Introductory Notes on the Trace Formula[1]

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Abstract

We survey some aspects of the trace formula and a few of its possible applications.

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1 Introduction

The purpose of these notes is to give the reader a rough idea about the trace formula and its applications. The trace formula was introduced by Selberg in his seminal work [Sel56]. Selberg mostly developed the trace formula for quotients of the hyperbolic plane by a Fuchsian group Γ of the first kind (both in the co-compact and the non co-compact case). One of his original motivations and applications was to show the existence of Maass forms with respect to $\Gamma = SL(2,\mathbb{Z})$. It was subsequently vastly generalized by Arthur in the context of adelic quotients $G(F)\backslash G(\mathbb{A})$ of a reductive group G over a number field F. Arthur's main driving force was the functoriality conjectures of Langlands. See [Gel] and [Art03] for more information about this.

Selberg's trace formula is a far reaching non-commutative generalization of the Poisson summation formula. It underlines a duality between geometric and spectral objects. Such a duality appears in other (not unrelated) contexts as well. For example the Lefschetz fixed-point theorem, in its various contexts, establishes a duality between fixed points and cohomology. This is a crucial ingredient in Grothendieck's approach, completed by Deligne, for the proof of the Weil conjectures ([FK88]). Another instance is Weil's explicit formula which, following Riemann, establishes in a precise way a duality between prime numbers and zeros

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of the Riemann zeta function. An intimate connection between Weil's explicit formula and the Selberg trace formula plays a dominant role in a relatively recent approach of Connes towards the Riemann Hypothesis ([Con99a], [Con99b], [Con00]; see also [Hej76a], [Gol89].) Other contexts of the trace formula include a discrete analogue by Terras ([Ter04], [Ter99]) and a guise in Mathematical Physics called the Gutzwiller trace formula ([Uri00], [Gut97], [Gut86]). An interesting connection between the Selberg trace formula and the L^2-index Theorem appears in the work of Stanton-Moscovici ([MS91], [MS89], [MS]). Finally, there is also the topological trace formula of Goresky-MacPherson [GM03].

Our goal in these notes is not to develop the trace formula from scratch, or even to discuss its derivation in any serious way, but rather to indicate what kind of problems the trace formula is suitable for. We will try to illustrate the techniques on several examples and along the way discuss the necessary background in various degrees of detail. For the interested reader we provide further references for a more in-depth discussion of the topics which are only mentioned here briefly.

There are already a number of excellent survey articles on the Arthur-Selberg trace formula. The most comprehensive one is a recent treatise by Arthur himself [Art05]. The present notes can hopefully serve as a preparation for [loc. cit.]. Additional sources include Gelbart [Gel96], Jacquet [Jac86], Knapp-Rogawski [KR97], Knightly-Li [KL06b], Konno [Kon00], Labesse [Lab86], Lai [Lai92], Langlands [Lan01], Shokranian [Sho92] and Tamagawa [Tam60]. The trace formula is also discussed in conjunction with other topics in Bump [Bum03], Gelbart [Gel75], Iwaniec [Iwa02], Langlands [Lan89], Terras [Ter85] and Venkov [Ven90]. For some survey articles on the relative trace formula we refer the reader to [Jac05a], [Jac97], [Lai88], [Lap06a].

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2 Poisson summation formula

Let us start with a very familiar case. Let f be a Schwartz function on $\mathbb{R}$. (In fact, it is sufficient to require that f is twice continuously differentiable and $f, f', f'' \in L^1(\mathbb{R})$.) Let $R(f)$ be the convolution operator on $L^2(\mathbb{T}) = L^2(\mathbb{Z}\backslash\mathbb{R})$. We can write it as an integral operator in the following way

$$\begin{aligned} R(f)\varphi(x) &= \int_{\mathbb{R}} f(y)\varphi(x+y)\ dy = \int_{\mathbb{R}} f(y-x)\varphi(y)\ dy \\ &= \int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} f(y+n-x)\varphi(y)\ dy = \int_{\mathbb{T}} K_f(x,y)\varphi(y)\ dy \end{aligned}$$

where $K_f(x,y) = \sum_{n\in\mathbb{Z}} f(y+n-x) \in C^\infty((\mathbb{Z}\backslash\mathbb{R})^2)$. We can compute the trace of $R(f)$ in two ways. On the one hand,

$$\operatorname{tr} R(f) = \int_{\mathbb{T}} K_f(x,x)\ dx = \sum_{n\in\mathbb{Z}} f(n).$$

On the other hand, we can diagonalize $R(f)$ using the orthonormal basis $e_n = e^{2\pi i n \cdot}$, $n \in \mathbb{Z}$. In fact, $R(f)e_n = \hat{f}(n)e_n$. Therefore $\operatorname{tr} R(f) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$. Altogether,

$$\boxed{\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).}$$

The Poisson summation formula admits very many applications. We will only describe a typical one. Taking the dilation of f by x we get

$$\sum_{n\in\mathbb{Z}} f\left(\frac{n}{x}\right) = x \sum_{n\in\mathbb{Z}} \hat{f}(nx).$$

If all the derivatives of f are in $L^1(\mathbb{R})$ (for example, if f is a Schwartz function) then $\hat{f}$ is rapidly decreasing and we get

$$\frac{1}{x}\sum_{n\in\mathbb{Z}} f\left(\frac{n}{x}\right) = \int_{\mathbb{R}} f(y)\ dy + O(x^{-N}) \tag{1}$$

for all $N > 0$. Thus, the Riemann sum on the left-hand side is an extremely good approximation for the integral of f.

3 The hyperbolic plane

We refer to [Iwa02] and [Ter85] for a more elaborate discussion. Let

$$\mathbb{H} = \{x + \mathrm{i}y : y > 0\}$$

be the upper half-plane. The group $G = SL(2,\mathbb{R})$ acts transitively by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

The stabilizer of the point i is $K = SO(2)$ and therefore we can identify $\mathbb{H}$ with $G/K = SL(2,\mathbb{R})/SO(2)$.

The space $\mathbb{H}$ is a Riemannian manifold with the metric $ds^2 = \frac{dx^2+dy^2}{y^2}$, and volume form $\frac{dx\ dy}{y^2}$. The effect of the factor y^2 in the denominator is dramatic: under this metric $\mathbb{H}$ is of constant *negative* curvature, and its geometry is very different from the Euclidean one. The metric and the volume form defined above are invariant under the action of G. The Laplacian of $\mathbb{H}$ is the second order differential operator given by

$$\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

It is negative-definite and commutes with the action of G. In fact, any differential operator on $\mathbb{H}$ which commutes with the G-action is a polynomial in Δ.

3.1 Spectral analysis on $\mathbb{H}$

Just like for the Euclidean plane, one can write down a basis of eigenfunctions for Δ and obtain a rather explicit spectral decomposition with respect to this basis. In fact, there are (at least) two natural ways to do it, corresponding to the rectangular and polar coordinates of $\mathbb{H}$. One solves the partial differential equation $\Delta f + \lambda f = 0$ by separating the variables. The rectangular coordinates give rise to the solutions

$$W_s(z) = \sqrt{y} K_s(2\pi y) e^{2\pi \mathrm{i} x}, \quad s \in \mathbb{C}$$

where K_s is the Bessel function. More precisely, W_s is an eigenfunction for Δ with $\lambda = \frac{1}{4} - s^2$. Similarly, $W_s(rz)$, $r > 0$. Any $f \in C_c^\infty(\mathbb{H})$ can be expanded as

$$f(z) = \frac{1}{2\pi} \int_0^\infty \int_0^\infty (f, W_{\mathrm{i}t}(r\cdot))_{\mathbb{H}} W_{\mathrm{i}t}(rz) t \sinh \pi t \; dt \; \frac{dr}{r}.$$

The polar coordinates give rise to an alternative expansion

$$f(z) = \sum_{m \in \mathbb{Z}} \int_0^\infty (f, U_{\mathrm{i}t}^m) U_{\mathrm{i}t}^m(z) t \tanh \pi t \; dt$$

where the U_s^m are given in terms of Legendre functions. This is especially useful when f is K-invariant, that is, it depends only on $\rho(z, \mathrm{i})$, where ρ is the hyperbolic distance. In this case only the term $m = 0$ appears.

Suppose that Γ is a discrete subgroup (say, torsion-free) of G which is co-compact in $\mathbb{H}$. (That is, Γ is a uniform lattice in G.) Let X be the quotient space $\Gamma\backslash\mathbb{H}$. It is called a compact hyperbolic surface. By the uniformization theorem for surfaces, the hyperbolic surfaces are precisely the compact Riemann surfaces of genus > 1. The Laplacian Δ on $\mathbb{H}$ descends to X, being invariant under G. However, in contrast to $\mathbb{H}$, the (unbounded) operator Δ has pure point spectrum on X, since X is compact. How to analyze its spectrum?

The problem is that it is very difficult to write down eigenfunctions explicitly. A natural procedure would be to average an eigenfunction of $\mathbb{H}$ over Γ. However, this will rarely converge. Even when it does, it may be very difficult to show that it is not zero!

Nevertheless, one can show that eigenfunctions do exist and give a precise asymptotics for their number. In fact, this is known in great generality (for any compact Riemannian manifold and any elliptic differential (or even pseudo-differential) operator) – see [Hör68].

We will outline a proof of this fact using the trace formula, which we introduce next.

4 Selberg's trace formula – Group theoretic formulation

Let G be any locally compact group and Γ a uniform lattice in G. (In particular, G is unimodular.) Let R be the regular representation of G on $L^2(\Gamma\backslash G)$

$$[R(g)\phi](x) = \phi(xg) \quad g \in G, x \in \Gamma\backslash G.$$

We can extend R to bounded measures on G. In particular, fixing a Haar measure dg on G we define a representation of the algebra $L^1(G)$ (with respect to convolution) by

$$R(f)\phi(x) = \int_G f(g)\phi(xg)\ dg = \int_G f(x^{-1}g)\phi(g)\ dg.$$

Suppose that $f \in C_c^\infty(G)$. By splitting the integral, we can write

$$R(f)\phi(x) = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma g)\phi(g)\ dg = \int_{\Gamma\backslash G} K_f(x,y)\phi(y)\ dy.$$

Thus, $R(f)$ is an integral operator with smooth kernel

$$K_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y). \tag{2}$$

In particular, $R(f)$ is trace class (X is compact!) and we can compute its trace in two ways. First we can write

$$\operatorname{tr} R(f) = \int_{\Gamma\backslash G} K_f(x,x)\ dx = \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma} f(x^{-1}\gamma x)\ dx.$$

We can break the sum over γ into conjugacy classes of Γ. The conjugacy class

$$[\gamma] = \{\delta^{-1}\gamma\delta : \delta \in \Gamma_\gamma\backslash\Gamma\}$$

where Γ_γ is the centralizer of γ in Γ contributes

$$\int_{\Gamma\backslash G} \sum_{\delta\in\Gamma_\gamma\backslash\Gamma} f(x^{-1}\delta^{-1}\gamma\delta x)\ dx = \int_{\Gamma_\gamma\backslash G} f(x^{-1}\gamma x)\ dx = \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) I(\gamma, f)$$

where $I(\gamma, f)$ is the orbital integral

$$I(\gamma, f) = \int_{G_\gamma\backslash G} f(x^{-1}\gamma x)\ dx.$$

Note that Γ_γ is a uniform lattice in G_γ (exercise). All in all,

$$\operatorname{tr} R(f) = \sum_{[\gamma]} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) I(\gamma, f).$$

Alternatively, we can compute $\operatorname{tr} R(f)$ by recalling that according to a result due to Gelfand, Graev and Piatetski-Shapiro, $L^2(\Gamma\backslash G)$ decomposes discretely into a direct sum of irreducible representations of G, each occurring with finite multiplicity ([GGPS90]). Thus,

$$\operatorname{tr} R(f) = \sum_{\pi\in\hat{G}} m(\pi) \operatorname{tr} \pi(f)$$

where $\hat{G}$ is the unitary dual of G, $m(\pi)$ is the multiplicity of π in $L^2(\Gamma\backslash G)$ and $\operatorname{tr}\pi(f)$ is the trace of the operator $\int_G f(x)\pi(x)\,dx$ in the space of π.

Comparing the two we obtain the trace formula equality

$$\boxed{\sum_{\{\gamma\}} \operatorname{vol}(\Gamma_\gamma\backslash G_\gamma) I(\gamma, f) = \sum_{\pi\in\hat{G}} m(\pi)\operatorname{tr}\pi(f)}$$

underlying the duality between conjugacy classes and irreducible representations.

Note that in the left-hand (geometric) side the first factor depends on Γ but not on f while the second depends on f but not on Γ. Similarly for the right-hand (spectral) side.

The distributions $I(\gamma, f)$ and $\operatorname{tr}\pi(f)$ are *invariant* in the sense that they vanish on any commutator (with respect to convolution); alternatively, they are invariant under conjugation of f by an element of G.

5 Geometric interpretation

To make the trace formula useful we have to understand the relation between the invariant distributions $I(\gamma, f)$ and $\operatorname{tr}\pi(f)$, and to cast them in differential geometric terms.

We specialize to the case $G = SL(2,\mathbb{R})$. We first recall the well-known classification of the irreducible representations of G. (See [HT92] for more details and beautiful applications of the representation theory of $SL(2,\mathbb{R})$.) Recall that there is a unique irreducible representation of G of dimension n for $n = 1, 2, \dots$ namely the symmetric $n-1$ power Sym^{n-1} of the standard two-dimensional representation. Consider the representation of G on the space of (smooth) functions on $\mathbb{R}^2 \setminus \{0\}$ (with G acting on the right). This representation is far from irreducible. In fact, for any (not necessarily unitary) character χ of $\mathbb{R}^*$ we can consider the subspace π_χ of those functions such that $f((rx, ry)) = (\chi(r)|r|)^{-1} f((x, y))$ for all $r \in \mathbb{R}^*$. (The factor $|r|$ is a convenient normalization factor.) It turns out that the π_χ's completely describe the irreducible representations of G. The characters of $\mathbb{R}^*$ are of the form $\chi(r) = |r|^s$ or $|r|^s \operatorname{sgn} r$ where sgn denotes the signum function. Since π_χ and $\pi_{\chi^{-1}}$ have the same character it is enough to consider the case $\operatorname{Re}(s) \geqslant 0$. The representation π_χ is irreducible unless χ is one of the characters $\chi_n(r) = r^n \operatorname{sgn}(r)$, $n \in \mathbb{Z}$. For $\chi = \chi_n$, $n \geqslant 0$ the representation π_χ has length three (two if $n = 0$), namely it has a unique irreducible quotient (for $n \neq 0$), equivalent to Sym^{n-1}, and two inequivalent irreducible subrepresentations $\pi_n^\pm$. Conversely, if π is an irreducible representations of G then exactly one of the following holds

1. π is of finite dimension n, in which case it is equivalent to the irreducible quotient Sym^{n-1} of π_{χ_n}, or,
2. π is equivalent to π_χ, $\chi \neq \chi_n$, $n \in \mathbb{Z}$; χ is uniquely determined up to taking inverse, or,
3. π is equivalent to $\pi_n^\pm$ for a unique $n \in \mathbb{Z}_{\geqslant 0}$ and sign $\pm$.

Note that the finite-dimensional representations of G (except for the 1-dimensional one) are not unitarizable. For any $s \in \mathbb{C}$ the representation $\pi_{|.|^s}$ admits a unique irreducible subquotient π_s which has a vector fixed under $SO(2)$, and such a vector (called *spherical vector*) is unique up to scalar. In fact, $\pi_s = \pi_{|\cdot|^s}$ unless s is an odd integer, in which case $\pi_s \simeq \mathrm{Sym}^{|s|-1}$. Note that $\pi_s \simeq \pi_{s'}$ if and only if $s = \pm s'$. Moreover, π_s is unitarizable if and only if $s \in \mathrm{i}\mathbb{R} \cup [-1, 1]$. The representations $\pi_n^{\pm}$, $n \in \mathbb{Z}_{>0}$ are exactly the irreducible subrepresentations of $L^2(G)$. They are therefore unitarizable and constitute the *discrete series* of G. (The representations $\pi_0^{\pm}$ are sometimes called *limits of discrete series*. They are also unitarizable.)

To tie the representation theoretic context to our previous discussion we note that eigenfunctions of the Laplacian on X with eigenvalue $(1-s^2)/4$ correspond to isometric embeddings of π_s in $L^2(\Gamma\backslash G)$. This is a simple application of Frobenius reciprocity, together with the fact that the action of Δ corresponds to the action of the Casimir element (the generator of the center of the universal enveloping algebra) on the spherical vector. Thus the multiplicity of $(1 - s^2)/4$ is exactly $m(\pi_s)$.

Suppose that $f \in C_c^\infty(G//K)$, that is f is bi-K-invariant. We can think of f as a K-invariant compactly supported function on $\mathbb{H}$. As such it depends only on $\rho(\cdot, \mathrm{i})$. On the spectral side $\operatorname{tr} \pi(f)$ is non-zero only if $\pi = \pi_s$ for some $s \in \mathbb{C}$. In fact, $\operatorname{tr} \pi(f)$ is simply the scalar by which $\pi(f)$ acts on the spherical vector.

Let $A = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t > 0 \right\}$ be the subgroup of positive diagonal elements and $N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$. Then NA is an index two subgroup of the group of upper triangular matrices and $G = NAK$ (Gram-Schmidt). We define the Abel transform of f by

$$\mathcal{A}(f)(t) = \int_{\mathbb{R}} f\left(\begin{pmatrix} e^{t/2} & x \\ 0 & e^{-t/2} \end{pmatrix} \right) dx.$$

This is closely related to the hyperbolic orbital integral of f. In fact, by a change of variable and the NAK decomposition we have for $t \neq 0$

$$\begin{aligned}
\mathcal{A}(f)(2t) &= |e^t - e^{-t}| \int_N f\left(n^{-1} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} n \right) dn \\
&= |e^t - e^{-t}| \int_{T\backslash G} f\left(g^{-1} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} g \right)\ dg \\
&= |e^t - e^{-t}| I\left(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, f \right).
\end{aligned}$$

It turns out that $\mathcal{A}$ is an isomorphism of algebras between $C_c^\infty(G//K)$ and $C_c^\infty(\mathbb{R})^{even}$. We can recover f from its Abel transform h (or rather, from the Fourier transform $\hat{h}$ of h) by the Plancherel inversion formula

$$f(e) = \int_{\mathbb{R}} \hat{h}(r) r \tanh r\ dr.$$

Moreover, it is easy to see from the polar decomposition $G = KAK$ that $\hat{h}(r) = \operatorname{tr} \pi_{2\mathrm{i}r}(f)$.

Finally, the non-trivial conjugacy classes in Γ exactly correspond to the closed geodesics in $\Gamma\backslash X$. (Since Γ was assumed to be torsion free, all conjugacy classes are hyperbolic.) If γ is conjugate to $\begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ then the length $l(\gamma)$ of the corresponding closed geodesic is t. Moreover, $\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma) = \log(l(\gamma_0))$ where $\gamma = \gamma_0^k$, $\gamma_0 \in \Gamma$ and k is maximal with respect to this property (i.e., γ_0 is primitive). Altogether, Selberg's trace formula takes the form

$$\sum \hat{h}(r_n) = \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H})}{2\pi} \int_{\mathbb{R}} \hat{h}(r) r \tanh(\pi r)\, dr + \sum \frac{\log l(\gamma_0)}{e^{l(\gamma)/2} - e^{-l(\gamma)/2}} h(l(\gamma))$$

where on the left-hand side $\frac{1}{4} + r_n^2$ ranges over the eigenvalues of the Laplacian while on the right-hand side γ ranges over the non-trivial conjugacy classes of Γ (i.e. closed geodesics) and γ_0 is as before.

Thus, viewed as an identity of distributions on h the Selberg trace formula is similar in shape to the Poisson summation formula. The non-identity geometric terms and the spectral terms are atomic distributions of h and $\hat{h}$ respectively. The roles of the integers are played by the Laplace eigenvalues on the left-hand side and by the length of closed geodesics on the right-hand side (and there are also weights which have to be taken into account).

6 Application: Weyl's law

We first recall Weyl's classical result. Let Ω be a bounded region in the Euclidean plane with smooth boundary $\partial\Omega$. Consider the Euclidean Laplacian $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ on the plane. We want to count solutions of the differential equation

$$\Delta\phi + \lambda\phi = 0, \quad \lambda \geqslant 0$$

with Dirichlet boundary condition $\phi|_{\partial\Omega} = 0$. Let $N_\Omega(R)$ be the counting function for the number of linearly independent solutions with $\lambda \leqslant R$. Weyl's result is

$$N_\Omega(R) \sim \frac{\mathrm{Area}(\Omega)}{4\pi} R \text{ as } R \to \infty.$$

In a more modern setting we consider a compact Riemannian d-manifold (M, g) and its Laplacian $\Delta = \operatorname{div}\operatorname{grad}$. This is a negative definite (unbounded) operator on $L^2(M, g)$. Let N_T be the counting function for the number of eigenfunctions with $\lambda \leqslant T^2$. In this context Weyl's law is

$$N_T \sim \frac{\mathrm{vol}(M)}{(4\pi)^{d/2}\mathbf{\Gamma}(d/2+1)} T^d \text{ as } T \to \infty.$$

This was proved by Minakshisundaram and Pleijel ([MP49]). As mentioned before, a stronger statement with remainder term $O(T^{d-1})$ was later proved by Hörmander, in a much more general context. (See also [DG75].)

Consider for example the sphere S^2. The eigenfunctions of Δ are the *spherical harmonics*. The eigenvalues are $n(n+1)$, $n = 0, 1, 2, \ldots$ with multiplicity $2n+1$.

Next, consider the two-dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$. Once again, we can write down the eigenfunctions explicitly, in this case as periodic exponential functions. The eigenvalues are $4\pi(n^2+m^2)$. We can approximate $\#\{(n,m) : n^2+m^2 < R^2\}$ by the area of the circle of radius R. A trivial upper bound for the remainder term is $O(R)$. To find the exact order of magnitude of the remainder term is a much more serious question called Gauss' circle problem.

Now we return to the compact hyperbolic surfaces M of the form $\Gamma\backslash\mathbb{H}$ where Γ is a uniform lattice of $SL(2,\mathbb{R})$. Let $\lambda_0 = 0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ be the eigenvalues of Δ acting in $L^2(\Gamma\backslash\mathbb{H})$. It is useful to set $\lambda_j = \frac{1}{4} + r_j^2$ with $r_j \in \mathbb{R}_{\geqslant 0} \cup \left[0, \frac{1}{2}\right]$ i, and $r_{-j} = -r_j$. Thus, $N_T = \#\{j : |r_j| \leqslant T\}$. We will use the trace formula only for $h \in C_c^\infty(\mathbb{R})^{even}$ supported near 0 so that their inverse Abel transform vanishes on any conjugate of a non-trivial element of Γ. In this case Selberg's trace formula simplifies to

$$\sum_{j=-\infty}^{\infty} \hat{h}(r_j) = \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H})}{2\pi} \int_{\mathbb{R}} \hat{h}(r) r \tanh(\pi r)\, dr.$$

We can assume that h as well as its Fourier transform $\hat{h}$ are non-negative and $\hat{h}$ is also non-negative on $[-\frac{1}{2}, \frac{1}{2}]$ after analytic continuation. For $t \in \mathbb{R}$ let

$$h_t(a) = \frac{1}{2} h(a)(e^{\mathrm{i}at} + e^{-\mathrm{i}at})$$

so that

$$\widehat{h_t}(r) = \frac{1}{2}(\hat{h}(t-r) + \hat{h}(t+r)).$$

We use the trace formula with h_t to get

$$\sum_j \hat{h}(t-r_j) = \mathrm{Area}(M) f_t(1) = \frac{\mathrm{Area}(M)}{4\pi} \int_{\mathbb{R}} \hat{h}(t-r) r \tanh(\pi r)\, dr. \qquad (3)$$

Since $|\tanh| \leqslant 1$ the right-side is $O(T)$. We infer that

$$\#\{j : |r_j - T| \leqslant 1\} = O(T). \qquad (4)$$

Next, we integrate the equality (3) over $t \in [-T, T]$. On the left-hand side,

$$\int_{-T}^{T} \sum_j \hat{h}(t-r_j) dt = N_T + \sum_{|r_j|>T} \int_{-T}^{T} \hat{h}(t-r_j)\, dt - \sum_{|r_j| \leqslant T} \int_{T}^{\infty} \hat{h}(t-r_j) dt.$$

Using (4) it is easy to see that this is $N_T + O(T)$. On the other hand, on the right-hand side it is easy to see by integration by parts that

$$\int_{-T}^{T} \int_{\mathbb{R}} \hat{h}(t-r) r \tanh(\pi r)\, dr\, dt = T^2 + O(T).$$

Altogether $N_T = \frac{\mathrm{Area}(M)}{4\pi} T^2 + O(T)$ as required.

What happens for $\Gamma = SL(2,\mathbb{Z})$? This is a case of a non-uniform lattice. The fundamental domain is the familiar hyperbolic triangle

$$\{z \in \mathbb{H} : |z| \geqslant 1 \text{ and } |\mathrm{Re} z| \leqslant 1/2\}.$$

It has a cusp at ∞, so it is not compact; yet, $\mathrm{vol}(\Gamma\backslash G) < \infty$. If we try to apply the trace formula approach there are several important differences. The operator $R(f)$, while still an integral operator with smooth kernel given by (2), is *not* of trace class, or even compact, since $K_f(x,y)$ is not in $L^2(\Gamma\backslash G\times\Gamma\backslash G)$. Likewise, the kernel is not integrable over the diagonal. Moreover, $L^2(\Gamma\backslash G)$ does not decompose discretely – it admits continuous spectrum as well. Finally, the terms $\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)$ appearing on the geometric side, could be infinite.

In order to deal with these issues Selberg regularized the integral of the kernel over the diagonal by truncating the kernel in a suitable way.

The upshot is an identity with additional terms coming from the continuous spectrum on the one hand and parabolic conjugacy classes on the other hand. To describe the continuous contribution we define the Eisenstein series

$$E(z;s) = \sum_{(m,n)=1} \frac{y^{s+\frac{1}{2}}}{|mz+n|^{2s+1}} = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} y(\gamma z)^{s+\frac{1}{2}} \quad z \in \mathbb{H}$$

where $\Gamma_\infty = \left\{\begin{pmatrix} 1 & n \\ 0 & 1\end{pmatrix} : n \in \mathbb{Z}\right\}$. The series converges for $\mathrm{Re}\, s > \frac{1}{2}$ and $E(\gamma z;s) = E(z;s)$ for all $\gamma \in \Gamma$. Also, $\Delta E(\cdot;s) = (\frac{1}{4} - s^2)E(\cdot;s)$ since this is true for the function $y^{s+\frac{1}{2}}$ and E is obtained from it by averaging. Less evident is the fact that $E(z,\cdot)$ admits meromorphic continuation to $\mathbb{C}$ and a functional equation

$$E(z;s) = \phi(s)E(z;-s), \quad \phi(s) = \frac{\sqrt{\pi}\Gamma(s)\zeta(2s)}{\Gamma(s+\frac{1}{2})\zeta(2s+1)}.$$

While $E(z,s)$ admits many poles for $\mathrm{Re}(s) < 0$, it is holomorphic on $\mathrm{Re}\, s \geqslant 0$ except for a simple pole at $s = \frac{1}{2}$ with a constant residue (as a function of z).

Let

$$L^2(\Gamma\backslash\mathbb{H}) = L^2_{disc}(\Gamma\backslash\mathbb{H}) \oplus L^2_{cont}(\Gamma\backslash\mathbb{H})$$

be the spectral decomposition of Δ into the discrete and continuous parts respectively. Then the map $L^2(\mathbb{R})^{even} \to L^2(\Gamma\backslash\mathbb{H})$ given by

$$f \mapsto Ef = \int f(t)E(z;\mathrm{i}t)\, dt$$

is an isometry onto $L^2_{cont}(\Gamma\backslash\mathbb{H})$ and

$$\Delta(Ef) = E\left(\left(\frac{1}{4}+t^2\right)f\right).$$

Alternatively, any $f \in L^2(\Gamma\backslash\mathbb{H})$ admits a decomposition

$$f(z) = \sum_j (f,u_j)u_j(z) + \frac{1}{4\pi}\int_{-\infty}^{\infty} (f,E(\cdot;\mathrm{i}t))E(z;\mathrm{i}t)dt$$

into eigenfunctions where the first sum is taken over an orthonormal basis of eigenfunctions of Δ in $L^2(\Gamma\backslash\mathbb{H})$. Equivalently,

$$||f||_2^2 = \sum_j |(f, u_j)|^2 + \frac{1}{4\pi}\int_{-\infty}^{\infty} |(f, E(\cdot; \mathrm{i}t))|^2 dt.$$

The main additional term in the trace formula coming from the continuous spectrum is

$$-\frac{1}{2\pi}\int_{\mathbb{R}} \hat{h}(r)\frac{\phi'}{\phi}(\mathrm{i}r)dr.$$

In the general case ϕ has to be interpreted as the determinant of the scattering matrix. (There are additional terms in the trace formula, which we won't specify.) Using the trace formula (in the non-compact case) Selberg showed

$$N_T + M_T \sim \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H})}{4\pi}T^2$$

where M_T is the winding number ϕ, namely $-\frac{1}{4\pi}\int_{-T}^{T}\frac{\phi'}{\phi}(\mathrm{i}t)\ dt$. Roughly, it is the counting function for the poles of ϕ (on $\mathrm{Re}(s) < 0$) with imaginary part $< T$. This is the contribution of continuous spectrum.

In the case where Γ is a congruence subgroup Selberg proved that ϕ has order 1 and hence $M_T = o(T^{1+\varepsilon})$ for every $\varepsilon > 0$. Thus, the Weyl law holds. (See [Sel89, p. 668], [Hej76b]). This is remarkable because for $SL(2,\mathbb{Z})$ we do not know how to write down explicitly a single Maass form (=eigenfunction of Δ) in a closed form. (See [BSV06] and [BS07] for numerical aspects of Maass forms and the trace formula.)

What happens for non-uniform lattices which are not congruence subgroups?

Selberg believed that the Weyl law should hold for them as well. However, the work of Phillips and Sarnak on the dissolution of cusp forms under deformation of congruence subgroups suggests quite the contrary. In fact, it may very well be the case that the discrete spectrum is finite for a generic Γ in a family! (See [PS92] and the literature cited there for more details. Also cf. Wolpert [Wol94] and Luo [Luo01] where results toward the invalidity of Weyl's law are proved.)

The error term in Weyl's law trivially gives a bound on the multiplicity of a given Laplace eigenvalue. Remarkably, this bound is difficult to substantially improve. See [Sar03] for this and related problems.

The Weyl law was obtained by localizing the geometric side of the trace formula. What happens if we localize the spectral side instead? We obtain information on the object dual to the Laplacian spectrum, namely the *length spectrum* – the set of lengths of closed geodesics of M. In fact, one obtains an analogue of the Prime Number Theorem for the lengths of closed geodesics. A remarkable example was obtained by Sarnak in his thesis, where he considered the case of $SL(2,\mathbb{Z})$ and obtained the asymptotic behavior of the average of class numbers of real quadratic fields, ordered by the size of the fundamental unit ([Sar82]).

Pushing the analogy between prime numbers and closed geodesics further, Selberg defined a zeta function, which bears his name, which is an "Euler product" (in a non-orthodox sense) over the closed geodesics, and proved analytic properties of it [Sel56]. For more details see [Vig78]. There are some attempts in the literature to generalize the Selberg zeta functions to higher dimensional situations, but a lot remains to be done.

7 Eichler-Selberg trace formula

In the same celebrated 1956 paper of Selberg [Sel56] he developed, along with the trace formula for Maass forms, a formula for traces of Hecke operators acting on modular forms of weight k. In fact, a special case of this formula was published in the preceding issue of the same Journal by Eichler ([Eic55]). The difference from the previous case is that instead of functions on $X = \Gamma\backslash\mathbb{H}$ we look at sections of a certain line bundle on X. Roughly speaking the Maass form case corresponds to $k = 0$. We can treat both cases in a uniform way group theoretically by considering test functions on G with K-type k (with $k = 0$ corresponding to spherical functions). To describe Eichler-Selberg trace formula let $\Gamma = SL(2, \mathbb{Z})$, $S_{k+1}(\Gamma)$ the space of cuspidal modular forms of weight $k + 1$, and let $T(n)$ be Hecke operator on $S_{k+1}(\Gamma)$. Then

$$-2\mathrm{tr}\, T(n) = \sum_{r:r^2\leqslant 4n} P_k(r,n)H(4n-r^2) + \sum_{dd'=n} \min(d,d')^k \tag{5}$$

where $P_k(r,n)$ is the coefficient of x^{k-1} in $(1 - rx + nx^2)^{-1}$ and $H(n)$ is the Hurwitz class number, counting (not necessarily primitive) binary quadratic forms of discriminant $-n$ up to equivalence. Note that

$$P_k(r,n) = \frac{\rho^k - \overline{\rho}^k}{\rho - \overline{\rho}} \text{ where } \rho^2 - r\rho + n = 0.$$

We refer to [Lan95] for the derivation of the trace formula in this context.

Let us describe a curious application. Recall that the L-function of a Hecke eigenform is given by

$$L(s,f) = \sum_{n=1}^{\infty} \lambda_n(f) n^{-(s+\frac{k}{2})}$$

where $\lambda_n = \lambda_n(f)$ is the eigenvalue of f under T_n. Thus,

$$\sum_{n=1}^{\infty} \mathrm{tr}\, T(n) n^{-(s+\frac{k}{2})} = \sum_f L(s,f)$$

where f ranges over an orthonormal basis of Hecke eigenforms of $S_{k+1}(\Gamma)$. By (5) this is related to

$$\sum_{m=1}^{\infty} H(m) m^{-s} \left(\frac{1}{\sqrt{m}} \sum_{r\in\mathbb{Z}} \frac{\mathrm{Im}(\frac{r}{\sqrt{m}} + \mathrm{i})^k}{|\frac{r}{\sqrt{m}} + \mathrm{i}|^{2s+k}} \right).$$

(We will suppress the other term on the right-hand side of (5), although it can be handled as well.) By the Poisson summation formula (cf. (1)) the Riemann sum in the parentheses of the expression above is equal to

$$\int_{\mathbb{R}} \frac{\operatorname{Im}(x+\mathrm{i})^k}{(x^2+1)^{s+\frac{k}{2}}}\,dx + O(m^{-N})$$

for all $N > 0$, where the implied constant depends only on N and k. The integral can be evaluated explicitly in terms of $\mathbf{\Gamma}$-functions. Thus, the analytic properties of $L(s,f)$ are governed by

$$\sum_{d=1}^{\infty} H(d)d^{-s}. \tag{6}$$

This is a well-known object whose analytic properties can be studied from several points of view. It is essentially the Shintani zeta function pertaining to the prehomogeneous space of binary quadratic forms (see [Shi75], [Yuk93]) for a general setup). It is also the Mellin transform of a weight 3/2 Eisenstein series. We refer to [Fri] for more information about this and related objects. The real quadratic analogue of (6) is closely related to Selberg's zeta function for $SL(2,\mathbb{Z})$ (cf. [Vig79]).

The analytic properties of Hecke L-functions of modular forms are of course well-known, and can be seen more directly. However, the meromorphic continuation of symmetric power L-functions of modular forms is still wide open. One may try to approach them in the same way as above. The picture becomes much more complicated, and the whole approach seems dubious. However, on a very crude level one can see that in a suitable sense, the main term in the geometric side "is" what is expected from the spectral side. For more details I refer to the notes "A different look at the trace formula" available on my home page. This approach is an outgrowth of an idea of Langlands who tried to analyze the logarithmic derivatives of higher symmetric power L-functions using the trace formula. We refer the interested reader to [Art05, p. 251-257] and [Lan07].

In companion with the Eichler-Selberg trace formula there is an older formula due to Petersson. It states that for any $m,n \geqslant 1$

$$\frac{(k-1)!}{(4\pi\sqrt{mn})^k}\sum_f a_m(f)\overline{a_n(f)} = \delta_{m,n} + \frac{2\pi}{\mathrm{i}^{k-1}}\sum_{c=1}^{\infty}\frac{S(n,m;c)}{c}J_k\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

where f ranges over an orthonormal basis in $S_{k+1}(\Gamma)$, $a_n(f)$ is the n-th Fourier coefficient, $S(n,m;c)$ is the Kloosterman sum

$$S(n,m;c) = \sum_{x,y\in(\mathbb{Z}/c\mathbb{Z})^*:xy=1} e^{\frac{2\pi\mathrm{i}(nx+my)}{c}}$$

and J_k is the k-th Bessel function

$$J_k(t) = \oint e^{t(\tau-\frac{1}{\tau})}\frac{d\tau}{\tau^{k+1}}.$$

This formula, originally derived from a computation of Fourier coefficients of Poincare series, has a representation theoretic interpretation, and a counterpart for Maass forms. This is known as the Kuznetsov trace formula. The advantage of Petersson's formula over the Eichler-Selberg trace formula is that it does not involve the class numbers, which are difficult to handle. Of course, the Kloosterman sums are also far from trivial objects, but there is a whole slew of algebraic geometric tools to work with them. It is therefore not surprising that it has a lot of applications. Unfortunately we will not discuss any of them here. We would however at least like to mention one of them, communicated to me by Akshay Venkatesh (unpublished). Let ϕ be an L^2-normalized Hecke-Maass eigenform on $SL(2,\mathbb{Z})\backslash\mathbb{H}$. There is an intriguing formula of Waldspurger expressing $\phi(\mathrm{i})$ in terms of L-functions. Roughly,

$$|\phi(\mathrm{i})|^2 \sim L\left(\frac{1}{2},\pi\right) L\left(\frac{1}{2},\pi\otimes\left(\frac{-4}{\cdot}\right)\right).$$

We can try to prove this formula using the Kuznetsov trace formula. The idea is to consider

$$\sum_n |\phi_n(\mathrm{i})| \hat{h}(\lambda_n) \tag{7}$$

as (the discrete part of) the kernel at the identity, i.e. as

$$\sum_{\gamma\in\Gamma} F_h(\rho(\gamma \mathrm{i},\mathrm{i}))$$

where F_h is an appropriate transform of h. On the other hand, we can write

$$\sum_\pi \hat{h}(\pi) L\left(\frac{1}{2},\pi\right) L\left(\frac{1}{2},\pi\otimes\left(\frac{-4}{\cdot}\right)\right)$$

in terms of Kuznetsov formulas by expanding the L-functions using the "approximate functional equation". By an ingenious argument one can prove the spectral comparison by comparing the geometric sides. The hypothetical equality boils down to a certain identity involving Kloosterman sums.

A similar idea appears in a work by Iwaniec on yet another formula of Waldspurger pertaining to half-integer modular forms [Iwa87]. It is also worth mentioning Venkatesh' thesis where he uses limiting forms of the Kuznetsov formula to obtain (previously known) cases of functoriality [Ven04].

In passing, let us note that studying the expression (7) from a different angle is related to the hyperbolic *lattice point problem.* A typical result is

$$\#\{\gamma\in SL(2,\mathbb{Z}) : ||\gamma||^2 \leqslant x\} \sim 6x$$

where $\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|^2 = a^2+b^2+c^2+d^2$. See [Ter85, §3.7, Example 1] and the references cited therein.

See [LY06] and [KL06a] for more about the Petersson and Kuznetsov trace formulas and applications to analytic number theory.

8 Jacquet-Langlands correspondence

One of the striking applications of the trace formula for $GL(2)$ is the Jacquet-Langlands correspondence between the automorphic representations of a quaternion algebra and those of $GL(2)$. In fact, a precursor of this correspondence appears in the works of Eichler ([Eic56]) and Shimizu ([Shi63]).

To motivate it, let us first recall that the irreducible representations of the compact group $SO(3)$ are determined (up to equivalence) by their dimension, which can be any odd positive integer. We write σ_n for the n-dimensional irreducible representation of $SO(3)$ (n odd). Similarly, the irreducible square-integrable representations of $PGL(2,\mathbb{R})$ are indexed by odd positive integers. We write them as π_n (with π_1 equal to the Steinberg representation). If we denote by χ_π the character of a representation (viewed as a function) then

$$\chi_{\pi_n}\left(\begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}\right) = -\chi_{\sigma_n}\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \theta\in[0,2\pi].$$

In fact, π_n together with the n-dimensional representation of $PGL(2,\mathbb{R})$ comprise the irreducible subquotients of an induced representation, whose character vanishes on the elliptic elements.

This picture is the Archimedean aspect of the local Jacquet-Langlands correspondence. To describe the p-adic aspect we recall that any p-adic field F admits a unique quaternion algebra D with center F. The multiplicative group $D^*=D\backslash\{0\}$ of D is an inner form of $G=GL(2,F)$. Moreover, the conjugacy classes of D^* correspond bijectively to the elliptic conjugacy classes of G. If x maps to g under this correspondence then the reduced trace of x is equal to the trace of g and the reduced norm Nrd of x is equal to the determinant of g. The local Jacquet-Langlands correspondence asserts a bijection between the equivalence classes of the irreducible representations of D^* and those of the square-integrable representations of G. If σ maps to π under this correspondence then $\chi_\sigma(x)=-\chi_\pi(g)$ whenever the conjugacy classes of x and g correspond. Philosophically this may seems strange at first because finite-dimensional representations correspond to infinite-dimensional ones!

There is a closely related, and even more striking, global statement. Recall that the quaternion algebras over $\mathbb{Q}$ are in one-to-one correspondence with finite subsets of places of $\mathbb{Q}$ (including the Archimedean one) with even cardinality. This correspondence is achieved by assigning to D the set S of places where it ramified, that is for which $D\otimes_{\mathbb{Q}}\mathbb{Q}_p$ is a quaternion algebra, rather than $M_2(\mathbb{Q}_p)$. Let D (and therefore S) be given. Set $G'=D^*$, and write $G'(\mathbb{A})$ for the locally compact group of invertible elements of the ring $D\otimes_{\mathbb{Q}}\mathbb{A}$. (The topology on $D^*(\mathbb{A})$ is the one induced from $(D\otimes\mathbb{A})^2$ under the map $x\mapsto(x,x^{-1})$.) Let

$$G'(\mathbb{A})^1=\{x\in G'(\mathbb{A}):|\mathrm{Nrd}(x)|=1\}.$$

Thus, $G'(\mathbb{A})^1$ is a normal subgroup of $G'(\mathbb{A})$ and the quotient is $\mathbb{R}_{>0}$. Then $G'\backslash G'(\mathbb{A})^1$ is compact and therefore $L^2(G'\backslash G'(\mathbb{A})^1)$ decomposes discretely into a direct sum of irreducible representations. The one-dimensional constituents are the characters $\chi \circ \mathrm{Nrd}$ where χ is a Dirichlet character of $\mathbb{Q}^* \backslash \mathbb{I}$. They naturally correspond to the one-dimensional representations $\chi \circ \det$ of $L^2(G(F)\backslash G(\mathbb{A})^1)$.

The global Jacquet-Langlands correspondence asserts the following statements.

1. $L^2(G' \backslash G'(\mathbb{A})^1)$ is multiplicity free, i.e., all irreducible constituents occur with multiplicity one.
2. Suppose that $\sigma = \otimes_v \sigma_v$ is an irreducible constituent which is not one-dimensional. Define $\pi = \otimes_v \pi_v$ where $\pi_v = \sigma_v$ if $v \notin S$ and π_v corresponds to σ_v under the local Jacquet-Langlands correspondence if $v \in S$. Then π is a cuspidal representation of $G(\mathbb{A})$.
3. Conversely, any cuspidal representation $\pi = \otimes \pi_v$ of $G(\mathbb{A})$ such that π_v is square-integrable for all $v \in S$ is obtained from an automorphic representation of G' by the above procedure.

Since the main terms in the spectral side of the trace formula are traces of representations, it seems very plausible to try to prove these assertions using the trace formula. In fact, the idea would be to compare the trace formula for the two groups G' and G. On the geometric side, the basic fact to bear in mind is that the conjugacy classes of $G'(F)$ are in canonical bijection with the conjugacy classes of $G(F)$ which are elliptic for all $v \in S$.

To explain how the comparison goes we had better rephrase the spectral theory for $GL(2)$ in the adelic setting. Let R be the right regular representation of $G(\mathbb{A})$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Let $f \in C_c^\infty(G(\mathbb{A}))$. That is, f is compactly supported, smooth in the Archimedean variable and bi-invariant under an open subgroup of $G(\mathbb{A}_f)$ where $\mathbb{A}_f$ denotes the finite adeles. The operator $R(f)$ is integral with kernel

$$K_f(x,y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y).$$

The spectral theory for $GL(2)$ gives

$$K_f(x,y) = K_f^{disc}(x,y) + K_f^{cont}(x,y)$$

where

$$K_f^{disc} = \sum_{\{\varphi\}} R(f)\varphi(x)\overline{\varphi(y)},$$

the sum being taken over an orthonormal basis of the discrete part (the sum of the irreducible subrepresentations) of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})^1)$, and

$$K_f^{cont}(x,y) = \sum_{\chi} \sum_{\{\varphi\}} \int_{-\infty}^{\infty} E(x, I(f,\chi,\mathrm{it})\varphi, \mathrm{it})\overline{E(y,\varphi,\mathrm{it})}dt$$

where the sum is over pairs $\chi = (\chi_1, \chi_2)$ of Dirichlet characters and over an orthonormal basis $\{\varphi\}$ of the space

$$I(\chi) = \left\{ \varphi : G(\mathbb{A}) \to \mathbb{C} | \varphi\left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} g \right) = \chi_1(t_1)\chi_2(t_2) \left| \frac{t_1}{t_2} \right|^{\frac{1}{2}} \varphi(g) \right\}$$

with the inner product

$$(\varphi_1, \varphi_2) = \int_{\left\{\left(\begin{smallmatrix} at_1 & * \\ 0 & at_2 \end{smallmatrix}\right): a \in \mathbb{R}_{>0}, t_1, t_2 \in \mathbb{Q}\right\} \backslash G(\mathbb{A})} \varphi_1(g)\overline{\varphi_2(g)}\, dg.$$

On this space there is a family of representations $I(\chi, s)$ given by

$$I(g, \chi, s)\varphi(\cdot) = (\varphi_s(\cdot g))_{-s}$$

where φ_s is defined by

$$\varphi_s\left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} k\right) = \left|\frac{t_1}{t_2}\right|^s \varphi(k).$$

The (adelic) Eisenstein series $E(\varphi, s)$ is defined as the meromorphic continuation of the sum

$$E(g, \varphi, s) = \sum_{\gamma \in B(\mathbb{Q}) \backslash G(\mathbb{Q})} \varphi_s(\gamma g)$$

which converges for $\mathrm{Re}(s) > 1$. Whenever regular it defines an intertwining map from $I(\chi, s)$ to the space of automorphic forms on G.

The idea is to compare the trace of the regular representation on the quotients $G(F)\backslash G(\mathbb{A})$ and $G'(F)\backslash G'(\mathbb{A})$ for matching functions f and f' on $G(\mathbb{A})$ and $G'(\mathbb{A})$ respectively. This is done through the geometric sides of both trace formulas, and ultimately yields the spectral comparison. More precisely, suppose that f' is given. Without loss of generality we can assume that f' is decomposable, that is $f' = \otimes f'_v$ where $f'_v \in C_c^\infty(G'(\mathbb{Q}_v))$. We choose $f = \otimes_v f_v$ on $G(\mathbb{A})$ in the following manner. For places $v \notin S$ we simply take $f'_v = f_v$ once we fix a splitting of D over v to identify $G(F_v)$ with $G'(F_v)$. The orbital integrals of f_v and f'_v will coincide (independently of the choice of splitting). At the places $v \in S$ we take f_v whose hyperbolic orbital integrals vanish and whose elliptic orbital integrals exactly match those of f'_v. (Of course, one has to show that this is possible.)

Some explicit arithmetic consequences of the Jacquet-Langlands correspondence are mentioned in a short note by Langlands (available on his electronic archive).

Another curious outcome of the Jacquet-Langlands correspondence is the existence of non-isometric compact hyperbolic surfaces with the same Laplacian spectrum! ([Vig80]).

9 Higher dimension

What are the analogues of the hyperbolic plane in higher dimension? The clue is in terms of the Riemannian structure (rather than the complex structure, which is special to the hyperbolic plane). The relevant notion is that of a *symmetric space* – a classical notion which goes back to Cartan. We will not give a precise definition. Roughly speaking, it is a Riemannian manifold with a lot of symmetries. In particular, the group of isometries acts transitively with a compact stabilizer. Cartan classified the symmetric spaces. There are three types (according to curvature)

1. The Euclidean plane (sectional curvature 0),
2. Compact type (positive sectional curvature),
3. Non-compact type (negative sectional curvature).

An example of the second type is the sphere S^n, which can be identified with $SO(n+1)/SO(n)$. Examples of the third type (which is the most important to us) include real or complex hyperbolic n-spaces, and the spaces of positive-definite quadratic or hermitian forms up to homothety in $n \geqslant 2$ variables. The general example of the third type is $X = G/K$ where G is a semisimple Lie group and K is its maximal compact subgroup (uniquely determined up to conjugation). The Riemannian metric comes from the Killing form which induces a positive-definite form on $\mathfrak{g}/\mathfrak{k}$ where $\mathfrak{g}$ (resp. $\mathfrak{k}$) is the Lie algebra of G (resp. K).

By definition, a locally symmetric space is a space whose universal cover is a symmetric space. This is especially interesting in the third case – it has the form $\Gamma\backslash G/K$ where G, K is as above and Γ is a torsion free (not necessarily uniform) lattice of G. Of course the Laplacian on G/K commutes with the G action and therefore descends to $\Gamma\backslash G/K$.

What is the known about the Weyl Law in this context?

Let us first consider the compact case. As mentioned before, the Weyl law, with a remainder term, is known for any compact Riemannian manifold.

We recall some facts about harmonic analysis of symmetric spaces. The standard reference is Helgason's books, e.g. [Hel00]; see also [Ter88] and [JL05]. The differential operators $\mathcal{D}$ on X commuting with G form an algebra isomorphic to the polynomial algebra in r variables. Here r is the rank of G (the dimension of a maximal split torus, or alternatively, the dimension of a maximal flat in X). The invariant differential operators descend to the quotient $\Gamma\backslash X$. Therefore it makes sense to ask for joint distributions of eigenvalues, i.e. consider f such that $Df = \lambda(D)f$ for a character λ of $\mathcal{D}$ and consider the multiplicity $m(\lambda)$. Fixing a maximal split torus with connected part A and letting $\mathfrak{a}$ be its Lie algebra, we can identify the spectrum of D with $\mathfrak{a}^*/W$ where W is the Weyl group (the quotient of the normalizer of A by its centralizer). We can therefore view λ as an element of $\mathfrak{a}^*/W$. In particular, $\lambda(\Delta) = \|\lambda\|^2$.

Let N be the unipotent radical of a minimal parabolic subgroup containing A. Then the product map $N \times A \times K \to G$ is a diffeomorphism. The decomposition $G = NAK$ is called the Iwasawa decomposition. Set $H(nak) = \log a \in \mathfrak{a}$ for $a \in A$, $n \in N$, $k \in K$.

The role of the exponential functions in the Euclidean space is played by Harish-Chandra's spherical functions. For each λ there is a unique bi-K-invariant eigenfunction $\phi_\lambda : G \to \mathbb{C}$ with eigenvalue λ and $\phi_\lambda(1) = 1$. In fact, ϕ_λ is given by

$$\phi_\lambda(g) = \int_K e^{\langle \lambda+\rho, H(kg)\rangle}\, dk.$$

If $\lambda \in \mathrm{i}\mathfrak{a}^*$ then $|\phi_\lambda(g)| \leqslant 1$ for all $g \in G$.

The Abel transform is the map

$$\mathcal{A} : C_c^\infty(G//K) \xrightarrow{\mathcal{A}} C_c^\infty(A)^W$$

given by

$$\mathcal{A}(f)(a) = \delta_0(a)^{\frac{1}{2}} \int_N f(an)\, dn$$

where δ_0 is the modulus function of AN. In fact, $\mathcal{A}$ is an isomorphism of topological algebras (under convolution). We also have the Fourier-Laplace transform

$$\hat{}: C_c^\infty(A)^W \to \mathcal{PW}(\mathfrak{a}_{\mathbb{C}}^*)^W$$

where $\mathcal{PW}$ denotes the Paley-Wiener space. The spherical inversion asserts that for any $f \in C_c^\infty(G//K)$ we have

$$f(1) = \int_{\mathrm{i}\mathfrak{a}^*/W} \widehat{\mathcal{A}(f)}(\lambda)\beta(\lambda)\, d\lambda$$

where β is the *Plancherel measure*. The latter is given by

$$\beta(\lambda) = |c(\lambda)|^{-2}$$

where $c(\lambda)$ is Harish-Chandra's c-function which can be expressed in terms of the $\mathbf{\Gamma}$-function via the Gindikin-Karpelevic formula.

Roughly, $\beta(\mathrm{i}\lambda) \sim \|\lambda\|^{d-r}$ where $d = \dim X$ for λ away from the walls.

Consider the case of $G = SL(n)$. Then $K = SO(n)$, $r = n-1$, $d = \frac{n(n+1)}{2} - 1$, A is the subgroup of diagonal matrices with positive entries, W is the symmetric group on n letters, N is the group of upper triangular matrices with 1's on the diagonal and the Iwasawa decomposition amounts to the Gram-Schmidt process. We can identify $\mathfrak{a}^*$ with $\{(\lambda_1, \dots, \lambda_n) : \sum \lambda_i = 0\}$. The c-function is given by

$$c(\lambda) = \prod_{i<j} \frac{\mathbf{\Gamma}_{\mathbb{R}}(\lambda_i - \lambda_j)}{\mathbf{\Gamma}_{\mathbb{R}}(\lambda_i - \lambda_j + 1)}, \quad \mathbf{\Gamma}_{\mathbb{R}}(s) = \pi^{-s/2}\mathbf{\Gamma}(s/2)$$

Using the trace formula, Duistermaat, Kolk and Varadarajan proved the following remarkable Theorem.

Theorem 1 ([DKV79]). *Let M be a compact quotient of $X = G/K$. Suppose that $\Omega \subset \mathrm{i}\mathfrak{a}^*$ is a W-invariant compact domain with piecewise C^2-boundary. Then*

$$\sum_{\lambda \in t\Omega} m(\lambda) = \frac{\mathrm{vol}(M)}{|W|} \int_{t\Omega} \beta(\lambda)\, d\lambda + O(t^{d-1}) \quad \text{as } t \to \infty$$

In particular, when Ω is the unit ball, we recover Weyl's law with an error term à la Hörmander.

The proof of Theorem 1 follows the same guidelines of the proof in the case of $SL(2)$ described above. Of course, it is technically more complicated, but nonetheless it uses Selberg's trace formula only for test functions for which on the geometric side the contribution is only from the identity element.

What about the non-compact case?

It turns out that it is more natural to consider here not the entire discrete spectrum, but a subspace called the *cuspidal* part. We will not define this important concept here. It is expected that the cuspidal part exhausts most of the discrete part, although this is not known in general. At any rate one can ask about the validity of Weyl's law for the cuspidal spectrum. Sarnak conjectured that this is true for congruence subgroups. It should be noted that by a well-known result of Margulis all finite volume locally symmetric spaces of rank > 1 are arithmetic ([Mar91]).

Donnelly gave the correct *upper* bound for the cuspidal part of the spectrum of any finite volume locally symmetric space ([Don82]).

As for lower bounds, Reznikov proved the Weyl law for arithmetic real or complex hyperbolic n-manifolds (a rank one situation) ([Rez93]).

The first non-compact higher rank situation was treated by Miller who showed that the Weyl law holds for $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$ ([Mil01]).

This was extended by Müller to $\Gamma_N\backslash SL(n,\mathbb{R})/SO(n)$, $n \geqslant 2$ where

$$\Gamma_N = \{\gamma \in SL(n,\mathbb{Z}) : \gamma \equiv 1 \pmod N\}, \quad N \geqslant 1$$

is the principal congruence subgroup ([Mül07]).

Recently, Lindenstrauss and Venkatesh showed that for any Chevalley group G (e.g. $SL(n)$, $Sp(n)$, and the like) and any congruence subgroup Γ of $G(\mathbb{Z})$ the locally symmetric space $\Gamma\backslash G(\mathbb{R})/K$ obeys Weyl's law ([LV07]) proving Sarnak's conjecture, at least in that case. Their proof is beautiful, very accessible and surprisingly short. It uses the existence of Hecke operators in a crucial way. The method of proof should carry over to any quotient by a congruence subgroup without too much difficulty.

What about the error term?

Recently, we were able to obtain an analogue of Theorem 1 (with a slightly weaker error term) in the case of $GL(n)$.

Theorem 2 ([LM]) *Let* $X = SL(n,\mathbb{R})/SO(n), M = \Gamma_N\backslash X, N \geqslant 3$. (*The latter guarantees that* Γ_N *is torsion free.*) *Then*

$$\sum_{\lambda\in t\Omega} m(\lambda) = \frac{\mathrm{vol}(M)}{|W|}\int_{t\Omega} \beta(\lambda)\ d\lambda + O(t^{d-1}(\mathrm{log} t)^n)$$

as $t \to \infty$

Let us briefly indicate what goes into the proof of Theorem 2. Roughly, the argument follows that of [DKV79]. However, there are important differences. First, because of the non-compactness, in order to get started we have to use Arthur's trace formula instead of Selberg's trace formula described above. Before describing its shape we have to say a few words about the continuous spectrum of $L^2(\Gamma\backslash G)$. This was carried out in a seminal work of Langlands ([Lan76]) following Selberg [Sel63]; cf. [MW95] for a more detailed account of Langlands' work (and [Wal03] for the analogous local statement). We will describe it for $G = GL(n)$, and only in rough terms. Let $(n_1,\dots,n_k)$ be a partition of n, i.e. $n = n_1+\dots+n_k$ and write any matrix in blocks according to this partition. Let P be the parabolic subgroup consisting of matrices whose lower diagonal blocks $n_i \times n_j$, $i > j$ are all zero. We can write $P = MU$ where M is the group of diagonal block matrices isomorphic to $GL(n_1) \times \cdots \times GL(n_k)$ and U is the unipotent radical (in which all diagonal blocks are the identity matrices). Let M^1 be the normal subgroup consisting of elements whose diagonal blocks have determinant ± 1. Then, roughly speaking, $L^2(\Gamma\backslash G)$ can be decomposed into a direct sum over partitions up to permutation. The part corresponding to an equivalence class of a partition is isomorphic to the integral over $\mathbb{R}^k$ of the representation parabolically induced from the discrete spectrum of $\Gamma_M\backslash M^1$. To make this more precise one has to work in the adelic setting and introduce Eisenstein series in full generality, but we shall not do it here.

Arthur's trace formula in its original form, developed in the late 70's and the 80's, is an identity of the form

$$\sum_{\mathfrak{o}} J_{\mathfrak{o}}(f) = \sum_{\chi} J_{\chi}(f)$$

where $\mathfrak{o}$ ranges over semi-simple conjugacy classes of $G(\mathbb{Z})$ and χ ranges over spectral data. (Again, this is an oversimplification. One should really work in the adelic setting; see [Art05, §§1-21].)

Unlike in the co-compact case the distributions $J_{\mathfrak{o}}$ are given in terms of *weighted orbital integrals*

$$\int_{G_\gamma\backslash G} f(g^{-1}\gamma g)w(g)\ dg$$

of elements γ whose semi-simple part is in $\mathfrak{o}$. In particular, they are non-invariant. Similarly, the distributions on the spectral side are built from weighted traces.

The weighted orbital integrals are hard to define, but they have a reasonably simple qualitative description. For example, for the regular unipotent classes it has the form

$$\int_N f(n)w(n)\ dn$$

where w is a linear combination of products of $\log(|p(n)|)$ where p is a non-zero polynomial in the entries of n.

Unlike in the co-compact case, we can not localize the geometric side to the identity element itself. However, we can still localize to the terms corresponding to $\mathfrak{o} = \{e\}$, that is to the unipotent conjugacy classes. It turns out the mild logarithmic singularities of the weight factor enable the analysis of the weighted orbital integrals of the non-trivial unipotent conjugacy classes to be carried out using a technique from another paper by Duistermaat, Kolk and Varadarajan [DKV83], namely the method of stationary phase. (See [Var97] for a very accessible account on the method of stationary phase and its application.) The crucial fact is that the function $\langle\mu, H(kn)\rangle$ on $K \times N$ is a Morse function. That is, its critical points are isolated (in this case $(w, 1)$, $w \in W$) and the Hessian is non-singular at any critical point. (By the Morse Lemma, around each critical point one can choose local coordinates in which the function is quadratic.)

What are the issues on the spectral side? Here, we have to control the contribution of the continuous spectrum, just like in the rank one situation. The combinatorics and the analysis is much more complicated. Just to give a feeling, the mere fact that Arthur's expansion of the spectral side is absolutely convergent (in an appropriate strong sense) and that it can be written (like the spectral expansion itself) in terms of discrete data is a very recent development ([FLM]). It relies on refining Arthur's expression in terms of (suitably defined) winding numbers as in Selberg's case. Among other things, this enables us to have more lax conditions on the test functions for which the trace formula is applicable, namely functions which are L^1 together with sufficiently many derivatives. This is in accordance with the situation for the Poisson summation formula.

To make a long story short, what we need in addition is the fine analytic behavior of the Rankin-Selberg L-functions studied by Jacquet, Shalika and others. It is here where the restriction to $GL(n)$ is essential.

What is the import of estimating the error term? It turns out that for some application it is not only necessary to find the main term, but we need also a power saving (not matter how small) in the error term. A typical case is the distribution of low-lying zeros of L-functions. It is a general principle that the zeros of an automorphic L-function (normalized in an appropriate way according to the analytic conductor [IS00]) are distributed according to a universal law called GUE, namely, they are spaced like eigenvalues of a large unitary matrix. A finer look at the zeros suggests that we sample them by

$$D(f;\phi) = \sum_{\gamma} \phi(\gamma)$$

where γ ranges over the (normalized) zeros of the L-function and ϕ is a test function. For this to be meaningful we have to average over a sufficiently rich family $\mathcal{F}$ of L-functions (depending on a parameter Q going to ∞), namely to consider

$$E(\mathcal{F}(Q);\phi) = \frac{1}{|\mathcal{F}(Q)|} \sum_{f\in\mathcal{F}(Q)} D(f;\phi).$$

This will pick up the distribution of low-lying zeros of L-functions in the family. The sensible question is what is the limit W of this distribution as $Q \to \infty$?

The Katz-Sarnak philosophy suggests that there are four possible limiting distributions W corresponding to the type of symmetry of the family (unitary, symplectic, even or odd orthogonal) — see [KS99b].

In the function field case there is a spectral interpretation of the zeros as eigenvalues of Frobenius acting on cohomology. Their distribution is governed by the monodromy group of the family ([KS99a]).

In the number field case we don't have such a spectral interpretation (which is partly responsible for our lack of knowledge of the Riemann Hypothesis). Nevertheless, there are partial results, conditional on GRH, concerning the low lying zeros of certain families of modular forms for $GL(2)$ ([ILS00]).

To extend these results to the context of (certain families of) automorphic forms of $GL(m)$, $m > 2$ one needs to show that

$$\operatorname{tr} T_n(\mathcal{F}(Q)) = \delta_n(\mathcal{F})|\mathcal{F}(Q)| + O(|\mathcal{F}(Q)|^{1-\varepsilon} n^k)$$

for some $\varepsilon > 0$ and k. The coefficient $\delta_n(\mathcal{F})$ (which could vanish for many n's) governs the distribution of low-lying zeros in the family. The case $n = 1$ is just estimating the size of the family with a power saving in the remainder term.

10 Other applications of the trace formula

We now mention additional higher rank applications of the trace formula; however, we will not go into much detail.

10.1 Generalized Jacquet-Langlands correspondence

Let G' be the multiplicative group of a central simple algebra of degree n over a number field F and $G = GL(n)$. The discrete spectrum of $L^2(G'(F)\backslash G'(\mathbb{A})^1)$ is "contained" in $L^2(G(F)\backslash G(\mathbb{A})^1)$. This is somewhat implicit in [AC89]. It is made more explicit in recent work by Badulesco ([Ba]). The corresponding local statement had been proved earlier by Rogawski ([Rog83]) – see also [Hen06].

This is the case of a relation between automorphic representations of a group and its quasi-split inner form. Such a relation is of course a very special case of what Langlands functoriality predicts. When dealing with groups other than $GL(n)$ the problem of stabilization arrises. This is a key notion introduced by Langlands with far reaching consequences ([Lan83])[2]. We will not discuss it here but refer the reader to [Art05, §§27-29]. Let me just mention a major recent breakthrough by Ngô who proved one of the main stumbling blocks to stabilization, namely the "Fundamental Lemma" ([Ngô]; cf. [Ngô06], [Lau06]). Also, it should be noted here that one of the first steps of stabilization, carried out by Kottwitz, already yields an important arithmetic payoff, namely, the solution of Weil's conjecture on the Tamagawa number ([Kot88]). We refer to [Clo89] for more details and the exciting history of this problem.

10.2 Base change

Let G be a reductive group over a number field F and E/F be a finite extension. One of the basic instances of functoriality predicts a transfer of automorphic representations of $G(\mathbb{A}_F)$ to those of $G(\mathbb{A}_E)$. Very little is known about this if E/F is not solvable (but see [Clo95]). On the other hand, if E/F is cyclic then we can hope to characterize the image as the automorphic representations of $G(\mathbb{A}_E)$ which are equivalent to their Galois twist. Henceforth assume that E/F is cyclic. Langlands, building on earlier work by Doi-Naganuma ([DN70]), Saito ([Sai75]) and Shintani ([Shi79]), established the correspondence between automorphic representations of $GL(2, \mathbb{A}_F)$ and Galois invariant automorphic representations of $GL(2, \mathbb{A}_E)$ ([Lan80]). Remarkably, he used it to prove a new case of the the Artin conjecture. Namely, he showed that to any irreducible two-dimensional representation $\rho : \mathrm{Gal}(\bar{F}/F) \to GL(2, \mathbb{C})$ with a finite solvable image of a certain

[2]Historically, stabilization and the related notion of endoscopy were conceived by Langlands while studying the zeta functions of Shimura varieties; see below.

kind corresponds an automorphic cuspidal representation of $GL(2,\mathbb{A})$, and hence ρ admits an entire Artin L-function. Later, this was extended by Tunnell to any continuous representation with a solvable image ([Tun81]), after results of Jacquet, Piatetski-Shapiro and Shalika became available ([JPSS81]). For more details see [Rog97]. These cases of the Artin's conjecture were extremely useful (years later) in the work of Wiles on Fermat's last theorem [Wil95]. In a sense they were used to jump-start the modularity argument. (See [Tay04] for more about Galois representations and modularity.)

At the heart of the trace formula approach for base change is the comparison between the usual trace formula for $GL(2,\mathbb{A}_F)$ and the twisted trace formula for $GL(2,\mathbb{A}_E)$. The latter is, by definition (a suitable regularization of)

$$\int_{GL(2,E)\backslash GL(2,\mathbb{A}_E)^1} K_f(x,x^\sigma)\ dx$$

where σ is a generator for the Galois group. One of the main difficulties is to find sufficiently many pairs of matching functions. (This is much harder than in the Jacquet-Langlands case, because here the groups are genuinely distinct.) The same idea in principle carries over to $GL(n)$, but this had to await for the development of Arthur's trace formula. (There is also an approach of Labesse which uses a less refined version of the trace formula – cf. [Lab99].) Ultimately, Arthur and Clozel extended Langlands' result to $GL(n)$ ([AC89], [Jac89]). This is an extremely important and useful result which marks one of the biggest victories of the trace formula so far. Among other things it was used (again, much later) in the proof of the Sato-Tate conjecture by Clozel, Harris, Shepherd-Barron and Taylor ([CHT], [HSBT], [Tay]).

10.3 Formula for traces of Hecke operators

We already saw that the trace formula is a very natural tool in studying the asymptotic behavior of eigenvalues. However, by the uncertainty principle, it is hopeless for the trace formula to tell the dimension of a single eigenvalue. On the other hand, there is a closed formula for the dimension of cusp forms of a given weight > 1 and level. This is obtained either by the Riemann-Roch theorem or from the Eichler-Selberg trace formula. (In contrast, the cusp forms of weight 1 are much more subtle. Deligne and Serre constructed an odd two-dimensional representation of the Galois group from any weight 1 cusp form [DS74]. Recently Khare and Wintenberger proved the converse statement, namely, that any odd Galois representation is modular. This results from their proof of Serre's conjecture ([KWa], [KWb]).)

To reconcile the apparent contradiction we recall that the cuspidal modular forms of weight > 1 correspond to automorphic representations whose component at the Archimedean place is square-integrable. The fact that the Plancherel measure is atomic on square-integrable representations enables us to obtain exact formulas for the dimension. This is true in higher dimension as well.

Harish-Chandra's most well-known work is the classification of square-integrable representations of a semi-simple group G. (For more details see [Her91], [Var00] and the references cited therein.) There is a close connection between square-integrable representations and representations of compact groups. A necessary and sufficient condition for G to have a square-integrable representation is that G admits a compact torus T which is a maximal torus. (That is, the rank of G over $\mathbb{C}$ coincides with the rank of K. Alternatively, G is an *inner* form of its compact form.) In this case, the square-integrable representations are parameterized by regular characters of T up to conjugation by the Weyl group W of K. If λ is the Harish-Chandra parameter of π then the character χ_π of π on T is given by

$$\chi_\pi(\gamma) = \frac{\sum_{w \in W} (-1)^{l(w)} \lambda(w\gamma)}{\Delta(\gamma)} \tag{8}$$

where W is the Weyl group of K and Δ is Weyl's discriminant. This is of course a generalization of Weyl's character formula in the compact case.

Suppose that π is a square-integrable representation of G which admits an integrable matrix coefficient ξ. The last condition amounts to a regularity condition on the Harish-Chandra parameter of π. (For $PGL(2)$ it excludes the Steinberg representation, which corresponds to weight 2 modular forms.) By applying the trace formula to ξ (which is possible since ξ is L^1) Langlands obtained a closed formula for the multiplicity $m(\pi)$ of π in the automorphic spectrum of any uniform lattice Γ of G. In fact, we see that

$$m(\pi) = \sum_\gamma \mathrm{vol}(\Gamma_\gamma \backslash G_\gamma) \int_{G_\gamma \backslash G} \xi(g^{-1}\gamma g)\ dg.$$

By Harish-Chandra, the orbital integral of ξ vanishes unless γ is elliptic (i.e., belongs to a compact subgroup of G), in which case it is equal to $\chi_\pi(\gamma^{-1})$. The character is given explicitly by (8).

Arthur generalized this to give a closed formula for the trace of a Hecke operator on any (stable) isotypic component of a square-integrable representation (L-packet) with a regular parameter. It is given by a finite sum of geometric terms, but now there are additional boundary terms coming from parabolic subgroups. We refer the interested reader to [Art05, §24] and [Art89] for more details. There is also a geometric counterpart of this formula [GKM97].

10.4 Automorphic forms of classical groups

A major current development in the trace formula is work in progress by Arthur aiming at understanding the relation between self-dual automorphic representations of $GL(n, \mathbb{A}_F)$ and automorphic representations of classical groups. More precisely, self-dual cuspidal representations of $GL(2n)$ should correspond to representations of either $SO(2n+1)$ or $SO(2n)$ (depending on whether the exterior-square or the symmetric-square L-function has a pole), while the self-dual cuspidal representations of $GL(2n+1)$ (in which case the symmetric square L-function has a pole) should correspond to $Sp(2n)$. The idea is to compare the twisted trace formula for $GL(n)$ with respect to the Cartan involution with trace formula for classical groups. This is a very serious undertaking which involves stabilization on both sides. It relies on suitable versions of the fundamental lemma which are expected to be proved soon, following the afore-mentioned work by Ngô. See [Art05, §30] for more details.

10.5 Shimura varieties and Langlands correspondence

The Eichler-Shimura correspondence (cf. [Shi94]) attaches to any weight two Hecke eigenform on $\Gamma_0(N)$ with integer coefficients a_n, an elliptic curve E over $\mathbb{Q}$ such that for almost all primes p the number of points of the reduction of E modulo p is $p+1-a_p$. (The converse statement – the modularity of elliptic curves, is the famous Taniyama-Shimura-Weil conjecture which was eventually solved by Breuil, Conrad, Diamond and Taylor [BCDT01] following the work of Wiles and Taylor.)

The Eichler-Shimura correspondence in essence describes the action of the Galois group on the ℓ-adic cohomology of the modular curve $X_0(N)$ in terms of Hecke correspondences. The modular curves admit a vast generalization, namely, Shimura varieties. A Shimura variety is, roughly speaking, a system $G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f$ where K_f ranges over open subgroups of $G(\mathbb{A}_f)$ where G is a reductive group over $\mathbb{Q}$. The fundamental fact, due to Shimura, is that for certain G's, we get this way "nice" varieties defined over a number field E. (We refer the reader to [Mil05] for an excellent introduction to Shimura varieties.) The (étale) cohomologies of Shimura varieties inherit an action of the Galois group, and passing to the inductive limit over K_f, this action commutes with the action of $G(\mathbb{A}_f)$. It was Langlands' idea that this may be used to obtain certain cases of the global Langlands correspondence (or, reciprocity laws) by decomposing the ensuing representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)\times G(\mathbb{A}_f)$. The idea is to compare the trace of the action of an element σ of the Galois group and a test function h on $G(\mathbb{A}_f)$ with the spectral side of the trace formula (with a suitable function at $G(\mathbb{R})$) – see [Lan77], [Lan79b]. Such a comparison is done in two steps. The first, using a version of the Lefschetz trace formula, is to express the trace of pairs (σ, h) acting on the cohomology with compact support (with suitable coefficients) in terms of orbital and twisted orbital integrals, namely the geometric side of the trace formula. The second step, using Arthur's trace formula, is to recast these traces in terms of automorphic forms – see [Kot90], [Kot84], [Clo93].

The intimate relation, envisioned by Langlands, between automorphic forms and Shimura varieties, is a two-sided road. On the one hand, it gives (at least in principle) deep arithmetic information about Shimura varieties (such as, relating their Hasse-Weil zeta functions to automorphic L-functions). On the other hand, it gives (again, at least in theory) invaluable information (such as the Ramanujan-Petersson conjecture) about automorphic representations which are of algebraic type (an integrality condition on the Archimedean component) – cf. [Clo91]. Any serious discussion of these matters lies well beyond the scope of these notes (and the author's expertise). We refer the reader to [Del79], [Cas79] and to the books [LR92], [CM90a], [CM90b], [Mor] to get a feeling of the wide ramifications of the problems involved. A major on-going project in Paris (including four books in various stages of writing) aims to solve these problems in great generality.

As pointed out before, not all automorphic representations correspond to

Galois representations and therefore they cannot all be realized through Shimura varieties. (See [Lan79a], [Clo90] for a hypothetical substitute.) However, Shimura varieties give sufficiently many automorphic representations in order to obtain at least the local Langlands correspondence ([HT01], [Hen00], [Car00]; see [BH06] for a more direct approach for $GL(2)$).

For global fields of positive characteristic (function fields of curves over finite fields) the analogues of Shimura varieties are Drinfeld moduli spaces of elliptic modules. Among other things, they provide a means to prove the local Langlands correspondence in the equal characteristic case ([LRS93], [Fal94]) and substantial information toward the global Langlands correspondence. We refer the reader to the volumes [Lau96], [Lau97] for an excellent exposition of this circle of ideas. (See also Rogawski's featured review on the MathSciNet.)

To achieve the global Langlands conjecture in full generality (in the function field case) one has to use the so-called Shtukas, also invented by Drinfeld. This was done by Drinfeld for $n = 2$ ([Dri80]) and extended to $GL(n)$ by Lafforgue ([Laf02], [Lau02b], [Lau02a]). Once again, the method is to compare the Lefschetz trace formula and the Arthur-Selberg trace formula. (For a state of the art result in the context of the Lefschetz trace formula see [Var07].) It should be noted however that Arthur developed the trace formula over number fields only, so that additional work had to be done by Lafforgue in order to apply it to the function field case ([Laf96a], [Laf96b]). Finally, a different approach (originally also due to Drinfeld) for the global Langlands conjecture in the function field case is the geometric Langlands correspondence. For more about this see [Lau03], [Fre07], [FGV02], [Gai03], [Fre04].

11 Other forms of the trace formula; the relative trace formula

Arthur's trace formula admits various other forms. We already encountered one of them, namely the Petersson/Kuznetsov trace formula. There is also an important local counterpart of the trace formula, also developed by Arthur [Art91]. Yet another variant is the trace formula for Lie algebras developed by Waldspurger in the local case ([Wal95]) and by Chaudouard in the global case ([Cha02]). The article [Kot05] by Kottwitz provides an excellent exposition and a thorough discussion of the local trace formula for Lie algebras and its applications to harmonic analysis.

We will end our discussion on the trace formula by saying a few words on the relative trace formula and its background.

The symmetric spaces of Cartan admit a generalization – namely pseudo-Riemannian homogeneous spaces. It is not clear to what extent one can speak about automorphic forms in this context. (See [KY05] for an elaborate discussion about related topics.) However, there has been a lot of work, starting with Flensted-Jensen, aiming at generalizing Harish-Chandra's work on $L^2(G)$ to the context of $L^2(G/H)$ where H the fixed point subgroup of an involution of H. (See [Del02] for more details.)

There is a (yet not completely polished) global counterpart of this setup, to which the relative trace formula is pertinent.

We will focus on the following example as a model for the relative trace formula. (For more details see [Off].)

Let E/F be quadratic extension of number fields and σ the Galois involution. Let G be the group $GL(n)$ over E. Consider for each hermitian form in n variables the unitary group

$$H_\xi = \{g \in G : \ {}^t\sigma(g)\,\xi g = \xi\}$$

as a group defined over F. We say that a cuspidal representation π of $G(\mathbb{A})$ is *distinguished* with respect to a unitary group if the *period integral*

$$\int_{H_\xi(F)\backslash H_\xi(\mathbb{A})} \varphi(h)\ dh$$

does not vanish on the space of π for some hermitian form ξ.

Can we describe the distinguished representations? The question was posed, and eventually answered, by Jacquet.

Theorem 3 ([Jac05b]). *π is distinguished with respect to a unitary group if and only if $\sigma(\pi) = \pi$.*

Note that by Arthur-Clozel the characterization coincides with the image of base change from $G' = GL(n)/F$. In fact, what Jacquet showed more directly is that the distinguished representations are exactly those arising from base change.

To describe his approach, consider the variety $\mathcal{H}_n$ of hermitian forms with the G-action $g \star \xi = {}^t\sigma(g)\,\xi g$. For $\Phi \in \mathcal{S}(\mathcal{H}_n(\mathbb{A}))$ define the "relative kernel"

$$K_\Phi(g) = \sum_{\xi \in \mathcal{H}_n(F)} \Phi(g \star \xi) \quad g \in G(\mathbb{A}_E).$$

Let N_n be the group of upper triangular matrices, ψ' a non-degenerate character of $N_n(F)\backslash N_n(\mathbb{A})$ and $\psi = \psi' \circ \mathrm{tr}_{E/F}$. The idea is to compare

$$RTF(\Phi) = \int_{N_n(E)\backslash N_n(\mathbb{A}_E)} K_\Phi(u)\psi(u)\ du$$

with the Kuznetsov trace formula

$$KTF(f') = \int_{(N_n(F)\backslash N_n(\mathbb{A}))^2} K_{f'}({}^t u_1^{-1}, u_2)\psi'(u_1 u_2^{-1})\ du_1\ du_2$$

for matching $\Phi \in \mathcal{S}(\mathcal{H}_n(\mathbb{A})) \leftrightarrow f' \in \mathcal{S}(G'(\mathbb{A}))$.

Building on our previous experience we will expand both RTF and KTF geometrically according to double cosets. In the RTF side we expand along $N_n(E)$-orbits of $\mathcal{H}_n(F)$. The regular orbits (those for which the stabilizer is trivial) are parameterized by the F-points of the diagonal torus T' of G'. Ignoring the singular terms we have

$$RTF(\Phi) \sim \sum_{a \in T'(F)} \Omega[\Phi; a]$$

where

$$\Omega[\Phi; a] = \int_{N_n(\mathbb{A}_E)} \Phi({}^t\sigma(u)au)\psi(u)\ du.$$

Similarly, on the KTF side, the regular orbits of $N_n(F) \times N_n(F)$ contribute

$$\sum_{a \in T'} \Omega'[f'; a] := \int_{(N_n(\mathbb{A}_F))^2} f'(u_1^t a u_2)\psi'(u_1 u_2)\ du_1\ du_2.$$

The matching condition is

$$\Omega_v[\Phi_v; a_v] = \eta(a_v)\Omega'_v[f'_v; a_v]$$

for a certain quadratic character $\eta = \otimes_v \eta_v$ of $T'(F)\backslash T'(\mathbb{A})$.

As before, one needs to show the existence of sufficiently many matching pairs of functions and compatibility with Hecke algebra base change homomorphism. (The latter is the fundamental lemma in this context.) On the spectral side, one has to write down the expansion in terms of cuspidal data in order to apply Langlands' argument on the separation of the discrete and continuous part. We refer to [Jac05b] and [Lap06b] for exact statements and proofs.

The characterization of distinguished representations naturally leads to the following question: what is the value of the unitary period?

We describe a typical result. Let $F = \mathbb{Q}$, $D < 0$ a fundamental discriminant, $E = \mathbb{Q}(\sqrt{D})$ and $\mathcal{O}_D$ the ring of integers of E. Let X be the locally symmetric space $X = GL(n, \mathcal{O}_D)\backslash GL(n, \mathbb{C})/U(n)$ and φ a cuspidal L^2-normalized Hecke eigenform on X. Let

$$\Lambda = GL(n, \mathcal{O}_D)\backslash GL(n, E)U(n)/U(n) \subset X$$

be the (finite) genus of the standard hermitian form $\|\cdot\|^2$.

Theorem 4 ([LO07]). *The (weighted) sum* $\sum^*_{x\in\Lambda} \varphi(x)$ *vanishes unless* φ *is obtained as a base change from a cusp form* φ' *on* $\Gamma_0(D)\backslash GL(n, \mathbb{R})/O(n)$*. In this case, if* π *is the cuspidal representation of* $GL(n, \mathbb{A})$ *corresponding to* φ'*, then*

$$\left| \sum_{x\in\Lambda}{}^{*} \varphi(x) \right|^2 = *(local\ factors)\frac{L(1, \pi \times \tilde{\pi} \times \chi_D)}{\mathrm{Res}_{s=1} L(s, \pi \times \tilde{\pi})}$$

where χ_D *is the quadratic character of conductor* D *and* $L(s, \pi \times \pi')$ *is the Rankin-Selberg* L*-function.*

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