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A GENERAL ALGORITHM FOR THE INVERSE TRANSFORMATION OF MAP PROJECTIONS USING JACOBIAN MATRICES

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Abstract- Coordinate transformations refer to mathematical processing that enables overlay of maps that use different coordinate reference systems, that is, map projections. The conversion from geographical to plane coordinates is the normal practice in cartography, which is called forward transformation. The inverse transformation, which yields geographical coordinates from map coordinates, is a more recent development, due to the need for transformation between different map projections especially in Geographic Information Systems (GIS). For the projections that have complex functions for forward transformation defining the invers projection is not easy. This paper describes a general iteration algorithm to derive the inverse equations of map projections using Jacobian matrices. The algorithm is applied to three cartographic projections, namely Aitoff-Hammer, Winkel-Tripel and Mollweide, which are commonly used for world maps.

Key Words- Map Projections, Inverse Equations, Newton's Iteration, Jacobian Matrix

1. INTRODUCTION

The Cartesian coordinates (X,Y) of a point on a map are calculated from latitude (j) and longitude (l) using the functions

$X = f_x(j, l)$

$Y = f_y(j, l)$

X-axis denotes the Equator positive to the east and Y-axis denotes the central meridian positive to the north. The functions or equations define a map projection in general. This conversion or transformation from geographical to plane coordinates is called forward transformation and is the normal practice in cartography. A frequently occured problem in cartography is to derive the geographical coordinates from the forward projection equations. This process is commonly called "inverse mapping". Although the inverse equations for many projections are already in existence, in some cases they must be developed [1]. But developing the inverse equations has sometimes proved difficult due to the complex projection equations. In this study, a general algorithm is described for the inverse solution of the map projections using partial derivatives, which could easily be applied to all kinds of projections. The algorithm is applied to three famous map projections.

2. NEWTON-RAPHSON ITERATION USING JACOBIAN MATRIX

This section derives the geographical latitude and longitude values from the plane coordinates of a map projection. An iterative algorithm using partial derivatives of the projection equations is developed for this purpose. The method is based on the solution of non-linear equations by inverting the Jacobian matrix of partial derivatives, which is well known in numerical analysis [2], [3]. The particular solution chosen in this study is a generalization of the Newton-Raphson iteration method [4].

Consider a point selected on the projection with plane coordinates X and Y (on the map plane). The problem is to find the geographical coordinates (l, j) of this point. We define the vectors $\mathbf{Q}_{i+1}$ and $\mathbf{Q}_i$ (i=1,2, …) with the elements of geographical coordinates for the iteration as follows

$$\mathbf{Q}_{i+1} = \begin{bmatrix} j_{i+1} \\ l_{i+1} \end{bmatrix} \tag{1}$$

$$\mathbf{Q}_i = \begin{bmatrix} j_i \\ l_i \end{bmatrix} \tag{2}$$

where (i) denotes the actual step of the iteration, and j_{i+1} and l_{i+1} indicate the coordinates obtained for the consequent step of the iteration using j_i and l_i.

A vector **F** consists of the mapping functions given by

$$\mathbf{F} = \begin{bmatrix} f_1(j_i, l_i) \\ f_2(j_i, l_i) \end{bmatrix} \tag{3}$$

where;

$$f_1(j_i, l_i) = X_i - X = 0 \tag{4}$$

$$f_2(j_i, l_i) = Y_i - Y = 0 \tag{5}$$

The iteration procedure can be written in matrix form as follows;

$$\mathbf{Q}_{i+1} = \mathbf{Q}_i - \Delta\mathbf{Q}$$

(6)

where;

$$\Delta\mathbf{Q} = \mathbf{J}^{-1}\mathbf{F} \tag{7}$$

The absolute value of (7) is compared with an accuracy level e

$$|\Delta\mathbf{Q}| \leq \begin{bmatrix} e \\ e \end{bmatrix} \tag{8}$$

Here, e is a convergence value and can be taken as 10^{-12}. If the condition being defined with equation (8) is realized then the iteration stops. This means (X_i ,Y_i) is sufficiently close to the selected coordinates (X,Y) at this iteration step [5].

Newton's iteration (6) needs an initial guess $\mathbf{Q}_o$ composed of initial latitude and longitude approximating the given X and Y through the forward projection equations The initial guess is based upon the functions f_1 and f_2 as defined by equations (4) and (5). These functions are used to examine the change in X_i and Y_i to X and Y, respectively, for a given assumption of j_i and l_i. Equation (7) is Newton's correction term. The absolute value of this term is compared to an accuracy level e. If the change between $\mathbf{Q}_i$ and $\mathbf{Q}_{i+1}$ is less than this convergence value, the iteration stops and the final j_i and l_i solve the inverse problem for the given X and Y. The matrix of partial derivatives, namely the Jacobian matrix, is defined as

$$\mathbf{J}=\begin{bmatrix}\left(\frac{\partial f_1}{\partial j}\right)_{j_i,l_i} & \left(\frac{\partial f_1}{\partial l}\right)_{j_i,l_i}\\ \left(\frac{\partial f_2}{\partial j}\right)_{j_i,l_i} & \left(\frac{\partial f_2}{\partial l}\right)_{j_i,l_i}\end{bmatrix} \tag{9}$$

The inverse of the Jacobian matrix is solved by taking the ratio of the adjoint matrix to the determinant of the Jacobian matrix,

$$\mathbf{J}^{-1}=\frac{Adj\mathbf{J}}{Det\mathbf{J}} \tag{10}$$

The adjoint matrix can be written for the two dimensional case as;

$$Adj\mathbf{J}=\begin{bmatrix}\left(\frac{\partial f_2}{\partial l}\right)_{j_i,l_i} & -\left(\frac{\partial f_1}{\partial l}\right)_{j_i,l_i}\\ -\left(\frac{\partial f_2}{\partial j}\right)_{j_i,l_i} & \left(\frac{\partial f_1}{\partial j}\right)_{j_i,l_i}\end{bmatrix} \tag{11}$$

and the determinant of the Jacobian matrix is;

$$Det\mathbf{J}=\left(\frac{\partial f_1}{\partial j}\right)_{j_i,l_i}\left(\frac{\partial f_2}{\partial l}\right)_{j_i,l_i}-\left(\frac{\partial f_2}{\partial j}\right)_{j_i,l_i}\left(\frac{\partial f_1}{\partial l}\right)_{j_i,l_i} \tag{12}$$

If we substitute (11) and (12) in (10) we can write;

$$\mathbf{J}^{-1}=\frac{1}{\left(\frac{\partial f_1}{\partial j}\right)_{j_i,l_i}\left(\frac{\partial f_2}{\partial l}\right)_{j_i,l_i}-\left(\frac{\partial f_2}{\partial j}\right)_{j_i,l_i}\left(\frac{\partial f_1}{\partial l}\right)_{j_i,l_i}}\begin{bmatrix}\left(\frac{\partial f_2}{\partial l}\right)_{j_i,l_i} & -\left(\frac{\partial f_1}{\partial l}\right)_{j_i,l_i}\\ -\left(\frac{\partial f_2}{\partial j}\right)_{j_i,l_i} & \left(\frac{\partial f_1}{\partial j}\right)_{j_i,l_i}\end{bmatrix} \tag{13}$$

Substituting (32) and (16) in (26) and (14) and (15) in (25) we can write

$$\begin{bmatrix}j_{i+1}\\ l_{i+1}\end{bmatrix}=\begin{bmatrix}j_i\\ l_i\end{bmatrix}-\frac{1}{\left(\frac{\partial f_1}{\partial j}\right)_{j_i,l_i}\left(\frac{\partial f_2}{\partial l}\right)_{j_i,l_i}-\left(\frac{\partial f_2}{\partial j}\right)_{j_i,l_i}\left(\frac{\partial f_1}{\partial l}\right)_{j_i,l_i}}\begin{bmatrix}\left(\frac{\partial f_2}{\partial l}\right)_{j_i,l_i} & -\left(\frac{\partial f_1}{\partial l}\right)_{j_i,l_i}\\ -\left(\frac{\partial f_2}{\partial j}\right)_{j_i,l_i} & \left(\frac{\partial f_1}{\partial j}\right)_{j_i,l_i}\end{bmatrix}\begin{bmatrix}f_1(j_i,l_i)\\ f_2(j_i,l_i)\end{bmatrix} \tag{14}$$

If we write the matrix elements seperately as a result, then we have [5], [6]:

$$j_{i+1}=j_i-\frac{f_1(j_i,l_i)\left(\frac{\partial f_2}{\partial l}\right)_{j_i,l_i}-f_2(j_i,l_i)\left(\frac{\partial f_1}{\partial l}\right)_{j_i,l_i}}{\left(\frac{\partial f_1}{\partial j}\right)_{j_i,l_i}\left(\frac{\partial f_2}{\partial l}\right)_{j_i,l_i}-\left(\frac{\partial f_2}{\partial j}\right)_{j_i,l_i}\left(\frac{\partial f_1}{\partial l}\right)_{j_i,l_i}} \tag{15}$$

$$l_{i+1}=l_i-\frac{f_2(j_i,l_i)\left(\frac{\partial f_1}{\partial j}\right)_{j_i,l_i}-f_1(j_i,l_i)\left(\frac{\partial f_2}{\partial j}\right)_{j_i,l_i}}{\left(\frac{\partial f_1}{\partial j}\right)_{j_i,l_i}\left(\frac{\partial f_2}{\partial l}\right)_{j_i,l_i}-\left(\frac{\partial f_2}{\partial j}\right)_{j_i,l_i}\left(\frac{\partial f_1}{\partial l}\right)_{j_i,l_i}} \tag{16}$$

3. APPLICATION

The algorithm presented above is applied to the three famous map projections which are commonly used for mapping the whole world. These are the Aitoff-Hammer projection, the Winkel Tripel projection and the Mollweide projection. In the sections below, the projection characteristics are summarized, the forward mapping functions and the partial derivatives of the Jacobian matrix are given for those selected map projections. The radius of curvature is assumed as one unit (R=1).

3.1 Aitoff-Hammer Projection

Russian Cartographer David A. Aitoff (1854-1933) devised an elementary but very appealing modification of one hemisphere of the equatorial aspect of the azimuthal equidistant projection. The Aitoff projection soon inspired Hammer [7] to invent a world map looking very much Aitoff's, but maintaining equal area instead, with prominent credit to Aitoff in both the title and text of Hammer's paper [8].

The projection is presented by Hammer, is an equal-area modified azimuthal projection. Central meridian is a straight line half the length of the Equator. Other meridians are complex curves intersecting at the poles, unequally spaced along the Equator and concave toward the central meridian. Equator is shown as a straight line. Other parallels are complex curves, unequally spaced along the central meridian and concave toward the nearest pole. The projection is symmetrical about the central meridian and the Equator.

The forward mapping functions for the Aitoff-Hammer projection are as follows [9], [10], [11];

$$f_1(\varphi,\lambda)=\frac{\sqrt{2}\sin\varphi}{\sqrt{1+\cos\varphi\,\cos\dfrac{\lambda}{2}}}-Y=0 \tag{17}$$

$$f_2(\varphi,\lambda)=\frac{2\sqrt{2}\cos\varphi\,\sin\dfrac{\lambda}{2}}{\sqrt{1+\cos\varphi\,\cos\dfrac{\lambda}{2}}}-X=0 \tag{18}$$

The partial derivatives for this mapping functions are

$$\frac{\partial f_1}{\partial\varphi}=\frac{(1+p)\cos\varphi+\cos(\lambda/2)}{p^{3/2}\sqrt{2}} \tag{19}$$

$$\frac{\partial f_1}{\partial\lambda}=\frac{\sqrt{2}}{4}\frac{\sin\varphi\,\cos\varphi\,\sin(\lambda/2)}{p^{3/2}} \tag{20}$$

$$\frac{\partial f_2}{\partial\varphi}=-\sqrt{2}\,\frac{(1+p)\sin\varphi\,\sin(\lambda/2)}{p^{3/2}} \tag{21}$$

$$\frac{\partial f_2}{\partial\lambda}=\frac{\sqrt{2}}{2}\frac{(1+p)\cos\varphi\,\cos(\lambda/2)+\cos^2\varphi}{p^{3/2}} \tag{22}$$

where,

$$p = 1 + \cos\varphi\cos(\lambda / 2) \tag{23}$$

The inverse solution for the Aitoff-Hammer projection can be obtained substituting equations (17) to (22) into equation (15) and (16) [5].

3.2 Winkel Tripel Projection

The Winkel Tripel projection was developed by Oswald Winkel (1873-1953) from Germany in 1921 averaging the cylindrical equidistant (equirectangular) and Aitoff projections [12]. Winkel himself applied the German term “Tripel” (in english “triple”), because he considered it a “compromise of the properties of three elements”-area, angle and distance-which resulted in a lower distortion distributed uniformly overall [17]. After analyzing the Winkel Tripel’s distortion characteristics, cartographers have suggested that it is suitable for whole-world applications [13], [14], [15].

The Winkel Tripel projection is a modified azimuthal projection that is neither conformal nor equal area, like the Winkel I and II. However, by using L. P. Lee's definitions the Tripel can also be classified as a Polyconic [16].

The projection functions for the Winkel Tripel projection are presented as follows [9], [10], [11]:

$$f_1(j_i, l_i) = \frac{1}{2}\left[\frac{2D}{C^{1/2}} cos\, j_i\, sin\frac{l_i}{2} + l_i\, cos\, j_0\right] - X = 0 \tag{24}$$

$$f_2(j_i, l_i) = \frac{1}{2}\left[\frac{D}{C^{1/2}} sin\, j_i + j_i\right] - Y = 0 \tag{25}$$

where;

$$D = arccos\left(cos\, j\; cos\frac{l}{2}\right) \tag{26}$$

$$C = 1 - cos^2 j\; cos^2\frac{l}{2} \tag{27}$$

φ_0 is the standard parallel chosen by Winkel as 50°28′ in the equidistant cylindrical component of the projection [12], [13], [17].

The partial derivatives for these functions are [6], [13], [18].

$$\left(\frac{\partial f_1}{\partial j}\right)_{j_i, l_i} = \frac{sin\, l_i\, sin\, 2j_i}{4C} - \frac{D}{C^{3/2}} sin\, j_i\, sin\frac{l_i}{2} \tag{28}$$

$$\left(\frac{\partial f_1}{\partial l}\right)_{j_i, l_i} = \frac{1}{2}\left[\frac{cos^2 j_i\, sin^2\frac{l_i}{2}}{C} + \frac{D}{C^{3/2}} cos\, j_i\, cos\frac{l_i}{2} sin^2 j_i + cos\, j_0\right] \tag{29}$$

$$\left(\frac{\partial f_2}{\partial j}\right)_{j_i, l_i} = \frac{1}{2}\left[\frac{sin^2 j_i\, cos\frac{l_i}{2}}{C} + \frac{D}{C^{3/2}}\left(1 - cos^2\frac{l_i}{2}\right) cos\, j_i + 1\right] \tag{30}$$

$$\left(\frac{\partial f_2}{\partial l}\right)_{j_i,l_i} = \frac{1}{8}\left[\frac{sin\,2j_i\; sin\frac{l_i}{2}}{C} - \frac{D}{C^{3/2}}\, sin\, j_i\, cos^2\, j_i\, sin\, l_i\right] \tag{31}$$

The geographical latitude and longitude values can be calculated using the coordinates of an arbitrary point selected on a map produced in the Winkel Tripel projection by substituting equations (24), (25), (28), (29), (30), and (31) into equations (15) and (16).

3.3 Mollweide Projection

To introduce a three-dimensional application of the algorithm the Mollweide projection is selected. This projection is presented by Carl B. Mollweide (1774-1825) from Germany in 1805. It is an equal-area pseudocylindrical projection. Central meridian is a straight line half as long as the equator. Meridians 90^{o} east and west of the central meridian form a circle and others are equally spaced semiellipses. Meridians are intersected at the poles and concave toward the central meridian. Parallels are unequally spaced straight lines, farthest apart near the equator and perpendicular to the central meridian. The projection is symmetrical about the central meridian or the equator. Scale is true along latitudes $40°44'$ north and south, constant along any given latitude and same for the latitude of opposite sign.

The projection functions for the Mollweide projection can be written as follows [5], [9], [10], [11]:

$$f_1(j\,,\,l\,,t) = \frac{2\sqrt{2}}{p}\, l\,\cos t - X = 0 \tag{32}$$

$$f_2(j\,,\,l\,,t) = \sqrt{2}\,\sin\, t - Y = 0 \tag{33}$$

$$f_3(j\,,l\,,t\,) = 2t + sin\,2t - p\;sin\,j\; = 0 \tag{34}$$

In this case the Jacobian matrix is

$$\mathbf{J} = \begin{bmatrix} \left(\frac{\P f_1}{\P j}\right)_{j_i} & \left(\frac{\P f_1}{\P l}\right)_{l_i} & \left(\frac{\P f_1}{\P t}\right)_{t_i} \\ \left(\frac{\P f_2}{\P j}\right)_{j_i} & \left(\frac{\P f_2}{\P l}\right)_{l_i} & \left(\frac{\P f_2}{\P t}\right)_{t_i} \\ \left(\frac{\P f_3}{\P j}\right)_{j_i} & \left(\frac{\P f_3}{\P l}\right)_{l_i} & \left(\frac{\P f_3}{\P t}\right)_{t_i} \end{bmatrix} = \begin{bmatrix} 0 & \frac{2\sqrt{2}}{p} cos\,t & -\frac{2\sqrt{2}}{p}\, l\; sin\,t \\ 0 & 0 & \sqrt{2}\,cos t \\ -p\; cos\,j & 0 & 4cos^2\, t \end{bmatrix} \tag{35}$$

and the inverse of this jacobian matrix can be written

$$\mathbf{J}^{-1} = -\frac{1}{4cos\,j\;cos^2\,t}\begin{bmatrix} 0 & -\frac{8\sqrt{2}}{p} cos^3\,t & -\frac{4}{p} cos^2\,t \\ -\sqrt{2}p\;cos\,j\;\,cos\,t & -2\sqrt{2}\,l\;cos\,j\;\,sin\,t & 0 \\ 0 & -2\sqrt{2}\,cos\,j\;\,cos\,t & 0 \end{bmatrix} \tag{36}$$

The inverse equations for the Mollweide projection [5]

$$j_{i+1} = j_i - \frac{1}{p\ cos\ j_i}\left[2\sqrt{2}\,cos\,t_i\, f_2(j_i, l_i, t_i) + f_3(j_i, l_i, t_i)\right] \quad (37)$$

$$l_{i+1} = l_i - \frac{1}{2\sqrt{2}\,cos\,t_i}\left[p_i\, f_1(j_i, l_i, t_i) + 2\,l_i\,tan\,t_i\, f_2(j_i, l_i, t_i)\right] \quad (38)$$

$$t_{i+1} = t_i - \frac{f_2(j_i, l_i, t_i)}{\sqrt{2}\,cos\,t_i} \quad (39)$$

4. CONCLUSION

For the projections handled in this article forward projection functions are complex. To derive the inverse projections fuctions, a specific method is required. The algorithm presented here is Newton-Rapson iteration with Jacobian matrices. It is applied to three world projections. It can also be adapted for all cartographic projections in order to derive inverse equations.

Since the manual calculation using this algortihm is not easy, a computer program is needed. With such a program, data capture from analog maps produced using the projections above can be possible. So captured data can also be easily integrated into any GIS system.

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