James Ferris
10/20/2003

*The work of Euclid, a paradigm of the mathematics of ancient Greece*

Euclid of Alexandria is one of those among the mathematicians of ancient Greek culture about whom little biographically is known. The date and city of his birth are not preserved for posterity, although modern historians assign approximate dates based on the extant literature of the period in which he worked. Archimedes of Syracuse mentions Euclid during the reign of Ptolemy I of Egypt (306-283 B.C.E), giving the modern expositor some idea of Euclid's historical framework.[i] In the literature Proclus (circa 450 C.E) records Archimedes and Euclid as contemporaries, "He [Euclid] is therefore younger than the pupils of Plato, but older than Eratosthenes and Archimedes. For these men were contemporaries, as Eratosthenes somewhere says."[ii] Archimedes lived from 287 until 212 B.C.E and Plato died in 347 B.C.E, fitting Euclid somewhere in the interval between the two, which also corresponds historically to the reign of Ptolemy I.[iii] Most historical references on Euclid are reconstructed from the historical period—it is hypothesized that Euclid studied at Plato's Academy in Athens due to the great body of geometric thought and study which existed in the city at the time.[iv] Euclid of Alexandria is sometimes confused with Euclid of Megara (circa 400 B.C.E) who was a contemporary of Plato[v] and a student of Socrates.[vi]

Ptolemy I gained political control of Egypt after the Macedonian empire collapsed following the death of Alexander the Great in 323 B.C.E. As a royal act Ptolemy I enabled the existence of a royal institution, the Museum (circa 300 B.C.E), for the advancement of his royal prestige and the general body of knowledge at the time. It is known that Euclid taught mathematics at the Museum of Alexandria[vii]and because of this the title Euclid of Alexandria is given to him by historians to distinguish him from other historical Euclids. Pappus (circa 310

C.E) wrote that Apollonius of Perga (circa 230 B.C.E) ‘spent a very long time with the pupils of Euclid at Alexandria[viii], which entails that Euclid had some pedagogic position at the Museum at Alexandria. It is also known that Euclid produced mathematical work; his largest piece is still extant (*Stoichia*, or *Elements*) and some commentators of his work claim that this work is surpassed in circulation in pre-modern times only by the Bible.[ix] During the course of Euclid’s career he produced academic works in diverse fields—Euclid is known for his work in mathematics, especially geometry, but he produced works in other fields, such as mechanics, music and optics, some of which still exist. Euclid’s *Phaenomena*, a work on spherical geometry in terms of technical astronomy is still extant in the Greek (235 Greenwood) as is his *Optics* (in which he espouses a view opposing the optical theories of Aristotle, namely, Euclid declares that the eyes emit beams which reflect from the object and return to eye). Euclid’s *Elements of Music*, mentioned by Proclus[x] is no longer present as one body, and the authenticity of the remaining treatises are debated by current scholars of the period.[xi]

The *Elements* is Euclid’s singularly most recognizable work—this classic geometrical disquisition, certainly constructed as a manual for study by ancient students of geometry, perhaps students at the Museum of Alexander—is used in the modern classroom as a foundation for high school geometry. The *Elements* is a composition incorporating the work of Greek mathematicians from the 6th to 4th century B.C.E, from Thales of Milatus to Eudoxus of Cnidus. Proclus in his *Summary* mentions Euclid thusly “ [it was he who] arranging in order many of Eudoxus’s theorems, perfecting many of Theaetetus’s, and also bring to irrefutable demonstration the things which had only been loosely proved by his predecessors.”[xii] Indeed, some of the most well known theorems are presented in Euclid’s *Elements*—the Pythagorean identity, relating the square of the hypotenuse in a right triangle to the sum of the squares of the bases is contained in Book I, Proposition 47. Three of Thales' theorems are presented in Book I, covered by propositions I, 5; I, 15; and I, 26. Proclus mentions that Hippocrates of Chios (circa 450 B.C.E) was the first geometer to compile the elements of geometry as such[xiii], and although that work is now lost[xiv], it may be plausible given the mathematical progression of Greek thought at the time that the first and perhaps second book of the *Elements* are based on the work of Hippocrates of Chios. The tenth book of the *Elements* is based, according to Pappus[xv], on the work of Theaetetus of Athens (circa 400 B.C.E). His work on the incommensurable relations and measurability of intervals (namely that of the irrational numbers to any other number) is based historically on the work of the Pythagorean sect, who expounded on commensurability of ratio in their theorems[xvi]. [1]

The *Elements* of Euclid does not begin with any preface to the body of the work, instead beginning the technical exposition with a set of definitions, followed by postulates and common notions. The definitions are content-specific with respect to the subject matter of each book, and are not always invoked during the following discourse. (Books VIII, IX, XII and XIII lack introductory definitions entirely). The postulates and common notions are specific only to the

[1] It is clear to the modern investigator that the body of the *Elements* is a compilation of pre-Euclidean mathematics of ancient Greece. In order to give the reader some idea of the mathematical heritage and lineage of the era, a brief chronology of Greek mathematicians (mostly geometers) will be described. Thales of Milatus (circa approx. 625-545 B.C.E) is recorded as having set forth some of the first work on proof and geometry. Pythagoras of Samos (approx. 570-475 B.C.E) ran a religious / mathematics cult and, along with his followers, is credited with important mathematical theorems, discoveries and general advancement of mathematical philosophy. Anaxagoras of Clazomenae (approx. circa 500 – 430 B.C.E) is credited with the introduction of philosophy to the Athenian polis upon his arrival in 480 B.C.E. Proclus records him as having done substantial work in the field of geometric investigation, and he is also known for his astronomical work. Hippocrates of Chios (approx. 470-410 B.C.E) is known to have written his own *Elements of Geometry*, which Euclid may have used, as well as other works on mathematical problems of the time. Theodorus of Cyrene (approx. circa 465-398 B.C.E) was the tutor of Plato and Theaetetus, a member of the Pythagorean sect, and did substantial work on the theory of incommensurable numbers. While no mathematical theorems are attributed to Plato of Athens (approx. 430-350 B.C.E), he did important work on the philosophy of mathematics and the idea of proof. Theaetetus of Athens (circa 420-370 B.C.E) is credited with major work on solid geometry and polyhedra—expanding on the work of the Pythagorean sect in the field. He is also known to have done work on the Greek theory of irrational numbers and expanded on the work of his tutor, Theodorus of Cyrene. Books X and XIII of the *Elements* are generally attributed to Theaetetus of Athens. Eudoxus of Cnidus (approx. 410-355 B.C.E) did important work on the theory of proportion and the method of exhaustion, a primitive form of integral calculus. Eudemus of Rhodes (circa 350-290 B.C.E) wrote several mathematical histories as well as some mathematical work.

first book—no other books of the *Elements* contain postulates or notions relative to their subject matter. The postulates (*aitemata* in Ancient Greek)[xvii] of the *Elements* are axioms which can be divided into two classes: constructability axioms and existence axioms. The first three postulates of Book I allow for construction of figures using a compass and a straightedge, whereas the fourth postulate states the equality of right angles and the fifth, the defining postulate of Euclidean geometry, allows only one line to be described parallel to another line through any point not on that line (presented here equivalently as Playfair's axiom). These postulates assume certain fundamental properties of the space in which the construction is to be worked, e.g., that the curvature of the space is zero. It is of course possible that this postulate not be satisfied, i.e., there exist no parallels to a line at a given point or more than one, but this illustration is possible only in a curved space, viz., if there are no parallels on a point to a given line the surface must be elliptical, if more than one parallel exists the surface must be hyperbolic, but these geometries (Riemannian and Lobachevskian, respectively) are not on 'flat' surfaces.

Euclid proceeds to unite the books of the *Elements* synthetically, building a mathematical foundation of two-dimensional plane geometry in Book I and enlarging the structure to create an edifice which he can use as a basis for an expansion onto a three-dimensional geometry of solids, which are treated in Books XI, XII and XIII. The first six books cover planar geometry, with the first two books covering properties of straight lines and parallels, triangles and quadrangles. Book II is primarily composed of what may be called geometric algebra [xviii]—the geometric propositions regarding the areas of rectangles of Book II can be rephrased in algebraic terms: the algebraic interpretation of Book II, proposition 1 would be something as follows:

For every natural number $a$ such that,

$$a = \sum^{\kappa} a_{\mathrm{i}}$$

$$i = \alpha$$

$$ab = b(a_{\alpha} + a_{\beta} + \ldots + a_{\kappa}) = ba_{\alpha} + ba_{\beta} + \ldots + ba_{\kappa}$$

that is, that multiplication is distributive over the natural numbers. Other basic algebraic identities are proven geometrically by propositions in Book II. Book II, proposition 4 gives a geometric proof for the binomial expansion for the power of two, and further identities are shown geometrically. Propositions 5, 6, 11 and 14 are usable in conjunction to solve quadratic equations with positive roots. Interestingly, two propositions (12 and 13) appended to Book II give what is very similar to a geometric interpretation of the law of cosines. Euclid of course, interprets all of this geometrically—it would not be until the work of Descartes (circa 1630 C.E.) that the realization began to form that there is an algebraic description accompanying every geometric object and that there is some geometric object determined by every algebraic formula.

Book III gives a treatment of circles in planar geometry—the properties and ratios of angles, chords, radii, tangents, arcs and sectors. Propositions 3, 4 and 9 treat the bisection of chords and diameters. Propositions 16, 17, 18 and 19 treat on properties of lines tangential to circles and Proposition 17 gives a method for construction of any line tangent to a circle. Book IV utilizes these propositions for the construction of polygonal figures inscribed and circumscribed by circles, and conversely, circles inscribed and circumscribed by polygonal figures. All of the propositions given in Book IV are propositions of construction, Proposition 1 and 10 being the only proposition not directly related to the construction of these nested figures, in which the polygons have 3, 4, 5, 6 and 15 sides respectively. It has been suggested that the 15-sided polygon was included in the *Elements* as an astronomical reference[xix]: the angle between the axis of the earth is approximately 23.5 degrees away from a line perpendicular to the ecliptic plane of Earth's orbit, and $^{1}/_{15}$ of 360 degrees is 24 degrees, which is the approximation of the obliquity of the ecliptic used by ancient astronomers. It has been held (ibid.) that the entire body of Book IV is a collection of mathematics from the Pythagorean sect.

Books XI, XII, and XIII are works on geometry in three dimensions and are very similar in text and flavor to the work in Books I-IV. The definitions given at the beginning of Book XI are comparable in style to those given in Book I. As Book XI follows I chronologically, material treated in previous books has been included in XI; Book XI has theorems which relate to the work in Book VI, which is a discussion of the previous books of planar geometry in terms of the theory of proportion given in Book V. The first half of Book XI deals with parallelepiped solids, that is, solid bodies described by parallelograms, while the latter half of the chapter deals with parallelepiped congruence, divisions and bisections of such solids. Book XII increases the breadth of solid geometry, considering pyramids, cylinders, cones, spheres and the ratios and relationships thereof. An important principle used in Book XII is the principle of exhaustion, which gives a means for approximating a shape by the sequence of similar figures contained by that shape. Propositions 2, 5, 10-12, and 18 employ this principle: XII, 5 uses it to show that pyramids with triangular bases of equal height are proportional to their bases, while XII, 18 uses the principle of exhaustion to show that spheres are similar to one another by a triple ratio of their diameter. Archimedes is recorded in his *On the Sphere and the Cylinder* as giving credit to Eudoxus of Cnidus for this method of exhaustion on solids—it is possible that much of the work in Book XII is due to Eudoxus.[xx] The method of exhaustion has some relationship with integral calculus[xxi], namely that that a certain series (in this case the inscribed figures) converges to some value, that is, the external perimeter of the figure. Certainly, neither Eudoxus nor Euclid formulated this methodology in terms of calculus, as the work in the *Elements* is entirely geometrical, and indeed the concept of limit would not be defined until the time of Newton (his *Principia Mathematica* was published in 1687).

Book XIII is the culminating point of the geometric aspect of the *Elements*. A scholium to this book, believed to by penned by Geminus (circa 20-50 C.E), attributes Book XIII of the *Elements* with regard to the work on the Platonic solids to Theaetetus of Athens[xxii], to whom the proofs of incommensurability in Book X are attributed. The latter half of Book XII describes the five platonic solids—the pyramid, cube, octahedron, dodecahedron and icosahedron, those with 4, 6,8,12 and 20 faces, all respectively regular. Euclid gives methods for the construction of each of these platonic solids and the construction of a sphere in which the body would rest in three-dimensional space, and closes the *Elements* with a remark (and proof) that no other solid could be constructed with the qualities of the Platonic solids, i.e., that no other regular polyhedra can be constructed. This is, of course, false—a counterexample may be given by constructing a regular hexahedra formed by two regular pyramids with base ABC and vertices D and D' on opposite sides of ABC.[xxiii] However, if the implicit definition of a regular solid given in this remark is amended to contain a clause that each solid angle of the figure is contained by the same number of base angles, the closing remark of the *Elements* remains true. Proclus[xxiv] has it that the construction of these Platonic solids is the *telos*, or goal of the entire *Elements*—it can been seen that Book XIII is dependant on all other books in the *Elements* except XII, and though it would seem that XIII makes little use of VII-IX it is dependant on them because X is, and Book X is employed rather heavily in the construction of the Platonic solids.[xxv]

Save postulates and common notions, Book V is logically independent from the preceding books, that is, the material in Book V is logically consistent in and of itself. The material considers issues of proportion, magnitudes of figures and ratios of magnitude. Euclid always employs the length of lines as examples of magnitude in these propositions, but the term, given by definitions 1 and 2, could instead be represented by any plane figure.

> V, Def. 1. A magnitude is a *part* of a magnitude, the less of the greater, when it measures the greater.
> V, Def. 2. The greater is a *multiple* of the less when it is measured by the less[xxvi]

Euclid then goes on to assert by ratio what he has proved by rectangle in Book II. Proposition I, Book V states that multiplication distributes over the sum of magnitudes in a fashion similar to algebraic proof for multiplicative distribution given in Book II, Proposition 1. The remaining propositions give proofs for the equality of ratios over magnitudes and the similitude of ratios. An anonymous writer of an addendum to an edition of the *Elements*, whom Heath suggests may be Proclus[xxvii], claims that Book V of the *Elements* is the work of Eudoxus of Cnidus (circa 365 B.C.E). Eudoxus's theory of proportion (also referred to in Archimedes's *On the Sphere and the* Cylinder) makes the study of commensurable and incommensurable lengths possible, and Euclid investigates the properties of commensurability in Book X of the *Elements*. Work was done previously in the Athenian Academy on the approximation of irrational numbers[xxviii], such as square root of two, expanding on the Pythagorean theory of commensurability as the previous investigations on proportion could not include the irrational numbers (the work of that sect extended only to commensurable ratios). Book VI of the *Elements* gives the work on planar geometry of Books I-IV in terms of the newly introduced theory of proportion, giving methods to construct proportional lines and figures and determining ratio between plane figures.

The Pythagoreans are credited with the discovery of incommensurable ratio sometime circa 500-400 B.C.E[xxix], that is, there exist ratios which exist which cannot be expressed as some rational number.

> X, Def 1: Those magnitudes are said to be *commensurable* which are measured by the same measure, and those *incommensurable* which cannot have any common measure.

Theodorus of Cyrene (circa 430 B.C.E) showed that the square roots of all non-square odd numbers less than or equal to 17 are irrational; this is recorded in Plato's *Theaetetus*.[xxx] According to a commentary on Book X translated into Arabic (according to some by Pappus)[xxxi], his student, Theaetetus of Athens, completed the work on incommensurability and put the concept of the αλογος (*alogos*), or irrational number on a more rigorous footing[xxxii]; in particular, Book X, Proposition 9 stems directly from his work[xxxiii]. Any line which is not measurable to some other measurable line is termed irrational

> X, Def 4: Let then the assigned straight line be called *rational*, and those straight lines which are commensurable with it, whether in length and in square, or in square only, *rational*, but those that are incommensurable with it *irrational*.

which is quite similar to the modern definition of irrationality, i.e., a number which is irrational is one which cannot be expressed as any ratio of integers.

Books VII through IX give a basis for higher arithmetic in geometric terms, and indeed can be read as their own set, dependant only on Book V, and perhaps only tangentially as such. Euclid proceeds with a definition of the unit:

> VII, Def 1: A *unit* is that by virtue of which each of the things that exist is called one.

which, though suitably obscure, gives a method for the construction of number:

> VII, Def 2: A *number* is a multitude composed of units.

in terms of the magnitude given in Book V, i.e., number can be taken as a magnitude of a line, which is the tact that Euclid takes in proving further theorems on arithmetic in geometric terms (similar in fashion to Book V and X). Propositions 11-19 are on the proportions of numbers and have analogues in Book V. Propositions 5-10 explore the properties of fractions of numbers. Euclid proceeds to define even, odd and prime numbers:

> VII, Def 6: An even number is that which is divisible into two equal parts.
> VII, Def 7: An *odd number* is that which is not divisible into two equal parts, or that which differs by a unit from an even number
> VII, Def 11: A *prime number* is that which is measured by a unit alone

Euclid defines multiplication and division of numbers in VII (VII 16 and VII 3, 4, 5 respectively), defines relatively prime numbers, composite numbers, perfect numbers and the advances into theorems regarding numbers. A reader of the *Elements* must keep the fact in mind that a unit is not a number—chiefly, that Euclid, intending 1 to be a unit in terms of magnitude (Book V) by which other numbers are comprised. This entails that 1 is not a number, which implies by VII, d.11 that 1 is not prime, as 1 is not a number. Furthermore, the theorems of Books VII through IX can apply only to natural numbers—Euclid bears no conception of numbers less than zero as his numbers are constructed geometrically, and it would make no sense for the magnitude of a geometric figure to be less than zero. The Archimedean property of the natural numbers (and indeed integers) is assumed in Euclid's number theory, and although never stated as such as a postulate it is invoked as early as Book VII, Proposition 2 as an implicit axiom—its use here is to prevent an infinite chain of successive subtraction.

Euclid goes on to prove some of the more fundamental theorems of higher arithmetic, indeed, Book VII Proposition 2 is known as the Euclidean algorithm; VII, Proposition 2 (resting in proof on Proposition 1) is used to find the greatest common divisor of two numbers. Proposition 30, that:

$(\forall\, a, b) \in \mathbf{N}$, $(\forall\, p)$ a prime $\in \mathbf{N}$,
$[p \mid ab \Rightarrow (p \mid a) \wedge (p \mid b)]$

is sometimes known as Euclid's lemma. Propositions 14, Book IX; 30, Book VII; and 32, Book VII imply the fundamental theorem of arithmetic, or unique factorization theorem, that is, all numbers factor into a unique product of primes, although Euclid never stated this in the *Elements*:

> VII, Prop. 30: If two numbers, multiplied by one another make some number, and any prime number measures the product, then it also measures one of the original numbers.
> VII, Prop. 32: Any number is either prime or is measured by some prime number.
> IX, Prop. 14: If a number is the least that is measured by prime numbers, then it is not measured by any other prime number except those originally measuring it.

A modern statement of the fundamental theorem of arithmetic would be:

$(\forall\, x) \in \mathbf{Z}$,

$$x = \prod_{i=1}^{j} \chi_j^{e_j} \quad \text{for } \chi_j \text{ prime}$$

Furthermore, this factorization is unique, namely,

$$\prod_{i=1}^{j} \psi_j^{e_j} = \prod_{i=1}^{k} \omega_k^{e_k} \Rightarrow \psi_i = \omega_i$$

Book VIII continues the work on proportionality and number, continuing to build on the edifice established in Book VII. Book IX gives a treatment of the properties of odd and even numbers and operations thereof, relationships between powers of numbers and prime and relatively prime numbers. A centrally important part of the *Elements*, at least in terms of number theory, is Book IX Proposition 20, that is, that the set of prime numbers is infinite in size (but denumerable).

IX, Prop. 20: Prime numbers are more than any assigned multitude of prime numbers.

A brief sketch of this proposition in a somewhat modern flavor (a bit anachronistic to the *Elements*, but capturing the flair of the proof) is as follows:

*Let S be a finite ordered set comprised of all the prime numbers. Take the least common multiple of all the elements of S, that is, for* $p_1, p_2, p_3, \ldots, p_n \in S$. *Call it p'.*

$$p' = lcm\,(p_1, p_2, p_3, \ldots, p_n) = \prod_{i=1}^{n} p_i\,,$$ *as all elements of S are prime.*

*Consider* $(p'+1)$. *If* $p'+1$ *is prime, it would be a member of S, which it is not, as it is greater than* $p_n$. *Consider then the case when* $p'+1$ *is not a member of S, with the implication that some prime* $p_\varphi \mid p'+1$. $p_\varphi \notin S$ *as* $[(\forall p) \in S, (p \mid p') \wedge (p \mid p'+1) \Rightarrow (p \mid 1)]$. *But* $p_\varphi \in S$ *as S contains all primes.* $\Rightarrow\Leftarrow$ □

The remaining propositions of Book IX continue to prove theorems about odd and even numbers. Interestingly, the last proposition of Book IX, Proposition 36, gives a method for the construction of perfect numbers, that is, numbers for which the product of all the divisors of the number equal the number itself. The few examples of perfect numbers are 6, 28, 496, 8126, 33550336, 8589869056, 137438691328, and 2305843008139952128. A modern statement of IX, Prop36. would be that:

*For p prime, when* $2^p - 1$ *is a prime number, then* $(2^p - 1)\, 2^{p-1}$ *is a perfect number*

Prime numbers of the form $(2^p - 1)$ are known as Mersenne primes. So 2305843008139952128 can be found by applying IX, 30 with the following result:

$(2^{31} - 1)(2^{30}) = 2305843008139952128\,,\ p = 31$

Indeed the methodology of Euclid is worthy of some examination—Euclid in his mathematic work has set a historical precedent for an analytic approach to mathematics, and one may examine the exposition of his demonstrations in order to gain some insight into the evolution of the mathematical processes of proof, axiomatization and demonstration. The demonstrations given for the propositions of the *Elements* consist in the Greek of six steps[xxxiv]: (Proposition 6, Book IV)[xxxv]

*Protasis*: statement of the proposition
*To inscribe a square in a given circle.*
*Ecthesis*: existence of the fundamental elements necessary for the conclusion of the proposition
*Let ABCD be the given circle*
*Diorismos*: Statement of fact of the necessary steps to conclude the demonstration
*It is required to inscribe a square in the circle ABCD*
*Kataskeve*: Construction of elements in addition to the e*cthesis* which are necessary to the conclusion of the demonstration.
*Draw two diameters AC and BD of the circle ABCD at right angles to one another, and join AB, BC, CD, and DA.*
*Apodeixis*: Deduction of the desired conclusion from the construction thus given, the proof of previous propositions and common notions and postulates.
*For the same reason each of the straight lines BC and CD also equals each of the straight lines AB and AD. Therefore the quadrilateral ABCD is equilateral. I say next that it is also right-angled…*
*Sumperasma*: Logical conclusion affirming the *protasis*.
*But it was also proved equilateral, therefore it is a square, and it has been inscribed in the circle ABCD. Therefore the square ABCD has been inscribed in the given circle. Q.E.F*
.

As this is the general method of deduction in the *Elements*, it is clear that Euclid adds some concept of mathematical rigor to historical body of mathematics existing at the time. It is important that Euclid selected a process which seeks to minimize the explicit place of intuition in mathematics, even though Euclid occasionally makes intuitive assumptions about the material which he treats in his *Elements*.

Euclid's work on the *Elements* comprises some of the best-established mathematics known since ancient times. The survival of the *Elements* and popularity of the work suggest something about the quality and presentation of the material—in the literature of ancient Greece Euclid is referred to as the "author of the *Elements*".[xxxvi] The deductive approach of the

*Elements* solidifies in some sense a stronger logical foundation for the body of mathematics, and indeed the axiomatic approach adopted by Euclid in his arrangement of the *Elements* would be embraced by mathematicians of the 19th century in their quest for rigor, consistency and elegance in mathematics. To be sure it is this axiomatic approach which gives the *Elements* a place in both history and mathematics, as the course of modern mathematics has been to approach axiomatization and formalism as both a means for removing reliance on intuition and increasing mathematical rigor. Although the exposition in Euclid's *Elements* is concise and succinct, there exist any number of logical holes, unstated assumptions, definitions and demonstrations which are at times intuitive, and sometimes important postulates have been assumed in the body of the work—David Hilbert, publishing his *Grundlagen der Geometrie* in 1899, would correct some of the deficits of rigor in the *Elements* and place the work on a much more certain logical framework. Truly Euclid's grand achievement in his arrangement of the *Elements* is that the work begins to place mathematics on a deductive basis, beginning the process of mathematical formalization as he proceeds to deduce his propositions from set of common notions and postulates.

The majority of the proofs given in the *Elements* are of a direct proofs of a synthetic nature; proof in the elements largely relies on previous theorems and material from anteceding books. In this case Euclid constructs his proofs directly, giving a clear demonstration through his six deductive steps and reaching a desired conclusion. Euclid creates his mathematical structure by building on his previous propositions—Books I through IV serve as a basis for XI through XIII, Books I-IV and V support Book VI, Book V supports book X, and indeed the logical structure of the subchapters of the *Elements*, I-IV, VII-IX, and XI-XIII have an internal hierarchy of dependence, with material in the early books of the segments supporting material in the latter. Other proofs in the *Elements* are given indirectly, i.e., through Eudoxus's method of exhaustion or *reductio ad absurdum* arguments. Generally in the *Elements* indirect methods proceed the ecthesis of the proposition[xxxvii], and serve (as they still do in modern mathematics) to proof a conjecture when a direct method is burdensome, inelegant or fraught with difficulty.

The dependence on geometry in the *Elements* appears to be a historical artifact of the period and mathematical community in which Euclid worked. Post-Platonic thought at the time suggests that the corpus of mathematics can be represented in a geometric means, and it would not be until the time on Diophantus of Alexandria (circa 250 C.E) that Greek mathematics would be divorced from Greek geometry.[xxxviii] Greek mathematics up until that point had no conception of number separate from geometry, and could not consider the existence of negative numbers, as a negative geometric value is absurd. As Euclid could not but be a product of his times, his work on the *Elements* is a reflection on mathematical thought of the period. It is arguable that the arrangement of the material reflects more or less the chronology of geometric discovery and mathematical evolution in ancient Greece[xxxix], and as such the *Elements* presents a document which is valuable for two historical reasons: it gives the modern investigator some insight into the development of the mathematical thought and culture of ancient Greece, and it highlights an exemplary attempt of the earliest members of the mathematical community to put the body of mathematics on a logically sound, rigorous foundation.

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