monotone in β, $G(\beta) > G(1) = 1$ so the constant in the theorem is always larger than 1, consistent with the obvious $\|T\| \le \|T^*\|$.

2. This result is trivial if $q = \infty$, since in that case

$$\|T^* f\|_\infty = \sup_\alpha \|T(\chi_\alpha f)\|_\infty \le \|T\| \sup_\alpha \|\chi_\alpha f\|_p = \|T\| \, \|f\|_p$$

Proof. Fix $f \in L^p$ with $\|f\|_p = 1$. Define

$$G(\alpha) = \|f\chi_\alpha\|_p^p \tag{9.1.6}$$

By (9.1.4), $\alpha \mapsto G(\alpha)$ is a continuous function and by (9.1.3), it obeys $\lim_{\alpha \to -\infty} G(\alpha) = 0$, $\lim_{\alpha \to \infty} G(\alpha) = 1$.

For $m = 1, 2, \ldots$ and $j = 1, \ldots, 2^m - 1$, define α_j^m by $G(\alpha_j^m) = j/2^m$ and α_j^m is the smallest such solution. Since G is continuous and monotone with $G(-\infty) = 0$ and $G(\infty) = 1$, such α_j^m exist.

Let $\chi_{-\infty} = 0$, $\chi_\infty = 1$, and $\alpha_0^m = -\infty$, $\alpha_{2^m}^m = \infty$. Define for $j = 1, \dots, 2^m$,

$$f_j^m = f(\chi_{\alpha_j^m} - \chi_{\alpha_{j-1}^m})$$

Then

$$\|f_j^m\|_p^p = \frac{1}{2^m} \tag{9.1.7}$$

Given α real in $[0,1]$, write $\alpha = \sum_{m=1}^\infty k_m/2^m$ with each $k_m = 0$ or 1 as its base 2 decimal expansion. If $\alpha = 1$, take $k_1 = k_2 = \cdots = 1$. Let $j_m(\alpha) = \sum_{\ell=1}^m k_\ell 2^{m-\ell}$ so, by the continuity of G and monotonicity of χ_α in α,

$$f\chi_\alpha = \sum_{\substack{m=1 \\ k_m=1}}^\infty f_{j_m(\alpha)}^m \tag{9.1.8}$$

modulo sets of measure zero. The sum is only over those m iwth $k_m = 1$.

It follows that

$$\begin{aligned} |(T^* f)(x)| &\le \sum_{m=1}^\infty \left[\sup_{1\le j\le 2^m} |T(f_j^m)(x)| \right] \\ &\le \sum_{m=1}^\infty \left(\sum_{j=1}^{2^m} |T(f_j^m)(x)|^q \right)^{1/q} \end{aligned} \tag{9.1.9}$$

since $\sum_{\ell=1}^L |a_\ell|^q \ge \max(|a_\ell|)^q$.

Call the term inside the m summand in (9.1.9) $h_m(x)$. Then

$$\begin{aligned} \|h_m\|_q^q &= \sum_{j=1}^{2^m} \int |Tf_j^m(x)|^q \, d\mu(x) \\ &= \sum_{j=1}^{2^m} \|Tf_j^m\|_q^q \\ &\le 2^m \|T\|^q \|f_j^m\|_p^q \\ &= 2^m 2^{-qm/p} \|T\|^q \end{aligned}$$

by (9.1.7). Thus, by the triangle inequality on the L^q norm,

$$\begin{aligned} \|T^* f\|_q &\le \sum_{m=1}^\infty \|h_m\|_q \\ &\le \sum_{m=1}^\infty \|T\| 2^{-m\beta} \end{aligned}$$

$$= \|T\| \frac{2^{-\beta}}{(1-2^{-\beta})}$$

$\square$

Remarks. 1. It is a well-known result that if $T : L^1(M, d\mu) \to L^\infty(M, d\mu)$, then $(Tf)(x) = \int K(x,y) f(y)\, d\mu(y)$ where $K \in L^\infty(M \times M)$ and $\|K\|_\infty = \|T\|$. For such an operator, it is trivial that $\sup_{\alpha,x} |T(f\chi_\alpha)(x)| \leq \|T\| \, \|f\|_1$. As $p \to 1$, $p \to \infty$, the constant $\beta \to 1$ and $2^{-p}/(1-2^{-p}) \to 1$, which is an indication of how good our constant is.

2. The measure space for the range need not be the same as the domain.

Here is a procedure for extending this result to some discrete cases. Let $\mathbb{Z}_+ = \{n \in \mathbb{Z} \mid n > 0\} = \{1, 2, 3, \dots, \}$. Let χ_n be the characteristic function of $\{1, \dots, n\} \subset \mathbb{Z}_+$. Then

Theorem 9.1.2. *Let $p < q < \infty$. Let T map $\ell^p(\mathbb{Z}_+)$ to $L^q(M, d\mu)$. Let*

$$(T^* f)(x) = \sup_{n=1,2,\dots,} |T(f\chi_n)(x)|$$

Then T^* maps $\ell^p(\mathbb{Z}_+)$ to $L^q(M, d\mu)$ and

$$\|T^* f\|_q \leq 2^{-\beta}(1-2^{-\beta})^{-1} \|T\| \, \|f\|_p \tag{9.1.10}$$

where $\beta = p^{-1} - q^{-1}$.

Proof. We will essentially linearly interpolate T to define a map from $L^p(\mathbb{R})$. Given a function $g \in L^p(\mathbb{R})$, define $Sg \in \ell^p(\mathbb{Z}_+)$ by

$$(Sg)(n) = \int_{n-1}^{n} g(y)\, dy$$

Also define $\tilde{S}$ (essentially a dual of S) taking $\ell^p(\mathbb{Z}_+)$ to $L^p(\mathbb{R}, dx)$ by

$$\begin{aligned} (\tilde{S} f)(x) &= f(n) && \text{if} \quad n-1 \leq x < n \\ &= 0 && \text{if} \quad x < 0 \end{aligned}$$

Then $\|\tilde{S} f\|_p = \|f\|_p$ and $\|Sg\|_p \leq \|g\|_p$ since $|\int_{n-1}^{n} g(y)\, dy| \leq (\int_{n-1}^{n} |g(y)|^p \, dy)^{1/p}$ by Hölder's inequality. Moreover, applied to functions on $\mathbb{Z}_+$,

$$S\tilde{S} = 1 \tag{9.1.11}$$

Define $H = TS : L^p(\mathbb{R}, dx) \to L^q(M, d\mu)$. By Theorem 9.1.1, for any $g \in L^p(\mathbb{R}, dx)$,

$$\begin{aligned}\|H^* g\|_q &\le 2^{-\beta}(1-2^{-\beta})^{-1}\|H\|\,\|g\|_p \\ &\le 2^{-\beta}(1-2^{-\beta})^{-1}\|T\|\,\|g\|_p\end{aligned}$$

since $\|S\| = 1$. Thus for any $f \in \ell^p(\mathbb{Z}_+)$,

$$\|H^*\tilde{S}f\|_q \le 2^{-\beta}(1-2^{-\beta})^{-1}\|T\|\,\|f\|_p \tag{9.1.12}$$

since $\|\tilde{S}\| = 1$. But by (9.1.11),

$$\begin{aligned}(T^*f)(x) &= \sup_{n=1,2,\dots} |[TS\tilde{S}(f\chi_n)](x)| \\ &\le \sup_{n=1,2,\dots} |[H\chi_n(\tilde{S}f)](x)| \\ &\le (H^*\tilde{S}f)(x)\end{aligned}$$

so (9.1.12) implies (9.1.10). □

As for applications of these inequalities, we start with a result on Fourier transforms. We exploit the fact that we will prove in Section **??** (see Theorem 1.7.14.TK) that if x-ref?

$$\hat{f}(k) = (2\pi)^{-\nu/2}\int e^{ik\cdot x} f(x)\, d^\nu x \tag{9.1.13}$$

initially on functions $f \in L^1(\mathbb{R}^\nu, d^\nu x)$, then one has an a priori bound for $1 \le p < 2$, $q = (1-p)^{-1}[> p$ since $p < 2]$,

$$\|\hat{f}\|_q \le \|f\|_p$$

initially for $f \in L^1 \cap L^p$ but then allowing $\hat{f}$ to be defined on $L^p(\mathbb{R}^\nu, d^\nu x)$.

Theorem 9.1.3. *Let $1 \le p < 2$. Then for any $f \in L^p(\mathbb{R}^\nu, d^\nu x)$,*

$$\sup_{0<R<\infty}\left|(2\pi)^{-\nu/2}\left[\int_{|x|<R} e^{-ik\cdot x} f(x)\, dx\right]\right| = (\mathcal{F}_* f)(k) \tag{9.1.14}$$

lies in $L^q(\mathbb{R}, d^\nu x)$. For each such f and a.e. $k \in \mathbb{R}^\nu$,

$$\lim_{R\to\infty}(2\pi)^{-\nu/2}\int_{|x|<R} e^{-ik\cdot x} f(x)\, dx = \hat{f}(k) \tag{9.1.15}$$

Remark. The point, of course, is that for $f \in L^p$ but $f \notin L^1$, the integral in (9.1.13) does not converge absolutely. (9.1.15) says it does converge conditionally (over balls) for a.e. k and defines the Fourier transform.

Proof. Let
$$A_\alpha = \{x \in \mathbb{R}^\nu \mid |x| < \alpha\}$$
By Theorem 9.1.1, $\mathcal{F}_* f \in L^q$ and $\|F_* f\|_q \leq 2^{-\beta}(1-2^{-p})\|f\|_p$. Such a maximal inequality, together with the existence of the limit if $f \in L^1 \cap L^p$, implies (9.1.15) by the usual strategy of Theorem 7.2.2. $\square$

We can apply Theorem 9.1.2 to general convergence of Fourier series with an arbitrary orthonormal basis. We need the following preliminary:

Lemma 9.1.4. *Let a_n be a sequence indexed by $\mathbb{Z}_+$ and let $\|a\|_p = (\sum_n |a_n|^p)^{1/p}$, which may be infinite. Then for $p > q$,*

$$\|a\|_p \leq \|a\|_q \tag{9.1.16}$$

Proof. We need only prove this for $\{a_n\}_{n=1}^\infty$ a finite sequence and then take limits. For such a sequence,

$$|a_j|^p \leq |a_j| \left(\sum_n |a_j| \right)^{p-1}$$

Summing over j, we find
$$\|a\|_p^p \leq \|a\|_1^p$$
proving (9.1.16) for $q = 1$. For general p, q, a, let $\tilde{a} = |a|^q$, $\tilde{p} = p/q > 1$, and note
$$\|a\|_p^p = \|\tilde{a}\|_{\tilde{p}}^{\tilde{p}} \leq \|\tilde{a}\|_1^{\tilde{p}} = \|a\|_q^p$$
proving (9.1.16) in general. $\square$

Theorem 9.1.5. *Let $\{\varphi_n\}_{n=1}^\infty$ be an arbitrary orthonormal basis for $L^2(M, d\mu)$. For a sequence $\{c_n\}_{n=0}^\infty$, define*

$$c^*(x) = \sup_{n=1,\dots,m} \left| \sum_{j=1}^n c_j \varphi_j(x) \right|$$

Then for any $p < 2$,

$$\|c^*\|_{L^2(M,d\mu)} \leq 2^{-\beta}(1-2^{-\beta})^{-1} \|c\|_{\ell^p} \tag{9.1.17}$$

with $\beta = p^{-1} - \frac{1}{2}$. *In particular, if* $c \in \ell^p$ *for* $p < 2$, *and* $\varphi(x) = \sum_{j=1}^\infty c_j \varphi_j(x)$ (L^2-*sum*), *then for a.e.* x,

$$\lim_{n\to\infty} \sum_{j=1}^n c_j \varphi_j(x) = \varphi(x) \tag{9.1.18}$$

Proof. By the lemma, if $Tc(x) = \sum_{j=1}^\infty c_j \varphi_j(x)$ (L^2-sum), then

$$\|Tc\|_{L^2(M,d\mu)} = \|c\|_2 \le \|c\|_p$$

Thus, by Theorem 9.1.2, (9.1.17) holds and then (9.1.18) follows by the usual strategy (see Theorem 7.2.2). $\square$

9.2 Harmonic and Subharmonic Functions

Definition. Let $\mathcal{O} \subset \mathbb{R}^\nu$ be a connected open set. A function, u, from $\mathcal{O}$ to $\mathbb{R}$ is called *harmonic* if and only if

(i) u is continuous.

(ii) For any compact set, $K \subset \mathcal{O}$, there is an $\varepsilon > 0$ so for all $x \in K$, $B_x^\varepsilon = \{y \mid |y - x| < \varepsilon\} \subset \mathcal{O}$ and

$$u(x) = \int_{S^{\nu-1}} u(x + r\omega)\, d\omega \tag{9.2.1}$$

for all $r \in (0, \varepsilon)$.

Here $S^{\nu-1}$ is the unit sphere in $\mathbb{R}^\nu$, $S^{\nu-1} = \{x \mid |x| = 1\}$, and $d\omega$ is the normalized spherical measure, that is, $\omega(S^{\nu-1}) = 1$ and ω is the unique rotation invariant measure. Then, for example,

$$\int_{B_x^\varepsilon} f(y)\, dy = \tau_\nu \int_0^\varepsilon dr \left[r^{\nu-1} \int f(x + r\omega)\, d\omega \right]$$

with the τ_ν, the area of the unit sphere $= \nu\times$ volume of the unit ball.

Definition. Let $\mathcal{O} \subset \mathbb{R}^\nu$ be a connected open set. A function, u, from $\mathcal{O}$ to $-\infty \cup \mathbb{R}$ is called *subharmonic* if and only if

(i) u is lsc.

(ii) For any compact set $K \subset \mathcal{O}$, there is an $\varepsilon > 0$ so for all $x \in K$, $B_x^\varepsilon = \{y \mid |y-x| < \varepsilon\} \subset \mathcal{O}$ and for $r < \varepsilon$,

$$\int_{S^{\nu-1}} |u(x+r\omega)|\, d\omega < \infty \tag{9.2.2}$$

$$u(x) \le \int_{S^{\nu-1}} u(x+r\omega)\, d\omega \tag{9.2.3}$$

Remarks. 1. lsc and usc are defined in Section 5.2.

2. Any usc function takes its maximum value on any compact set, so in (9.2.2), the integral could only diverge to $-\infty$. That we assume it doesn't, eliminates the case $u \equiv -\infty$ since (9.2.2) implies $u(x) < \infty$ for a.e. $x \in \mathcal{O}$.

3. The usual definition makes a stronger hypothesis; namely, that (9.2.1) or (9.2.3) holds for any x and r with $\overline{B_x^r} \subset \mathcal{O}$. We will prove this below for u's that obey the weaker hypothesis. If u is assumed C^2, we can suppose only that for each x, (9.2.1) holds for some ε without requiring a uniformity for $x \in K$.

4. For u harmonic, the ideas in Theorem 9.2.4 show that u continuous can be replaced by a measurable and bounded on compact sets.

5. u is called *superharmonic* if $-u$ is subharmonic. u is harmonic if and only if it is both subharmonic and superharmonic.

6. In Example 9.2.21, we will construct a subharmonic function which is only usc and not continuous.

Example 9.2.1. Let $\mathcal{O}$ be a convex set in $\mathbb{R}^\nu$ and $F : \mathcal{O} \to \mathbb{R}$ a convex function. Then, we claim that F is subharmonic for F is continuous and so usc. Moreover, if $\overline{B_x^r} \subset \mathcal{O}$ and $\omega \in S^{\nu-1}$,

$$F(x) \le \tfrac{1}{2}\, F(x+r\omega) + \tfrac{1}{2}\, F(x-r\omega) \tag{9.2.4}$$

since $x = \frac{1}{2}(x+r\omega) + \frac{1}{2}(x-r\omega)$. Averaging over ω, (9.2.4) implies (9.2.3). Notice in case $\nu = 1$, $S^{\nu-1} = \{\pm 1\}$ and (9.2.3) is just midpoint convexity. So, in one dimension, subharmonic is the same as convex (and harmonic is the same as affine). □

The following pair of results makes it clear why subharmonicity inclues a usc condition.

Proposition 9.2.2. *Let u be a subharmonic function on $\mathcal{O} \subset \mathbb{R}^\nu$. For any $x \in \mathcal{O}$,*

$$u(x) = \lim_{r \downarrow 0} \int u(x+r\omega)\, d\omega \tag{9.2.5}$$