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IFOR

Spring term 2011

Introduction to Optimization — Serie 4

http://www.ifor.math.ethz.ch/teaching/lectures/intro_ss11

Exercise 11: LP Modeling

Consider the polyhedron $P = \{x \in \mathbf{R}^n \mid a_i'x \le b_i,\ i = 1, \ldots, m\}$. A ball B with center y and radius r is defined as the set of all points within Euclidean distance r from y, i.e. $B = \{x \mid \|y - x\|_2 \le r\}$. We are interested in finding a ball with the largest possible radius, which is entirely contained in P. The center of such a ball is called the *Chebychev center* of P.

a) Provide a Linear Program for the problem of finding the Chebyshev center of P.

b) Solve your LP with a Linear Optimization software (e.g. CPLEX) for the polyhedron $P = \{x \in \mathbb{R}^3 \mid Ax \le b\}$, where

$$A = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & 2 \\ 2 & 4 & -4 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 4 \\ 16 \\ 16 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solution:

(a) We introduce a variable vector $y \in \mathbb{R}^n$ representing the coordinates of the Chebychev center of P and a nonnegative variable $r \in \mathbb{R}_{\ge 0}$ representing the radius of the (Euclidean) ball with center y.

For finding a ball with the largest possible radius, which is entirely contained in P, we have to ensure that for each P-defining inequality $a_i'x \le b_i$ the distance between the center y of the ball to the hyperplane given by $a_i'x = b_i$ is less than or equal to the radius r. For this, we consider the points $z^i \in \mathbb{R}^n$, $i = 1, \ldots, m$, where each z^i is located on the line that is defined by the center y and the direction a_i and that has the distance r to the Chebyshev center y, i. e. $z^i = y + \frac{1}{||a_i||_2} a_i\, r$.

It now suffices to guarantee that each inequality $a_i'x \le b_i$ is valid for the point z^i. This yields

$$b_i \quad \ge \quad a_i'z^i = a_i'(y + \tfrac{1}{||a_i||_2} a_i\, r) = a_i'y + \tfrac{a_i'a_i}{||a_i||_2} r = a_i'y + ||a_i|| r, \qquad \text{for } i = 1, \ldots, m.$$

Thus, finding the Chebychev center of the polyhedron P can be formulated as the following linear optimization problem

$$\max \ r, \quad \text{s. t. } a_i'y + ||a_i||_2 r \le b_i, \ \text{ for all } i = 1 \ldots, m, \ \text{ and } r \ge 0.$$

(b) For the given polyhedron P, the LP formulation for finding the Chebychev center is given by

$$\max \ r, \quad \text{s. t.} \begin{pmatrix} 4 & -2 & 4 & 6 \\ -2 & 1 & 2 & 3 \\ 2 & 4 & -4 & 6 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ r \end{pmatrix} \quad \le \quad \begin{pmatrix} 4 \\ 16 \\ 16 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad r \ge 0.$$

The cplex interface computes $y^* = (0.75, 3.25, 0.75)$ and $r^* = 0.75$ with optimal value 0.75. This means the ball with largest possible radius, which is entirely contained in P is given by the ball with center $(0.75, 3.25, 0.75)$ and radius 0.75.

Exercise 12: LP Optimality

Find all necessary conditions on the parameters $s, t \in \mathbf{R}$ such that the Linear Program given by

$$\begin{array}{rl} \text{maximize} & x_1 + x_2 \\ \text{subject to} & sx_1 + tx_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

a) is unbounded, c) has multiple optimal solutions,

b) is infeasible, d) has one unique optimal solution.

In particular, argue for each case why your conditions obtained are the only ones.

Solution:

(a)$s \leq 0$ or $t \leq 0$ or both.

(b)never.

(c)$s = t$ and $s > 0$.

(d)$s > 0$ and $t > 0$ and $s \neq t$.