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Theory of Intersections on the Arithmetic Surface

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1. Let K be an algebraic number field, $\Lambda \subset K$ be the ring of integers in K, X be a curve of genus g over K and f: $V \to \operatorname{Spec} \Lambda$ be its nonsingular model. Here we shall describe a method which yields a very close analogy between a two-dimensional scheme V and compact algebraic surface.

For the sake of simplicity we shall supopse that the fibration family f: $V \to \operatorname{Spec}\Lambda$ has no degenerate fibres. We shall denote an inclusion of our field K in the complex number field by the symbol ∞. We choose one inclusion from each pair of complex conjugated inclusions. From now on the symbol $\sum_\infty$ means that every real inclusion and one of every pair of complex conjugated inclusions is present in our sum. Let us denote the Riemann surface of an algebraic curve $X \otimes_\infty \boldsymbol{C}$ by X_∞. First of all, we shall define a notion which is analogous to the notion of a divisor on the compact algebraic surface.

DEFINITION. A compactified divisor or c-divisor is a formal linear combination

$$D = \sum_i k_i C_i + \sum_\infty \lambda_\infty X_\infty.$$

Here C_i is an irreducible closed subset in V of codimension 1, $k_i \in \boldsymbol{Z}$, $\lambda_\infty \in \boldsymbol{R}$. To avoid confusion we shall call a usual divisor on our scheme V a finite divisor, or f-divisor, and write accordingly a letter "c" or "f" near the corresponding symbol. We shall often consider a finite divisor as a c-divisor with $\lambda_\infty = 0$. All c-divisors form a group which we denote by $\operatorname{Div}_c(V)$.

To define a principal divisor, it is necessary to fix on each surface X_∞ a hermitian metric ds^2_∞. We shall assume the corresponding volume element to satisfy the following condition: $\int_{X_\infty} d\mu_\infty = 1$.

Let $\varphi \in K(X)$ be a rational function. The divisor of the function φ is defined by the formula

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$$(\varphi) = \sum_{C \subset V} \nu_C(\varphi) \cdot C + \sum_{\infty} \nu_\infty(\varphi) \cdot X_\infty.$$

Here $\sum \nu_C(\varphi) \cdot C = (\varphi)_f$ is a usual divisor of the function φ and

$$\nu_\infty(\varphi) = - \int_{X_\infty} \log|\varphi| \, d\mu_\infty.$$

All c-divisors modulo principal divisors form a group which we denote by $\mathrm{Pic}_c(V)$.

2. There exists a theory of intersections for c-divisors. For two c-divisors D_1, D_2 their real intersection index $(D_1, D_2) \in \boldsymbol{R}$ can be defined. It is bilinear, symmetrical and is invariant under c-equivalence: For $\varphi \in K(X)$, $((\varphi)_c, D) = 0$. When written in terms of the finite divisors the invariance property has the following form:

$$(D_f + (\varphi)_f, D'_f) = (D_f, D'_f) + \deg D' \cdot \sum_{\infty} \int_{X_\infty} \log|\varphi| \, d\mu_\infty.$$

Here and below $\deg D$ means the degree of D on the general fibre. It follows from this relation that when restricted on the divisors of degree 0 our index is an invariant under usual linear equivalence. In this case such an index is equal to Neron's index.

The intersection index depends on the choice of the metrics ds^2_∞. However, there exists one metric on the curve X_∞ which is the most convenient for the theory of intersections. To define it, let us consider the Jacobian J_∞ of the curve X_∞ and let $ds^2_{J,\infty}$ be the invariant metric on J_∞ defined by its Θ-polarisation.

Definition. The canonical metric ds^2_∞ is a restriction of the metric $ds^2_{J,\infty}$, under the canonical inclusion $X_\infty \to J_\infty$.

From this moment we shall consider every metric ds^2_∞ to be a canonical one. It is interesting to note that there exists a c-divisor class $\mathscr{K} \in \mathrm{Pic}_c(X)$ which is an analogue of the canonical divisor class on the algebraic surface and which has the following property: If $C \subset V$ is an irreducible horizontal curve on V and δ_C is an absolute discriminant of its ring of regular functions, then

$$(C, C) + (C, \mathscr{K}) = \log|\delta_C|. \qquad (*)$$

3. Now we shall formulate an analogue of the Riemann-Roch theorem. First of all we shall describe an interpretation of c-divisor classes which is analogous to the interpretation of usual divisor classes as linear bundles. Let $\mathscr{L}$ be an invertible sheaf on V. Then $\mathscr{L}$ defines for each ∞ a complex linear bundle L_∞ over the surface X_∞. If every bundle L_∞ is provided with a hermitian metric $\| \ \|_\infty$ on it, whose curvature form is proportional to the form $d\mu_\infty$, we shall call $\mathscr{L}$ a c-bundle. The group of c-bundles is isomorphic to the group $\mathrm{Pic}_c(V)$.

Let us denote by $V_\infty(\mathscr{L})$ the complex linear space of sections of the bundle L_∞ for the complex ∞ and the corresponding real space for the real ∞. The metric $\| \ \|_\infty$ defines a positive function F_∞ on the $V_\infty(\mathscr{L})$ by the following formula:

$$\log F_\infty(s) = \int_{X_\infty} \log \|s\|_\infty \, d\mu_\infty \quad \text{for } s \in V_\infty(\mathscr{L}).$$

Let us define on the space $V(\mathscr{L}) = \bigoplus_\infty V_\infty(\mathscr{L})$ a "norm" function $F = \prod_\infty F_\infty^{\alpha_\infty}$. Here α_∞ equals 1 for a real ∞ and 2 for a complex ∞. If $\deg \mathscr{L} > 2g - 2$, then the Λ

module $\Omega_{\mathscr{L}} = \Gamma(V, \mathscr{L})$ is a projective module of rank $m + 1 - g$ and has a natural inclusion as a lattice into the space $V(\mathscr{L})$. We can define the density of $\Omega_{\mathscr{L}}$ with respect to the norm function F by the formula

$$\mathfrak{r}(\Omega_{\mathscr{L}}) = \left(\prod_{\infty} \nu(B_{\infty})\right) \Big/ \nu(\Omega_{\mathscr{L}}).$$

Here B_{∞} is a unit ball of the norm function F_{∞}, $\nu(B_{\infty})$ is its volume measured by any euclidean metric E_{∞} on the space V_{∞} and $\nu(\Omega_{\mathscr{L}})$ is a volume of the fundamental cube of the lattice $\Omega_{\mathscr{L}}$ measured by the metric $E = \bigoplus E_{\infty}$ on the $V(\mathscr{L})$.

CONJECTURE 1.

$$\log \mathfrak{r}(\Omega_{\mathscr{L}}) = \tfrac{1}{2}(\mathscr{L}, \mathscr{L} - \mathscr{K}) + \tfrac{1}{2}\deg \mathscr{L} \cdot \log|\partial| + d.$$

Here ∂ is the discriminant of the field K and $d = d(X)$ is some invariant of the curve X.

CONJECTURE 2. Invariant $d(X)$ has an interpretation as the height of the point which corresponds to X in the moduli variety of curves of genus g.

Let us put $\tilde{d}(\mathscr{L}) = \log \mathfrak{r}(\Omega_{\mathscr{L}}) - \frac{1}{2}(\mathscr{L}, \mathscr{L} - \mathscr{K}) - \frac{1}{2}\deg \mathscr{L} \cdot \log|\partial|$. Conjecture 1 asserts that $\tilde{d}(\mathscr{L})$ does not depend on $\mathscr{L}$.

It is possible to prove that $\tilde{d}(\mathscr{L})$ is absolutely bounded if $\deg \mathscr{L}$ is fixed. For an elliptic curve X it is possible to prove that $\tilde{d}(\mathscr{L}) = \text{const}$ if $\deg \mathscr{L}$ is fixed.

4. In the last section we shall consider two questions. For the first one let $\xi \in \operatorname{Pic}(X/K)$ be a divisor class of degree $m > 2g - 2$ on X. Suppose, for the sake of simplicity, that the class number of K equals 1. Then ξ corresponds to the single invertible sheaf $\mathscr{L}_{\xi}$ on V, and all effective divisors on V of class ξ are interpreted as classes of equivalent elements of the lattice $\Omega_{\xi} = \Gamma(V, \mathscr{L}_{\xi})$ under the action of the group of units of K.

Let us consider those vectors of the lattice Ω_{ξ} which correspond to irreducible divisors on V. The discriminant δ of a prime divisor is a function which is defined only on such vectors, but it can be extended on the whole lattice Ω_{ξ} by the formula (*). In this way we get a function δ on the lattice Ω_{ξ} which is proportional to the $(m + 2g - 2)$th power of the norm function. Using Conjecture 1 we can compute the density $\mathfrak{r}_{\delta}(\xi)$ of the lattice Ω_{ξ} with respect to the function $\delta^{(m+2g-2)^{-1}}$. Here is the asymptotic behavior of this density when $\deg \xi = m$ is fixed:

$$\log \mathfrak{r}_{\delta}(\xi) = -\text{const} \cdot (g - 1) \cdot B(\xi, \xi) + o(B(\xi, \xi)).$$

Here const is positive and $B(\xi, \xi)$ is the quadratic part of the height of the point on the Jacobian of the curve X with respect to Θ-polarisation. So we can see that if $g > 1$ than $\mathfrak{r}_{\delta}(\xi) \to 0$ when $\xi \to \infty$.

The second question is about the distribution of the divisors of degree $m > 2g - 2$ with regard to height.

Let H be any c-divisor of degree 1 and D be an f-divisor. We shall call the magnitude $N_H(D) = \exp(D \cdot H)$ a height of a divisor D with regard to H. It is possible to prove that the series

$$\sum_{D>0;\ \deg D=m} \frac{1}{N_H(D)^s} = \zeta_H^{(m)}(s)$$

converges if $s > m + 1 - g$ and that the limit

$$T = \lim_{s \to m+1-g} (s - (m + 1 - g)) \cdot \zeta_H^{(m)}(s)$$

exists and does not equal zero. Using Conjecture 1 we can compute that

$$T = \text{const}(m) \cdot R \cdot \frac{h}{\mu} \cdot |\delta|^{m/2} \exp \alpha(H) \exp d(X) \cdot \sum_{\xi \in \text{Pic}(X/K)} \exp(-F(\xi)).$$

Here $\alpha(H) = \frac{1}{2} m(m + 2 - 2g)(H, H) + \frac{1}{2} m(H, \mathscr{K})$, h is the class number of K, R is its regulator and μ is the number of roots of unity of K. At last $F(\xi)$ is the sum of the quadratic and linear parts of the height on the Jacobian of X in regard to Θ-polarisation.

References

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