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Applications of Slowly Changing Functions in the Estimation of Growth Properties of Composite Entire Functions on the Basis of their Maximum Terms and Maximum Moduli

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Abstract

In the paper we prove some comparative growth properties of composite entire functions on the basis of their maximum terms and maximum moduli using generalised L^*-order and generalised L^*-lower order.

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1 Introduction, Definitions and Notations.

Let $\mathbb{C}$ be the set of all finite complex numbers and f be an entire function defined in $\mathbb{C}$. The maximum term $\mu(r,f)$ of $f=\sum\limits_{n=0}^{\infty}a_n z^n$ on $|z|=r$ is defined by $\mu(r,f)=\max\limits_{n\geq 0}|a_n|r^n$ and the maximum modulus $M(r,f)$ of f on $|z|=r$ is defined as $M(r,f)=\max\limits_{|z|=r}|f(z)|$ .We use the standard notations and definitions in the theory of entire functions which are available in [11]. In the sequel we use the following notation :

$\log^{[k]} x=\log\left(\log^{[k-1]} x\right)$ for $k=1,2,3,....$ and $\log^{[0]} x=x$.

To start our paper we just recall the following definitions :

Definition 1 *The order ρ_f and lower order λ_f of an entire function f are defined as*

$$\rho_f=\limsup_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log r} \quad and \quad \lambda_f=\liminf_{r\to\infty}\frac{\log^{[2]} M(r,f)}{\log r}.$$

Extending this notion, Sato [6] defined the generalised order and generalised lower order of an entire function as follows :

Definition 2 *[6]Let m be an integer ≥ 2. The generalised order $\rho_f^{[m]}$ and generalised lower order $\lambda_f^{[m]}$ of an entire function f are defined by*

$$\rho_f^{[m]}=\limsup_{r\to\infty}\frac{\log^{[m]} M(r,f)}{\log r} \text{ and } \lambda_f^{[m]}=\liminf_{r\to\infty}\frac{\log^{[m]} M(r,f)}{\log r}$$

respectively.

For $m=2$, Definition 2 reduces to Definition 1.

If $\rho_f<\infty$ then f is of finite order. Also $\rho_f=0$ means that f is of order zero. In this connection Datta and Biswas [2] gave the following definition :

Definition 3 *[2]Let f be an entire function of order zero. Then the quantities ρ_f^{**} and λ_f^{**} of f are defined by:*

$$\rho_f^{**}=\limsup_{r\to\infty}\frac{\log M(r,f)}{\log r} \text{ and } \lambda_f^{**}=\liminf_{r\to\infty}\frac{\log M(r,f)}{\log r}.$$

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [7] defined it in the following way:

Definition 4 *[7]A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon\,(>0)$,*

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon} \;\; for \; r \geq r(\varepsilon) \;\; and$$

uniformly for $k\,(\geq 1)$.
If further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0 \; .$$

Somasundaram and Thamizharasi [8] introduced the notions of L-order (L-lower order) for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. The more generalised concept for L-order (L-lower order) for entire function are L^*-order (L^*-lower order). Their definitions are as follows:

Definition 5 *[8]The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]} \;\; and \;\; \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}.$$

In the line of Sato [6] , Datta and Biswas [2] one can define the generalised L^*-order $\rho_f^{[m]L^*}$ and generalised L^*-lower order$\lambda_f^{[m]L^*}$ of an entire function f in the following manner :

Definition 6 *Let m be an integer ≥ 1. The generalised L^*-order $\rho_f^{[m]L^*}$ and generalised L^*-lower order $\lambda_f^{[m]L^*}$ of an entire function f are defined as*

$$\rho_f^{[m]L^*} = \limsup_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\log\left[re^{L(r)}\right]} \;\; and \;\; \lambda_f^{[m]L^*} = \liminf_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\log\left[re^{L(r)}\right]}$$

respectively.

Datta, Biswas and Hoque [3] reformulated Definition 6 in terms of the maximum terms of entire functions in the following way:

Definition 7 *[3] The growth indicators $\rho_f^{[m]L^*}$ and $\lambda_f^{[m]L^*}$ for an entire function f are defined as*

$$\rho_f^{[m]L^*} = \limsup_{r\to\infty} \frac{\log^{[m]} \mu(r, f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{[m]L^*} = \liminf_{r\to\infty} \frac{\log^{[m]} \mu(r, f)}{\log\left[re^{L(r)}\right]}$$

respectively where m be an integer ≥ 1.

Lakshminarasimhan [4] introduced the idea of the functions of L-bounded index. Later Lahiri and Bhattacharjee [5] worked on the entire functions of L-bounded index and of non uniform L-bounded index. In this paper we would like to investigate some growth properties of composite entire functions on the basis of their maximum terms and maximum moduli using generalised L^*-order and generalised L^*-lower order .

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 *[9] Let f and g be any two entire functions with $g(0) = 0$. Then for all sufficiently large values of r,*

$$\mu(r, f \circ g) \geq \frac{1}{2}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|, f\right).$$

Lemma 2 *[1] If f and g are any two entire functions then for all sufficiently large values of r,*

$$M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \leq M(r, f \circ g) \leq M(M(r, g), f).$$

3 Theorems.

In this section we present the main results of the paper.

Theorem 1 *Let f and g be any two entire functions such that $0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$ where $m \geq 1$ and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$. Then for every constant A and real number x,*

$$\lim_{r\to\infty} \frac{\log^{[m]} \mu(r, f \circ g)}{\left\{\log^{[m]} \mu(r^A, f)\right\}^{1+x}} = \infty .$$

Proof. If x is such that $1+x \leq 0$, then the theorem is obvious. So we suppose that $1+x>0$.
Now in view of Lemma 1, we get for all sufficiently large values of r that

$$\begin{aligned} \mu(r, f \circ g) &\geq \frac{1}{2}\mu\left(\frac{1}{16}\mu\left(\frac{r}{2}, g\right), f\right) \\ i.e.,\ \log^{[m]}\mu(r, f \circ g) &\geq O(1) + \log^{[m]}\mu\left(\frac{1}{16}\mu\left(\frac{r}{2}, g\right), f\right) \end{aligned}$$

$$i.e.,\ \log^{[m]}\mu(r, f \circ g) \geq O(1) + \left(\lambda_f^{[m]L^*} - \varepsilon\right)\left[\log\left\{\frac{1}{16}\mu\left(\frac{r}{2}, g\right)\right\} + L\left(\frac{1}{16}\mu\left(\frac{r}{2}, g\right)\right)\right]$$

$$i.e.,\ \log^{[m]}\mu(r, f \circ g) \geq O(1) + \left(\lambda_f^{[m]L^*} - \varepsilon\right)\left[\log M\left(\frac{r}{2}, g\right) + O(1) + L\left(\frac{1}{16}\mu\left(\frac{r}{2}, g\right)\right)\right]$$

$$\begin{aligned} &i.e.,\ \log^{[m]}\mu(r, f \circ g) \geq O(1) \\ &+ \left(\lambda_f^{[m]L^*} - \varepsilon\right)\left\{\left(\frac{r}{2}\right)e^{L(r)}\right\}^{\lambda_g^{L^*} - \varepsilon} + O(1) + L\left(\frac{1}{16}\mu\left(\frac{r}{2}, g\right)\right) \end{aligned} \tag{1}$$

where we choose $0 < \varepsilon < \min\left\{\lambda_f^{[m]L^*}, \lambda_g^{L^*}\right\}$.
Also for all sufficiently large values of r, we obtain that

$$\begin{aligned} \log^{[m]}\mu\left(r^A, f\right) &\leq \left(\rho_{L(f)}^{[m]L^*} + \varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\} \\ i.e.,\ \log^{[m]}\mu\left(r^A, f\right) &\leq \left(\rho_f^{[m]L^*} + \varepsilon\right)\log\left\{r^A e^{L\left(r^A\right)}\right\} \\ i.e.,\ \left\{\log^{[m]}\mu\left(r^A, f\right)\right\}^{1+x} &\leq \left(\rho_f^{[m]L^*} + \varepsilon\right)^{1+x}\left(A\log r + L\left(r^A\right)\right)^{1+x}. \end{aligned} \tag{2}$$

Therefore from (1) and (2) it follows for all sufficiently large values of r that

$$\frac{\log^{[m]}\mu(r, f \circ g)}{\left\{\log^{[m]}\mu\left(r^A, f\right)\right\}^{1+x}}$$

$$\geq \frac{O(1) + \left(\lambda_f^{[m]L^*} - \varepsilon\right)\left\{\left(\frac{r}{2}\right)e^{L(r)}\right\}^{\lambda_g^{L^*} - \varepsilon} + O(1) + L\left(\frac{1}{16}\mu\left(\frac{r}{2}, g\right)\right)}{\left(\rho_f^{[m]L^*} + \varepsilon\right)^{1+x}\left(A\log r + L\left(r^A\right)\right)^{1+x}}. \tag{3}$$

Thus the theorem follows from (3). ■

In the line of Theorem 1, we may establish the following theorem for the right factor of the composite entire function :

Theorem 2 *Let f and g be any two entire functions with $0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$ and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ where $m \geq 1$. Then for every constant A and real number x,*

$$\lim_{r\to\infty} \frac{\log^{[m]} \mu(r, f \circ g)}{\left\{\log^{[2]} \mu(r^A, g)\right\}^{1+x}} = \infty \ .$$

The proof is omitted.

Theorem 3 *Let f and g be any two entire functions such that $0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$ and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ where $m \geq 1$. Then for any two positive integers α and β,*

$$\lim_{r\to\infty} \frac{\log^{[m+1]} \mu(\exp(\exp(r^\alpha)), f \circ g)}{\log^{[m]} \mu(\exp(r^\beta), f) + L(\exp(\exp(r^\alpha)))} = \infty \ ,$$

where $K(r, \alpha; L) = \begin{cases} 0 \ \text{if } r^\beta = o\{L(\exp(\exp(r^\alpha)))\} & \text{as } r \to \infty \\ L(\exp(\exp(r^\alpha))) & \text{otherwise} \ . \end{cases}$

Proof. Taking $x = 0$ and $A = 1$ in Theorem 1, we obtain for $K > 1$ and for all sufficiently large values of r that

$$\begin{aligned} \log^{[m]} \mu(r, f \circ g) &> K \log^{[m]} \mu(r, f) \\ i.e., \ \log^{[m-1]} \mu(r, f \circ g) &> \left\{\log^{[m-1]} \mu(r, f)\right\}^K \\ i.e., \ \log^{[m-1]} \mu(r, f \circ g) &> \left\{\log^{[m-1]} \mu(r, f)\right\}^K \\ i.e., \ \log^{[m-1]} \mu(r, f \circ g) &> \log^{[m-1]} \mu(r, f) \end{aligned} \tag{4}$$

Therefore from (4) we get for all sufficiently large values of r that

$$\log^{[m]} \mu(\exp(\exp(r^\alpha)), f \circ g) > \log^{[m]} \mu(\exp(\exp(r^\alpha)), f)$$

$$\begin{aligned} & i.e., \ \log^{[m]} \mu(\exp(\exp(r^\alpha)), f \circ g) \\ > & \left(\lambda_f^{[m]L^*} - \varepsilon\right) . \log\{\exp(\exp(r^\alpha)) . \exp L(\exp(\exp(r^\alpha)))\} \end{aligned}$$

$$
\begin{aligned}
& i.e., \;\; \log^{[m]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right) \\
> \;\; & \left(\lambda_f^{[m]L^*} - \varepsilon\right). \left\{\left(\exp\left(r^{\alpha}\right)\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
& i.e., \;\; \log^{[m]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right) \\
> \;\; & \left(\lambda_f^{[m]L^*} - \varepsilon\right). \left\{\left(\exp\left(r^{\alpha}\right)\right)\left(1 + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\left(\exp\left(r^{\alpha}\right)\right)}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
i.e., \;\; \log^{[m+1]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right) \;\; > \;\; & O(1) + \log\exp\left(r^{\alpha}\right) \\
& + \log\left\{1 + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\left(\exp\left(r^{\alpha}\right)\right)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
i.e., \;\; \log^{[m+1]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right) > O(1) + r^{\alpha} & \\
+ \log\left\{1 + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\left(\exp\left(r^{\alpha}\right)\right)}\right\} &
\end{aligned}
$$

$$
\begin{aligned}
i.e., \;\; \log^{[m+1]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right) > O(1) + r^{\alpha} + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) & \\
- \log\left[\exp\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\}\right] & \\
+ \log\left[1 + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\exp\left(\mu r^{\alpha}\right)}\right] &
\end{aligned}
$$

$$
\begin{aligned}
i.e., \;\; \log^{[m+1]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right) > O(1) + r^{\alpha} + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) & \\
+ \log\left[\frac{1}{\exp\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\}}\right. & \\
\left. + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\exp\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\}.\exp\left(r^{\alpha}\right)}\right] &
\end{aligned}
$$

$$
\begin{aligned}
i.e., \;\; \log^{[m+1]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right) > O(1) + r^{(\alpha-\beta)}.r^{\beta} & \\
+ L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) \;\; . &
\end{aligned} \tag{5}
$$

Again we have for all sufficiently large values of r that

$$
\begin{aligned}
\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) &\le \left(\rho_f^{[m]L^*} + \varepsilon\right) \log\left\{\exp\left(r^{\beta}\right) e^{L\left(\exp\left(r^{\beta}\right)\right)}\right\} \\
i.e., \;\; \log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) &\le \left(\rho_f^{[m]L^*} + \varepsilon\right) \left\{\log\exp\left(r^{\beta}\right) + L\left(\exp\left(r^{\beta}\right)\right)\right\} \\
i.e., \;\; \log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) &\le \left(\rho_f^{[m]L^*} + \varepsilon\right) \left\{r^{\beta} + L\left(\exp\left(r^{\beta}\right)\right)\right\}
\end{aligned}
$$

$$i.e., \ \frac{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) - \left(\rho_f^{[m]L^*} + \varepsilon\right) L\left(\exp\left(r^{\beta}\right)\right)}{\left(\rho_f^{[m]L^*} + \varepsilon\right)} \leq r^{\beta} \ . \tag{6}$$

Now from (5) and (6) it follows for all sufficiently large values of r that

$$\begin{aligned}&\log^{[m+1]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right) \\ &\geq O(1) + \left(\frac{r^{(\alpha-\beta)}}{\rho_f^{[m]L^*} + \varepsilon}\right)\left[\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) - \left(\rho_f^{[m]L^*} + \varepsilon\right) L\left(\exp\left(r^{\beta}\right)\right)\right] \\ &\qquad + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\end{aligned} \tag{7}$$

$$\begin{aligned}i.e., \ \frac{\log^{[m+1]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right)}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)} &\geq \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) + O(1)}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)} \\ &+ \frac{r^{(\alpha-\beta)}}{\rho_f^{[m]L^*} + \varepsilon}\left\{1 - \frac{\left(\rho_f^{[m]L^*} + \varepsilon\right) L\left(\exp\left(r^{\beta}\right)\right)}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)}\right\} \ .\end{aligned} \tag{8}$$

Again from (7) we get for all sufficiently large values of r that

$$\begin{aligned}&\frac{\log^{[m+1]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right)}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} \\ &\qquad \geq \frac{O(1) - r^{(\alpha-\beta)} L\left(\exp\left(r^{\beta}\right)\right)}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} \\ &\qquad + \frac{\left(\frac{r^{(\alpha-\beta)}}{\rho_f^{[m]L^*} + \varepsilon}\right) \log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} \\ &\qquad\qquad + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}\end{aligned}$$

$$\begin{aligned}i.e., \ \frac{\log^{[m+1]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right)}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} &\geq \frac{\frac{O(1) - r^{(\alpha-\beta)} L\left(\exp\left(r^{\beta}\right)\right)}{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}}{\frac{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)}{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} + 1} \\ &+ \frac{\left(\frac{r^{(\alpha-\beta)}}{\rho_f^{[m]L^*} + \varepsilon}\right)}{1 + \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)}} + \frac{1}{1 + \frac{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)}{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}} \ .\end{aligned} \tag{9}$$

Case I. If $r^{\beta} = o\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\}$ then it follows from (8) that

$$\liminf_{r \to \infty} \frac{\log^{[m+1]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right)}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)} = \infty \ .$$

Case II. If $r^{\beta} \neq o\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\}$ then two sub cases may arise:
Sub case (a). If $L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) = o\left\{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)\right\}$, then we get from (9) that

$$\liminf_{r\to\infty} \frac{\log^{[m+1]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right)}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} = \infty \ .$$

Sub case (b). If $L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) \sim \log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)$ then

$$\lim_{r\to\infty} \frac{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)} = 1$$

and we obtain from (9) that

$$\liminf_{r\to\infty} \frac{\log^{[m+1]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right)}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} = \infty \ .$$

Combining Case I and Case II we obtain that

$$\lim_{r\to\infty} \frac{\log^{[m+1]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right)}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} = \infty \ ,$$

where $K\left(r, \alpha; L\right) = \begin{cases} 0 \text{ if } r^{\mu} = o\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\} \text{ as } r \to \infty \\ L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) \text{ otherwise} \ . \end{cases}$
This proves the theorem. ■

Theorem 4 *Let f and g be any two entire functions with $0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$ and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ where $m \geq 1$. Then for any two positive integers α and β,*

$$\lim_{r\to\infty} \frac{\log^{[m+1]} \mu\left(\exp\left(\exp\left(r^{\alpha}\right)\right), f \circ g\right)}{\log^{[2]} \mu\left(\exp\left(r^{\beta}\right), g\right) + L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)} = \infty \ ,$$

where $K\left(r, \alpha; L\right) = \begin{cases} 0 \textit{ if } r^{\beta} = o\left\{L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right)\right\} \textit{ as } r \to \infty \\ L\left(\exp\left(\exp\left(r^{\alpha}\right)\right)\right) \textit{ otherwise} \ . \end{cases}$

The proof is omitted because it can be carried out in the line of Theorem 3.

Remark 1 *In view of Lemma 2 , the results analogous to Theorem 1, Theorem 2, Theorem 3 and Theorem 4 can also be derived in terms of maximum moduli of composite entire functions.*

Theorem 5 *Let f and g be any two entire functions such that $0 < \rho_g^{L^*} < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$ where $m \geq 1$. Then for any $\beta > 1$,*

$$\lim_{r\to\infty} \frac{\log^{[m]} \mu(r, f \circ g)}{\log^{[m]} \mu(r, f) \cdot K(r, g; L)} = 0 \ ,$$

where $K(r,g;L) = \begin{cases} 1 \text{ if } L(\mu(\beta r, g)) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\} \text{ as } r \to \infty \\ \qquad\qquad \text{and for some } \alpha < \lambda_f^{[m]L^*} \\ L(\mu(\beta r, g)) \text{ otherwise.} \end{cases}$

Proof. In view of Lemma 2 and taking $R = \beta r$ in the inequality $\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f)$ $\{cf.$ [10] $\}$, we have for all sufficiently large values of r that

$$\mu(r, f \circ g) \leq M(r, f \circ g) \leq M(M(r,g), f)$$

$$i.e., \ \log^{[m]} \mu(r, f \circ g) \leq \log^{[m]} M(M(r,g), f)$$

$$i.e., \ \log^{[m]} \mu(r, f \circ g) \leq \left(\rho_f^{[m]L^*} + \varepsilon\right) \left[\log M(r,g) e^{L(M(r,g))}\right]$$

$$i.e., \ \log^{[m]} \mu(r, f \circ g) \leq \left(\rho_f^{[m]L^*} + \varepsilon\right) \left[\log M(r,g) + L(M(r,g))\right] \tag{10}$$

$$i.e., \ \log^{[m]} \mu(r, f \circ g) \leq \left(\rho_f^{[m]L^*} + \varepsilon\right) \left[\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*} + \varepsilon\right)} + L\left(\frac{\beta}{(\beta - 1)}\mu(\beta r, g)\right)\right]$$

$$i.e., \ \log^{[m]} \mu(r, f \circ g) \leq \left(\rho_f^{[m]L^*} + \varepsilon\right) \left[\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*} + \varepsilon\right)} + L(\mu(\beta r, g))\right]. \tag{11}$$

Also we obtain for all sufficiently large values of r that

$$\log^{[m]} \mu(r, f) \geq \left(\lambda_f^{[m]L^*} - \varepsilon\right) \log\left[re^{L(r)}\right]$$

$$i.e., \ \log^{[m]} \mu(r, f) \geq \left(\lambda_f^{[m]L^*} - \varepsilon\right) \log\left[re^{L(r)}\right]$$

$$i.e., \ \log^{[m]} \mu(r, f) \geq \left[re^{L(r)}\right]^{\left(\lambda_f^{[m]L^*} - \varepsilon\right)} . \tag{12}$$

Now from (11) and (12) we get for all sufficiently large values of r that

$$\frac{\log^{[m]} \mu(r, f \circ g)}{\log^{[m]} \mu(r, f)} \leq \frac{\left(\rho_f^{[m]L^*} + \varepsilon\right) \left[\left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)} + L\left(\mu\left(\beta r, g\right)\right)\right]}{\left[re^{L(r)}\right]^{\left(\lambda_f^{[m]L^*}-\varepsilon\right)}} \; . \tag{13}$$

Since $\rho_g^{L^*} < \lambda_f^{[m]L^*}$, we can choose $\varepsilon\,(> 0)$ in such a way that

$$\rho_g^{L^*} + \varepsilon < \lambda_f^{[m]L^*} - \varepsilon \; . \tag{14}$$

Case I. Let $L\left(\mu\left(\beta r, g\right)\right) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some $\alpha < \lambda_f^{[m]L^*}$. As $\alpha < \lambda_f^{[m]L^*}$, we can choose $\varepsilon\,(> 0)$ in such a way that

$$\alpha < \lambda_f^{[m]L^*} - \varepsilon \; . \tag{15}$$

Since $L\left(\mu\left(\beta r, g\right)\right) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\}$ as $r \to \infty$ we get on using (15) that

$$\frac{L\left(\mu\left(\beta r, g\right)\right)}{r^{\alpha} e^{\alpha L(r)}} \to 0 \text{ as } r \to \infty$$

$$i.e., \; \frac{L\left(\mu\left(\beta r, g\right)\right)}{\left[re^{L(r)}\right]^{\left(\lambda_f^{[m]L^*}-\varepsilon\right)}} \to 0 \text{ as } r \to \infty \; . \tag{16}$$

Now in view of (13), (14) and (16) we obtain that

$$\lim_{r\to\infty} \frac{\log^{[m]} \mu(r, f \circ g)}{\log^{[m]} \mu(r, f)} = 0 \; . \tag{17}$$

Case II. If $L\left(\mu\left(\beta r, g\right)\right) \neq o\left\{r^{\alpha} e^{\alpha L(r)}\right\}$ as $r \to \infty$ and for some $\alpha < \lambda_f^{[m]L^*}$ then we get from (13) that for a sequence of values of r tending to infinity,

$$\frac{\log^{[m]} \mu(r, f \circ g)}{\log^{[m]} \mu(r, f)\, L\left(\mu\left(\beta r, g\right)\right)} \leq \frac{\left(\rho_f^{[m]L^*} + \varepsilon\right) \left\{re^{L(r)}\right\}^{\left(\rho_g^{L^*}+\varepsilon\right)}}{\left[re^{L(r)}\right]^{\left(\lambda_f^{[m]L^*}-\varepsilon\right)} L\left(\mu\left(\beta r, g\right)\right)} + \frac{\left(\rho_f^{[m]L^*} + \varepsilon\right)}{\left[re^{L(r)}\right]^{\left(\lambda_f^{[m]L^*}-\varepsilon\right)}} \; . \tag{18}$$

Now using (14) it follows from (18) that

$$\lim_{r\to\infty} \frac{\log^{[m]} \mu(r, f \circ g)}{\log^{[m]} \mu(r, f)\, L\left(\mu\left(\beta r, g\right)\right)} = 0 \; . \tag{19}$$

Combining (17) and (19) we obtain that

$$\lim_{r\to\infty} \frac{\log^{[m]} \mu(r, f\circ g)}{\log^{[m]} \mu(r, f)\cdot K(r, g; L)} = 0\ ,$$

$$\text{where } K(r,g;L) = \begin{cases} 1 \text{ if } L(\mu(\beta r, g)) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\} \text{ as } r\to\infty \\ \qquad\qquad\qquad \text{and for some } \alpha < \lambda_f^{[m]L^*} \\ L(\mu(\beta r, g)) \text{ otherwise.} \end{cases}$$

Thus the theorem is established. ■

The following theorem can be carried out in the line of Theorem 5 and therefore its proof is omitted :

Theorem 6 *Let f and g be any two entire functions with $0 < \rho_g^{L^*} < \rho_f^{[m]L^*} < \infty$ where $m \geq 1$. Then for any $\beta > 1$,*

$$\liminf_{r\to\infty} \frac{\log^{[m]} \mu(r, f\circ g)}{\log^{[m]} \mu(r, f)\cdot K(r, g; L)} = 0\ ,$$

$$\textit{where } K(r,g;L) = \begin{cases} 1 \textit{ if } L(\mu(\beta r, g)) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\} \textit{ as } r\to\infty \\ \qquad\qquad\qquad \textit{and for some } \alpha < \rho_f^{[m]L^*} \\ L(\mu(\beta r, g)) \textit{ otherwise.} \end{cases}$$

Replacing maximum term by maximum modulus in Theorem 5 and Theorem 6 we respectively get Theorem 7 and Theorem 8 and therefore their proofs are omitted.

Theorem 7 *Let f and g be any two entire functions such that $0 < \rho_g^{L^*} < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$ where $m \geq 1$. Then*

$$\lim_{r\to\infty} \frac{\log^{[m]} M(r, f\circ g)}{\log^{[m]} M(r, f)\cdot K(r, g; L)} = 0\ ,$$

$$\textit{where } K(r,g;L) = \begin{cases} 1 \textit{ if } L(M(r, g)) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\} \textit{ as } r\to\infty \\ \qquad\qquad\qquad \textit{and for some } \alpha < \lambda_f^{[m]L^*} \\ L(M(r, g)) \textit{ otherwise.} \end{cases}$$

Theorem 8 *Suppose f and g be any two entire functions with $0 < \rho_g^{L^*} < \rho_f^{[m]L^*} < \infty$ where $m \geq 1$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m]} M(r, f\circ g)}{\log^{[m]} M(r, f)\cdot K(r, g; L)} = 0\ ,$$

$$\textit{where } K(r,g;L) = \begin{cases} 1 \textit{ if } L(M(r, g)) = o\left\{r^{\alpha} e^{\alpha L(r)}\right\} \textit{ as } r\to\infty \\ \qquad\qquad\qquad \textit{and for some } \alpha < \rho_f^{[m]L^*} \\ L(M(r, g)) \textit{ otherwise.} \end{cases}$$

Theorem 9 *Let f and g be any two entire functions with $\rho_f^{[m]L^*} < \infty$, $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$ where m is any positive integer. Then for any $\beta > 1$,*
(a) If $L(\mu(\beta r, g)) = o\left\{\log^{[2]} \mu(r, g)\right\}$ then

$$\limsup_{r\to\infty} \frac{\log^{[m+1]} \mu(r, f\circ g)}{\log^{[2]} \mu(r, g) + L(\mu(\beta r, g))} \leq \frac{\rho_g^{L^*}}{\lambda_g^{L^*}}$$

and (b) if $\log^{[2]} \mu(r, g) = o\{L(\mu(\beta r, g))\}$ then

$$\lim_{r\to\infty} \frac{\log^{[m+1]} \mu(r, f\circ g)}{\log^{[2]} \mu(r, g) + L(\mu(\beta r, g))} = 0 \ .$$

Proof. Taking $R = \beta r$ in the inequality

$$\mu(r, f) \leq M(r, f) \leq \frac{R}{R-r}\mu(R, f) \{cf.\ [10]\ \}$$

and also using $\log\left\{1 + \frac{O(1)+L(\mu(\beta r,g))}{\log\mu(\beta r,g)}\right\} \sim \frac{O(1)+L(\mu(\beta r,g))}{\log\mu(\beta r,g)}$, for all sufficiently large values of r we obtain from (10) that

$$\begin{aligned} &\log^{[m]} \mu(r, f\circ g) \\ \leq\ & \left(\rho_f^{[m]L^*} + \varepsilon\right)\left[\log\mu(\beta r, g) + O(1) + L\left(\frac{\beta}{(\beta-1)}\mu(\beta r, g)\right)\right] \end{aligned}$$

$$i.e.,\ \ \log^{[m]} \mu(r, f\circ g) \leq \left(\rho_f^{[m]L^*} + \varepsilon\right)\log\mu(\beta r, g)\left[1 + \frac{O(1) + L(\mu(\beta r, g))}{\log\mu(\beta r, g)}\right]$$

$$\begin{aligned} i.e.,\ \ \log^{[m+1]} \mu(r, f\circ g) \leq\ & \log\left(\rho_f^{[m]L^*} + \varepsilon\right) + \log^{[2]} \mu(\beta r, g) \\ & + \log\left\{1 + \frac{O(1) + L(\mu(\beta r, g))}{\log\mu(\beta r, g)}\right\} \end{aligned}$$

$$\begin{aligned} i.e.,\ \ \log^{[m+1]} \mu(r, f\circ g) \leq\ & \log\left(\rho_f^{[m]L^*} + \varepsilon\right) + \left(\rho_g^{L^*} + \varepsilon\right)\log\left\{\beta r e^{L(\beta r)}\right\} \\ & + \log\left\{1 + \frac{O(1) + L(\mu(\beta r, g))}{\log\mu(\beta r, g)}\right\} \end{aligned}$$

$$\begin{aligned} i.e.,\ \ \log^{[m+1]} \mu(r, f\circ g) \leq\ & \log\left(\rho_f^{[m]L^*} + \varepsilon\right) + \left(\rho_g^{L^*} + \varepsilon\right)\log\left\{\beta r e^{L(r)}\right\} \\ & + \log\left\{1 + \frac{O(1) + L(\mu(\beta r, g))}{\log\mu(\beta r, g)}\right\} \end{aligned}$$

$$i.e., \;\; \log^{[m+1]} \mu\left(r, f \circ g\right) \leq O\left(1\right) + \left(\rho_g^{L^*} + \varepsilon\right) \left\{\log \beta r + L\left(r\right)\right\} + \frac{O(1) + L\left(\mu\left(\beta r, g\right)\right)}{\log \mu\left(\beta r, g\right)}$$

$$\begin{aligned} i.e., \;\; \log^{[m+1]} \mu\left(r, f \circ g\right) &\leq O\left(1\right) + \left(\rho_g^{L^*} + \varepsilon\right) \left\{\log r + L\left(r\right)\right\} \\ &\quad + \left(\rho_g^{L^*} + \varepsilon\right) \log \beta + \frac{O(1) + L\left(\mu\left(\beta r, g\right)\right)}{\log \mu\left(\beta r, g\right)} \; . \end{aligned} \tag{20}$$

Again from the definition of L^*-lower order, we get for all sufficiently large values of r that

$$\begin{aligned} &\log^{[2]} \mu\left(r, g\right) \geq \left(\lambda_g^{L^*} - \varepsilon\right) \log\left[re^{L(r)}\right] \\ i.e., \;\; &\log^{[2]} \mu\left(r, g\right) \geq \left(\lambda_g^{L^*} - \varepsilon\right) \log\left[re^{L(r)}\right] \\ i.e., \;\; &\log^{[2]} \mu\left(r, g\right) \geq \left(\lambda_g^{L^*} - \varepsilon\right) \left[\log r + L\left(r\right)\right] \\ i.e., \;\; &\log r + L\left(r\right) \leq \frac{\log^{[2]} \mu\left(r, g\right)}{\left(\lambda_g^{L^*} - \varepsilon\right)} \; . \end{aligned} \tag{21}$$

Hence from (20) and (21) it follows for all sufficiently large values of r that

$$\begin{aligned} &\log^{[m+1]} \mu\left(r, f \circ g\right) \\ \leq \;\; &O\left(1\right) + \left(\frac{\rho_g^{L^*} + \varepsilon}{\lambda_g^{L^*} - \varepsilon}\right) \cdot \log^{[2]} \mu\left(r, g\right) + \left(\rho_g^{L^*} + \varepsilon\right) \log \beta + \frac{O(1) + L\left(\mu\left(\beta r, g\right)\right)}{\log \mu\left(\beta r, g\right)} \end{aligned}$$

$$\begin{aligned} i.e, \;\; &\frac{\log^{[m+1]} \mu\left(r, f \circ g\right)}{\log^{[2]} \mu\left(r, g\right) + L\left(\mu\left(\beta r, g\right)\right)} \\ \leq &\frac{O\left(1\right) + \left(\rho_g^{L^*} + \varepsilon\right) \log \beta}{\log^{[2]} \mu\left(r, g\right) + L\left(\mu\left(\beta r, g\right)\right)} + \left(\frac{\rho_g^{L^*} + \varepsilon}{\lambda_g^{L^*} - \varepsilon}\right) \cdot \frac{\log^{[2]} \mu\left(r, g\right)}{\log^{[2]} \mu\left(r, g\right) + L\left(\mu\left(\beta r, g\right)\right)} \\ &\qquad + \frac{O(1) + L\left(\mu\left(\beta r, g\right)\right)}{\left[\log^{[2]} \mu\left(r, g\right) + L\left(\mu\left(\beta r, g\right)\right)\right] \log \mu\left(\beta r, g\right)} \end{aligned}$$

$$\begin{aligned} i.e, \;\; \frac{\log^{[m+1]} \mu\left(r, f \circ g\right)}{\log^{[2]} \mu\left(r, g\right) + L\left(\mu\left(\beta r, g\right)\right)} &\leq \frac{\frac{O(1) + \left(\rho_g^{L^*} + \varepsilon\right) \log \beta}{L(\mu(\beta r, g))}}{\frac{\log^{[2]} \mu(r, g)}{L(\mu(\beta r, g))} + 1} + \frac{\left(\frac{\rho_g^{L^*} + \varepsilon}{\lambda_g^{L^*} - \varepsilon}\right)}{1 + \frac{L(\mu(\beta r, g))}{\log^{[2]} \mu(r, g)}} \\ &\quad + \frac{1}{\left[1 + \frac{\log^{[2]} \mu(r, g)}{L(\mu(\beta r, g))}\right] \log \mu\left(\beta r, g\right)} . \end{aligned} \tag{22}$$

Since $L(\mu(\beta r, g)) = o\left\{\log^{[2]} \mu(r,g)\right\}$ as $r \to \infty$ and $\varepsilon\,(>0)$ is arbitrary, we obtain from (22) that

$$\limsup_{r\to\infty} \frac{\log^{[m+1]} \mu(r, f \circ g)}{\log^{[2]} \mu(r,g) + L(\mu(\beta r, g))} \le \frac{\rho_g^{L^*}}{\lambda_g^{L^*}} \ . \tag{23}$$

Again if $\log^{[2]} \mu(r,g) = o\{L(\mu(\beta r, g))\}$ then from (22) we get that

$$\lim_{r\to\infty} \frac{\log^{[m+1]} \mu(r, f \circ g)}{\log^{[2]} \mu(r,g) + L(\mu(\beta r, g))} = 0 \ . \tag{24}$$

Thus the theorem follows from (23) and (24). ■

Corollary 1 *Let f and g be any two entire functions with $\rho_f^{[m]L^*} < \infty$ and $0 < \rho_g^{L^*} < \infty$ where $m \ge 1$. Then for any $\beta > 1$,*
(a) if $L(\mu(\beta r, g)) = o\left\{\log^{[2]} \mu(r,g)\right\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[m+1]} \mu(r, f \circ g)}{\log^{[2]} \mu(r,g) + L(\mu(\beta r, g))} \le 1$$

and (b) if $\log^{[2]} \mu(r,g) = o\{L(\mu(\beta r, g))\}$ then

$$\liminf_{r\to\infty} \frac{\log^{[m+1]} \mu(r, f \circ g)}{\log^{[2]} \mu(r,g) + L(\mu(\beta r, g))} = 0 \ .$$

We omit the proof of Corollary 1 because it can be carried out in the line of Theorem 7.

Remark 2 *The equality sign in Theorem 5 and Corollary 1 cannot be removed as we see in the following example:*

Example 1 *Let $f = g = \exp z$, $m = 2$, $\beta = 2$ and $L(r) = \frac{1}{p}\exp\left(\frac{1}{r}\right)$ where p is any positive real number.*
Then

$$\rho_f^{L^*} = \lambda_g^{L^*} = \rho_g^{L^*} = 1.$$

Now

$$\begin{aligned} \log \mu(r, f \circ g) &\le \log M(r, f \circ g) = \exp r, \\ \textit{and } 2\mu(2r, g) &\ge M(r, g) = \exp r \ . \end{aligned}$$

Also

$$\begin{aligned}\log\mu(r,f\circ g) &\geq \log M\left(\frac{r}{2},f\circ g\right)+O(1)=\exp\left(\frac{r}{2}\right)+O(1),\\ \text{and } \mu(r,g) &\leq M(r,g)=\exp r\ .\end{aligned}$$

So

$$L(M(r,g))=L(\exp r)=\frac{1}{p}\exp\left(\frac{1}{\exp r}\right)\ .$$

Hence

$$\limsup_{r\to\infty}\frac{\log^{[3]}\mu(r,f\circ g)}{\log^{[2]}\mu(r,g)+L(\mu(\beta r,g))}\leq\limsup_{r\to\infty}\frac{\log r}{\log r+O(1)+\frac{1}{p}\exp\left(\frac{1}{\exp r}\right)}=1$$

and

$$\liminf_{r\to\infty}\frac{\log^{[3]}\mu(r,f\circ g)}{\log^{[2]}\mu(r,g)+L(\mu(\beta r,g))}\geq\liminf_{r\to\infty}\frac{\log r+O(1)}{\log r+\frac{1}{p}\exp\left(\frac{1}{\exp r}\right)}=1\ .$$

Therefore

$$\liminf_{r\to\infty}\frac{\log^{[3]}\mu(r,f\circ g)}{\log^{[2]}\mu(r,g)+L(\mu(\beta r,g))}=\limsup_{r\to\infty}\frac{\log^{[3]}\mu(r,f\circ g)}{\log^{[2]}\mu(r,g)+L(\mu(\beta r,g))}=1\ .$$

Theorem 10 *Let f and g be any two entire functions with $\rho_f^{[m]L^*}<\infty$ and $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$ where m is any positive integer. Then*
(a) if $L(M(r,g))=o\left\{\log^{[2]}M(r,g\right\}$ then

$$\limsup_{r\to\infty}\frac{\log^{[m+1]}M(r,f\circ g)}{\log^{[2]}M(r,g)+L(M(r,g))}\leq\frac{\rho_g^{L^*}}{\lambda_g^{L^*}}$$

and (b) if $\log^{[2]}M(r,g)=o\{L(M(r,g))\}$ then

$$\lim_{r\to\infty}\frac{\log^{[m+1]}M(r,f\circ g)}{\log^{[2]}M(r,g)+L(M(r,g))}=0\ .$$

Corollary 2 *Let f and g be any two entire functions with $\rho_f^{[m]L^*}<\infty$ and $0<\rho_g^{L^*}<\infty$ where $m\geq1$. Then for any $\beta>1$,*
(a) if $L(M(r,g))=o\left\{\log^{[2]}M(r,g)\right\}$ then

$$\liminf_{r\to\infty}\frac{\log^{[m+1]}M(r,f\circ g)}{\log^{[2]}M(r,g)+L(M(r,g))}\leq1$$

and (b) if $\log^{[2]} M(r,g) = o\{L(M(r,g))\}$ *then*

$$\liminf_{r\to\infty} \frac{\log^{[m+1]} M(r, f\circ g)}{\log^{[2]} M(r,g) + L(M(r,g))} = 0 \ .$$

We omit the proof of Theorem 10 and Corollary 2 because in view of Lemma 2 it can be carried out in the line of Theorem 9 and Corollary 1 respectively.

Remark 3 *Considering* $f = g = \exp z$, $m = 2$ *and* $L(r) = \frac{1}{p}\exp\left(\frac{1}{r}\right)$ *where* p *is any positive real number, one can easily verify that the equality sign in Theorem 10 and Corollary 2 cannot be removed.*

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