Lecture 6.4

Supercurrent and critical currents

All about critical currents, zero resistance, and flux quantization

This lecture gathers several diverse answers to the question "When (or why) does a superconductor superconduct?" We previously (re)defined "superconductivity" as "the existence of long-range order of the order parameter $\Psi(\mathbf{r})$ (in particular, of the phase field $\theta(\mathbf{r})$); we (re)defined "supercurrent" $\mathbf{J}_s(\mathbf{r})$ as "the collective current of the condensate" described by $\Psi(\mathbf{r})$. Thus, although $\mathbf{J}_s$ was called "supercurrent", we haven't seen why it has with zero resistance; or how that property follows from the long-range order.

Understanding zero resistivity really means understanding *how it breaks down*, e.g. what is the *critical current*, which occupies most of this chapter. Four different ways of killing superflow are presented:

(a) intrinsic limit on the phase gradient $\nabla_{\mathbf{A}}\theta$;

(b) limit from $\mathbf{B}$ field due to the current;

(c) decay of the persistent current in a ring, or dissipation in a thin wire, due to passage of a vortex across the superconductor;

(d) Landau's critical velocity, depending on the dispersion law of elementary excitations.

The first and second scenario correspond, roughly, to the two fields of G-L theory ($\Psi(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$; which breakdown comes first depends on the sample geometry.

In Sec. 6.4 C (item (c) above) we get at what really makes the material superconducting, when that state is stable. I first remind that that the condensate's equations of motion imply a ballistic response to an electric field, hence a superconductor cannot maintain a voltage drop *in the steady state*.

Then, we see why the magnetic flux through any closed ring of superconductor must be quantized. (This will be the basis of the SQUID device, to be discussed in Lec. 6.5 , and of the vortices in Type II superconductors discussed in Lec. 6.6 and Lec. 6.7 .) The "stiffness" of the phase, as evidenced in flux quantization, is ultimately the explanation of zero resistivity; a particular manifestation is the "persistent current" in a superconducting ring.

6.4 A Ginzburg-Landau picture of critical current

How does Landau's microscopic v_c fit into the macroscopic Ginzburg-Landau picture? The only way to represent instability of the superconducting state is that n_s, or equivalently the order parameter magnitude, is driven to zero. The GL picture doesn't explicitly include any microscopic elementary excitations, but thermal excitations are implicit in the reduction of the order parameter amplitude $|\Psi|$ at $T > 0$. [1] *To rephrase the ending of the preceding section, as we bring $\epsilon_{eff}(\mathbf{q})$ closer to zero at the critical $\mathbf{q}$ point, more such excitations appear (as thermal excitations, or as quantum zero-point type fluctuations at $T = 0$) and $|\Psi|$ gets reduced.* (For a general discussion of the relationship of GL and BCS – for equilibrium statics only – see Lec. 7.8 [omitted].)

I will next show that, all by themselves, the GL equations imply a critical current. The Landau and GL critical velocities are respectively the microscopic and macroscopic formulations of the same phenomenon; and the following derivation shows that the two answers are same to within a factor of order unity.

Digression: boundary condition subtleties

Let's assume space-independent fields; the external constraint is the phase gradient, so we take that fixed and write $k_\theta \equiv |\nabla\theta|$ for short. (Naively we might have tried instead to constrain the total current. But physically, we're considering the stability against a very local fluctuation: in that case, certainly the external phase difference is the constraint.)

[the rest of this subsection expands the above statements, in perhaps overmuch detail]

The question we will set up first is, given a boundary condition with a net phase change $\Delta\theta = \theta(L) - \theta(0)$ across our sample in (say) the x direction, what is the order parameter reduction and the current? We'll proceed by minimizing the free energy given this boundary condition, $\int_0^L dx k_\theta(x) = \Delta\theta$, where $k_\theta \equiv |\nabla\theta|$ for short. If the sample is not too thick, we can have a strong current density without making a large magnetic field, so we neglect the magnetic field energy as well as the vector potential in the gauge-invariant gradients. As a preliminary note, it can be verified that the minimum free energy solution is always to have $k_\theta(x)$ uniform.

Assume k_θ varies; to conserve current, $n_s(x)$ must vary correspondingly. Expand $F_L()$ to second order as a function of n_s. The first order term will cancel because $\int \delta n_s(x) \propto -\int \delta k_\theta(x) = 0$ due to the $\Delta\theta$ boundary condition. The second order term is proportional to $d^2F_L/d{n_s}^2[\delta k_\theta(x)]^2$, which is positive definite since $d^2F_L/d{n_s}^2 = \beta >$ 0.

Since $v = \hbar k_\theta/m_*$, the assumption of fixed and uniform k_θ is evidently proper when we want a critical *velocity*, e.g. for a superfluid put in motion relative to a channel containing it. If on the other hand we want a critical *current*, it's a rather subtle piece of thermodynamics that this is correct thermodynamically, and that it's wrong to fix $J = n_s\hbar k_\theta/m_*$. *See Tinkham, Sec. 4.4.* The physical reason is that, to change $\Delta\theta$ (while keeping a fixed J), we'd transiently have a nonzero time derivative $(\dot\theta(L) - \dot\theta(0))$. It can be shown (Lec. 6.5) that $\dot\theta(x,t) = e_*V(x,t)/\hbar$, so we'd have a transient voltage across the superconductor, meaning that work would be done on it by our current source.

[1] Fluctuations reduce order parameters, as we first computed in the case of phonon fluctuations and crystal order (see Lec. 1.6 , Debye-Waller factor.) For a general discussion of the relationship of GL and BCS – for equilibrium statics only – see Lec. 7.8 [omitted].

Energy minimization

The gradient free energy under these assumptions can be written

$$F_{grad} = \frac{\hbar^2}{2m_*}|\nabla\Psi|^2 = \frac{\hbar^2}{2m_*}n_s|\nabla\theta|^2 = |\alpha|\ \xi^2 {k_\theta}^2\ |\Psi|^2 \tag{6.4.1}$$

where we used $n_s \equiv |\Psi|^2$ and rewrote the coefficient using the definition of ξ (see Lec. 6.0 and Lec. 6.1). (We've assumed uniformity in space, so $\nabla|\Psi| = 0$.) The total free energy density is

$$F_{grad} + F_L = -|\alpha|(1 - {k_\theta}^2\xi^2)|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4. \tag{6.4.2}$$

Minimizing the free energy (6.4.2), with respect to $n_s \equiv |\Psi|^2$ (as done in Lec. 6.1 for the gradient-free case) ,

$$n_s = |\Psi|^2 = (1 - {k_\theta}^2\xi^2)n_{s0} \tag{6.4.3}$$

where $n_{s0} = |\Psi_0|^2$. Thus

$$J_s = e^*\frac{\hbar k_\theta}{m_*}(1 - {k_\theta}^2\xi^2)n_{s0}. \tag{6.4.4}$$

At $k_\theta = 1/\sqrt{3}\xi$ this has its maximum

$$J_c = \frac{2}{3\sqrt{3}}e^*\frac{\hbar}{m_*\xi}n_{s0} \tag{6.4.5}$$

I've checked that numbers from Landau's critical velocity agrees with the last result, to factors of order unity (see Ex. 6.4.2).

In samples larger than λ, the magnetic field mechanism of Sec. 6.4 B comes into play and we must consider some sort of "intermediate state."

Another viewpoint on the GL critical current – Since k_θ is analogous to the strain in a solid, the k_θ maximizing (6.4.4) is analogous to the elastic limit of a pure solid: if we "stretch" the phase variation too much, the superconducting state "breaks" by going normal. Like the mechanical limits of real solids, the critical currents of real superfluids are usually determined by defects or specially weak places. For classic superconductors, a typical value[2] is

$$J_c \sim 10^4\ \text{Amp/cm}^2. \tag{6.4.6}$$

Digression on Galilean invariance

In his celebrated book on superconductivity, de Gennes asserted the choice of m_* in the GL theory is arbitrary; if so, the superfluid velocity $\mathbf{v}_s$ is not a physical observable but just a convenient way to parametrize a current. However, the velocity *is* physically meaningful as is clear from microscopic formulas such as that (above) for the $\epsilon_{eff}(k)$. Of course, Galilean invariance *isn't* exact in a solid: the electron dispersion relation not exactly of the form $\hbar^2k^2/2m_e$ and hence is changed by a Galilean boost. [3]

The GL theory, which applies even to neutral superfluids said in (6.4.3) that $|\Psi|$ gets reduced in a moving superfluid. That's an apparent violation of Galilean invariance! To address this, we need to use a two-fluid model: the superfluid velocity represents an

[2]Lifted from W. A. Harrison, *Solid State Physics*.

[3]In addition, **k** really means the crystal momentum, not the real momentum, so our use of Newton's conservation laws is valid only when we can neglect umklapp, e.g. at low temperatures when thermal phonons all have small wavevectors.

underlying superfluid state with a nonzero phase gradient; the normal fluid velocity $\mathbf{v}_n$ *accounts for the effects of a gas of elementary excitations in this background, which are in equilibrium with some other degrees of freedom at velocity* $\mathbf{v}_n$*. The actual statement, then, is the order parameter reduction depends on* $\mathbf{v}_s - \mathbf{v}_n$*; the GL theory had implicitly taken* $\mathbf{v}_n = 0$.

Josephson critical current: preview

High T_c or granular superconductors may consist of many small grains, with the supercurrent propagated from grain to grain by coherent (pair) tunneling: this is a Josephson junction, which is the subject of the next lecture (Lec. 6.5). It will be shown there that the junction's current is

$$I_c \sin \Delta\theta \tag{6.4.7}$$

where $\Delta\theta$ is the jump in phase (of Ψ) between the two sides; clearly I_c is the maximum supercurrent of that junction. This is closely analogous to the order parameter critical current, since $\Delta\theta$ is a discrete analog of $k_\theta \equiv |\nabla\theta|$. (Notice how (6.4.4) and (6.4.7) have similar dependences on the phase difference, beginning linear and showing a maximum.)

6.4 B Critical current due to magnetic field

There is a second route to critical currents. Consider the following paradox: superconductivity and magnetic field are mutually exclusive (Meissner effect); but supercurrents make magnetic fields; ergo there are no supercurrents in superconductors! In fact this is essentially *true*: the paradox's resolution is that supercurrents only flow on the surface. More precisely they are the screening currents (see Lec. 6.2) of the magnetic fields they create. and decay in the same exponential fashion, so they are essentially confined to a layer that is about a penetration depth (λ) thick. If the current is so great as to produce a field that exceeds the critical field H_c, then it must drive the sample normal at that point, which might disconnect the domain of material in the superconducting phase.

Using the formulas for λ, Φ_0^*, and H_c in Table 6.1.1, Eq. (6.4.5) can be massaged into the form

$$J_c = \frac{\sqrt{2}c}{16\pi\lambda} H_c \tag{6.4.8}$$

Eq. (6.4.8) shows that the order-parameter mechanism of Sec. 6.4 A dominates for sample thicknesses small compared to λ, where the current density can be high while the total field produced is small compared to H_c. In a sample thicker than λ, we know the current is confined within $\sim \lambda$ of the surface; then the magnetic field limit of this section appears at the same J_c as (6.4.8) (within factors of order unity).

Partly restated the last paragraph: In the form (6.4.8), you see that along a domain wall (which always adjoins a normal region with the critical field H_c), the screening currents necessarily approach the critical current. It's also apparent that in a sample carrying the critical current density, the magnetic field produced (according to Ampère's law) becomes comparable to H_c only when the sample thickness is at least λ.

6.4 C Flux quantization

First take on zero resistance

Let's first approach "zero resistance" from the viewpoint of Ohm's law: that is, let's show the voltage drop is $V = 0$ while the current is nonzero. *A superconductor cannot maintain a voltage difference.*

Recall (see the London equations in Lec. 6.2) that, in the presence of an electric field, the supercurrent *accelerates*, just like an undamped charged particle. If the superconductor is in series with an ordinary resistance, that suffices to show that the equilibrium state must have zero voltage drop across the superconductor.

The same argument can be restated in the language of the phase function $\theta(\mathbf{r}, t)$. As a special corollary of the Time-Dependent Ginzburg-Landau equation introduced in Sec. 6.1 B , I claim

$$\frac{d\theta(\mathbf{r}, t)}{dt} = -\frac{1}{\hbar}\mu(\mathbf{r}, t) \tag{6.4.9}$$

where μ is the (electro)chemical potential at $\mathbf{r}$. *[You might guess (6.4.9) from the single-particle Schrödinger wavefunction, in which the phase angle rotates in time as $d\theta/dt = E/\hbar$, where E is the eigenenergy.]*

So, if you had a fixed electric potential drop between points $\mathbf{r}_1$ and $\mathbf{r}_2$, the phase difference $\theta_1 - \theta_2$ grows linearly with time, as must the phase gradient – which is entirely equivalent to saying the supercurrent $\mathbf{J}_s \propto \nabla\theta$ accelerates ballistically. If nothing else intervened, it would quickly reach the critical current (see below): thus, DC voltage is inconsistent with a steady state. [4]

Flux quantization and persistent current

Consider a ring (or cylinder) of superconductor (Fig. 6.4.1), pierced by a net flux Φ_B. In the superconductor,

$$\mathbf{J}_s = \frac{\hbar n_s e_*}{m_*}\left(\nabla\theta - \frac{e^*}{\hbar c}\mathbf{A}\right) \tag{6.4.10}$$

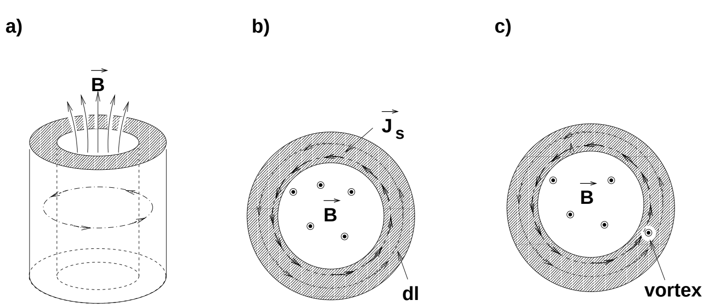

Figure 6.4.1: (a). A cylinder of superconductor pierced by flux. Loop integrals will be done along the dashed curve, which is deep within the superconducting bulk. (b). Top (end-on) view. Supercurrents $\mathbf{J}_s$ (indicated by arrows) flow only along inner surface; *dl* marks the contour for the loop integral of $\nabla\theta$. (c) A normal region contains one flux quantum, i.e. a *vortex*. The path integral would be smaller by 2π for a loop passing inside the vortex than for a loop passing outside the vortex.

[4] But AC $V(t)$ will be possible, as in the Josephson effect (Lec. 6.5).

But, deep within the superconductor's bulk – farther than $\sim \lambda$ from its surface – $\mathbf{J}_s = 0$ *[As first asserted in Sec. 6.2 C.* Hence

$$\nabla\theta = \frac{e^*}{\hbar c}\mathbf{A}. \tag{6.4.11}$$

Now let's do the loop integral of both sides of (6.4.11) along the curve $\mathbf{l}$ indicated in Fig. 6.4.1. On the one hand,

$$\oint d\mathbf{l} \cdot \nabla\theta = 2\pi n \tag{6.4.12}$$

for some integer n, since the phase factor $e^{i\theta}$ must be continuous at the end of end of the loop. On the other hand, by a Stokes identity

$$\frac{e^*}{\hbar c}\oint d\mathbf{l} \cdot \mathbf{A} = \frac{e^*}{\hbar c}\Phi_B \tag{6.4.13}$$

where Φ_B is the flux inside the loop. Combining (6.4.11),(6.4.12) and (6.4.13), we get

$$\Phi_B = n\Phi_0^* \tag{6.4.14}$$

where

$$\Phi_0^* \equiv \frac{2\pi\hbar c}{e^*} = 2.07 \times 10^{-7}\text{gauss-cm}^2 \tag{6.4.15}$$

is the (superconducting) *flux quantum.* Because it contains e_*, Φ_0^* is half as big as the flux quantum of mesoscopic transport (Lec. 2.2).

Vortices

Remember (from Lec. 6.3) the domains of Normal phase (containing flux) in the intermediate state of Type I superconductors? A corollary of (6.4.14) is that each such domain must contain an integral number of flux quanta. (We can simply take the loop integral around the normal domain instead of a physical hole.)

Consequently, too, there is a minimum value ($n = 1$ flux quanta) of the total flux through any normal domain. That smallest domain is a *quantized vortex* (also called a *flux line*), a line around which there is magnetic field and the order parameter is suppressed.

Notice that if θ changes through 2π along a loop encircling the vortex, there must be a place inside that loop at which θ is undefined. That is the centerline of the vortex. Furthermore, in order for $\Psi(\mathbf{r})$ to be continuous even on that line, we must have $|\Psi(\mathbf{r})| = 0$ there. The region in which $|\Psi(\mathbf{r})| \ll \Psi_0$ is called the vortex *core.* (Vortices are discussed in more detail in [the first two sections of] Lec. 6.6 .)

Vortices are found in Type II superconductors: remember that is the case $\lambda/\xi > 1/\sqrt{2}$). In Type I superconductors, vortex lines *attract* and would merge to form domains, as in the intermediate state (Sec. 6.3 C).

6.4 D Decay of a persistent current in a ring

A second way to parse "resistance" is: given a current, by what process does its velocity get damped; and its kinetic energy get dissipated?

SORRY – this could cut to the point more quickly.

Consider a ring (or hollow cylinder) at $T < T_c$, with a (super)current circulating, as in Sec. 6.4 C: how can this current decay? A lower energy state is always available (superconducting with $\mathbf{J}_s = 0$). But we'll find there's a large barrier to that lower energy state, so the persistent current is (very) metastable. Still, in any *finite* ring there is, in fact, a nonzero but very small energy dissipation, via quantized events related to flux quantization. (For a standard type I superconductor, an limit on resistivity $\rho \leq 3 \times 10^{-23}\Omega$ cm was measured. [5] Another measurement (runs of ~ 30 days) found time constants of roughly 10^5 years. [6])

The current is on the inner surface of the cylinder. Since there's no field deep in the bulk of the cylinder, there must be a uniform field in the hollow center which is being screened by the surface current, and therefore proportional to it. As argued in the Sec. 6.4 C, the net flux of that field must be a multiple $n\Phi_0^*$ of the flux quantum. We showed in Sec. 6.4 C too that $\oint(\nabla\theta \cdot d\mathbf{l}) = 2\pi n$. The only way to decrease the current is to decrease the encircled flux, such that $n \to n-1$, called a "phase slip" [7] since $\oint(\nabla\theta) \cdot d\mathbf{l}$ changes by 2π. That requires moving a quantum of flux from inside to outside the cylinder.

The energy cost of making a vortex line is proportional to its length, so the energy barrier is related to the thickness of the ring or cylinder; hence, thermal activation is possible and important in sufficiently small samples. More realistically, the barrier is against nucleating a single small closed *loop* of vortex. Once the loop is nucleated, every bit of it feels a "Magnus" force from the supercurrent (see Lec. 6.7) pushing the loop's diameter to expand – which it can do without any further barriers [8] until it stretches across the thickness of our ring. However, in a large sample, the expected frequency of phase slips can exceed than the age of the universe: in effect we have a *persistent current* that flows without dissipation.

In view of (6.4.9), the rate of phase slips *is* the voltage difference, in a superconducting type material:

$$\hbar \frac{d}{dt}(\theta_2 - \theta_1) = -(\mu_2 - \mu_1). \tag{6.4.16}$$

[I believe μ_i here denotes the pair chemical potential. Possibly this equation and the surrounding ideas belongs after Lec. 6.5 .] So if the flux decay rate happens to be proportional to the gauge-invariant gradient, or equivalently if the dissipation is proportional to $\mathbf{J}_s^2$, it means $V \propto I$ with I^2R Joule heating, and we have an Ohmic resistance. In that case, the current must decay exponentially with time, as in an ordinary RL circuit. Contrariwise, if the decay rate and dissipation scale more rapidly to zero when $\mathbf{J}_s \to 0$ – indeed we expect an activated form $\exp(-B/T)$ where B is the barrier mentioned above – we say the resistance is zero.

A current-carrying wire (or sheet) is no different – locally – from a cylinder or ring. In place of the multiple-connected topology, some boundary condition fixes the (gauge-invariant) phase difference between the two ends. Therefore, to change the current, we still have to pass a flux quantum's worth of flux across the wire, a phase-slip process which has the same huge barrier in any reasonable-sized samples, so we observe zero resistance.

In summary, whereas a normal current can be degraded just by scattering one independent electron, a supercurrent can be degraded only by a process which involves a large chunk of superconductor. It is due to the topological properties of the phase angle, and so it is possible only in a system which has undergone a broken symmetry.

[5] D. J. Quinn III and W. B. Ittner III, J. Appl. Phys. 33, 748 (1962)

[6] J. File and R. G. Mills, PRL 10, 93 (1963). *I need to check if this is the record.*

[7] Phase slips also occur in charge-density waves [Lec. 3.4] where the order parameter also has a phase, but they have a different relation to the current in that case.

[8] This probably occurs at special sites where the nucleation barrier is lowered, completely analogous to a Frank-Read source that nucleates *dislocation* loops repeatedly in crystalline solids.

As Anderson emphasizes in *Basic Notions*, the rigidity of ordinary solids is completely analogous. [9] *[As we approach the normal state continuously by letting $|\Psi|$ vanish – see next sections – the cost of making a vortex also vanishes. Thus it becomes easy to nucleate phase slips and dissipation will be seen.]*

At $T = 0$ quantum tunneling replaces activation over the barrier.

6.4 X Landau's critical velocity

Landau considered the instability of superflow with respect to creating elementary excitations. This fundamental mechanism for a critical current does not involve the magnetic field, so it applies in principle to neutral superfluids. It certainly applies in any superconducting slab or wire sufficiently thin compared to the penetration depth, so current doesn't make an appreciable magnetic field.

[Sec. 6.4 B works out a distinct mechanism, related to electromagnetism: by Ampère's law, a supercurrent necesssarily creates a magnetic field, but this tends to destroy superconductivity. The mechanism of the present section is more general since it applies to neutral superfluids too.

Basic idea

First note that by Galilean relativity, a neutral superfluid *in vacuum* can be boosted as fast as we like without affecting its superfluidity. We have a critical current only when our system is in contact with a *stationary* external world, which serves as a momentum reservoir: (i) the container walls (or sometimes a porous medium), in the case of a neutral superfluid (ii) the solid lattice, in the case of a superconductor. So $|\mathbf{v}_s|$, implicitly, is always measured relative to an environment with $\mathbf{v} = 0$.

Let's imagine we constrain the phase difference as a boundary condition. So long as the superfluid order parameter is nonzero we have a nonzero phase gradient k_θ hence a nonzero supercurrent. The alternative is a normal state, with zero order parameter; here, phase difference $\Delta\theta$ is undefined and we can have zero current. Compare the respective total energies: the SC state is higher by the KE of the supercurrent, but lower by the condensation energy. Hence, once the former exceeds the latter, the system goes normal.

Within the Ginzburg-Landau picture (see Sec. 6.4 A), as $|\mathbf{v}_s|$ increases (always measured relative to the environment), the order parameter is reduced, until superconductivity disappears at a critical $|\mathbf{v}_s|$, or equivalently at a critical current J_c.

Doppler effect for elementary excitations

Consider any elementary excitation in any medium with velocity $\mathbf{v}$. (It's a generalized quasiparticle as described in Lec. 1.7 , which might be either a boson or a fermion.) Let its energy dispersion be $\epsilon(\mathbf{q})$ as a function of the wavevector, in a comoving frame (in which the superfluid appears at rest). Claim: measured in the "lab frame" (in which the metal ions or channel walls are at rest), the effective dispersion

$$\epsilon_{eff}(\mathbf{q}) = \epsilon(\mathbf{q}) + \mathbf{v} \cdot \hbar\mathbf{q}. \tag{6.4.17}$$

This is the Galilean transformation of the dispersion law, or equivalently the Doppler effect, as we shall see.

[9] See N. P. Ong's website for a popular-level explanation of the rigidity (2007).

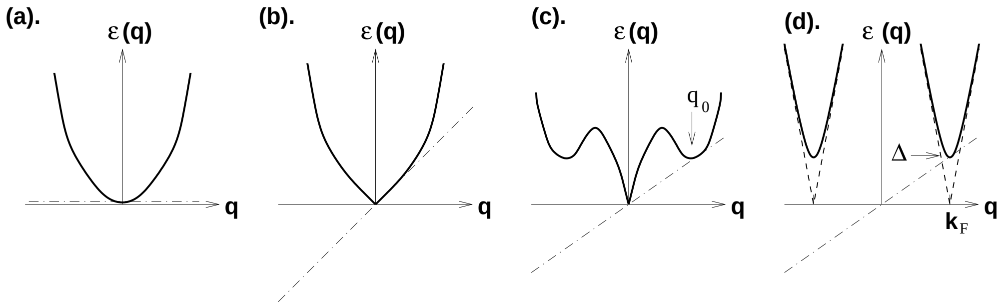

Figure 6.4.2: Landau's critical velocity. In each panel, the bold line is the dispersion curve and the dot-dashed line, tangent to it, has slope $\hbar v_c$ where v_c is the critical velocity. (a). Dispersion $\hbar^2 q^2/2\mathbf{m}$ for a free particle, giving $v_c = 0$. (b) Phonon dispersion in dilute superfluid gas (c) Phonon/roton excitations in ^{4}He; the "roton" refers to the minimum occuring at $|\mathbf{q}| = q_0$. (d) Fermion dispersion for a superconductor. The heavy dashed line shows the electron/hole dispersion of a normal Fermi liquid as in Lec. 1.8(no superconductivity), which implies $v_c = 0$. The solid curve shows the Bogoliubov quasiparticle dispersion as in Lec. 7.3 ; Δ_0 is the superconducting gap.

Let's see how (6.4.17) is justified. According to Galilean relativity, an object with energy E and momentum $\mathbf{p}$ in a frame moving at velocity $\mathbf{v}$, relative to the lab is transformed to $E' = E + \mathbf{v}\cdot\mathbf{p}$ in the stationary reference frame. Transcribing $E \to \epsilon(\mathbf{q})$ and $\mathbf{p} \to \hbar\mathbf{q}$, we claim the effective dispersion relation is (6.4.17). [10]

A second approach to rationalize (6.4.17) could be called the "Doppler shift" argument. Write the wavefunction, for the quasiparticle in the comoving frame of the fluid.

$$\psi(\mathbf{r}', t) = e^{-i\omega' t + i\mathbf{q}\cdot\mathbf{r}'} \tag{6.4.18}$$

where $\mathbf{r}'$ is measured in the comoving frame, and $\hbar\omega \equiv \epsilon(\mathbf{q})$, Just substituting $\mathbf{r} = \mathbf{r}' + \mathbf{v}t$ into (6.4.18), we obtain $\psi(\mathbf{r}, t) = \exp(-i\omega t + i\mathbf{q}\cdot\mathbf{r})$, with

$$\omega = \omega' - \mathbf{v}\cdot\mathbf{q}. \tag{6.4.19}$$

When $\psi(x)$ represents the amplitude of a sound wave, (6.4.19) is precisely the Doppler effect; when it is a Schrödinger amplitude in quantum mechanics, we identify $\hbar\omega' = \epsilon(\mathbf{q})$ and $\hbar\omega = \epsilon_{eff}(\mathbf{q})$, and (6.4.19) becomes (6.4.17). *RESTATING: We obtain* $\psi = e^{i(\hbar\mathbf{q}\cdot\mathbf{r} - \epsilon_{eff}(\mathbf{q})t/\hbar)}$ *which is just (6.4.18) with* $\epsilon(\mathbf{q}) \to \epsilon_{eff}(\mathbf{q})$ *as defined by (6.4.17).* [11]

[This justification has some analogies to the argument in Lec. 6.3 , as to why our effective field energy given a background field $\mathbf{H}$*, reduced to* $|\mathbf{B} - \mathbf{H}|^2$*.]*

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[10] Another version from (classical) Newtonian mechanics, in terms of the excitation's momentum $\mathbf{p}$. If that is changed by $\Delta\mathbf{p}$ in a time interval Δt, then a momentum $-\Delta\mathbf{p}$ is transferred to the stationary reservoir and exerts a force $\mathbf{f} = -\Delta\mathbf{p}/\Delta t$ during that interval. The site where this force acts is displaced $\delta\mathbf{r} = \mathbf{v}\Delta t$, so the work done is $\Delta W = \mathbf{f}\cdot\Delta\mathbf{r} = -\Delta\mathbf{p}\cdot\mathbf{v}$. By integrating this, one obtains an energy term $-\mathbf{v}\cdot\mathbf{p}$ which (after identifying $\mathbf{p} = \hbar\mathbf{q}$) is the second term of (6.4.17).

[11] An alternative way to frame the "Doppler" argument starts with the definition of group velocity in the comoving frame $\mathbf{v}_g = \hbar^{-1}\nabla_{qk}\epsilon(\mathbf{q})$ (as derived in basic solid state theory, for example). Then the group velocity in the stationary frame, determined from $\epsilon_{eff}(\mathbf{q})$ in the same fashion, ought to be just $\mathbf{v} + \mathbf{v}_g$, and (6.4.17) is the only formula that gives this.

Landau's critical velocity

Now, if $\epsilon_{eff}(\mathbf{q}) \leq 0$ for some $\mathbf{q}$, then the system is unstable to emitting excitations at that wavevector. This always happens for large enough $\mathbf{v}$, as is clear when (6.4.17) is represented graphically, in the (q, ϵ) plane by the difference between the $\epsilon(q)$ curve and a line at slope $\hbar v$ as in Fig. 6.4.2). Then Landau's critical velocity v_c is the first velocity where this happens. It is simply $1/\hbar$ times the slope of a line from the origin tangent to the bottom of the dispersion curve (see Fig. 6.4.2.)

If you had the dispersion relation $\hbar^2 q^2/m$ of an ordinary particle (e.g. one free atom of ^{4}He), the line would be tangent at $q = 0$ so $v_c = 0$. For an ordinary metal $v_c = 0$ too, since we have $\epsilon(k_F) = 0$ for electron or hole excitations. (It is proper to measure from the Fermi level (as introduced in Lec. 1.7 .)

However, in a superconductor the Landau critical slope is nonzero because a gap develops in the dispersion relation, $\epsilon(k_F) = \Delta$.[12]

In a neutral superfluid, the low-lying excitations are phonons with dispersion $\omega(q) = v_s q$, with v_s the speed of sound, so the Landau criterion would say $v_c = v_s$ due to phonons, as in Fig. 6.4.2(a). In ^{4}He, as is well known, the curve $\omega(q)$ bends downwards again at larger q and has the so-called "roton" minimum around $q = q_0 \approx 2\pi/(3\text{nm})$, shown in Fig. 6.4.2(b); the line would actually be tangent near to this minimum yielding a much smaller Landau v_c around 60 m/s. The real critical velocity is about 1/100 smaller than *that*; it is believed to be due to not-so-elementary excitations such as nucleation of vortex loops at irregularities along the surface.

The main point of the Landau criterion, then, is that (i) it provides a strict upper bound on v_c, and (ii) it extends the idea that superflow persists because the system must overcome an energy barrier to reach a state of no superflow.

Landau's critical velocity and $T > 0$

At finite temperatures, the superfluid/cuperconductor contains a gas of thermal excitations (fermions called quasiparticles in a superconductor, or phonons in superfluid helium) in equilibrium with the unmoving walls of the container, or (in a solid) with the lattice (and perhaps the defects which are fixed in it). In view of (6.4.17) the excitations with wavevector antiparallel to the flow are favored over those with wavevector aligned with it, indeed it turns out the excitations carry a current contribution opposite to the superflow. Thus $\mathbf{J}$ is decreased while the phase gradient $\nabla\theta$ is unchanged, so that n_s^* is reduced from its $T = 0$ value. [One can profitably develop this picture into a "two-fluid" model, with some transport due to a "normal" fluid of quasiparticles that behave much like ordinary carriers in (say) a semiconductor.]

Now let's note what happens to the order parameter for a velocity close to Landau's critical velocity, v_c. Recall that by definition, at $v = v_c$ the effective dispersion curve $\epsilon_{eff}(\mathbf{q})$ has a zero-energy excitation at a certain wavevector $\mathbf{q}_c$. Such excitations will have a large thermal population, if $T > 0$, and even in the ground state there will be large zero-point fluctuations. Consequently, as $v \to v_c$ and $\epsilon_{eff}(\mathbf{q}_c) \to 0$, the order parameter magnitude $|\Psi| \to 0$.

Having $\epsilon_{eff}(\mathbf{q}_c) = 0$ is much like having a soft phonon mode in an elastic lattice (see Lec. 3.0 and Lec. 3.4). The reduction of $|\Psi|$ by fluctuations in a superfluid or superconductor is just like Debye-Waller factor reduction of the harmonic crystal's order

[12] M. Tinkham (*Introduction to Superconductivity*, p. 119 (1st ed.), says this v_c is called the "depairing velocity" since (as just noted) the condensate of pairs will emit quasiparticles until it decays to zero. He refers us to J. Bardeen, *Rev. Mod. Phys.* 34, 667 (1962) for a review.

parameter due to fluctuations (Lec. 1.6). [13]

6.4 Y Intermediate state in wire due to current?

Consider a wire of radius $R \gg \lambda$ at $T < T_c$ carrying a current I; Now, the field just outside the surface is $B(R) = 2I/cR$ from Ampere's law. If $B(R) > H_c$, then the wire can no longer (all) stay superconducting, and a resistivity appears. (the "Silsbee effect"). Thus $I_c = \frac{c}{2} H_c R$

So what happens in the cylindrical wire for $I > I_c$? We encounter a paradox: for any cylindrically symmetrical current distribution, $B(r = 0) = 0$ so there must be some superconducting core along the wire's central axis. This core is a cylinder extending (say) to a radius $R' < R$. This core must carry all the current (it would have no longitudinal voltage drop, so there is nothing to drive a current in the normal-state shell surrounding the core.) But $B(R') > B(R) > H_c$, so we face the same old contradiction at the smaller radius R'.

The only way out is that there *is* a longitudinal voltage drop, and the superconducting core region must be broken up into disconnected domains (in the longitudinal direction). I believe the current passing from one domain to the next has such a high density that the metal gets driven normal by the current density, even though $B = 0$ along the central axis.

This is another form of the "intermediate state" made up of coexisting domains of superconductor and normal material (at field H_c.) The resistivity has been described by a sort of effective medium theory; that is, one coarse-grains to a scale bigger than the domains but smaller than the sample size, and finds an effective uniform resistivity for the mixture. Such theories are suspect, since the conductivity certainly depends on the spatial arrangement of conducting regions.

Similar contradictions appear when a slab geometry is considered.

I am somewhat puzzled whether the whole notion is well-posed. If the field adjacent to the surface is nearly H_c, then Ampère's law says $\nabla \times \mathbf{B} = (4\pi/c)\mathbf{J}$; since the fields are exponentially decaying this implies $Js = cH_c/4\pi\lambda$ is the current density adjacent to the surface. On the other hand, we see from (6.4.8) that (within G-L theory) a fundamental limit on the current density anywhere is $\sqrt{2}cH_c/16\pi\lambda$, which is certainly smaller.

An independent version of the same story

This version is more complete

Even this extremely simple geometry has a complicated solution. Imagine a (solid) cylindrical wire, carrying a current I. For sufficiently small currents, it flows as supercurrent. (Don't forget, all this current flows in a layer of thickness $\sim \lambda$ next to the surface.) However, when the wire's magnetic field exceeds H_c, the wire next to the surface cannot remain superconducting. Does the superconducting region moves inwards? That just makes it worse, since the magnetic field scales as I/R. Does the whole wire go normal? In that case, the current gets distributed uniformly throughout the cross-section, so that sufficiently close to the axis, $B < H_c$ and that part of the wire should go superconducting. Thus, we have a paradox, so long as we assume translational symmetry along the wire.

[13] This is a preview of calculations in Lec. 7.5 , which includes an exercise finding the order parameter reduction via BCS/Bogoliubov theory. Note that even at $T = 0$, I think there's a Ψ reduction (in a neutral superfluid): when $\epsilon_{eff}(\mathbf{q}) \to 0$ for such a phonon/roton mode, then its zero-point motion diverges, hence $\Psi \to 0$. Something similar must with Bogoliubov quasiparticles in that exercise.

The approximate solution was found by London: it consists of a stack of superconducting domains, with a conical shape, with separation Δz. See Fig. 6.4.3. They do not quite touch, and there is a voltage drop ΔV from each to the next. We assume (this is an approximation) that within the normal parts, the current points exactly in the longitudinal (z) direction. Consequently, within the normal portions, $B = B(r)$ independent of z. Furthermore, where B touches the superconducting domain, it must be equal to H_c: thus B is independent of r. If

$$I(r) = 2\pi \int_0^r J(r')dr' \tag{6.4.20}$$

is the net current within r, then by Ampère's law, $I(r) \propto r$ which requires $J(r) \propto 1/r$, namely $J(r) = I/2\pi Rr$ to get total current I. At radius r, the normal current path has length $r(\Delta z)/R$ so that, by Ohm's law, the voltage drop is

$$\Delta V = \frac{r(\Delta z)}{R}\frac{I}{2\pi Rr}\rho = \frac{\Delta z}{2\pi R^2}\rho = \frac{\Delta V_{\text{normal}}}{2} \tag{6.4.21}$$

where ρ is the normal-state resistivity. The cancellation of the r factors is the justification for the conical shape assumed. Also, in (6.4.21) ΔV_{normal} is the voltage drop, if the sample were all in the normal state (since its cross-section is πR^2). Thus, this predicts a discontinuous drop of the resistance by a universal factor on transitioning from the completely normal state to this intermediate state. (It is actually a factor of ~ 0.7 due to domain-wall energy costs, which we ignored.)

The above argument doesn't tell us what Δz should be. *That was done later. Presumably it depends on those domain wall costs.*

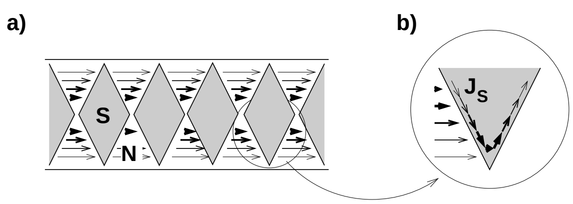

Figure 6.4.3: (a). Intermediate state solution for cylindrical wire. For current above a certain critical value, wire develops a stack of superconducting domains, offset by Δz and not quite touching at the centers. Geometry has cylindrical symmetry around the center axis. The current density (arrows) in the normal regions increases toward the axis. (b). In the superconducting parts, the current flows as a screening current around the edges.

Exercises

Ex. 6.4.1 Landau's critical velocity in a superconductor I (T)

Try the dispersion of a particle in free space, $\epsilon(\mathbf{q}) = |\hbar\mathbf{q}|^2/2m_*$, in (6.4.17): check you get $|\hbar\mathbf{q} - m_*\mathbf{v}|^2 + const$. Is this sensible?

Ex. 6.4.2 Landau's critical velocity in a superconductor II (T)

The dispersion relation of a quasiparticle in a BCS superconductor at $T = 0$ has a dispersion law with a sharp minimum at $\epsilon(k_F) = \Delta_0$.

(a) Write the critical velocity v_c this implies according to Landau's argument.

(b) Note also that the microscopic theory gives a BCS coherence length as $\xi_0 = \hbar v_F/\pi\Delta_0$. Thus show that your answer from (a) agrees with the Ginzburg-Landau answer (within a factor of order unity!) provided we identify the G.-L. and BCS coherence lengths, $\xi \sim \xi_0$.

Metal	T_c(K)	Δ(meV)	H_c(mT)	ξ_0(μm)	λ (μm)
Sn	3.72	11.5	30.9	23	3.4
Al	1.14	3.4	10.5	160	1.6
Pb	7.19	27.3	80.3	8.3	3.7
Nb	9.50	30.5	198*	3.8	3.9
Nb_3Ge	23.2				

Table 6.4.1: Data for standard metal and alloy superconductors [All taken from Kittel.]. * means a type II superconductor: thermodynamic critical field is shown, note $H_{c1} < H_c < H_{c2}$.

Ex. 6.4.3 Phase slips in a wire I: variational approach

Imagine a wire with the superconducting phase constrained at both ends. For example, it is bent into a loop and has a net current around it, corresponding to a phase change of $2\pi n$. The way to change that is if, at some point along the wire, the order parameter magnitude momentarily goes to zero; at that moment, we can change the phase on one side of that node (and not the other), with no gradient cost since there is no order parameter. Then the order parameter comes back, in the reverse process, but the net phase change is $2\pi(n-1)$. As explained in the text, the macroscopic voltage is $h/2e_*$ times the phase-slip rate, i.e. the effective resistivity is proportional to the phase-slip rate per unit length of the wire. The phase-slip rate should have an activated temperature dependence, $\exp(-U_B/T)$ where U_B is an energy barrier.

The aim of this exercise is an estimate of this energy barrier.

(a). Show that, in the G-L free energy density, we can write

$$F_L(\Psi) - F_L(\Psi_0) = |F_{\rm cond}|\left(1 - \frac{|\Psi|^2}{\Psi_0^2}\right)^2 \tag{6.4.22a}$$

$$F_{\rm grad} = |F_{\rm cond}|\xi^2 \frac{|\nabla\Psi|^2}{\Psi_0^2} \tag{6.4.22b}$$

It will be handy to scale the energy, order parameter, and length in this fashion.

(b). We have a one dimensional geometry with coordinate x. The free energy per unit length is $FA_\perp$ where $A_\perp$ is the cross-sectional area.

Consider a case with no current; there will be random phase slips of either sign. The ground state would have $\Psi = \Psi_0$ everywhere. Consider a trial state where

$$\Psi(x=0)/\Psi_0 = \phi_1, \qquad 0 \le \phi_1 \le 1. \tag{6.4.23}$$

We want to map out a function $U(\phi)$ equal to the energy cost of the *best possible* G-L configuration, constrained by the condition in Eq. (6.4.23). To do so, we assume a variational state

$$\Psi(x) = \begin{cases} \Psi_0(1 - \Delta\phi|x|/\ell), & |x| < \ell; \\ \Psi_0, & |x| > \ell. \end{cases} \tag{6.4.24}$$

where $\Delta\phi = 1 - \phi$.

Work out the total added cost $\Delta F_{\rm tot}$ a sum of the Landau and gradient contributions; they favor short and long ℓ, respectively. The Landau term should come to $|F_{\rm cond}|\ell(\Delta\phi)^2 C(\Delta\phi)$, where $C(...)$ is a second-order polynomial.

Find the optimal ℓ and substitute to find $\Delta F_{\rm tot}(\phi, \ell) = U(\phi)$.

(c) *optional* Now redo it all in the presence of a (super)current density J_s. The key result is that now, the barrier is lower and the maximum occurs *before* $\Psi = 0$ is reached.

Don't worry about magnetic field energies. Hints: (1) J_s must be independent of x, by current conservation, so this enters as a spatially constant parameter. (2) The amplitude part of $F_{\rm grad}$, which we had in the earlier parts, separates from the phase part of $F_{\rm grad}$. (3) The phase part is proportional to $J_s^2/nsup(x)$, where $n_s(x) \equiv |\Psi(x)|^2$ as usual. (4) Since $J_s \propto n_s(x)(d\theta/dx)^2$, evidently reducing n_s in an interval around $x = 0$ increases the overall phase change across that interval.

An important subtlety is that, as we vary Ψ_1, we should *not* keep J_s the same. Instead, we should imagine the phase difference at distant points is being held fixed. Since we increased the phase offset around $x = 0$, we decreased the phase offset everywhere else (so the overall current J_s is diminished compared to the original current, but remember that at any stage J_s is uniform in x.) The diminished phase offset decreases the phase-gradient energy everywhere else. This change can be calculated by analogy to the path used in Sec. 6.3 A to estimate magnetic field energy.

Ex. 6.4.4 Phase slips in a wire II: exact solution

We can use the calculus of variations to get the exact differential equation for $\Psi(x)$, which is simple so long as there is no magnetic field and no current.

(a). First set to zero the variational derivative of $F_{\rm tot}$ (based on, perhaps, (6.4.22)) with respect to $\Psi(x)$: your result should have the form

$$\text{const}\; d^2\Psi/dx^2 - F_L'(\Psi) = 0 \tag{6.4.25}$$

where $F_L'(\Psi) \equiv dF_L(\Psi)/d\Psi$. Next, multiply both sides of (6.4.25) by $d\Psi/dx$ and notice that each term can be integrated, so the whole thing can be written $d\tilde{H}/dx = 0$, where

$$\tilde{H} \equiv \frac{1}{2}\text{const}\Big(\frac{d\Psi}{dx}\Big)^2 + V(\Psi). \tag{6.4.26}$$

Mathematically, if we make the mapping $x \to$ time, $\Psi \to$ coordinate, $V(\Psi) \to$ potential energy, and $\tilde{H} \to$ Hamiltonian, this is exactly how you integrate Newton's equations of motion for a particle in that one-dimensional potential. (What is the mathematical relation of $V(\Psi)$ to $F_L(\psi)$?)

(b). You can now integrate this; to fix the "constant of motion" $\tilde{H}$, note that far away from the fluctuation, $\Psi(x) = \Psi_0$. (You also know the value $\Psi(0) = \phi_1\Psi_0$, but $d\Psi/dx|_{x=0}$ is not determined.) You should find something proportional to $\tanh(x - x_0)$; what determines x_0?

(c). Now, insert your answer into the energy densities (6.4.22)(a,b) and integrate over x. (Hint: if you didn't work (c), there is enough information in the last paragraph of (c) to guess the full solution.) You should get $|F_{\rm cond}|\xi A_\perp$ times a very simple function of $\Delta\phi \equiv \Delta\Psi/\Psi_0$.

Ex. 6.4.5 How small to see phase slips?

(a). Let's guess that the energy barrier to a (transient) fluctuation in which the order parameter vanishes in a wire is $|F_{\rm cond}|\xi A_\perp$, where $A_\perp$ is the cross-sectional area.

That is plausible, within a factor unity, by dimensional analysis: the cost of suppressing Ψ to zero uniformly is $|F_{\rm cond}|$ (by definition), and the healing length over which the order parameter returns to its bulk value is $\sim \xi$.

Based on the H_c and T_c of either Al or Pb, what would be the diameter of a wire such that $U_B = 10 T_c$? (This is based on a rough notion that e^{-10} is large enough to give an important phase slip rate; obviously, to be quantitative, one needs the attempt frequency, which is harder.)

Hints: (i) you can get $|F_{\rm cond}|$ from the H_c. (ii) for units: $({\rm Tesla})^2/8\pi = 10^{-7}{\rm J/m}^3$.

(b). If the wire shows a voltage of 1μV, how many phase slips per second are occurring? (Hint: you don't need any dimensions or material parameters.)