On topological graphs with at most four crossings per edge

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Abstract

We show that if a graph G with $n \geq 3$ vertices can be drawn in the plane such that each of its edges is involved in at most four crossings, then G has at most $6n - 12$ edges. This settles a conjecture of Pach, Radoičić, Tardos, and Tóth, and yields a better bound for the famous *Crossing Lemma*: The *crossing number*, $\mathrm{cr}(G)$, of a (not too sparse) graph G with n vertices and m edges is at least $c\frac{m^3}{n^2}$, where $c > 1/29$. This bound is known to be tight, apart from the constant c for which the previous best bound was $1/31.1$.

As another corollary we obtain some progress on the *Albertson conjecture*: Albertson conjectured that if the chromatic number of a graph G is r, then $\mathrm{cr}(G) \geq \mathrm{cr}(K_r)$. This was verified by Albertson, Cranston, and Fox for $r \leq 12$, and for $r \leq 16$ by Barát and Tóth. Our results imply that Albertson conjecture holds for $r \leq 18$.

1 Introduction

A *topological graph* is a graph drawn in the plane with its vertices as points and its edges as Jordan arcs that connect corresponding points and do not contain any other vertex as an interior point. Any two edges of a topological graph have a finite number of intersection points. Every intersection point of two edges is either a vertex that is common to both edges, or a crossing point at which one edge passes from one side of the other edge to its other side. Throughout this paper we assume that no three edges cross each other at a single crossing point. A topological graph is *simple* if every pair of its edges intersect at most once.

For a topological graph D we denote by $\mathrm{cr}(D)$ the *crossing number* of D, that is, the number of crossing points in D. The crossing number of of an abstract graph G, $\mathrm{cr}(G)$, is the minimum value of $\mathrm{cr}(D)$ taken over all drawings D of G as a topological graph. The following result was proved by Ajtai, Chvátal, Newborn, Szemerédi [6] and, independently, Leighton [16].

Theorem 1 ([6, 16])**.** *There is an absolute constant $c > 0$ such that for every graph G with n vertices and $m > 4n$ edges we have $cr(G) \geq c\frac{e^3}{n^2}$.*

This celebrated result is known as the *Crossing Lemma* and has numerous applications in combinatorial and computational geometry, number theory, and other fields of mathematics.

The Crossing Lemma is tight, apart from the multiplicative constant c. This constant was originally very small, and later was shown to be at least $1/64 \approx 0.0156$, by the probabilistic proof of the Crossing Lemma due to Chazelle, Sharir, and Welzl [3]. Pach and Tóth [20] proved that $0.0296 \approx 1/33.75 \leq c \leq 0.09$ (the lower bound applies for $m \geq 7.5n$). Their lower bound was later improved by Pach, Radoičić, Tardos, and Tóth [19] to $c \geq 1024/31827 \approx 1/31.1 \approx 0.0321$ (when $m \geq \frac{103}{16}n$). Both improved lower bounds for c were obtained using

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the same approach, namely, finding many crossings in sparse graphs. To this end, it was shown that topological graphs with few crossings per edge have few edges.

Denote by $e_k(n)$ the maximum number of edges in a topological graph with $n > 2$ vertices in which every edge is involved in at most k crossings. Let $e^*_k(n)$ denote the same quantity for *simple* topological graphs. It follows form Euler's Polyhedral Formula that $e_0(n) \leq 3n - 6$. Pach and Tóth showed that $e^*_k(n) \leq 4.108\sqrt{k}n$ and also gave the following better bounds for $k \leq 4$.

Theorem 2 ([20]). *$e^*_k(n) \leq (k+3)(n-2)$ for $0 \leq k \leq 4$. Moreover, these bounds are tight when $0 \leq k \leq 2$ for infinitely many values of n.*

Pach et al. [19] observed that the upper bound in Theorem 2 applies also for not necessarily simple topological graphs when $k \leq 3$, and proved a better bound for $k = 3$.

Theorem 3 ([19]). *$e_3(n) \leq 5.5n - 11$. This bound is tight up to an additive constant.*

By Theorem 2, $e^*_4(n) \leq 7n - 14$. Pach et al. [19] claim that similar arguments to their proof of Theorem 3 can improve this bound to $(7 - \frac{1}{9})n - O(1)$. They also conjectured that the true bound is $6n - O(1)$. Here we settle this conjecture on the affirmative, also for not necessarily simple topological graphs.

Theorem 4. *Let G be a topological graph with $n \geq 3$ vertices. If every edge of G is involved in at most four crossings, then G has at most $6n - 12$ edges. This bound is tight up to an additive constant.*

Using the bound in Theorem 4 and following the footsteps of [19, 20] we obtain the following linear lower bound for the crossing number.

Theorem 5. *Let G be a graph with $n > 2$ vertices and m edges. Then $cr(G) \geq 5m - \frac{139}{6}(n - 2)$.*

This linear bound is then used to get a better constant factor for the bound in the Crossing Lemma, by plugging it into its probabilistic proof, as in [17, 19, 20].

Theorem 6. *Let G be a graph with n vertices and m edges. Then $cr(G) \geq \frac{1}{29}\frac{m^3}{n^2} - \frac{35}{29}n$. If $m \geq 6.95n$ then $cr(G) \geq \frac{1}{29}\frac{m^3}{n^2}$.*

Albertson conjecture. The *chromatic number* of a graph G, $\chi(G)$, is the minimum number of colors needed for coloring the vertices of G such that none of its edges has monochromatic endpoints. In 2007 Albertson conjectured that if $\chi(G) = r$ then $\mathrm{cr}(G) \geq \mathrm{cr}(K_r)$. That is, the crossing number of an r-chromatic graph is at least the crossing number of the complete graph on r vertices.

If G contains a *subdivision*[1] of K_r then clearly $\mathrm{cr}(G) \geq \mathrm{cr}(K_r)$. A stronger conjecture (than Albertson conjecture and also than *Hadwiger conjecture*) is therefore that if $\chi(G) = r$ then G contains a subdivision of K_r. However, this conjecture, which was attributed to Hajós, was refuted for $r \geq 7$ [9, 11].

Albertson conjecture is known to hold for small values of r: For $r = 5$ it is equivalent to the Four Color Theorem, whereas for $r = 6$, $r \leq 12$, and $r \leq 16$, it was verified respectively by Oporowskia and Zhao [18], Albertson, Cranston, and Fox [7], and Barát and Tóth [8]. By using the new bound in Theorem 5 and following the approach in [7, 8], we can now verify Albertson conjecture for $r \leq 18$.

Theorem 7. *Let G be an n-vertex r-chromatic graph. If $r \leq 18$ or $r = 19$ and $n \neq 37, 38$, then $cr(G) \geq cr(K_r)$.*

[1]A subdivision of K_r consists of r vertices, each pair of which is connected by a path such that the paths are vertex disjoint (apart from their endpoints).

Organization. The bulk of this paper is devoted to proving Theorem 4 in Section 2. In Section 3 we recall how the improved crossing numbers are obtained, and their consequences.

2 Proof of Theorem 4

We understand a *multigraph* as a graph that might contain parallel edges but no loops. We may assume, without loss of generality, that the topological (multi)graphs that we consider do not contain self-crossing edges, for such crossing points can be easily eliminated by re-routing the self-crossing edge at a small neighborhood of the crossing point.

Let e_1 and e_2 be two intersecting edges in a topological multigraph G and let $x_1, x_2, \ldots, x_t$ be their intersection points, ordered as they appear along e_1 and e_2. If $t \geq 2$ then for every $i \in \{1, 2, \ldots, t-1\}$ the open Jordan region whose boundary consists of the edge-segments of e_1 and e_2 between and x_i and x_{i+1} is called a *lens*. We call x_i and x_{i+1} the *poles* of the lens. A lens is *empty* if it does not contain a vertex of G. We will need the following fact later.

Proposition 2.1. *Let l be an empty lens in a topological multigraph G that is bounded by two edge-segments s_1 and s_2. If s_1 is crossed by an edge e and l does not contain a smaller empty lens then s_2 is also crossed by e.*

Proof. Suppose that there is an edge e that crosses s_1 but not s_2. Since l does not contain any vertex of G, e must intersect s_1 at another point (either a crossing point or a vertex of G). But then l contains a smaller empty lens. □

The upper bound in Theorem 4 will follow from the next claim.

Theorem 8. *Let G be a topological multigraph with $n \geq 3$ vertices and no empty lenses. If every edge in G is involved in at most four crossings, then G has at most $6n - 12$ edges. This bound is tight for infinitely many values of n.*

To see that it is enough to prove Theorem 8, consider a topological graph G with $n \geq 3$ vertices in which every edge is involved in at most four crossings. We may assume that there is no other n-vertex topological graph with the latter property and more edges than G (otherwise, replace G by this graph). Furthermore, we may assume that there is no other n-vertex topological graph G' with at most four crossings per edge and the same number of edges as G, such that $\mathrm{cr}(G') < \mathrm{cr}(G)$ (otherwise, replace G by G'). We claim that G has no empty lenses, and therefore, by Theorem 8, it has at most $6n - 12$ edges.

Indeed, suppose that G has empty lenses and let l be an empty lens that contains no other empty lens. Let e_1 and e_2 be the edges that form l and let s_1 and s_2 be the edge-segments of e_1 and e_2, respectively, that bound l. Denote by x_1 and x_2 the endpoints of s_1 and s_2. At least one of x_1 and x_2 is a crossing point, for otherwise e_1 and e_2 are parallel edges in G. Let G' be the graph we obtain by 're-routing' e_1 along s_2 and e_2 along s_1 and drawing them such that they do not intersect at the crossing points among x_1 and x_2 (see Figure 1 for an example). Note that since l contains no other empty lens, it follows from Proposition 2.1 that every edge that crosses s_1 must cross s_2, and vice verse. Therefore, every edge in G' is involved in at most four crossings. However, G' has the same number of edges as G, but has fewer crossings, and this contradicts our assumption on G.

We therefore turn now to proving Theorem 8. For a topological multigraph G we denote by $M(G)$ the plane map induced by G. That is, the vertices of $M(G)$ are the vertices and crossing points in G, and the edges of $M(G)$ are the crossing-free segments of the edges of G (where each such segment connects two vertices of $M(G)$). We say that a face f of $M(G)$ is *good* if $|f| = 3$ or f is incident to at most one vertex of G. Otherwise, f is *bad*. We call G *good* if every face of $M(G)$ is good. The next lemma will allow us to assume that G is good.

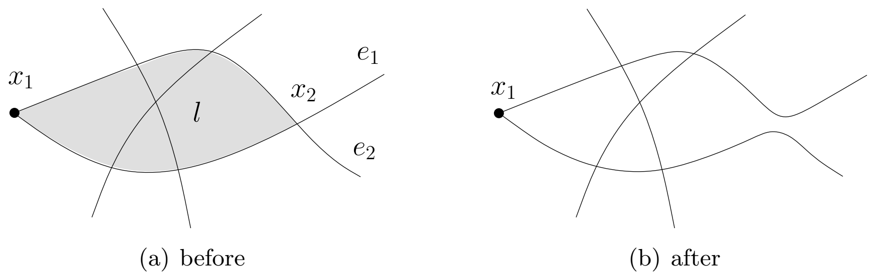


Figure 1: Getting rid of an empty lens.

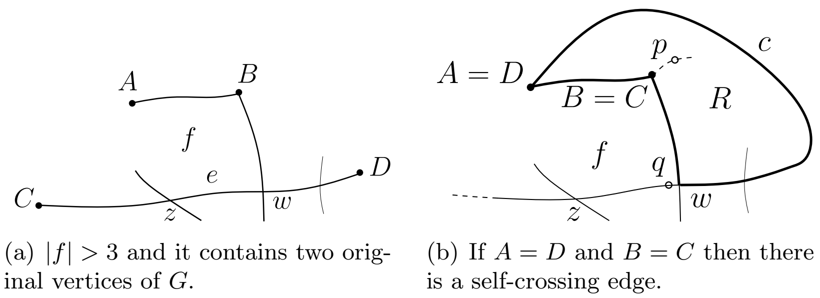


Figure 2: Illustrations for the proof of Lemma 2.2

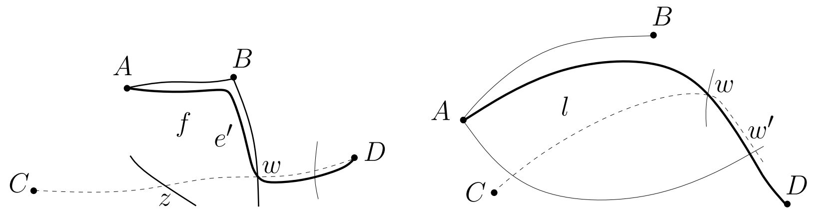
Lemma 2.2. *Let G be a topological multigraph with no empty lenses and at most k crossings per edge. Then there exists a good topological multigraph G' with no empty lenses and at most k crossing per edge such that: (1) $V(G) = V(G')$; (2) $|E(G)| \leq |E(G')|$; and (3) $cr(G) \geq cr(G')$.*

Proof. Since G has no empty lenses, every face in $M(G)$ is of size at least three. Suppose that $M(G)$ contains a face f, $|f| > 3$, that is incident to two vertices of G, denote them by A and B.

If A and B are not adjacent in f, then we can add an edge (a 'chord') between them within f. Observe that the new edge cannot form an empty lens. Indeed, suppose an empty lens is formed when $e = AB$ is added. Then, there must be another edge $e' = AB$. As e, the edge e' is also crossing-free by Proposition 2.1. However, if e' is crossing-free and forms an empty lens with e, then e' must be edge of f, and so A and B are adjacent in f.

We continue adding such 'chords' as long as possible, until the plane map contains no face with two vertices of G that are not adjacent in that face. Denote by G_1 the resulting topological graph and suppose that $M(G_1)$ has a face f such that $|f| > 3$ and f is incident to two vertices A, B of G_1 (that are adjacent in f). Assume that B follows A in a clockwise order of the vertices of f, and denote by w and z the following vertices after B. Notice that both w and z must be crossing points in G, since $|f| > 3$ and f contains no non-adjacent vertices of G. Let $e = CD$ be the edge of G_1 that contains the edge wz of f. Suppose that D is the endpoint of e such that w lies between D and z on e (refer to Figure 2(a)).

We wish to show that we can replace CD by a new edge AD or BC, such that the new graph has fewer crossings than G_1. To this end, we first show that $A \neq D$ or $B \neq C$, in order to avoid creating a loop in the underlying abstract multigraph. Suppose that $A = D$ and $B = C$, that is $CD = BA$ (we write BA to distinguish this edge from the edge AB of f). Observe that BA is not the same edge that contains the edge-segment Bw, since then it will cross itself at w. Therefore, there is a point p on BA near f, outside of it, and not on Bw or AB. Let q be a point near w on the edge-segment wB of BA. Consider the closed curve c that consists of AB, Bw, and the edge-segment wA of BA. Since AB and Bw are



(a) Replacing CD with AD.

(b) If G_2 has an empty lens, then so does G_1.

Figure 3: "Fixing" the bad face f when $A \neq D$.

crossing-free and wA cannot cross itself, c is a Jordan curve. Let R be the region bounded by c. Suppose that we traverse c clockwise (such that R is to our right). If we visit A, B, w in this order, then R must contain f (recall that this is the clockwise order of these vertices in f). Therefore, $p \notin R$ and $q \in R$. Otherwise, if f is not in R then $p \in R$ and $q \notin R$ (see Figure 2(b) for an example). It follows that the edge-segment pq of BA must cross the edge-segment wA of BA, but then BA crosses itself.

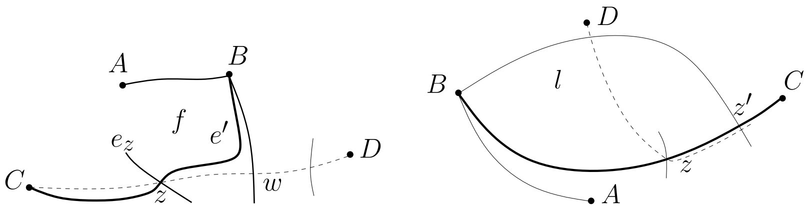
Suppose that $A \neq D$. Let G_2 be the topological graph we obtain by replacing e with a new edge $e' = AD$ as illustrated in Figure 3(a). That is, e' closely follows AB and Bw inside f, then it crosses CD and the edge containing Bw at w, and closely follows wD. Observe that G_2 has the same vertex set as G_1 and the same number of edges. The new edge e' is involved in at least one less crossing than e, and the number of crossings for every other edge can only decrease. Therefore, $\mathrm{cr}(G_2) < \mathrm{cr}(G_1)$ and we have not increased the maximum number of crossings per edge.

Next we show that G_2 does not contain an empty lens. Indeed, suppose that we have created an empty lens and let l be an empty lens that does not contain a smaller empty lens. Since the new edge AD follows the edge-segment wD of the old edge CD, it follows that A must be a pole of l, for otherwise G_1 also contains an empty lens. Denote by w' the other pole of l and observe that $w' = D$ or w' is some crossing point on the (closed) edge-segment wD of AD.

Orient the new edge-segment Aw' such that l is to its right, and denote the other (old) edge-segment the bounds l by $w'A$. Suppose that Aw' is oriented from w' to A. Since the edge AB is the edge that follows AD in a counterclockwise order of the edges around A, it follows that B must be the other pole of l (that is $w' = B = D$), for otherwise l contains B. However, AB is crossing-free and AD is not, so by Proposition 2.1 there is a smaller empty lens in l, which contradicts our choice of l.

Therefore, Aw' is oriented from A to w'. Observe that by the construction of the new edge AD, when traversing AD from A to D, at the point w we may turn right and follow the edge-segment wC of CD. Since l lies to the right of Aw' it follows that this edge-segment must be inside l. The lens l does not contain C, therefore the edge-segment wC intersects the edge-segment ww' of Aw' or the edge-segment $w'A$. (Note that it is possible that $A = C$ but it is impossible that $C = w' = D$ since G_1 contains no loops. It is also impossible that wC intersects the new crossing-free edge-segment Aw.) However, since AD follows CD from w to D, it follows that G_1 already contains an empty lens (see Figure 3(b) for an illustration).

Consider now the case that $A = D$, and therefore, $B \neq C$. Denote by e_z the edge that crosses CD at z. Let G_2 be the topological graph we obtain by replacing e with a new edge $e' = BC$ as illustrated in Figure 4(a). That is, e' closely follows Bw and wz inside f, then it crosses CD and e_z at z, and closely follows the edge-segment zC. Observe that G_2 has the same vertex set as G_1 and the same number of edges. The new edge e' is involved in



(a) Replacing CD with BC.

(b) If G_2 has an empty lens, then so does G_1.

Figure 4: "Fixing" the bad face f when $B \neq C$.

at least one less crossing than e, and the number of crossings for every other edge can only decrease. Therefore, $\mathrm{cr}(G_2) < \mathrm{cr}(G_1)$ and we have not increased the maximum number of crossings per edge.

Next we show that G_2 does not contain an empty lens. Indeed, suppose that we have created an empty lens and let l be an empty lens that does not contain a smaller empty lens. Since the new edge BC follows the edge-segment zC of the old edge CD, it follows that B must be a pole of l, for otherwise G_1 also contains an empty lens. Denote by z' the other pole of l and observe that $z' = C$ or z' is some crossing point on the (closed) edge-segment zC of BC.

Orient the new edge-segment Bz' such that l is to its left, and denote the other (old) edge-segment the bounds l by $z'B$. Suppose that Bz' is oriented from z' to B. Since the edge AB is the edge that follows BC in a clockwise order of the edges around B, it follows that A must be the other pole of l (that is $z' = A = C$), for otherwise l contains A. However, AB is crossing-free and BC is not, so by Proposition 2.1 there is a smaller empty lens in l, which is impossible.

Therefore, Bz' is oriented from B to z'. Observe that by the construction of the new edge BC, when traversing BC from B to C, at the point z we may turn left and follow the edge-segment zD of CD. Since l lies to the left of Bz' it follows that this edge-segment must be inside l. The lens l does not contain D, therefore the edge-segment zD intersects the edge-segment zz' of Bz' or the edge-segment $z'B$. (Note that it is possible that $B = D$ but it is impossible that $C = z' = D$ since G_1 contains no loops. It is also impossible that zD intersects the new crossing-free edge-segment Bz.) However, since BC follows CD from z to C, it follows that G_1 already contains an empty lens (see Figure 4(b) for an illustration).

It follows that we can add new 'chords' and replace edges as above, until a good topological multigraph is obtained. □

Let G be an n-vertex topological multigraph such that G has no empty lenses and every edge in G is involved in at most four crossings. As before, we choose G such that it has the maximum number of edges among the n-vertex multigraphs with those properties. Furthermore, we may assume that there is no other n-vertex topological multigraph G' such that G' has no empty lenses, G' has the same number of edges as G, and fewer crossings than G. Finally, by Lemma 2.2 we may assume that G is good.

If $n = 3$ then G has at most 6 edges. Indeed, otherwise there is a vertex v of degree at least five. Let x and y be the two other vertices, such that there are at least three edges between v and x. These three edges form at least two disjoint lenses, however, y can be in at most one of them, and therefore G has an empty lens. Thus, G has at most $6n - 12 = 6$ edges and the theorem holds when $n = 3$. Assume therefore that $n > 3$. We may also assume that the minimum degree in G is at least 7, for otherwise we can remove a vertex of degree at most 6, and conclude the theorem by induction.

We use the *Discharging Method* to prove Theorem 8. This technique, that was introduced and used successfully for proving structural properties of planar graphs (most notably, in the proof of the Four Color Theorem [4]), has recently proven to be a useful tool also for solving several problems in geometric graph theory [1, 2, 5, 15, 21]. The idea is to assign a *charge* to every face of the planar map $M(G)$ such that the total charge is $4n-8$. Then, redistribute the charges in several steps such that eventually the charge of every face is nonnegative and the charge of every vertex $v \in V(G)$ is $\deg(v)/3$. Hence, $2|E(G)|/3 = \sum_{v \in V(G)} \deg(v)/3 \leq 4n-8$ and we get the claimed bound on $|E(G)|$. Next we describe the proof in details. Unfortunately, as it often happens when using the discharging method, the proof requires considering many cases and sub-cases.

Charging. Let V', E', and F' denote the vertex, edge, and face sets of $M(G)$, respectively. For a face $f \in F'$ let $v(f)$ denote the number of vertices of G on the boundary of f. It is easy to see that $\sum_{f \in F'} v(f) = \sum_{u \in V(G)} \deg(u)$ and that $\sum_{f \in F'} |f| = 2|E'| = \sum_{u \in V'} \deg(u)$. Note also that every vertex in $V' \setminus V(G)$ is a crossing point of G and therefore its degree in $M(G)$ is four. Hence,

$$\sum_{f \in F'} v(f) = \sum_{u \in V(G)} \deg(u) = \sum_{u \in V'} \deg(u) - \sum_{u \in V' \setminus V(G)} \deg(u) = 2|E'| - 4\left(|V'| - n\right).$$

Assigning every face $f \in F'$ a charge of $|f| + v(f) - 4$, we get that total charge over all the faces is

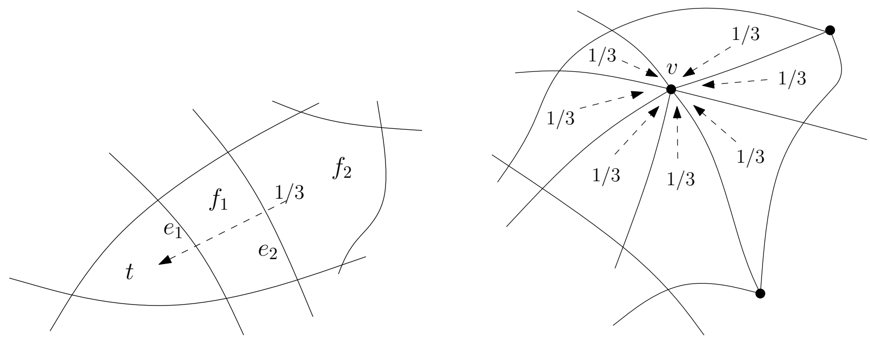
$$\sum_{f \in F'} (|f| + v(f) - 4) = 2|E'| + 2|E'| - 4\left(|V'| - n\right) - 4|F'| = 4n - 8,$$

where the last equality follows from Euler's Polyhedral Formula by which $|V'|+|F'|-|E'| = 2$.

Discharging. We will redistribute the charges in several steps. We denote by $ch_i(x)$ the charge of an element x (either a face in F' or a vertex in $V(G)$) after the ith step, where $ch_0(\cdot)$ represents the initial charge function. We will use the terms *triangles*, *quadrilaterals*, *pentagons* and *hexagons* to refer to faces of size 3, 4, 5 and 6, respectively. An integer before the name of a face, denotes the number of original vertices (vertices of G) on its boundary. For example, a 2-triangle is a face of size 3 that has 2 original vertices on its boundary. It follows from our choice of G (using Lemma 2.2) that if $v(f) > 1$ for a face f, then f is a triangle. Since G has no empty lenses, there are no faces of size 2 in F'. Therefore, initially, the only faces with a negative charge are 0-triangles.

Step 1: Charging 0-triangles. Let t be a 0-triangle, let e_1 be one of its edges, and let f_1 be the other face incident to e_1 (see Figure 5(a)). It must be that $|f_1| > 3$, for otherwise there would be an empty lens. If f_1 is not a 0-quadrilateral, then we move 1/3 units of charge from f_1 to t, and say that f_1 *contributed* 1/3 units of charge to t *through* e_1. Otherwise, if f_1 is a 0-quadrilateral, let e_2 be the opposite edge to e_1 in f_1, and let f_2 be the other face incident to e_2. We claim that f_2 cannot be a 0-quadrilateral. Indeed, suppose that f_2 is a 0-quadrilateral, let e_3 be the opposite edge to e_2 in f_2 and let f_3 be the other face that is incident to e_3. Let a and b be the two edges of G that cross at the vertex that is opposite to e_1 in t. Then a and b are already involved in four crossings, therefore f_3 must have at least two original vertices on its boundary (endpoints of a and b), since f_3 is not a triangle (this would imply an empty lens). However, we chose G such that such a face is impossible. Therefore if f_1 is a 0-quadrilateral, then f_2 contributes 1/3 units of charge to t through e_2. In a similar way t obtains 2/3 units of charge from the two other 'directions'. ⇝

After the first discharging step the charge of every 0-triangle is zero. Note that in at most one of the three 'directions' in which a 0-triangle t 'seeks' charge it can encounter a 0-quadrilateral. Indeed, two neighboring 0-quadrilaterals to t would imply that the third neighboring face has two original vertices and size greater than three and hence is not good.



(a) Step 1: f_2 contributes 1/3 units of charge to the 0-triangle t through e_2.

(b) Step 2: v receives 1/3 units of charge from each face incident to it.

Figure 5: The first two discharging steps.

Observation 2.3. *A face can contribute at most once through each of its edges in Step 1. Moreover, if a face contributes through one of its edges in Step 1 then the vertices of this edge are crossing points in G.*

Recall that according to our plan, the charge of every original vertex should be one third of its degree. The next discharging step takes care of this.

Step 2: Charging vertices of G. In this step every vertex of G takes 1/3 units of charge from each face it is incident to (see Figure 5(b)). ⇜⇝

It follows from Observation 2.3 and the discharging steps that $ch_2(f) \geq 2|f|/3+2v(f)/3-4$, for every face f. Therefore $ch_2(f) \geq 0$ if $|f| \geq 6$.

Observation 2.4. *Let f be a face in $M(G)$. Then*
- *if $|f| \geq 6$ then $ch_2(f) \geq 0$, and equality may hold only if f is a 0-hexagon;*
- *if f is a 1-pentagon then $ch_2(f) \geq 2/3$;*
- *if f is a 0-quadrilateral or a 0-triangle then $ch_2(f) = 0$;*
- *if f is a 1-quadrilateral then $ch_2(f) \geq 0$;*
- *if f is a 2-triangle then $ch_2(f) = 1/3$; and*
- *if f is a 1-triangle then $ch_2(f) = -1/3$.*

Note that we have not mentioned 0-pentagons. Showing that 0-pentagons end up with a nonnegative charge will be the most challenging task, and therefore we postpone the analysis of their charge until after all the discharging steps are described.

After the second discharging step the charge of every vertex $v \in V(G)$ is $\deg(v)/3$ and the only faces with a negative charge are 1-triangles (we will see later in Proposition 2.13 that the charge of 0-pentagons after Step 1 is nonnegative). In the next three steps we redistribute the charges such that the charge of every 1-triangle becomes zero.

Let f be a 1-triangle and let $v \in V(G)$ be the vertex of G that is incident to f. Let g_1 and g_2 be the two faces that share an edge of $M(G)$ with f and are also incident to v. We call g_1 and g_2 the *neighbors* of f (see Figure 6 for an example). Note that $g_1 \neq g_2$ since the degree of every vertex in G is at least 7. Next, we define the *wedge* and the *wedge-neighbor* of f. Let h_1 be the edge of f that is opposite to v and let f_1 be the other face that is incident to h_1. If f_1 is not a 0-quadrilateral then it is the wedge-neighbor of f. Otherwise, let h_2 be the opposite edge to h_1 in f_1 and let f_2 be the other face that is incident to h_2. Again, if f_2 is not a 0-quadrilateral then it is the wedge-neighbor of f. If f_2 is a 0-quadrilateral then let h_3 be the opposite edge to h_2 in f_2 and let f_3 be the other face that is incident to h_3. In this case, it is not hard to see (similarly to our observation in Step 1) that f_3 cannot be a 0-quadrilateral, and so it will be the wedge-neighbor of f. Suppose that f_j is the wedge-neighbor of f. Then the *wedge* of f consists of f and $\bigcup_{i=1}^{j-1} f_j$.

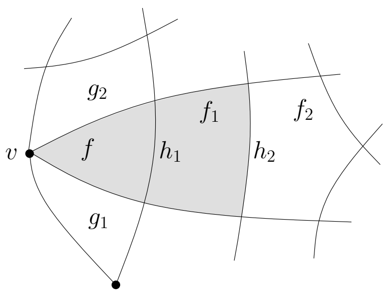


Figure 6: g_1 and g_2 are the neighbors of the 1-triangle f. f_2 is its wedge-neighbor and the wedge of f is $f \cup f_1$.

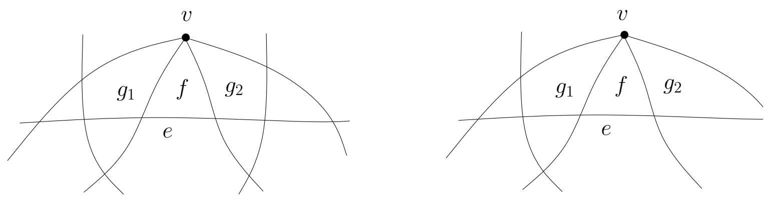


Figure 7: $ch_2(g_1), ch_2(g_2) \leq 0$ and g_1 is not a 1-triangle.

Proposition 2.5. *Let f be a* 1*-triangle and let g_1 and g_2 be its two neighbors. Then if $ch_2(g_1) \leq 0$ and $ch_2(g_2) \leq 0$ then g_1 and g_2 are* 1*-triangles.*

Proof. The neighbors of a 1-triangle must have at least one vertex of G on their boundary. Therefore, by Observation 2.4, if $ch_2(g_i) \leq 0$ then g_i is either a 1-triangle or a 1-quadrilateral, for $i = 1, 2$. Suppose without loss of generality that g_1 is a 1-quadrilateral and that $ch_2(g_i) \leq 0$ for $i = 1, 2$. Let v be the vertex of G that is incident to f and let e be the edge of G that contains the edge of f that is opposite to v in f. Then e must be crossed at least five times, see Figure 7. □

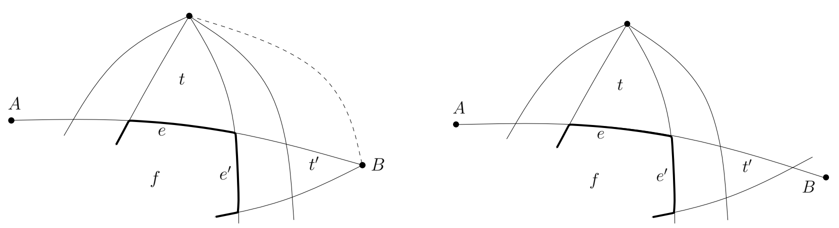
If the two neighbors of a 1-triangle have a nonpositive charge, and hence are 1-triangles by Proposition 2.5, then the 1-triangle obtains the missing charge from its wedge.

Step 3: Charging 1-triangles with poor neighbors. If f is a 1-triangle whose two neighbors are 1-triangles, then the wedge-neighbor of f contributes 1/3 units of charge to f through the edge of $M(G)$ that it shares with the wedge of f. ⇜⇝

Note that in Step 3, as in Step 1, charge is contributed only through edges whose both endpoints are crossing points. Moreover, a face cannot contribute through the same edge in Steps 1 and 3. Therefore, there if $ch_3(f) < 0$ for a face f, then f is either a 1-triangle or a 0-pentagon.

Proposition 2.6. *Let f be a face that contributes charge in Step 3 to a* 1*-triangle t through one of its edges e, such that e is an edge of t. Then f does not contributes charge in Step 1 or 3 through neither of its two edges that are incident to e.*

Proof. Let AB be the edge of G that contains e and let e' be an edge of f that is incident to e. Then AB contains four crossing points: the endpoints of e and two crossing points, one on each side of e on AB, since the neighbors of t must be 1-triangles. It is then impossible that f contributes charge through e' to a 1-triangle t' in Step 3, since one neighbor of such a triangle has to be a 2-triangle (see Figure 8(a)). It is also impossible that f contributes charge through e' to a 0-triangle t' in Step 1, since then AB has more than four crossings (see Figure 8(b)), or there is an empty lens. □



(a) f cannot contribute charge through e' in Step 3.

(b) f cannot contribute charge through e' in Step 1.

Figure 8: f contributes charge to t through e in Step 3 and e is an edge of t.

For 1-triangles with a negative charge after Step 3, the missing charge will come from either both neighbors or one neighbor and their wedge-neighbor. The next proposition shows that if the charge of one neighbor of such a 1-triangle is zero (implying that this neighbor is a 1-quadrilateral or a 1-triangle), then the other neighbor is able to contribute charge to the 1-triangle.

Proposition 2.7. *Let f be a 1-triangle and let g_1 and g_2 be its neighbors. If g_1 is a 1-triangle or a 1-quadrilateral such that $ch_3(g_1) = 0$ then g_2 is a 2-triangle.*

Proof. The claim clearly holds when g_1 is a 1-triangle, since the other neighbor of g_1 must also be a 1-triangle if $ch_3(g_1) = 0$. Therefore assume that g_1 is a 1-quadrilateral. We consider three cases based on $ch_2(g_1)$. If $ch_2(g_1) = 0$, then it is easy to see that g_2 is a 2-triangle, for otherwise there would be an edge of G that is crossed more than four times, or a bad face of $M(G)$ (see Figure 9(a)).

Suppose that $ch_2(g_1) = 1/3$. Let e and e' be the edges of G that contain the edges of g_1 that are not incident to v. Let t be the 1-triangle to which g_1 has contributed charge in Step 3 through an edge that is contained in e. If f is bounded by e, then e has four crossing points and the same arguments as in the previous case apply. Therefore, assume that f is bounded by e' and refer to Figure 9(b). Note first that it impossible that t and g_1 share an edge. Indeed, because the neighbors of t must be 1-triangles, this would imply that e is crossed more than four times (see Figure 9(b)). Therefore e' has four crossings and it follows, as above, that g_2 is a 2-triangle (see Figure 9(c)).

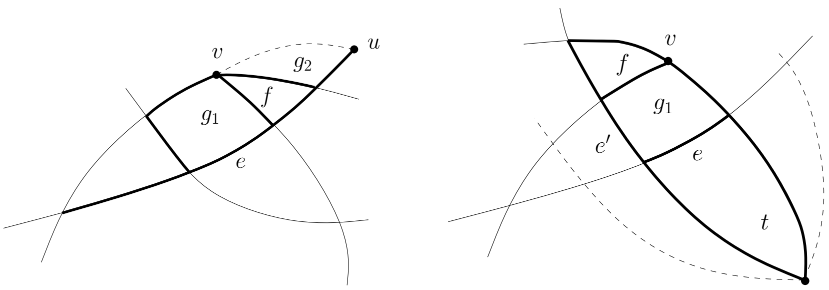
Suppose now that $ch_2(g_1) = 2/3$. Let t_1 and t_2 be the two 1-triangles to which g_1 has contributed charge in Step 3. By Proposition 2.6 neither t_1 nor t_2 share an edge of $M(G)$ with g_1. Let e be the edge of G that bounds f and g_1 (refer to Figure 9(d)). Then e has four crossings and therefore, as before, it follows that g_2 is a 2-triangle. □

Recall that after Step 3 the charge of every 1-triangle whose two neighbors have a non-positive charge (and hence are 1-triangles themselves) becomes zero.

Step 4: Charging 1-triangles with positive neighbors. Let f be a 1-triangle such that $ch_3(f) < 0$ and let g be a neighbor of f such that $ch_3(g) > 0$. Denote by g' the other neighbor of f. Then g contributes 1/6 units of charge to f through the edge of $M(G)$ that they share if: (1) g is not a 1-quadrilateral; (2) g is a 1-quadrilateral and $ch_3(g) \geq 2/3$; or (3) g is a 1-quadrilateral, $ch_3(g) = 1/3$, and g' is a 1-triangle or g' is a 1-quadrilateral whose charge after Step 3 is 1/3. ⇜⇝

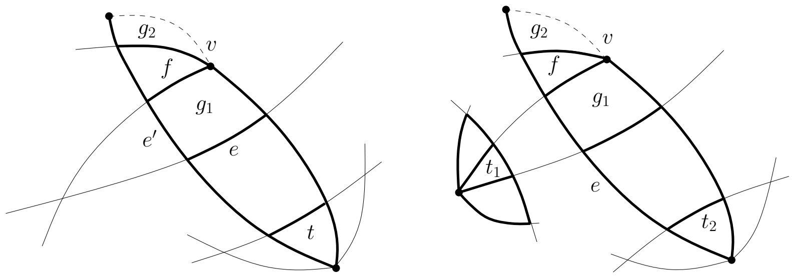
Proposition 2.8. *There is no face f such that $ch_3(f) \geq 0$ and $ch_4(f) < 0$.*

Proof. Clearly we only have to consider faces containing original vertices of G. If f is a 1-triangle, then $ch_3(f) \leq 0$ and so it cannot contribute charge in Step 4. If f is a 2-triangle, then $ch_3(f) = 1/3$ and it contributes to at most two 1-triangles in Step 4 and so $ch_4(f) \geq 0$.



(a) $ch_2(g_1) = 0$

(b) $ch_2(g_1) = 1/3$. If g_1 and t share an edge then e is crossed more than four times.



(c) $ch_2(g_1) = 1/3$ and g_1 and t do not share an edge.

(d) $ch_2(g_1) = 2/3$.

Figure 9: Illustrations for the proof of Proposition 2.7. If g_1 is a 1-quadrilateral such that $ch_3(g_1) = 0$, then the other neighbor of f is a 2-triangle.

If f is a 3-triangle then $ch_3(f) = 1$ and it does not contribute any charge in Step 4. If f is a 1-quadrilateral then it contributes to at most two 1-triangles only if $ch_3(f) \geq 1/3$ and therefore $ch_4(f) \geq 0$. If f is a face of size greater than four then it is easy to see that its charge remains positive. □

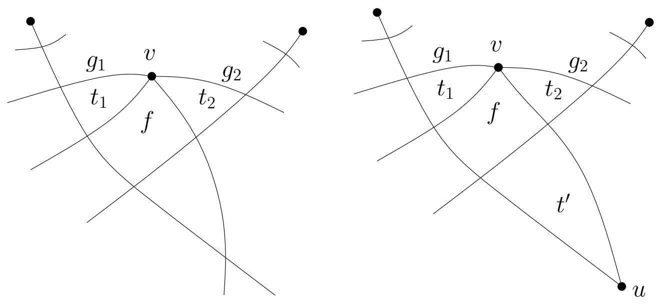
Proposition 2.9. *If f is a* 1*-triangle then $ch_4(f) \geq -1/6$.*

Proof. Let g_1 and g_2 be the neighbors of f. If $ch_4(f) < -1/6$ it means that f did not receive charge from neither g_1 nor g_2 in Step 4. The only faces containing an original vertex that have a non-positive charge after Step 3 are 1-triangles and 1-quadrilaterals. Suppose that g_1 is a 1-quadrilateral whose charge after Step 3 is zero. Then g_2 must be a 2-triangle by Proposition 2.7 and therefore contributes charge to f in Step 4. If g_1 is a 1-triangle then it cannot be that g_2 is also a 1-triangle because then after Step 3 we have $ch_3(f) = 0$. It is also impossible that g_2 is a 1-quadrilateral with a zero charge, by Proposition 2.7. Therefore, g_2 must contribute 1/6 units of charge to f in Step 4 in this case. □

Proposition 2.10. *If f is a* 1*-quadrilateral such that $ch_3(f) = 1/3$ then f contributes charge to at most one* 1*-triangle in Step 4.*

Proof. Suppose that f is a 1-quadrilateral such that $ch_3(f) = 1/3$ and f contributes charge to two 1-triangles t_1 and t_2 in Step 4. Let g_1 and g_2 be the other neighbors of t_1 and t_2, respectively. Note that according to Step 4, each of g_1 and g_2 must be either a 1-triangle or a 1-quadrilateral whose charge is 1/3 after Step 3. Observe also that it is impossible that $ch_2(f) = 1/3$, since this would imply an edge in G that is crossed more than four times, see Figure 10(a).

Therefore, assume that $ch_2(f) = 2/3$ and denote by t' the 1-triangle to which f has contributed charge in Step 3. However, t' must share an edge of $M(G)$ with f, and this implies that both of its neighbors are not 1-triangles (see Figure 10(b)). □



(a) If $ch_2(f) = 1/3$ then there is an edge with more than four crossings.

(b) If $ch_2(f) = 2/3$ then f does not contributes charge to t' in Step 3 since it is impossible that both neighbors of t' are 1-triangles.

Figure 10: f is a 1-quadrilateral such $ch_3(f) = 1/3$ that contributes charge to t_1 and t_2 in Step 4.

Step 5: Finish charging 1-triangles. Let f be a 1-triangle, let g' be the wedge-neighbor of f and let e' be the edge of $M(G)$ that is common to g' and the wedge of f. If $ch_4(f) < 0$ then g' contributes 1/6 units of charge to f through e'. ⇝

Observation 2.11. *Let f be a face and let e be an edge of f. Then f contributes charge through e at most once during the discharging steps 1–5.*

Proposition 2.12. *Let f be a face in $M(G)$. If $ch_5(f) < 0$ then f is a 0-pentagon.*

Proof. It follows from Proposition 2.9 and Step 5 that the charge of every 1-triangle is zero after the fifth discharging step. Suppose that f is a 1-quadrilateral. Then $ch_3(f)$ is either 0, 1/3, or 2/3. In the first case f does not contribute charge in Steps 4 and 5, and therefore $ch_5(f) = 0$. If $ch_3(f) = 2/3$ then clearly $ch_5(f) \geq 1/6$. If $ch_3(f) = 1/3$ then it follows from Proposition 2.10 that $ch_4(f) \geq 1/6$ and so $ch_5(f) \geq 0$. It is not hard to see, keeping Observation 2.11 in mind, that the charge of any other face but a 0-pentagon cannot be negative. □

Step 6: Charging 0-pentagons. Let f be a face such that $ch_5(f) > 0$ and let $B(f)$ be the set of 0-pentagons f' such that $ch_5(f') < 0$ and f' and f intersect at exactly one vertex of $M(G)$. If $B(f) \neq \emptyset$ then in the sixth discharging step f sends $ch_5(f)/|B(f)|$ units of charge to every 0-pentagon in $B(f)$ through their intersection point. ⇝

It follows from Proposition 2.12 and Step 6 that it remains to show that after the last discharging step the charge of every 0-pentagon is nonnegative. Note that a 0-pentagon can contribute either 1/3 or 1/6 units of charge (to a triangle) at most once through each of its edges. We first show, in Proposition 2.13, that the charge of a 0-pentagon after Step 1 is nonnegative and therefore is either 1, 2/3, 1/3 or 0. We then consider these cases separately in Lemmas 2.15, 2.16, 2.17 and 2.19, respectively.

First, we introduce some useful notations. Let f be a 0-pentagon. Let $e_0, \ldots, e_4$ be the edges on the boundary of f, listed in their clockwise cyclic order. The vertices of f are denoted by $v_0, \ldots, v_4$, such that v_i is incident to v_i and v_{i+1} (addition is done modulo 5). For every edge $e_i = v_i v_{i+1}$ of f we denote by $A_i B_i$ the edge of G that contains e_i, such that v_i is between A_i and v_{i+1} on $A_i B_i$. Denote by t_i the 1-triangle to which f sends charge through e_i, if such a triangle exists. Note that if t_i is a 1-triangle then one of its vertices is $A_{i-1} = B_{i+1}$. Its other vertices will be denoted by x_i and y_i such that x_i is contained in $A_{i-1}B_{i-1}$ and y_i is contained in $A_{i+1}B_{i+1}$. If t_i is a 0-triangle, then w_i denotes its vertex which is the

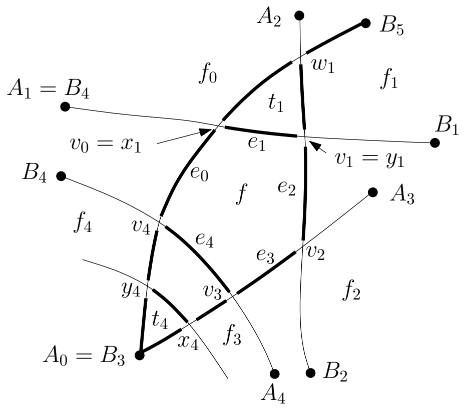


Figure 11: The notations used for vertices, edges, and faces near a 0-pentagon f. Bold edge-segments mark edges of $M(G)$.

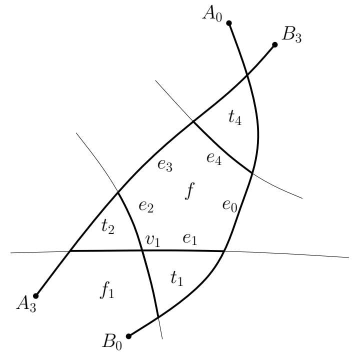


Figure 12: A 0-pentagon cannot contribute to three 0-triangles through non-consecutive edges.

crossing point of $A_{i-1}B_{i-1}$ and $A_{i+1}B_{i+1}$, and, as before, x_i and y_i denote its other vertices. Obviously, different notations might refer sometimes to the same point. Finally, we denote by f_i the face that is incident to v_i and is incident neither to e_i nor to e_{i+1}. That is, the intersection of f and f_i is exactly v_i, and thus $f \in B(f_i)$ if $ch_5(f) < 0$. See Figure 11 for an example of these notations. Note also that in all the figures bold edge-segments mark edges of $M(G)$.

Proposition 2.13. *Let f be a 0-pentagon. Then $ch_1(f) \geq 0$. Moreover, if $ch_1(f) = 0$ then f has contributed charge to three 0-triangles through three consecutive edges on its boundary.*

Proof. Suppose that f has contributed charge through three non-consecutive edges. Assume without loss of generality that these edges are e_1, e_2, e_4. Then the edges A_0B_0 and A_3B_3 have four crossings, and this implies that either f_1 is a bad face, or $A_3 = B_0$ and G has an empty lens (see Figure 12). □

Proposition 2.14. *Suppose that f is a 0-pentagon that contributes charge in Step 3 through e_i and e_{i+1}, for some $0 \leq i \leq 4$, such that the wedges of t_i and t_{i+1} each contain exactly one 0-quadrilateral. Then f_i contributes at least 1/3 units of charge to f in Step 6.*

Proof. Assume without loss of generality that $i = 1$ and refer to Figure 13. Let A_2y_1p be the neighbor of t_1 that shares an edge with f_1 and Let A_3x_2q be the neighbor of t_2 that shares an edge with f_1. Observe that $|f_1| \geq 5$ and that f_1 contributes at most 1/6 units of charge through y_1p and x_2q. Note also that f_1 contributes at most 1/6 units of charge

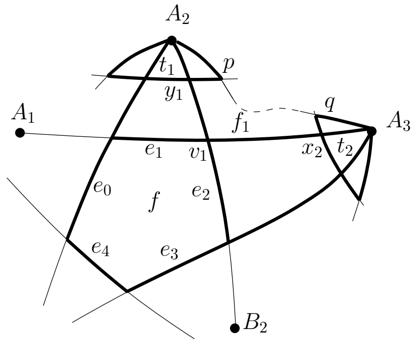


Figure 13: An illustration for the proof of Proposition 2.14: If f contributes charge to t_1 and t_2 in Step 3 and their wedges each contain exactly one 0-quadrilateral, then f_1 contributes at least 1/3 units of charge to f in Step 6.

through v_1x_2. Indeed, if it contributes charge through v_1x_2 in Step 1, then A_2B_2 would have more than four crossings, and if f_1 contributes charge through v_1x_2 in Step 3, then the wedge of t_2 would have two 0-quadrilaterals. Similarly, f_1 contributes at most 1/6 charge through v_1y_1. None of the vertices q, x_2, y_1, p can be the intersection of f_1 with a 0-pentagon. Note that if $|f_1| = 5$ then $B(f_1) = \{f\}$ and f_1 does not contribute charge through pq and so $ch_5(f_1) \geq 1/3$. Therefore f_1 sends 1/3 units of charge to f in Step 6 in this case.

If $|f_1| \geq 6$ then the clockwise chain from p to q contains $|f_1|-4$ edges and at most $|f_1|-5$ vertices through which f_1 might contribute charge in Step 6. Therefore every face in $B(f_1)$ receives from f_1 in Step 6 at least $\frac{|f_1|-4-4/6-(|f_1|-4)/3}{|f_1|-4} \geq 1/3$ units of charge. □

Lemma 2.15. *Let f be a 0-pentagon such $ch_1(f) = 1$. Then $ch_6(f) \geq 0$.*

Proof. Suppose that $ch_1(f) = 1$ and $ch_5(f) < 0$. Then f contributes charge either to exactly two or to at least three 1-triangles in Step 3. We consider each of these cases separately.
Case 1: $ch_3(f) = 1/3$ and $ch_5(f) = -1/6$. That is, f contributes 1/3 units of charge to two 1-triangles in Step 3 and contributes 1/6 units of charge to three 1-triangles in Step 5. We may assume without loss of generality that in Step 3 either f contributes charge to t_1 and t_2, or it contributes charge to t_1 and t_3.
Sub-case 1.1: f contributes charge to t_1 and t_2 in Step 3 and to t_3, t_4, t_0 in Step 5. Recall that by Proposition 2.6 e_1 cannot be an edge of t_1 and e_2 cannot be an edge of t_2.

Note that neither of the wedges of t_1 and t_2 contain two 0-quadrilaterals. Indeed, suppose that the wedge of t_2 contains two 0-quadrilaterals and refer to Figure 14. Then, since A_1A_3 has four crossings it follows that A_1v_0 is an edge in $M(G)$, and so $t_0 = A_1v_4v_0$. Similarly, $t_4 = A_0v_3v_4$. Because f_4 is a good face, it must be a 2-triangle, that is, $f_4 = A_1v_4A_0$. In Step 5 f sends charge to t_4, therefore the other neighbor of t_4, f_3, cannot be a 2-triangle, and so e_3 is not an edge of t_3. It follows that the wedge of t_1 contains exactly one 0-quadrilateral, and therefore the size of f_0 is at least four. f_0 may not contribute charge to t_0 in Step 4 only if $|f_0| = 4$ and $ch_3(f_0) \leq 1/3$. However, it is not hard to see that if $|f_0| = 4$ then $ch_3(f_0) = 2/3$. Therefore f_0 contributes charge to t_0 in Step 4 (as does f_4), and thus f does not contribute charge to t_0 in Step 5, a contradiction.

Therefore, each of the wedges of t_1 and t_2 contain exactly one 0-quadrilateral. Thus, by Proposition 2.14 the face f_1 contributes at least 1/3 units of charge to f in Step 6, and so $ch_6(f) \geq 0$.
Sub-case 1.2: f contributes charge to t_1 and t_3 in Step 3 and to t_2, t_4, t_0 in Step 5. Consider first the case that e_1 is an edge of t_1 and refer to Figure 15(a). Then the wedges of t_2

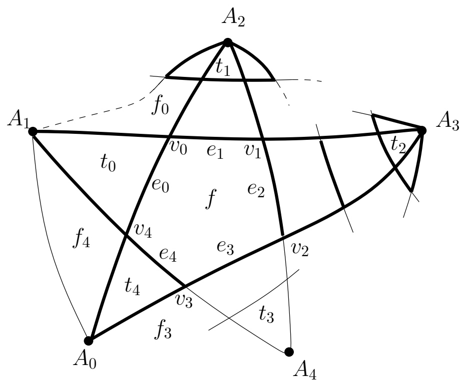
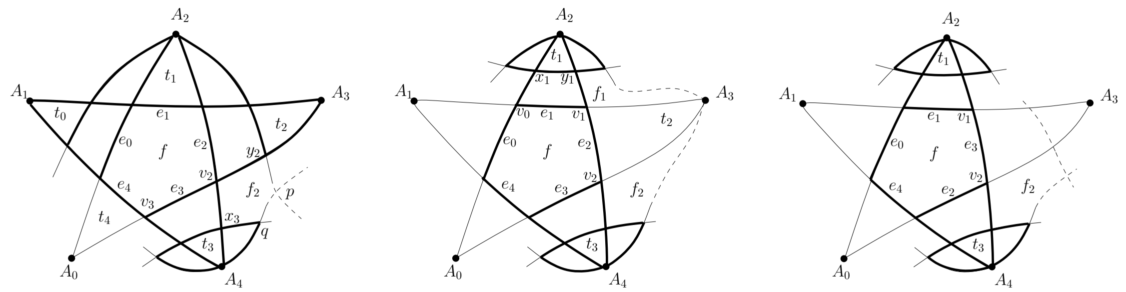


Figure 14: Sub-case 1.1 in the proof of Lemma 2.15. If the wedge of t_2 contains two 0-quadrilaterals then t_0 receives charge from both of its neighbors in Step 4, and no charge from f.



(a) e_1 is an edge of t_1. f_2 contributes charge to f in Step 6.

(b) If f_1 and f_2 are both 1-quadrilaterals then f does not contribute charge through e_2.

(c) If $|f_2| \geq 5$ then it contributes at least 1/6 units of charge to f in Step 6.

Figure 15: Sub-case 1.2 in the proof of Lemma 2.15: f contributes 1/3 units of charge in Step 3 through each of e_1 and e_3.

and t_0 contain one 1-quadrilateral each. It is impossible that $t_3 = A_4v_2v_3$ since its two neighbors should be 1-triangles and in this case if f_2 is a 1-triangle then there is an empty lens. Therefore the wedge of t_3 contains exactly one 0-quadrilateral (two 0-quadrilaterals imply more than four crossings on A_1A_4).

Let A_4x_3q be the 1-triangle that is a neighbor of t_3 and shares an edge with f_2. Consider the face f_2 and observe that $|f_2| > 4$ for if $|f_2| = 4$ then G has an empty lens. Suppose that $|f_2| = 5$ and let p be its fifth vertex (its other vertices are q, x_3, v_2, y_2). Refer to Figure 15(a) and note that it is not hard to see that $ch_5(f_2) \geq 1/6$ since the only edge through which f_2 might contribute 1/3 units of charge (in Step 3) is py_2, but in this case f_2 does not contribute charge through pq. Observe also that f_2 cannot intersect another 0-pentagon precisely at q, x_3 or y_2. It might intersect a 0-pentagon at p, but then it does not contribute any charge through pq and py_2, and then $ch_5(f_2) \geq 1/2$. Therefore in Step 6 f_2 contributes to f at least 1/6 units of charge and f ends up with a nonnegative charge.

If $|f_2| \geq 6$ then on the clockwise chain from y_2 to q there are $|f_2| - 3$ edges and at most $|f_2| - 4$ vertices through which f_2 contributes charge in Step 6. Since f_2 contributes at most 1/6 units of charge through v_2y_2, x_3q, and x_3v_2, it contributes at least $\frac{|f_2|-4-3/6-(|f_2|-3)/3}{|f_2|-3} \geq$ 1/6 units of charge to every face in $B(f_2)$ in Step 6.

The case that e_3 is an edge of t_3 is symmetric, therefore suppose now that e_1 is not an edge of t_1 and e_3 is not an edge of t_3 and refer to Figure 15(b). Observe that the wedges of t_1 and t_3 must contain exactly one 0-quadrilateral each, for otherwise A_2A_4 has more than four crossings. We claim that at least one of f_1 and f_2 is of size at least five. Indeed, suppose that both of them are of size four. Then it is impossible that both f_1 and f_2 are 0-quadrilateral, since this implies an empty lens. Assume without loss of generality that f_1 is not a 0-quadrilateral. If f_1 is a 1-quadrilateral, then A_3v_1 is an edge of $M(G)$, but then so is A_3v_2. This means that the two neighbors of $t_2 = A_3v_1v_2$ are 1-quadrilaterals, see Figure 15(b). Observe that these 1-quadrilaterals cannot contribute charge in Steps 1 and 3. Therefore, f_1 and f_2 contribute charge to t_2 in Step 4 and thus, f does not contribute charge to t_2 in Step 5.

Hence, we may assume without loss of generality that $|f_2| \geq 5$ (see Figure 15(c) for an example). It is not hard to see that, as in the case where we assumed that e_1 is an edge of t_1, f_2 sends at least 1/6 units of charge to f in Step 6.

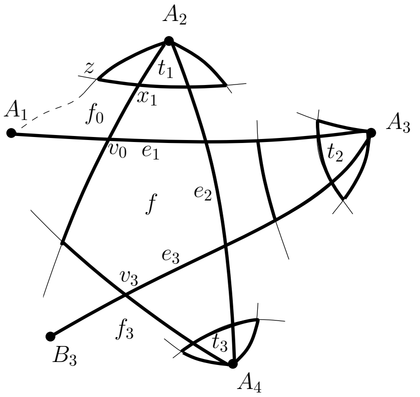
Case 2: $ch_3(f) \leq 0$ and $ch_5(f) < 0$. In this case f contributes 1/3 units of charge to at least three 1-triangles Step 3. By symmetry there are two sub-cases to consider.
Sub-case 2.1: f contributes charge through each of e_1, e_2, e_3 in Step 3. Observe that none of the 1-triangles t_1, t_2, t_3 can share an edge (of $M(G)$) with f according to Proposition 2.6. Moreover, the wedges of t_1 and t_3 must contain exactly one 0-quadrilateral, for otherwise A_2A_4 has more than four crossings.

If there is only one 1-quadrilateral in the wedge of t_2, then by Proposition 2.14 each of f_1 and f_2 contributes at least 1/3 units of charge to f in Step 6 and so $ch_6(f) \geq 0$.

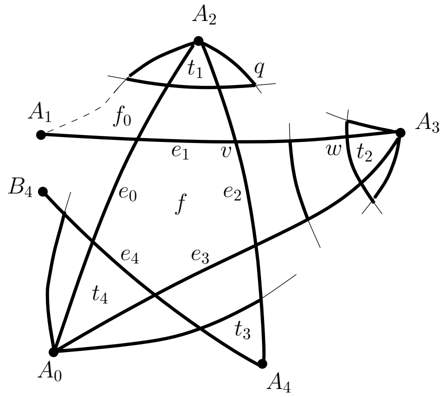
Therefore, assume that there are two 0-quadrilaterals in the wedge of t_2 and refer to Figure 16(a). It follows that A_1v_0 is an edge of f_0 and B_3v_3 is an edge of f_3. Consider f_0 and observe that $|f_0| \geq 4$. Note also that f_0 does not contribute any charge through x_1v_0, as this would imply that the edge of G that contains x_1y_1 has more than four crossings. Denote by z the other vertex (but v_0) that is adjacent to x_1 in f_0, and observe that f_0 contributes at most 1/6 units of charge through each of zx_1 and A_1v_0.

If f_0 is a 1-quadrilateral (as in Figure 16(a)), then it does not contribute charge through A_1z, and therefore $ch_5(f_0) \geq 1/3$. In this case $B(f_0) = \{f\}$ and so f_0 sends 1/3 units of charge to f is Step 6.

If $|f_0| \geq 5$ then consider the clockwise chain from A_1 to z, and observe that it contains



(a) Sub-case 2.1: f contributes charge to t_1, t_2, t_3 in Step 3. If there are two 0-quadrilateral in the wedge of t_2 then f_3 and f_0 each contribute at least 1/3 units of charge to f in Step 6.



(b) Sub-case 2.2: f contributes charge to t_1, t_2, t_4 in Step 3. If there are two 0-quadrilateral in the wedge of t_2 then f_0 contributes at least 1/3 units of charge to f in Step 6.

Figure 16: Illustrations for Case 2 in the proof of Lemma 2.15.

$|f_0|-3$ edges and at most $|f_0|-4$ vertices through which f_0 sends charge in Step 6. Therefore, every face in $B(f_0)$ (including f) receives from f_0 in Step 6 at least $\frac{|f_0|-4+1-2/6-1/3-(|f_0|-3)/3}{|f_0|-3} \geq$ 1/3 units of charge.

By symmetry f_3 also contributes at least 1/3 units of charge to f in Step 6 and therefore f ends up with a nonnegative charge.

Sub-case 2.2: In Step 3 f contributes charge through each of e_1, e_2, e_4, and does not contribute charge through e_3 and e_0 (otherwise we are back in Sub-case 2.1). Thus, $ch_5(f) \geq -1/3$. Observe that none of the 1-triangles t_1, t_2 can share an edge (of $M(G)$) with f according to Proposition 2.6. If the wedges of t_1 and t_2 each contain one 0-quadrilateral, then by Proposition 2.14 the face f_1 sends at least 1/3 units of charge to f in Step 6 and thus $ch_6(f) \geq 0$.

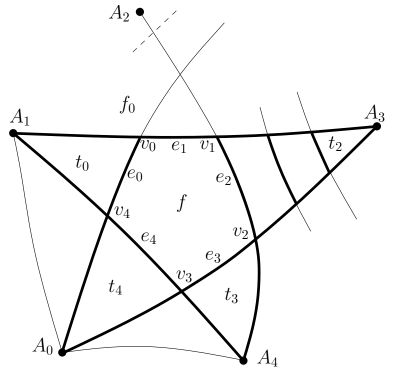
Thus, assume without loss of generality that the wedge of t_2 contains two 0-quadrilaterals and refer to Figure 16(b). Since A_3A_0 contains at most four crossings, e_4 must be an edge of t_4. It follows that $A_1 \neq B_4$, for otherwise, A_1A_3 would have more than four crossings.Therefore f does not contribute charge through e_0 and thus it must contribute charge through e_3. Hence, $A_4 = B_2$ and the wedge of t_1 must contain exactly one 0-quadrilateral. It is not hard to see, as in Sub-case 2.1, that f_0 contributes at least 1/3 units of charge to f in Step 6. □

Lemma 2.16. *Let f be a 0-pentagon such $ch_1(f) = 2/3$. Then $ch_6(f) \geq 0$.*

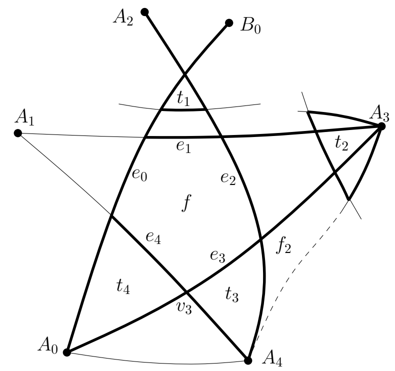
Proof. Assume without loss of generality that f contributes 1/3 units of charge in Step 1 to t_1 through e_1. There are two cases to consider, based on whether f contributes 1/3 units of charge to one or more 1-triangles in Step 3.

Case 1: $ch_3(f) = 1/3$ and $ch_5(f) = -1/6$. That is, f contributes 1/3 units of charge to exactly one 1-triangle t' in Step 3, and 1/6 units of charge to three 1-triangles in Step 5. Without loss of generality we may assume that either $t' = t_2$ or $t' = t_3$.

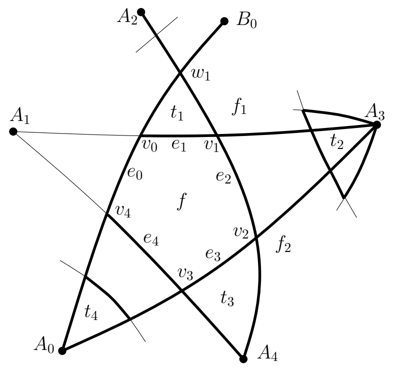
Sub-case 1.1: f sends 1/3 units of charge to t_2 in Step 3. We observe first that the wedge of t_2 cannot contain two 0-quadrilaterals. Indeed, suppose it does and refer to Figure 17(a). Since A_1A_3 and A_3A_0 have four crossings it follows that e_0 is an edge of t_0 and e_4 is an edge of t_4. Since f_0 is a good face, there must be two crossing points between A_2 and v_1 on A_2A_4. Therefore, e_3 is an edge of t_3. However, this implies that the two neighbors of t_4 are



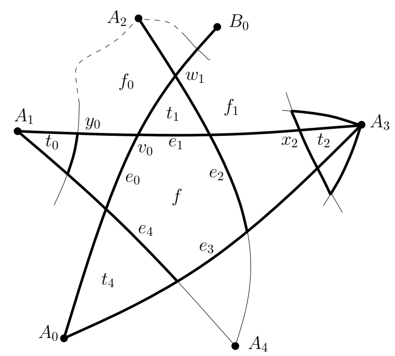
(a) If the wedge of t_2 contains two 0-quadrilaterals, then the two neighbors of t_4 are 2-triangles.



(b) If e_1 is not an edge of t_1, then t_3 receives 1/6 units of charge from each of its neighbors in Step 4.



(c) If A_2w_1 is not an edge in $M(G)$, then $t_3 = A_4v_2v_3$ and e_4 is not an edge of t_4. Thus the size of f_1 is at least five and it contains one original vertex.



(d) A_2w_1 is an edge in $M(G)$ and B_0w_1 is not.

Figure 17: Sub-case 1.1 in the proof of Lemma 2.16: f sends 1/3 units of charge to t_1 in Step 1 and 1/3 units of charge to t_2 in Step 3.

2-triangles, and therefore it will not receive any charge from f in Step 5. If the wedge of t_2 contains no 0-quadrilaterals, then either A_2A_4 has five crossings or there is an empty lens. Therefore, the wedge of t_2 must contain exactly one 0-quadrilateral.

Next, we observe that e_1 must be an edge of t_1. Indeed, suppose it is not and refer to 17(b). Since A_2A_4 and A_0B_0 have four crossings, e_3 is an edge of t_3 and e_4 is an edge of t_4. Therefore one neighbor of t_3 is a 2-triangle ($f_3 = A_4A_0v_3$). Consider f_2, the other neighbor of t_3. Then either f_2 is a 1-quadrilateral whose charge after Step 3 is 2/3 (see Figure 17(b)) or a face of size at least five. In any case, t_3 receives 1/6 units of charge from each of its neighbors in Step 4 and therefore does not receive any charge from f in Step 5.

Consider now the edge-segment A_2w_1. Suppose that it contains a crossing point between A_2 and w_1 and refer to Figure 17(c). It follows that e_3 is an edge of t_3. The face f_2 is a neighbor of t_3. Its size is at least five or it is a 1-quadrilateral whose charge after Step 3 is 2/3. Therefore, the other neighbor of t_3 cannot be a 2-triangle (otherwise f does not contribute charge to t_3), and therefore e_4 is not an edge of t_4. It follows that B_0w_1 is an edge of f_1. Note that the size of f_1 is at least five and it contains one original vertex. It is not hard to see that f_1 contributes in Step 6 at least 1/6 units of charge to every 0-pentagon in $B(f_1)$ (including f), and therefore f ends up with a nonnegative charge.

It remains to consider the case that A_2w_1 is an edge in $M(G)$. If B_0w_1 is an edge in $M(G)$ then f_1 is a face of size at least five that contains one original vertex. Hence, as before, it is not hard to see that it contributes in Step 6 at least 1/6 units of charge to f. Suppose that the edge-segment B_0w_1 is crossed and refer to Figure 17(d). It follows that e_4 is an edge of t_4. Since f_0 is a good face, e_0 is not an edge of t_0. Consider the face f_0 and observe that $|f_0| \geq 4$ and it contributes 1/3 units of charge through v_0w_1, and at most 1/6 units of charge through v_0y_0. Suppose that f_0 is a 1-quadrilateral. Then it does not contribute charge through A_2w_1 in Step 4, since $ch_3(f_0) = 1/3$ and if the face that shares A_2w_1 with f_0 is a 1-triangle, then the other neighbor of this 1-triangle is a 2-triangle. Note also that the face that shares A_2y_0 with f_0 is a 2-triangle, and therefore f_0 does not contribute charge through this edge as well. Thus, $ch_5(f_0) \geq 1/6$ and $B(f_0) = \{f\}$ and so f receives as least 1/6 units of charge from f_0 in Step 6.

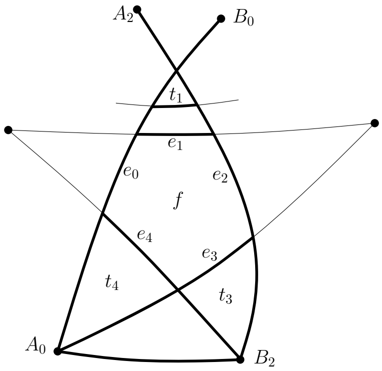
If $|f_0| \geq 5$ then f_0 might contribute 1/6 units of charge through A_2w_1. Consider the clockwise chain from y_0 to A_2, and observe that it contains $|f_0|-3$ edges and at most $|f_0|-4$ vertices through which f_0 sends charge in Step 6. Therefore, every face in $B(f_0)$ (including f) receives from f_0 in Step 6 at least $\frac{|f_0|-4+1-2/6-2/3-(|f_0|-3)/3}{|f_0|-3} \geq 1/6$ units of charge.

Sub-case 1.2: f sends 1/3 units of charge to t_3 in Step 3. We observe first that e_1 must be an edge of t_1. Indeed, suppose it does not and refer to Figure 18(a). Since each of A_2B_2 and A_0B_0 contain four crossings, e_3 is an edge of t_3 and e_4 is an edge of t_4. But then one neighbor of t_3 is a 2-triangle and therefore f could not have contributed charge to t_3 in Step 3.

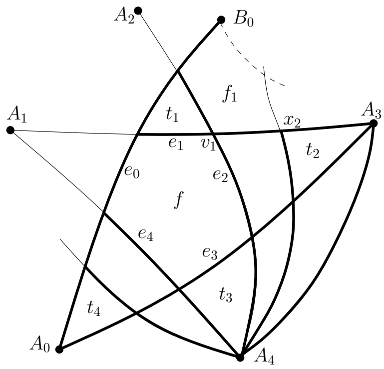
If e_3 is an edge of t_3, then it is easy to see that f ends up with a nonnegative charge. Indeed, refer to Figure 18(b) and observe that in this case one neighbor of t_2 is a 2-triangle which means that its other neighbor is either a 1-quadrilateral or a 1-triangle. This implies that the size of f_1 is at least five and it contains one original vertex. Thus, it is not hard to see that f_1 contributes at least 1/6 units of charge to f in Step 6.

Therefore, assume that e_3 is not an edge of t_3 and refer to Figure 18(c). Observe that e_0 is not an edge of t_0, since this would imply that f_0 is a bad face. Suppose that e_4 is an edge of t_4 and consider the face f_3. Let A_4y_3q be the 1-triangle that shares an edge with t_3 and f_3. Note that $|f_3| \geq 4$ and that $v(f_3) = 1$. Observe also that f_3 contributes at most 1/6 units of charge through A_0v_3, v_3y_3, and y_3q. If f_3 is a 1-quadrilateral then $ch_5(f_3) \geq 1/6$ since f_3 does not charge through A_0q (because the face sharing A_0q with f_3 is a 2-triangle). Since $B(f_3) = \{f\}$, f ends up with a nonnegative charge in this case.

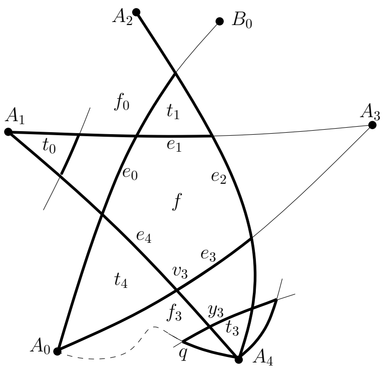
If $|f_3| \geq 5$ then consider the clockwise chain from q to A_0, and observe that it contains



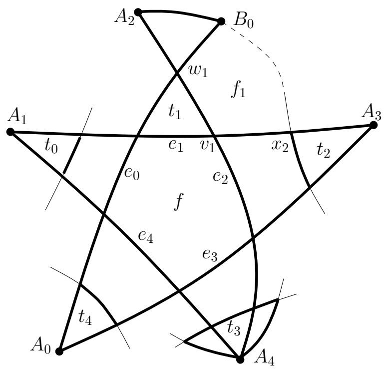
(a) If e_1 is not an edge of t_1 then one neighbor of t_3 is a 2-triangle.



(b) e_3 is an edge of t_3. f_1 sends charge to f in Step 6.

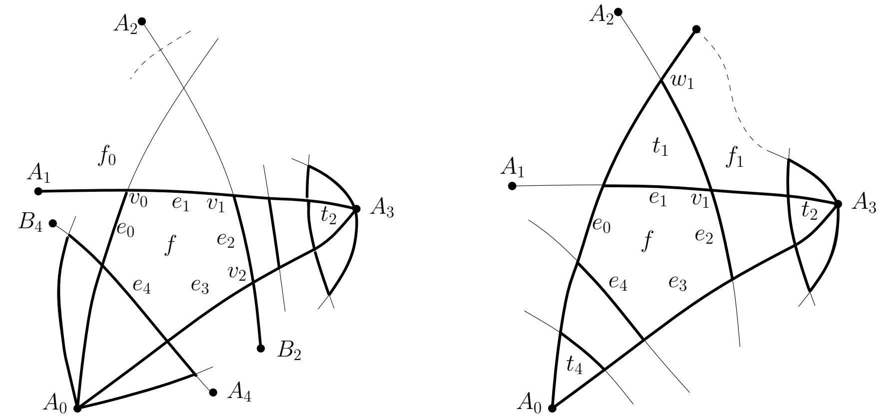


(c) e_3 is not an edge of t_3 and e_4 is an edge of t_4. f_1 sends charge to f in Step 6.



(d) e_3 is not an edge of t_3 and e_4 is not an edge of t_4. f_3 sends charge to f in Step 6.

Figure 18: Sub-case 1.2 in the proof of Lemma 2.16: f sends 1/3 units of charge to t_1 in Step 1 and 1/3 units of charge to t_3 in Step 3.



(a) If the wedge of t_2 contains two 1-quadrilaterals, then f does not contribute charge through e_3 and e_0.

(b) If e_4 is not an edge of t_4, then f' sends charge to f in Step 6.

Figure 19: Sub-case 2.1 in the proof of Lemma 2.16.

$|f_3|-3$ edges and at most $|f_3|-4$ vertices through which f_3 sends charge in Step 6. Therefore, every face in $B(f_3)$ (including f) receives from f_3 in Step 6 at least $\frac{|f_3|-4+1-3/6-1/3-(|f_3|-3)/3}{|f_3|-3} \geq$ 1/6 units of charge.

It remains to consider the case that e_4 is not an edge of t_4. Refer to Figure 18(d) and observe that e_2 cannot be an edge of t_2, because then f_1 would be a bad face. Similarly, e_0 is not an edge of t_0. Note that $|f_1| \geq 4$ and $v(f_1) = 1$. Observe that f_1 contributes 1/3 units of charge to B_0 and to t_1 and at most 1/6 units of charge through v_1x_2. Furthermore, f_1 does not contribute any charge through B_0w_1. If f_1 is a 1-quadrilateral (as in Figure 18(d)), then it also does not contribute any charge through B_0x_2, and therefore $ch_5(f_1) \geq 1/6$. We also have $B(f_1) = \{f\}$ in this case, and so f_1 sends at least 1/6 units of charge to f in Step 6.

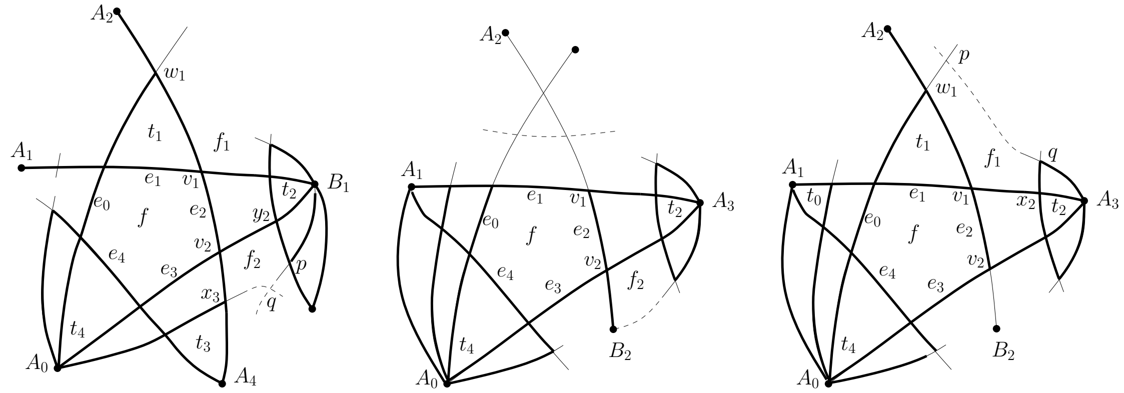
If $|f_1| \geq 5$ then consider the clockwise chain from B_0 to x_2, and observe that it contains $|f_1|-3$ edges and at most $|f_1|-4$ vertices through which f_1 sends charge in Step 6. Therefore, every face in $B(f_1)$ (including f) receives from f_1 in Step 6 at least $\frac{|f_1|-4+1-2/6-2/3-(|f_1|-3)/3}{|f_1|-3} \geq$ 1/6 units of charge.

Case 2: $ch_3(f) \leq 0$ and $ch_5(f) < 0$. That is, f contributes 1/3 units of charge to two 1-triangles in Step 3, and also sends charge to at least one more 1-triangle in Step 3 or Step 5. Recall that we assume without loss of generality that f sends 1/3 units of charge to t_1 in Step 1. Note that we may assume that $ch_5(f) \geq -1/3$, for otherwise f contributes charge to at least three 1-triangles in Step 3, and we have actually considered this scenario in Case 2 of Lemma 2.15.

If f sends charge to two 1-triangles in Step 3 through consecutive edges on its boundary, then by Proposition 2.14 it ends up with a nonnegative charge. Therefore, by symmetry, there are two remaining cases to consider.

Sub-case 2.1: f sends 1/3 units of charge to t_2 and t_4 in Step 3. Observe first that the wedge of t_2 must contain exactly one 1-quadrilateral. Indeed, if it contains no 1-quadrilateral (that is, e_2 is an edge of t_2) then the edge of G that contains e_2 has more than four crossings. Suppose that the wedge of t_2 contains two 1-quadrilaterals and refer to Figure 19(a). Then e_4 must be an edge of t_4. Since f_0 is a good face there is a crossing point between A_2 and v_1 on A_2B_2. Therefore B_2v_2 is an edge in $M(G)$ and it is impossible that $A_4 = B_2$ and f sends charge through e_3. Similarly, since A_1v_0 is an edge in $M(G)$, it is impossible that $A_1 = B_4$ and f sends charge through e_0. Therefore, $ch_5(f) \geq 0$, a contradiction.

Consider now the case that e_4 is not an edge of t_4, and refer to Figure 19(b). Notice



(a) If f contributes charge through e_3 then f_2 sends charge to f in Step 6.

(b) f sends charge through e_0 and there are two crossing points between A_2 and v_1. f_2 sends charge to f in Step 6.

(c) f sends charge through e_0 and there is one crossing point between A_2 and v_1. f_1 sends charge to f in Step 6.

Figure 20: Sub-case 2.1 in the proof of Lemma 2.16. e_4 is an edge of t_4.

that $|f_1| \geq 5$, $v(f) = 1$, and it is therefore not hard to see that f_1 sends at least 1/3 units of charge to f in Step 6 and so f ends up with a nonnegative charge.

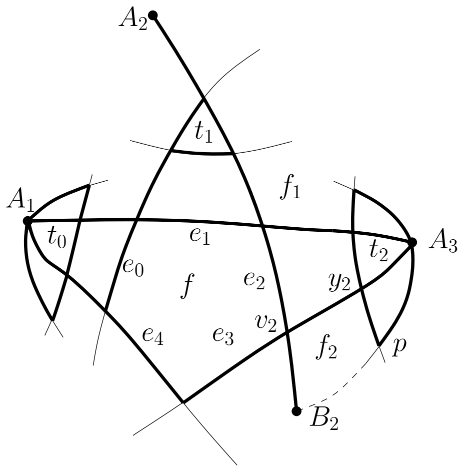
It remains to consider the case that e_4 is an edge of t_4. If $ch_5(f) < 0$ then f must have contributed charge through e_3 or e_0 in Step 5. Suppose that f sends 1/6 units of charge through e_3 in Step 5, and refer to Figure 20(a). Note that $|f_2| \geq 5$ and let x_3, v_2, y_2, p, q be (some of) its vertices listed in a clockwise order. Observe that f_2 contributes no charge through v_2y_2 and at most 1/6 units of charge through x_3v_2 and y_2p. If $|f_2| = 5$, then it might contribute at most 1/6 units of charge through qx_3 and pq. However, if f_2 contributes through one of these edges, then it does not contribute charge through q in Step 6. Therefore, f_2 sends at least 1/3 units of charge to f in Step 6.

If $|f_2| \geq 6$ then consider the clockwise chain from p to x_3, and observe that it contains $|f_2| - 3$ edges and at most $|f_2| - 4$ vertices through which f_2 sends charge in Step 6. However, if f_2 contributes charge through pq then it does not contribute charge through q in Step 6. Therefore, every face in $B(f_2)$ (including f) receives from f_2 in Step 6 at least $\min\left\{\frac{|f_2|-4-2/6-(|f_2|-3)/3}{|f_2|-4}, \frac{|f_2|-4-2/6-(|f_2|-4)/3}{|f_2|-3}\right\} \geq 1/3$ units of charge.

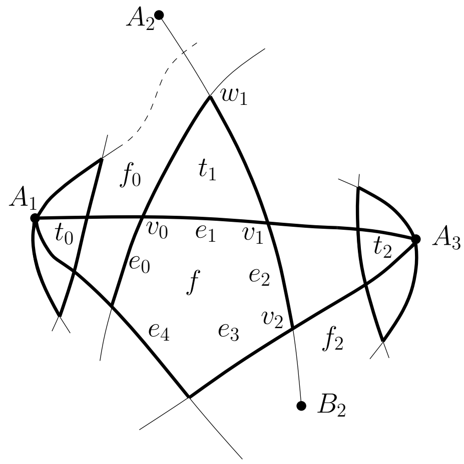
Finally, suppose that f does not contribute (at least 1/6 units of) charge through e_3 and sends 1/6 units of charge through e_0 in Step 5. If there are two crossing points between A_2 and v_1 on A_2B_2, then $v(f_2) = 1$ and $|f_2| \geq 4$, and it is not hard to see that f_2 contributes at least 1/3 units of charge to f in Step 6 (see Figure 20(b) for an example).

Otherwise, w_1 is the only crossing point between A_2 and v_1 on A_2B_2. Consider the face f_1 and note that its size is at least five. Let p be the other vertex of f_1 that is adjacent to w_1 but v_1, and let q be the other vertex of f_1 that is adjacent to x_2 but v_1. Refer to Figure 20(c) and observe that f_1 sends 1/3 units of charge through w_1v_1 and at most 1/6 units of charge through qx_2, x_2v_1, and w_1p. If $|f_1| = 5$ then f_2 cannot send charge through both pq and w_1p (because then the 1-triangle that gets the charge through pq has two 2-triangles for neighbors). Therefore $ch_5(f_1) \geq 1/6$. If $ch_5(f_1) = 1/6$ then $B(f_1) = \{f\}$. If $B(f_1)$ contains another face then this face must intersect f_1 exactly at p, but in this case f_1 does not contribute charge through pq and w_1p and so $ch_5(f_1) \geq 1/3$. Therefore, if $|f_1| = 5$ then f_1 sends at least 1/6 units of charge to f in Step 6.

If $|f_1| \geq 5$ then consider the clockwise chain from w_1 to q, and observe that it contains $|f_1| - 3$ edges and at most $|f_1| - 4$ vertices through which f_1 sends charge in Step 6. Recall that f_1 contributes at most 1/6 units of charge through w_1p. Therefore, every face in $B(f_1)$



(a) If e_1 is not an edge of t_1, then f_2 sends charge to f in Step 6.



(b) If e_1 is an edge of t_1 then f_0 or f_2 send charge to f in Step 6.

Figure 21: Sub-case 2.2 in the proof of Lemma 2.16.

(including f) receives from f_1 in Step 6 at least $\frac{|f_1|-4-3/6-1/3-(|f_1|-4)/3}{|f_1|-3} \geq 1/6$ units of charge. Note that one neighbor of t_0 is a 2-triangle, and therefore $ch_5(f) \geq -1/6$. Thus, f ends up with a nonnegative charge.

Sub-case 2.2: f sends $1/3$ units of charge to t_2 and t_0 in Step 3. Observe first that it is impossible for e_2 to be an edge of t_2, because then A_2B_2 has more than four crossings. Similarly, e_0 cannot be an edge of t_0. It follows that the wedges of t_2 and t_0 each contain exactly one 1-quadrilateral.

Suppose first that e_1 is not an edge of t_1 and refer to Figure 21(a). Consider the face f_2 and observe that its size is at least four and it contains one original vertex, B_2. Let A_3y_2p be the 1-triangle that is a neighbor of t_2 and shares an edge with f_2. Notice that f_2 contributes no charge through v_2y_2, at most $1/6$ units of charge through B_2v_2 and y_2p, and no charge through B_2p if it is an edge of f_2. It is therefore not hard to see that f_2 sends at least $1/3$ units of charge to f in Step 6.

Suppose now that e_1 is an edge of t_1, and refer to Figure 21(b). Consider the face f_0 and observe that its size is at least five. If A_2 is a vertex of f_0 then it is easy to see that this face contributes at least $1/3$ units of charge to f in Step 6. Otherwise, there must be a crossing point between A_2 and w_1 on A_2B_b and thus B_2v_2 is an edge of f_2 (see Figure 21(b)). In this case, as before, f_2 contributes at least $1/3$ units of charge to f in Step 6. □

Lemma 2.17. *Let f be a 0-pentagon such $ch_1(f) = 1/3$. Then $ch_6(f) \geq 0$.*

Proof. Suppose that $ch_1(f) = 1/3$ and $ch_5(f) < 0$. Assume without loss of generality that f contributes $1/3$ units of charge in Step 1 to t_1 through e_1. By symmetry, there are two cases to consider, according to whether the other edge through which f sends charge in Step 1 is e_2 or e_3.

Case 1: f contributes charge through e_2 in Step 1. Observe first that if f sends charge through e_4, then e_4 must be an edge of t_4. Indeed, suppose that f sends charge through e_4, and e_4 is not an edge of t_4. Then A_3B_3 and A_0B_0 contain four crossings each and it follows that f_1 is a bad face, see Figure 22.

Proposition 2.18. *If f sends charge through e_4, then f receives at least $1/3$ units of charge in Step 6.*

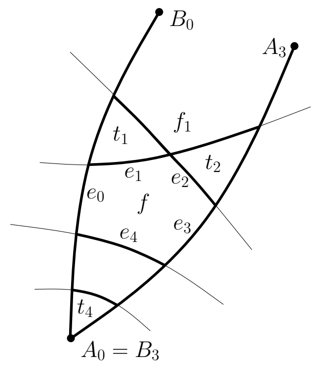
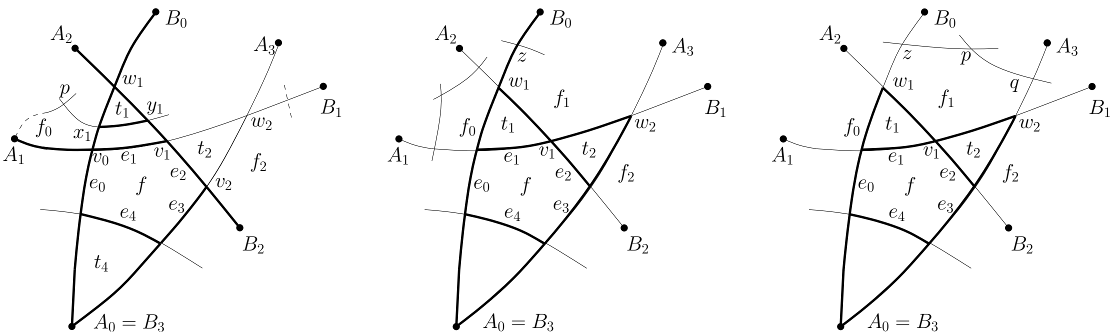


Figure 22: Case 1 in the proof of Lemma 2.17: f contributes charge through e_1 and e_2 in Step 1. If e_4 is not an edge of t_4, then f_1 is a bad face.



(a) If e_1 is not an edge of t_1 then f_0 sends at least 1/3 units of charge to f in Step 6.

(b) If f_0 is a 0-pentagon then f_2 is a bad face.

(c) f_1 is a 0-hexagon.

Figure 23: Illustrations for the proof of Proposition 2.18: f contributes charge through e_1 and e_2 in Step 1.

Proof. Suppose first that e_1 is not an edge of t_1 and refer to Figure 23(a). Since A_2B_2 has four crossings and f_2 is a good face, it follows that there is another crossing point (but w_2) on A_1B_1 between v_1 and B_1. Therefore A_1v_0 is an edge of f_0. Since A_1 and A_2 cannot be in the same face of size greater than three, it follows that $|f_0| \geq 4$. Let p be the other vertex of f_0 (except v_0) that is adjacent to x_1. Observe that $p \notin B(f_0)$ for otherwise B_0 and one endpoint of the edge of G that contains x_1y_1 are incident to a bad face. Note also that f_0 does not contribute charge through x_1p and x_1v_0 (the latter would imply a bad face containing A_3 and B_0). Therefore, if $|f_0| = 4$ then $ch_5(f_0) \geq 1/3$. Thus, if f_0 is a 1-quadrilateral then it sends at least 1/3 units of charge to f in Step 6.

If $|f_0| \geq 5$ then consider the clockwise chain from A_1 to p, and observe that it contains $|f_0|-3$ edges and at most $|f_0|-4$ vertices through which f_0 sends charge in Step 6. Therefore, every face in $B(f_0)$ (including f) receives from f_0 in Step 6 at least $\frac{|f_0|-4+1-1/6-1/3-(|f_0|-3)/3}{|f_0|-3} \geq$ 1/3 units of charge.

The case that e_2 is not an edge of t_2 is symmetric, so suppose now that e_1 is an edge of t_1 and e_2 is an edge of t_2. Since f_1 is a good face there is a crossing point on B_0w_1 or on A_3w_2. Suppose without loss of generality that there is a crossing point z on B_0w_1. Therefore, zw_1, w_1v_1, v_1w_2 are edges of f_1 and $|f_1| \geq 5$. Note that $f_0 \notin B(f_1)$ for otherwise f_2 is a bad face (see Figure 23(b)). Similarly, $f_2 \notin B(f_1)$ for otherwise f_0 is a bad face.

If $|f_1| = 5$ then A_3 must be a vertex of f_1, for otherwise B_0 and A_3 are two vertices of a bad face. By the previous observations $B(f_1) = \{f\}$ in this case, and so f receives at least

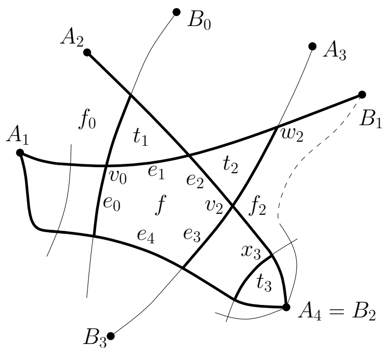


Figure 24: Case 1 in the proof of Lemma 2.17: f contributes charge through e_1 and e_2 in Step 1. If f sends 1/3 units of charge through e_3 in Step 3, then f_2 sends charge to f in Step 6.

1/3 units of charge from f_1 in Step 6. If A_3 is a vertex of f_1 and $|f_1| > 5$ it is not hard to see that it still holds that f_1 sends at least 1/3 units of charge to f in Step 6.

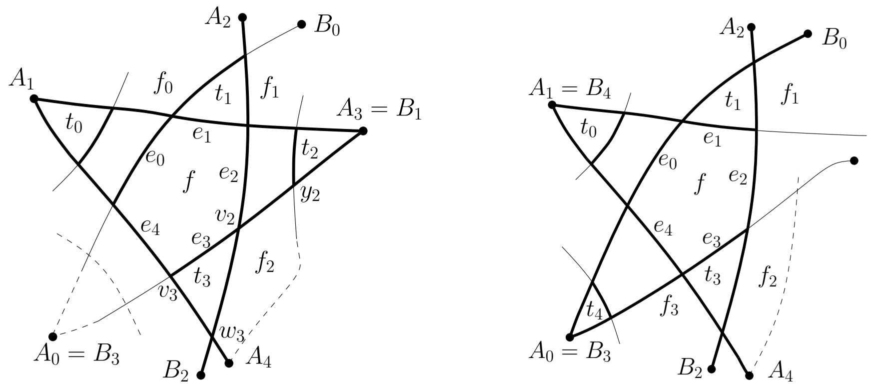
Suppose that A_3 is not a vertex of f_1, and therefore $|f_1| \geq 6$. Let $w_2, v_1, w_1, z, z_1, \ldots, z_t$ be the vertices of f_1 listed in their clockwise order ($t \geq 2$). f_1 cannot contribute charge through zz_1 in Step 1, since then A_0B_0 would have more than four crossings. If f_1 contributes charge through zz_1 in Step 3, then it follows from Proposition 2.6 that it cannot contribute charge through w_1z in Step 1 or Step 3. Therefore f_1 contributes a total of at most 1/2 units of charge through w_1z and zz_1. Similarly, it contributes a total of at most 1/2 units of charge through $z_{t-1}z_t$ and z_tw_2. Note also that if f_1 contributes charge through z_1 in Step 6, then it does not contribute charge through zz_1 for this would imply more than four crossings on A_0B_0. Moreover, if $|f_1| = 6$ (i.e., $t = 2$), then by symmetry f_1 does not contribute charge through z_1z_2 as well.

Therefore, if $|f_1| = 6$ then f_1 contributes at least $\min\left\{\frac{6-4-2/3-2\cdot\frac{1}{2}}{1}, \frac{6-4-4/3}{2}\right\} = 1/3$ to f in Step 6, and if $|f_1| \geq 7$ then f_1 contributes at least $\min\left\{\frac{|f_1|-4-2\cdot\frac{1}{2}-\frac{|f_1|-4}{3}}{|f_1|-5}, \frac{|f_1|-4-\frac{|f_1|-1}{3}}{|f_1|-4}\right\} \geq$ 1/3 to f in Step 6. □

Suppose now that f sends charge through e_4 to t_4. It follows from Proposition 2.18 that if f sends 1/6 units of charge through e_4 and 1/6 units of charge through at least one of e_3 and e_0, then $ch_6(f) \geq 0$. By Proposition 2.6, if f sends 1/3 units of charge through e_4 in Step 3, then it cannot send charge through e_3 or e_0 in Step 3, and hence f ends up with a nonnegative charge in this case as well.

Therefore it remains to consider the case that f sends 1/3 units of charge through at least one of the two edges e_3 and e_0, and at least 1/6 units of charge through the other edge among the two. Assume without loss of that f sends 1/3 units of charge through e_3 in Step 3, and at least 1/6 units of charge through e_0. Refer to Figure 24 and observe that e_3 is not an edge of t_3 by Proposition 2.6. It follows that there must be a crossing point on A_1v_0, for otherwise f_0 is a bad face. Therefore, x_3, w_2, B_1 are vertices of f_2 and its size is at least five. It is not hard to see that f_2 contributes at least 1/3 units of charge to f in Step 6, and so f ends up with a nonnegative charge.

Case 2: f contributes charge through e_3 in Step 1. Note that since A_2B_2 has four crossings it follows that e_1 and e_3 are edges of t_1 and t_3, respectively. By symmetry, we may assume that if $ch_5(f) < 0$ then f contributes charge through e_2 and e_0, or f contributes charge through e_4 and e_0.



(a) Sub-case 2.1: f contributes charge through e_2 and e_0 in Steps 3 and 5. f_2 contributes at least 1/3 units of charge to f in Step 6.

(b) Sub-case 2.2: f contributes charge through e_4 and e_0 in Steps 3 and 5. f_2 contributes at least 1/3 units of charge to f in Step 6.

Figure 25: Illustrations for Case 2 in the proof of Lemma 2.17: f contributes charge through e_1 and e_3 in Step 1.

<u>Sub-case 2.1:</u> Suppose that f contributes charge through e_2 and e_0, and refer to Figure 25(a). Observe that e_0 is not an edge of t_0, for otherwise f_0 would be a bad face. Since A_4B_4 has four crossings, it follows that A_4 is a vertex of f_2. Therefore, e_2 is not an edge of t_2, for otherwise f_2 would be a bad face.

Consider the face f_2 and observe that $|f_2| \geq 4$ and that f_2 contributes no charge through A_4w_3, 1/3 units of charge through w_3v_2, and no charge through v_2y_2 (the latter would imply that B_1 and B_0 are incident to a bad face). Note that if $|f_2| = 4$, then f_2 does not contribute any charge through A_4y_2 since the face that shares A_4y_2 with f_2 is a 2-triangle. Therefore, if $|f_2| = 4$ then f_2 contributes at least 1/3 units of charge to f in Step 6. If $|f_2| \geq 5$ then the clockwise chain from y_2 to A_4 contains $|f_2| - 3$ edges and at most $|f_2| - 4$ vertices through which f_2 sends charge in Step 6. Therefore, every face in $B(f_2)$ (including f) receives from f_2 in Step 6 at least $\frac{|f_2|-4+1-2/3-(|f_2|-3)/3}{|f_2|-3} \geq 1/3$ units of charge. Thus, if f does not contribute charge through e_4, then it ends with a nonnegative charge.

Suppose that f contributes charge through e_4. Then by symmetry, f_1 also contributes at least 1/3 units of charge to f in Step 6, and so $ch_6(f) \geq 0$.

<u>Sub-case 2.2:</u> Suppose that f contributes charge through e_4 and e_0, and refer to Figure 25(b). Note that e_4 is not an edge of t_4, for otherwise f_3 is a bad face. For the same reason, e_0 is not an edge of t_0. Considering the face f_2 it is not hard to see that as in Sub-case 2.1 it contributes at least 1/3 units of charge to f in Step 6. By symmetry, so does f_1 and therefore f ends up with a nonnegative charge. □

Lemma 2.19. *Let f be a 0-pentagon such $ch_1(f) = 0$. Then $ch_6(f) = 0$.*

Proof. It follows from Proposition 2.13 that f contributes charge through three consecutive edges in Step 1. If $ch_6(f) < 0$ then f must also contribute charge through at least one more edge in Step 3 or Step 5. Assume without loss of generality that f contributes charge through e_1, e_2, e_3 in Step 1 and through e_0 in Step 3 or 5, and refer to Figure 26. Observe that A_2B_2 has four crossings. Since f_0 is good face, e_0 is not an edge of t_0. However, this implies that B_1 and A_4 are vertices of f_2 and hence f_2 is a bad face. □

It follows from Proposition 2.12 and Lemmas 2.15, 2.16, 2.17 and 2.19 that the final charge of every face in $M(G)$ is nonnegative. Recall that the charge of every original vertex of G is

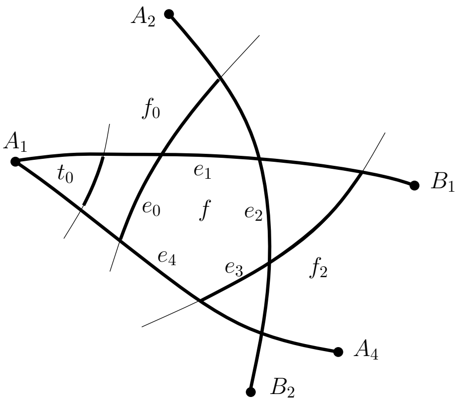
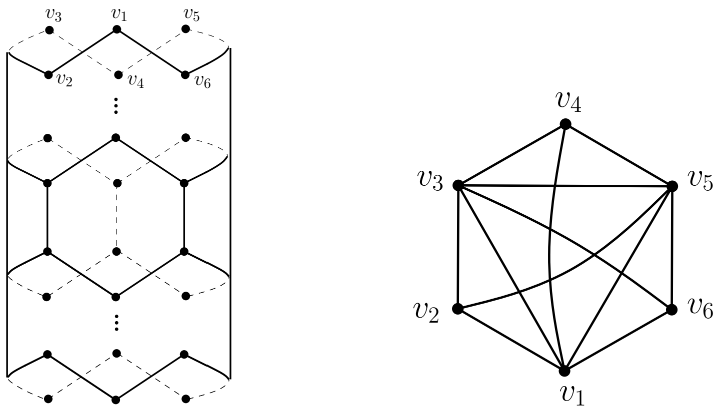


Figure 26: If f contributes charge through e_1, e_2, e_3 in Step 1 and also contributes charge to t_0, then f_2 is a bad face.



(a) Tiling a vertical cylinder surface with horizontal layers each consisting of three hexagons. The top and bottom are also tiled with hexagons.

(b) Drawing edges in the top face to get an almost tight lower bound for Theorem 4.

Figure 27: A lower bound construction.

1/3, and that the total charge is $4n - 8$. It follows that $2|E(G)|/3 = \sum_{v \in V(G)} \deg(v)/3 \leq 4n - 8$ and thus $|E(G)| \leq 6n - 12$.

To see that this bound in Theorem 8 is tight for infinitely many values of n we use the same construction of Pach et al. [19, Proposition 2.8]. That is, given $n = 6l$ we tile a vertical cylindrical surface with $l - 1$ horizontal layers each consisting of three hexagonal faces that are wrapped around the cylinder. The top and bottom of the cylinder are also tiled with hexagonal faces. See Figure 27(a) for an illustration of this construction. Note that every vertex is adjacent to exactly three hexagons, except for three vertices of the top face (v_1, v_3, v_5 in Figure 27(a)) and three vertices of the bottom face that are adjacent to two hexagons. Next, we draw for each hexagon all the possible diagonals. Thus, the degree of every vertex is 12, except for six vertices whose degree is 8. Hence, the number of edges is $(12(n-6) + 8 \cdot 6)/2 = 6n - 12$.

Observe that this construction contains parallel edges (but no empty lenses). For example, there are parallel edges between v_2 and v_4, v_2 and v_6, and v_4 and v_6 in Figure 27(a). By removing three edges from each of the top and bottom hexagons (as in Figure 27(b)), we obtain a topological graph with $6n-18$ edges. This shows that the bound of Theorem 4 is tight up to an additive constant.

3 Applications of Theorem 4

3.1 A better Crossing Lemma

Let G be a graph with $n > 2$ vertices and m edges. The following linear bounds on the crossing number $\mathrm{cr}(G)$ appear in [19] and [20].

$$\begin{aligned} \mathrm{cr}(G) &\geq m - 3(n-2) && (1)\\ \mathrm{cr}(G) &\geq \frac{7}{3}m - \frac{25}{3}(n-2) && (2)\\ \mathrm{cr}(G) &\geq 4m - \frac{103}{6}(n-2) && (3)\\ \mathrm{cr}(G) &\geq 5m - 25(n-2) && (4) \end{aligned}$$

Using Theorem 4 we can obtain a similar bound, as stated in Theorem 5:

$$\mathrm{cr}(G) \geq 5m - \frac{139}{6}(n-2) \quad (5)$$

Proof of Theorem 5: If $n = 3$ or $n = 4$ the statement trivially holds since $\mathrm{cr}(G) \geq 0$. If $n \geq 5$ and $m \leq 6(n-2)$ then the statement holds by (3). Suppose now that $m > 6(n-2)$ and consider a drawing of G. Remove an edge of G with the most crossings, and continue doing so as long as the number of remaining edges is greater than $6(n-2)$. It follows from Theorem 4 that each of the $m-6(n-2)$ removed edges was crossed by at least 5 other edges at the moment of its revomal. By (3), the number of crossings in the remaining graph is at least $4(6(n-2)) - \frac{103}{6}(n-2)$. Therefore, $\mathrm{cr}(G) \geq 5(m-6(n-2)) + 4(6(n-2)) - \frac{103}{6}(n-2) = 5m - \frac{139}{6}(n-2)$. □

Using the new linear bound it is now possible to obtain a better Crossing Lemma, by plugging it into its probabilistic proof, as in [17, 19, 20].

Proof of Theorem 6: Let G be a graph with n vertices and $m \geq 6.95n$ edges and consider a drawing of G with $\mathrm{cr}(G)$ crossings. Construct a random subgraph of G by selecting every vertex independently with probability $p = 6.95n/e \leq 1$. Let G' be the subgraph of G that is induced by the selected vertices. Denote by n' and m' the number of vertices and edges in G', respectively. Clearly, $\mathbb{E}[n'] = pn$ and $\mathbb{E}[m'] = p^2e$. Denote by x' the number of crossing in the drawing of G' inherited from the drawing of G. Then $\mathbb{E}[\mathrm{cr}(G')] \geq \mathbb{E}[x'] = p^4\mathrm{cr}(G)$. It follows from Theorem 5 that $\mathrm{cr}(G') \geq 5m' - \frac{139}{6}n'$ (note that this it true for any $n' \geq 0$), and this holds also for the expected values: $\mathbb{E}[\mathrm{cr}(G')] \geq 5\mathbb{E}[m'] - \frac{139}{6}\mathbb{E}[n']$. Plugging in the expected values we get that $\mathrm{cr}(G) \geq \left(\frac{5}{6.95^2} - \frac{139}{6 \cdot 6.95^3}\right)\frac{m^3}{n^2} = \frac{2000}{57963}\frac{m^3}{n^2} \geq \frac{1}{29}\frac{m^3}{n^2}$.

Consider now the case that $m < 6.95n$. Comparing the bounds (1)–(5) one can easily see that (1) is best when $3(n-2) \leq m < 4(n-2)$, (2) is best when $4(n-2) \leq m < 5.3(n-2)$, (3) is best when $5.3(n-2) \leq m < 6(n-2)$, and (5) is best when $6(n-2) \leq m$. If we consider the possible values of $m < 6.95$ according to these intervals and use the best bound for each interval, then we get that $\mathrm{cr}(G) \geq \frac{1}{29}\frac{m^3}{n^2} - \frac{35}{29}n$. □

The new bound for the Crossing Lemma immediately implies better bounds in all of its applications. We recall three such improvements from [19] and [20]. Since the computations are almost verbatim to the proofs in [19], we omit them.

Corollary 3.1. *Let G be an n-vertex multigraph with m edges and edge multiplicity t. Then* $cr(G) \geq \frac{1}{29}\frac{m^3}{mn^2} - \frac{35}{29}nt^2$.

Corollary 3.2. *Let G be an n-vertex simple topological graph. If every edge of G is crossed by at most k other edges, for some $k \geq 2$, then G has at most $3.81\sqrt{k}n$ edges.*

Corollary 3.3. *The number of incidences between m lines and n points in the Euclidean plane is at most $2.44m^{2/3}n^{2/3} + m + n$.*

The previous best constant in the last upper bound was 2.5. It is known [20] that this constant should be greater than 0.42.

3.2 Albertson conjecture

Recall that according to Albertson conjecture if $\chi(G) = r$ then $\mathrm{cr}(G) \geq \mathrm{cr}(K_r)$. A graph G is *r-critical* if $\chi(G) = r$ and the chromatic number of every proper subgraph of G is less than r. Obviously, if H is a subgraph of G then $\mathrm{cr}(H) \leq \mathrm{cr}(G)$. therefore, it is enough to prove Albertson conjecture for r-critical graphs. Recall also that it suffice to consider graphs with no subdivision of K_r. The next result shows that we may consider only graphs with at least $r + 5$ vertices.

Lemma 3.4 ([8, Corollary 11])**.** *An r-critical graph with at most $r + 4$ vertices contains a subdivision of K_r (and thus satisfies Albertson conjecture).*

The approach of [7] and [8] for proving Albetson conjecture is to plug lower bounds on the minimum number of edges in r-critical graphs into lower bounds on the crossing number and compare the results to an upper bound on $\mathrm{cr}(K_r)$. By using the same method with the new bounds on the crossing number, we can verify Albertson conjecture for further values of r.

Let $f_r(n)$ be the minimum number of edges in an n-vertex r-critical graph. Since K_r is the only r-critical graph with r vertices we have $f_r(r) = r(r-1)/2$. Another trivial bound is $f_r(n) \geq n(r-1)/2$, because the degree of every vertex in an r-critical graph must be at least $r - 1$. The study of $f_r(n)$ goes back to Dirac [10]. He proved that there is no r-critical graph on $r + 1$ vertices and that if $r \geq 4$ and $n \geq r + 2$ then

$$f_r(n) \geq n(r-1)/2 + (r-3)/2. \tag{6}$$

This was improved by Kostochka and Stiebitz [14] to

$$f_r(n) \geq n(r-1)/2 + (r-3), \tag{7}$$

when $n \neq 2r - 1$. Considering the case $n = 2r - 1$, Barát and Tóth [8] concluded

Lemma 3.5 ([8, Corollary 7])**.** *Let G be an n-vertex r-critical graph with m edges, such that $r \geq 4$. If G does not contain a subdivision of K_r then $m \geq n(r-1)/2 + (r-3)$.*

Gallai [12] found exact values of $f_r(n)$ for $6 \leq r + 2 \leq n \leq 2r - 1$:

$$f_r(n) = \frac{1}{2}(n(r-1) + (n-r)(2r-n) - 2). \tag{8}$$

He also characterized the graphs obtaining this bound. His results yield:

Lemma 3.6 ([8, Corollary 5])**.** *Let G be an n-vertex r-critical graph with m edges, such that $6 \leq r + 2 \leq n \leq 2r - 1$. If G does not contain a subdivision of K_r then $m \geq \frac{1}{2}(n(r-1) + (n-r)(2r-n) - 1)$.*

Instead of using the linear bound of Theorem 5 directly, we will use a more refined bound obtained from it using the probabilistic argument (as is done in [8]).

Lemma 3.7. *Let* $cr(n, m, p) = \frac{5m}{p^2} - \frac{139n}{6p^3} + \frac{139}{3p^4} - \frac{6n^2(1-p)^{n-2}}{p^4}$. *For every graph* G *with* $n \geq 9$ *vertices and* m *edges and every* $0 < p \leq 1$ *we have* $cr(G) \geq cr(n, m, p)$.

Proof. We will use the linear bound of Theorem 5, however it does not hold for $n \leq 2$. Therefore, for every graph G we define

$$\mathrm{cr}'(G) = \begin{cases} \mathrm{cr}(G) & \text{if } n \geq 3 \\ 5 & \text{if } n = 2 \\ 24 & \text{if } n = 1 \\ 47 & \text{if } n = 0 \end{cases}$$

Thus, for every graph G we have

$$\mathrm{cr}'(G) \geq 5m - \frac{139}{6}(n-2). \tag{9}$$

Let G be a graph with n vertices and m edges and let $0 < p \leq 1$. Consider a drawing of G with $\mathrm{cr}(G)$ crossings. Construct a random subgraph of G by selecting every vertex independently with probability p. Let G' be the subgraph of G that is induced by the selected vertices. Denote by n' and m' the number of vertices and edges in G', respectively. Consider the drawing of G' as inherited from the drawing of G, and let x' be the number of crossings in this drawing. Clearly, $\mathbb{E}[n'] = pn$, $\mathbb{E}[m'] = p^2 m$, and $\mathbb{E}[x'] = p^4 \mathrm{cr}(G)$. From (9) and the linearity of expectation we get:

$$\begin{aligned} \mathbb{E}[x'] &\geq \mathbb{E}[\mathrm{cr}(G')] - 5 \cdot \Pr(n' = 2) - 24 \cdot \Pr(n' = 1) - 47 \cdot \Pr(n' = 0) \\ &\geq 5p^2 m - \frac{139}{6}pn + \frac{139}{3} - 5\binom{n}{2}p^2(1-p)^{n-2} - 24np(1-p)^{n-1} - 47(1-p)^n \\ &\geq 5p^2 m - \frac{139}{6}pn + \frac{139}{3} - 6n^2p^2(1-p)^{n-2}. \end{aligned}$$

Dividing by p^4, the lemma follows. □

Before proving Theorem 7, let us recall the best known upper bound on the crossing number of K_r [13]:

$$\mathrm{cr}(K_r) \leq Z(r) = \frac{1}{4}\left\lfloor\frac{r}{2}\right\rfloor\left\lfloor\frac{r-1}{2}\right\rfloor\left\lfloor\frac{r-2}{2}\right\rfloor\left\lfloor\frac{r-3}{2}\right\rfloor. \tag{10}$$

Proof of Theorem 7: We follow the proof of Theorem 2 in [8]. Given r let G be an r-critical graph with n vertices and m edges. We assume that G does not contain a subdivision of K_r for otherwise we are done. By Lemma 3.4 we may assume that $n \geq r + 5$. Lemma 3.5 is used to get a lower bound on m, namely $m \geq (r-1)n/2 + (r-3)$. This bound is plugged into Lemma 3.7 and for an appropriate value of p we get a lower bound on $\mathrm{cr}(G)$ that is greater than $Z(r)$ for $n \geq n'$. Then it remains to verify the conjecture for each n in the range $r + 5, \ldots, n'$. This is done using a lower bound on m we get from either Lemma 3.5 or Lemma 3.6 and picking p such that $\mathrm{cr}(n, m, p) \geq Z(r)$. We will always have $n \geq 22$ and $p \geq 0.5$, therefore we may assume that

$$\mathrm{cr}(n, m, p) \geq \frac{5m}{p^2} - \frac{139n}{6p^3} + \frac{139}{3p^4} - 0.05 \tag{11}$$

$r = 18$, $\mathrm{cr}(K_{18}) \le 1008$			
n	m	p	$\lceil \mathrm{cr}(n,m,p) \rceil$
23	228	0.555	1073
24	240	0.556	1132
25	251	0.560	1176
26	261	0.567	1204
27	270	0.576	1217
28	278	0.586	1218
29	285	0.599	1206
30	291	0.613	1183
31	296	0.628	1151
32	300	0.646	1111
33	303	0.665	1064
34	305	0.686	1010

$r = 19$, $\mathrm{cr}(K_{19}) \le 1296$			
n	m	p	$\lceil \mathrm{cr}(n,m,p) \rceil$
24	251	0.523	1321
25	264	0.524	1397
26	276	0.527	1455
27	287	0.533	1495
28	297	0.540	1518
29	306	0.548	1527
30	314	0.558	1520
31	321	0.570	1501
32	327	0.583	1471
33	332	0.597	1430
34	336	0.613	1380
35	339	0.631	1322
36	341	0.650	**1259**
37	349	0.656	**1269**
38	358	0.659	**1292**

Table 1: Lower bounds on the number of edges and crossing numbers for specific values of n for $r = 18$ (left) and $r = 19$ (right).

1. Suppose that $r = 17$ and let G be an n-vertex 17-critical graph with m edges. By (10) we have $\mathrm{cr}(K_r) \le 784$. It follows from Lemmas 3.4 and 3.5 that we may assume that $n \ge 22$ and $m \ge 8n + 14$. From (11) we have $\mathrm{cr}(G) \ge \mathrm{cr}(n, 8n + 14, 0.727) \ge 15.38n + 298.25$. Therefore, if $n \ge \frac{784-298.25}{15.38} \ge 31.58$ the conjecture holds. Since Barát and Tóth [8] have already verified Albertson conjecture for $r = 17$ and $n \le 31$, we are done.

2. Suppose that $r = 18$ and let G be an n-vertex 18-critical graph with m edges. By (10) we have $\mathrm{cr}(K_r) \le 1008$. It follows from Lemmas 3.4 and 3.5 that we may assume that $n \ge 23$ and $m \ge 8.5n + 15$. From (11) we have $\mathrm{cr}(G) \ge \mathrm{cr}(n, 8.5n + 15, 0.69) \ge 18.74n + 361.88$. Therefore, if $n \ge \frac{1008-361.88}{18.74} \ge 34.47$ the conjecture holds. It remains to verify the conjecture for $n = 23, \ldots, 34$. Table 1 (left) shows the lower bound on m for each n, the value of p we choose, and the corresponding lower bound on the crossing number that we get. Note that since we are interested in values of n such that $r + 2 \le n \le 2r - 1$, we may use Lemma 3.6 instead of the Lemma 3.5.

3. Suppose that $r = 19$ and let G be an n-vertex 19-critical graph with m edges. By (10) we have $\mathrm{cr}(K_r) \le 1296$. It follows from Lemmas 3.4 and 3.5 that we may assume that $n \ge 23$ and $m \ge 9n + 16$. From (11) we have $\mathrm{cr}(G) \ge \mathrm{cr}(n, 9n + 16, 0.66) \ge 22.72n + 427.78$. Therefore, if $n \ge \frac{1296-427.78}{22.72} \ge 38.21$ then the conjecture holds. It remains to verify the conjecture for $n = 24, \ldots, 38$. Table 1 (right) shows the lower bound on m for each n, the value of p we choose, and the corresponding lower bound on the crossing number that we get.[2]

Therefore, the conjecture holds for $r = 19$ and every $n \notin \{36, 37, 38\}$. As is done in [8], we can handle the case $n = 36$ by using a result of Gallai [12], who proved that an r-critical graph with $2r-2$ vertices is the *join*[3] of two smaller critical graphs. Therefore, if $n = 36$ then G is the join of an r_1-critical graph $G_1 = (V_1, E_1)$ and an r_2-critical graph $G_2 = (V_2, E_2)$,

[2] The code of our calculations appears in Appendix A.

[3] A *join* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ consists of the two graphs and the edges $\{(v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2\}$.

such that $r_1 + r_2 = 19$. Let $n_i = |V_i|$ and $m_i = |E_i|$, for $i = 1, 2$. Then $n_1 + n_2 = 36$ and $m = m_1 + m_2 + n_1 n_2$.

We assume without loss of generality that $r_1 \le r_2$, and therefore have to consider the cases $r_1 = 1, \dots, 9$. Suppose that $r_1 = 1$, which implies that $G_1 = K_1$. If G_2 contains a subdivision of K_{18} then G contains a subdivision of K_{19} and we are done. Otherwise, by Lemma 3.5 we get $m_2 \ge 313$. Therefore, $m = n_1 n_2 + m_2 \ge 348$ when $r_1 = 1$.

Since G_1 is r_1-critical and G_2 is r_2-critical we have $m \ge f_{r_1}(n_1) + f_{r_2}(n_2) + n_1 n_2$. Note that $n_1 = 36 - n_2 \le 36 - r_2 = r_1 + 17$ and if $r_1 = 2$ then $G_1 = K_2$. A computer calculation using the trivial bound for $f_r(n)$ along with (7), reveals that $m \ge 348$ for every $r_1 = 2, \dots, 9$ and every $n_1 = r_1, \dots, r_1 + 17$ (ignoring cases where $n_1 = r_1 + 1$ or $n_2 = r_2 + 1$ since there are no such critical graphs). Therefore, we conclude that G has at least 348 edges. Picking $p = 0.635$ we get that $\mathrm{cr}(G) \ge \mathrm{cr}(36, 348, 0.635) = 1343 \ge \mathrm{cr}(K_{19})$. □

Recall that Barát and Tóth [8] showed that if Albertson conjecture is false, then the minimal counter-example is an r-critical graph with at least $r + 5$ vertices (Lemma 3.4). They also gave an upper bound of $3.57r$ on the number of vertices in such a minimal counter-example (improving a $4r$ bound due to Albertson et al. [7]). Using Theorem 5 we can improve upon this bound as well.

Lemma 3.8. *If G is an r-critical graph with $n \ge 3.03r$ vertices, then $cr(G) \ge cr(K_r)$.*

Proof. The proof is similar to the proof of Lemma 3 in [8]. We repeat it here for completeness, and because there is a small typo in the calculation in [8].

Let G be an r-critical graph with n vertices drawn in the plane with $\mathrm{cr}(G)$ crossings. We may assume that $r \ge 19$, since for $r \le 18$ the conjecture holds. If $n \ge 3.57r$ then it follows from [8] that $\mathrm{cr}(G) \ge \mathrm{cr}(K_r)$. Therefore, we assume that $n = \alpha r$ for some $3.03 \le \alpha < 3.57$. Note that $n \ge 3r \ge 57$. Let $5 \le k \le n$ be an integer and let $G_1, G_2, \dots, G_t$, $t = \binom{n}{k}$, be all the (inherited drawings of) subgraphs induced by exactly k vertices in G. Denote by m_i the number of edges in G_i, and note that by Theorem 5 we have $\mathrm{cr}(G_i) \ge 5m_i - \frac{139}{6}(k-2)$. Observe also that every crossing in G appears in $\binom{n-4}{k-4}$ subgraphs and every edge in G appears in $\binom{n-2}{k-2}$ subgraphs. Finally, recall that $m \ge n(r-1)/2$ since G is r-critical. Thus we have,

$$
\begin{aligned}
\mathrm{cr}(G) &\ge \frac{1}{\binom{n-4}{k-4}} \sum_{i=1}^{t} \mathrm{cr}(G_i) \ge \frac{1}{\binom{n-4}{k-4}} \sum_{i=1}^{t} \left(5m_i - \frac{139(k-2)}{6}\right) \\
&= 5m\frac{\binom{n-2}{k-2}}{\binom{n-4}{k-4}} - \frac{139(k-2)\binom{n}{k}}{6\binom{n-4}{k-4}} \\
&\ge \frac{5(r-1)n}{2}\frac{(n-2)(n-3)}{(k-2)(k-3)} - \frac{139n(n-1)(n-2)(n-3)}{6k(k-1)(k-3)} \\
&= \frac{n(n-2)(n-3)}{2(k-3)}\left(\frac{5(r-1)}{k-2} - \frac{139(n-1)}{3k(k-1)}\right) \\
&= \frac{\alpha^3 r(r-\frac{2}{\alpha})(r-\frac{3}{\alpha})}{2(k-3)}\left(\frac{5(r-1)}{k-2} - \frac{139(\alpha r-1)}{3k(k-1)}\right) \\
&\ge \frac{\alpha^3 r(r-2)((r-3)+2)}{2(k-3)}\left(\frac{5(r-1)}{k-2} - \frac{139(r-1)(\alpha+\frac{\alpha-1}{r-1})}{3k(k-1)}\right) \\
&= \frac{\alpha^3 r(r-1)(r-2)(r-3)}{2(k-3)}\left(\frac{5}{k-2} - \frac{139\alpha}{3k(k-1)}\right) + h(\alpha, r, k),
\end{aligned}
$$

where

$$\begin{aligned}
h(\alpha, r, k) &= \frac{\alpha^3 r(r-1)(r-2)}{2(k-3)}\left(\frac{10}{k-2} - \frac{139}{3k(k-1)}\left(2\alpha + 2\frac{\alpha-1}{r-1} + \frac{r-3}{r-1}(\alpha-1)\right)\right) \\
&\geq \frac{\alpha^3 r(r-1)(r-2)}{2(k-3)}\left(\frac{10}{k-2} - \frac{139}{3k(k-1)}\left(2\alpha + \frac{\alpha-1}{9} + (\alpha-1)\right)\right).
\end{aligned}$$

Suppose now that $3.17 \leq \alpha \leq 3.57$. Then for $k = 47 < n$ we have $h(\alpha, r, 47) \geq 0$ and therefore

$$\begin{aligned}
\mathrm{cr}(G) &\geq \frac{\alpha^3}{2 \cdot 44}\left(\frac{5}{45} - \frac{139\alpha}{3 \cdot 47 \cdot 46}\right) r(r-1)(r-2)(r-3) \\
&\geq \frac{1}{64} r(r-1)(r-2)(r-3) \geq \mathrm{cr}(K_r).
\end{aligned}$$

Suppose now that $3.05 \leq \alpha \leq 3.17$. Then for $k = 41 < n$ we have $h(\alpha, r, 41) \geq 0$ and therefore

$$\begin{aligned}
\mathrm{cr}(G) &\geq \frac{\alpha^3}{2 \cdot 38}\left(\frac{5}{39} - \frac{139\alpha}{3 \cdot 41 \cdot 40}\right) r(r-1)(r-2)(r-3) \\
&\geq \frac{1}{64} r(r-1)(r-2)(r-3) \geq \mathrm{cr}(K_r).
\end{aligned}$$

Finally, suppose that $3.03 \leq \alpha \leq 3.05$. Then for $k = 40 < n$ we have $h(\alpha, r, 40) \geq 0$ and therefore

$$\begin{aligned}
\mathrm{cr}(G) &\geq \frac{\alpha^3}{2 \cdot 37}\left(\frac{5}{38} - \frac{139\alpha}{3 \cdot 40 \cdot 39}\right) r(r-1)(r-2)(r-3) \\
&\geq \frac{1}{64} r(r-1)(r-2)(r-3) \geq \mathrm{cr}(K_r).
\end{aligned}$$

□

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A sage code of the calculations in the proof of Theorem 7

```
sage: Dirac(n,r)=((r-1)*n+r-3)/2
sage: KS(n,r)=((r-1)*n+2*r-6)/2
sage: Gallai(n,r)=((r-1)*n+(n-r)*(2*r-n)-2)/2
sage: BT_Gal(n,r)=Gallai(n,r)+0.5
sage: cr_prime(n,m,p)=5*m/p^2-139*n/(6*p^3)+139/(3*p^4)-0.05
sage: Z(r)=floor(r/2)*floor((r-1)/2)*floor((r-2)/2)*floor((r-3)/2)/4
sage: def proc1(r):
...       sols = solve([cr_prime(n,KS(n,r),p).diff(p)==0, cr_prime(n,KS(n,r),p)==Z(r)],n,p,
    solution_dict=True)
...       for s in sols:
...           if (s[n].imag()==0 and s[p].imag()==0): # output only real solutions
...               print "p=",s[p].n(),",n=",s[n].n()
sage: proc1(17)
p= 0.727523979840676 ,n= 31.5627659574468
sage: cr_prime(n,KS(n,17),0.727)
15.3896636507376*n + 298.258502516192
sage: proc1(18)
p= 0.690689920492434 ,n= 34.4659498207885
sage: cr_prime(n,KS(n,18),0.69)
18.7463154231188*n + 361.887598221377
sage: def proc2(n,r):
...       if n <= 2*r-2:
...           m = ceil(BT_Gal(n,r))
...       else:
...           m = ceil(KS(n,r))
...       sols = solve(diff(cr_prime(n,m,p),p)==0, p, solution_dict=True)
...       best_p= round(sols[1][p],3)
...       best_cr = ceil(cr_prime(n,m,best_p))
...       str = '\t\t'+repr(n)+' & '+repr(m)+' & '+repr(best_p.n())+' & '+repr(best_cr) + '
     \\\\'
...       print str
sage: for n in range(23,35):
...       proc2(n,18)
```

```
                23 & 228 & 0.555000000000000 & 1073 \\
                24 & 240 & 0.556000000000000 & 1132 \\
                25 & 251 & 0.560000000000000 & 1176 \\
                26 & 261 & 0.567000000000000 & 1204 \\
                27 & 270 & 0.576000000000000 & 1217 \\
                28 & 278 & 0.586000000000000 & 1218 \\
                29 & 285 & 0.599000000000000 & 1206 \\
                30 & 291 & 0.613000000000000 & 1183 \\
                31 & 296 & 0.628000000000000 & 1151 \\
                32 & 300 & 0.646000000000000 & 1111 \\
                33 & 303 & 0.665000000000000 & 1064 \\
                34 & 305 & 0.686000000000000 & 1010 \\
sage: proc1(19)
p= 0.659831121833534 ,n= 38.2051696284330
sage: cr_prime(n,KS(n,19),0.66)
22.7249538544304*n + 427.789066289688
sage: for n in range(24,39):
...       proc2(n,19)
                24 & 251 & 0.523000000000000 & 1321 \\
                25 & 264 & 0.524000000000000 & 1397 \\
                26 & 276 & 0.527000000000000 & 1455 \\
                27 & 287 & 0.533000000000000 & 1495 \\
                28 & 297 & 0.540000000000000 & 1518 \\
                29 & 306 & 0.548000000000000 & 1527 \\
                30 & 314 & 0.558000000000000 & 1520 \\
                31 & 321 & 0.570000000000000 & 1501 \\
                32 & 327 & 0.583000000000000 & 1471 \\
                33 & 332 & 0.597000000000000 & 1430 \\
                34 & 336 & 0.613000000000000 & 1380 \\
                35 & 339 & 0.631000000000000 & 1322 \\
                36 & 341 & 0.650000000000000 & 1259 \\
                37 & 349 & 0.656000000000000 & 1269 \\
                38 & 358 & 0.659000000000000 & 1292 \\
sage: def f(n,r): # lower bound for the number of edge in n-vertex r-critical graph
...       best=0
...       if n==r: # K_r
...           best=n*(n-1)/2
...       elif n>r+1:
...           best=ceil(n*(r-1)/2) # trivial
...           if (r>=4 and n>=r+2):
...               best=max(best,ceil(Dirac(n,r)))
...               if n!=2*r-1:
...                   best=max(best,ceil(KS(n,r)))
...               if n<=2*r-1:
...                   best=max(best,ceil(Gallai(n,r)))
...       return best
sage: # considering the case r=19, n=36
sage: min_m=348
sage: for r1 in range(2,10):
...       r2 = 19-r1
...       if r1==2:
...           max_n1=2
...       else:
...           max_n1=36-r2
...       for n1 in range(r1,max_n1+1):
...           n2 = 36-n1
...           if (n1!=r1+1 and n2!=r2+1):
...               curr = f(n1,r1)+f(n2,r2)+n1*n2
...               min_m = min(min_m,curr)
...
sage: print min_m
348
```