DISINTEGRATION OF MEASURES

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Definition 1. Let $(X,\mathfrak{M},\lambda),(Y,\mathfrak{N},\mu)$ be sigma-finite measure spaces and let $T\colon X\to Y$ be a measurable map. A (T,μ)-*disintegration* is a collection $\{\lambda_y\}_{y\in Y}$ of measures on $\mathfrak{M}$ such that:

(i) each λ_y is a sigma-finite measure and $\lambda_y(\{x\in X : T(x)\neq y\})=0$ for almost every x.

and for each nonnegative measurable function on f on X: :

(ii) The function $y\to\int f(x)\,d\lambda_y(x)$ is μ-measurable and

(iii) $\int_X f(x)\,d\lambda(x)=\int_Y\left(\int_X f(x)\,d\lambda_y(x)\right)d\mu(y)$.

Our aim is to establish the following:

Theorem 2. *Let λ be a sigma-finite regular Borel measure on a metric space X and let T be measurable map from X into $(Y,\mathfrak{M},\mu)$, where μ is sigma-finite and such that $T_*\lambda << \mu$. If $\mathfrak{M}$ is countably generated and contains all singletons, then λ has a (T,μ) disintegration. If $\{\lambda'_y\},\{\lambda_y\}$ are two such disintegrations then $\lambda_y=\lambda'_y$ almost everywhere, further if $T_*\lambda=\mu$, then λ_y is a probability measure for almost every y.*

Lemma 3. *Let X be a compact metric space, then there exists a countable set $C\subseteq C(X,[0,\infty))$ which is closed under addition and scaling by non-negative rational numbers and contains the constant function 1, with the following property. If $l\colon C\to[0,\infty)$ satisfies $l(f+g)=l(f)+l(g)$, and $l(qf)=ql(f)$ for every $q\in\mathbb{Q}\cap[0,\infty)$, $f,g\in C$ then there exists a unique positive regular Borel measure μ on X such that $l(f)=\int f\,d\mu$.*

Proof. Since X is compact metric we know that $C(X)$ and hence $C(X,[0,\infty))$ is separable (since subsets of separable metric spaces are separable) and thus we can find a countable dense subset of $C(X,[0,\infty))$ which contains 1, let C be the nonnegative functions in the $\mathbb{Q}$-linear span of such a set. Then C is countable and contained in $C(X,[0,\infty))$, is closed under addition and positive scaling, and the closed $\mathbb{R}$-linear span of C is $C(X,\mathbb{R})$. Suppose $l: C\to[0,\infty)$ is additive and $l(qf)=ql(f)$ for $q\in\mathbb{Q}\cap[0,\infty)$. Extend l to $C-C$ (which is the $\mathbb{Q}$-linear span of C) by

$$l(g-f)=l(g)-l(f)$$

for $f,g\in C$. If $g-f=g'-f'$, then $g+f'=g'+f$ and applying l gives

$$l(g)+l(f')=l(g')+l(f)$$

subtracting

$$l(g)-l(f)=l(g')-l(f')$$

so the extension is well-defined. It is easy to see that l is now $\mathbb{Q}$-linear and additive. We claim that there exists some constant A such that $|l(f)|\leq A\|f\|$, for all $f\in C$

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(where the norm is the supremum norm). Indeed for fixed $f \in C$ we can find $q \in \mathbb{Q}$ such that $\|f\| \leq q \leq 2\|f\|$, then $q - f$ is nonnegative. Since C is by construction the nonnegative elements in its closed linear span we have that $q - f \in C$ and so

$$0 \leq l(q - f) = l(q) - l(f) = ql(1) - l(f) \leq 2\|f\|l(1) - l(f)$$

thus $0 \leq l(f) \leq 2\|f\|l(1)$. It then follows that there exists a constant A (possibly different than the first) so that $|l(f)| \leq A\|f\|$ for all $f \in C - C$. This implies that l is a uniformly continuous map from $C - C$ into $\mathbb{R}$, and thus by completeness of $\mathbb{R}$ extends to a continuous map $l\colon C(X, \mathbb{R}) \to \mathbb{R}$. Existence of μ now follows from the Riesz Representation Theorem, and uniqueness is clear since the closed linear span of C is $C(X)$.

□

We now establish an easier version of Theorem 2, and reduce Theorem $T : disint$ to this case.

Theorem 4. *Let λ be a regular Borel measure on a compact metric space X and let T be measurable map from X into $(Y, \mathfrak{M}, \mu)$ wher μ is finite and such that $T_*\lambda << \mu$. If $\mathfrak{M}$ is countably generated and contains all singletons, then λ has a (T, μ) disintegration. If $\{\lambda'_y\}, \{\lambda_y\}$ are two such disintegrations then $\lambda_y = \lambda'_y$ almost everywhere, further if μ is a probability measures and $T_*\lambda = \mu$, then λ_y is a probability measure for almost every y.*

Proof. Define a measure ν on $\mathcal{B} \times \mathfrak{M}$ (here $\mathcal{B}$ denotes the Borel sets) by

$$\nu(E) = \int_X \chi_E(x, Tx) d\lambda(x)$$

using the monotone convergence theorem and that λ is finite it is easy to that ν is a finite measure. By standard arguments,

$$\int_{X \times Y} f \, d\nu(x, y) = \int_X f(x, Tx) d\lambda(x)$$

for any measurable $h\colon X \times Y \to [0, \infty)$.

Let $\Gamma = \{(x, y) \in X \times Y : y = T(x)\}$ we claim that Γ is measurable. To see this, let $\mathfrak{A}$ be a countable family which generates $\mathfrak{M}$. Let B be the algebra over $\mathbb{Q}$ generated by $\{\chi_A : A \in \mathfrak{A}\}$. Replacing $\mathfrak{A}$ with the collection of sets whose characteristic function is in B, we see that we may assume that $\mathfrak{A}$ is an algebra, i.e. is closed under finite unions, complements and contains $\varnothing$ and Y. We claim that

$$\{y\} = \cap_{B \in \mathfrak{A}, y \in B} B. \tag{1}$$

Indeed, let A be the set on the right-hand side. If $A \neq \{y\}$, let $z \in A \setminus \{y\}$, then every set in $\mathfrak{A}$ which contains y also contains z, but by taking complements it also easy to see that every set in $\mathfrak{A}$ which contains z must contain y. Thus δ_y and $\frac{1}{2}\delta_y + \frac{1}{2}\delta_z$ are two measures which agree on $\mathfrak{A}$. Since $\mathfrak{A}$ generates $\mathfrak{M}$ as a sigma-algebra, the uniqueness in Caratheodory's theorem implies that $\delta_y = \frac{1}{2}\delta_y + \frac{1}{2}\delta_z$ on $\mathfrak{M}$. But this is impossible since $\{y\} \in \mathfrak{M}$. This establishes (1), thus

$$\Gamma = \bigcap_{A \in \mathfrak{A}} \{(x, y) : x \in T^{-1}(A), y \in A\}.$$

Thus Γ and hence Γ^c is measurable and

$$\nu(\Gamma^c) = \int_X \chi_{\Gamma^c}(x, Tx) \, d\lambda(x) = 0.$$

Let $C \subseteq C(X, [0, \infty))$ be as in Lemma 3. For fixed $f \in C$ we have a measure ν_f on $\mathfrak{M}$ where

$$\nu_f(E) = \int f(x)\chi_E(y)\,d\nu(x,y) = \int_X f(x)\chi_E(Tx)\,d\lambda(x).$$

If $\mu(E) = 0$ then we have

$$\nu_f(E) \leq \|f\| \int_X \chi_E(Tx)\,d\lambda_x = \|f\|(T_*\lambda)(E) = 0$$

since $T_*\lambda << \mu$. Let $\lambda'_y f$ be the value of the Radon-Nikodym derivative of ν_f with respect to μ at y. Since C is countable we may choose a co-null subset Y_0 of Y such that $f \to \lambda_y f$ is additive and $\lambda_y(qf) = q\lambda_y f$ for all $q \in \mathbb{Q} \cap [0, \infty)$ and $y \in Y_0$. Thus by the Lemma, for $y \in Y_0$ there exists a unique measure λ_y on X such that

$$\int f(x) d\lambda_y(x) = \lambda'_y f.$$

In particular on Y_0 we have that

$$\lambda_y(X) = \lambda'_y 1 = \frac{d\nu_1}{d\mu} = \frac{dT_*\lambda}{d\mu}$$

which is 1 if $dT_*\lambda = d\mu$. So λ_y is a probability measure if $T_*\lambda = \mu$.

Setting $\lambda_y = 0$ off Y_0 we claim that $\{\lambda_y\}_{y\in Y}$ is the desired disintegration. We show a more general version of (ii) namely we show that for $f\colon X \to [0,\infty), g\colon Y \to [0, \infty)$ measurable that

$$(ii)' \int_X f(x)g(y)\,d\lambda_y(x)\,d\mu(y) \text{ is measurable}$$

and

$$(iii)' \int_Y \int_X f(x)g(y)\,d\lambda_y(x) d\mu(y) = \int_{X\times Y} f(x)g(y)\,d\nu(x,y)$$

Let us first verify that $(ii)'$ and $(iii)'$ hold for f continuous. By linearity, and our choice of C, we see that it suffices to show $(ii)'$ and $(iii)'$ for f in C. For $f \in C$, and almost every y

$$\int_X f(x)g(y) d\lambda_y(x) = g(y)\lambda'_y f$$

and is thus measurable. Further for $f \in C$,

$$\int_Y \int_X f(x)g(y) d\lambda_y(x) d\mu(y) = \int_Y g(y)\frac{d\nu_f}{d\mu}(y)\,d\mu(y) = \int_Y g(y) d\nu_f(y) =$$

$$\int_X f(x)g(y)\,d\nu(x,y).$$

This verifies $(ii)'$ and $(iii)'$ for $f \in C(X)$.

We now verify $(ii)'$ and $(iii)'$ for characteristic functions of Borel sets. Let $\mathfrak{N}$ be the class of Borel sets in X satisfying $(ii)'$ and $(iii)'$ for all $g\colon Y \to [0, \infty)$ measurable. We have that $X, \varnothing \in \mathfrak{N}$ since we verified $(ii)', (iii)'$ for constant functions. By linearity and the monotone convergence theorem we have that $\mathfrak{N}$ is closed under differences and disjoint unions so $\mathfrak{N}$ is a sigma-algebra. Thus we only have to verify $(ii)'$ and $(iii)'$ for χ_U, where $U \subseteq X$ is open, in fact since X is a compact metric space and thus second countable. We only have to verify $(ii)'$ and $(iii)'$ for $\chi_{B(x,\varepsilon)}$ where $x \in X$ and $\varepsilon > 0$. Since X is compact by Uryshon's Lemma we can find a sequence $(f_n)_{n\geq 1}$ in $C(X)$ such that $0 \leq f_n \leq 1, f_n\big|_{B(x,\varepsilon)} = 1$ and

$f_n = 0$ outside $B(x, \varepsilon + 1/n)$. Then $f_n \to \chi_{B(\varepsilon,x)}$ pointwise and is dominated by 1 thus a double use of the dominated convergence theorem implies $(ii)'$ and $(iii)'$ for $B(\varepsilon, x)$. An entirely similar argument using that $\mathcal{B} \otimes \mathfrak{M}$ is generated by sets of the form $A \times B, A \in \mathcal{B}, B \in \mathfrak{M}$ establishes the stronger claim that

$$\int_{X\times Y} f(x,y) = \int_Y \int_X f(x,y) d\lambda_y(x) d\mu(y)$$

for f non-negative and $\mathcal{B} \otimes \mathfrak{M}$ measurable. In particular by what we saw earlier

$$0 = \nu(\Gamma^c) = \int_Y \int_X \chi_{\Gamma^c}(x,y)\, d\lambda_y(x)\, d\mu(y) = \int_Y \lambda_y(\{x : T(x) \neq y\})\, d\mu(y)$$

so that $\lambda_y(\{x : T(x) \neq y\}) = 0$ for μ-almost every y.

Finally we have to establish uniqueness. If $\{\lambda_y\}, \{\widetilde{\lambda_y}\}$ are two disintegrations, by symmetry it suffices to show that $\lambda_y \geq \widetilde{\lambda_y}$ almost everywhere. Fix $f \in C$, and let $A \subseteq Y$ be measurable. Since $\lambda_y, \widetilde{\lambda_y}$ are concentrated on $\{x \in X : T(x) = y\}$ we have

$$\int_A \int_X f(x)\, d\lambda_y(x) d\mu(y) = \int_Y \int_X f(x)\chi_A(y)\, d\lambda_y(x)\, d\mu(y) = \int_Y \int_X f(x)\chi_A(Tx)\, d\lambda_y(x)\, d\mu(y) =$$

$$\int_X f(x)\chi_A(Tx)\, d\lambda(x) = \int_A \int_X f(x)\, d\widetilde{\lambda_y}(x).$$

So

$$\int_X f(x)\, d\lambda_y(x) = \int_X f(x) d\widetilde{\lambda_y}(x)$$

for μ almost all y. Since there are countably many $f \in C$ we can find a conull $Y_0 \subseteq Y$ such that the above holds for all $y \in Y_0$ and $f \in C$. Since the closed linear span of C is $C(X)$ we have that

$$\int_X f(x)\, d\lambda_y(x) = \int_X f(x) d\widetilde{\lambda_y}(x)$$

for all $y \in Y_0$. So $\lambda_y = \widetilde{\lambda_y}$ for all $y \in Y_0$. □

We are now in a position to prove Theorem 2.

Proof of Theorem 2. First assume that μ is finite. In this case, by regularity and since λ is sigma-finite we can find a null set N in X and countably many disjoint compact sets $K_1, K_2, \ldots$ such that

$$X = \cup_{n=1}^{\infty} K_n \cup N$$

applying Theorem 4 to each K_n gives a disintegration, add them all together to get a disintegration for X. Applying uniqueness to each K_n gives uniqueness of the disintegration as well. The sigma-finite case is similar, just reduce it to the finite case.

For the last part, assume that $T_*\lambda = \mu$. We have for all $A \subseteq Y$ that

$$\mu(A) = \int_Y \chi_A(y) dT_*\lambda(y) = \int_X \chi_A(Tx)\, d\lambda(x) = \int_Y \int_X \chi_A(Tx)\, d\lambda_y(x) d\mu(y)$$

since λ_y is concentrated on $\{x : T(x) = y\}$ the above is

$$\int_Y \int_X \chi_A(y)\, d\lambda_y(x)\, d\mu(y) = \int_Y \lambda_y(X)\chi_A(y)\, d\mu(y) = \int_A \lambda_y(X)\, d\mu(y).$$

Thus $\lambda_y(X)$ have the same integral over any set and thus are equal almost everywhere. $\square$