Hashing

Lecture #5 of Algorithms, Data structures and Complexity

Joost-Pieter Katoen, Ed Brinksma

Formal Methods and Tools Group

E-mail: `katoen@cs.utwente.nl`

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Overview

$\Rightarrow$ *Introduction*

- *Direct addressing*
- *Hashing*
  - Collision resolution using chaining
  - Complexity analysis of chaining
- *Open addressing*
  - Probing strategies
  - Complexity analysis of open addressing
- *Hash functions*

Introduction

- A *dictionary* ADT stores information that can be retrieved at any time
  - the set of items stored is dynamic
  - items have a key and information associated with that key
  - example: symbol table for a compiler where keys are strings (i.e., identifiers)
- A dictionary d supports the following operations:
  - *search*(k) looks up the information stored under key k in d
  - *insert*(e) stores information object e into d
  - *delete*(e) deletes information object e from d; requires e to be in d
- Which data structure is appropriate to implement a dictionary?
  - a heap: insertion and deletion are efficient, but how about search?
  - ordered array/list: insertion is linear in worst case
  - red-black tree: all operations are logarithmic in worst case

under reasonable assumptions a hash table takes $O(1)$ on average for all operations

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⇒ *Direct addressing*

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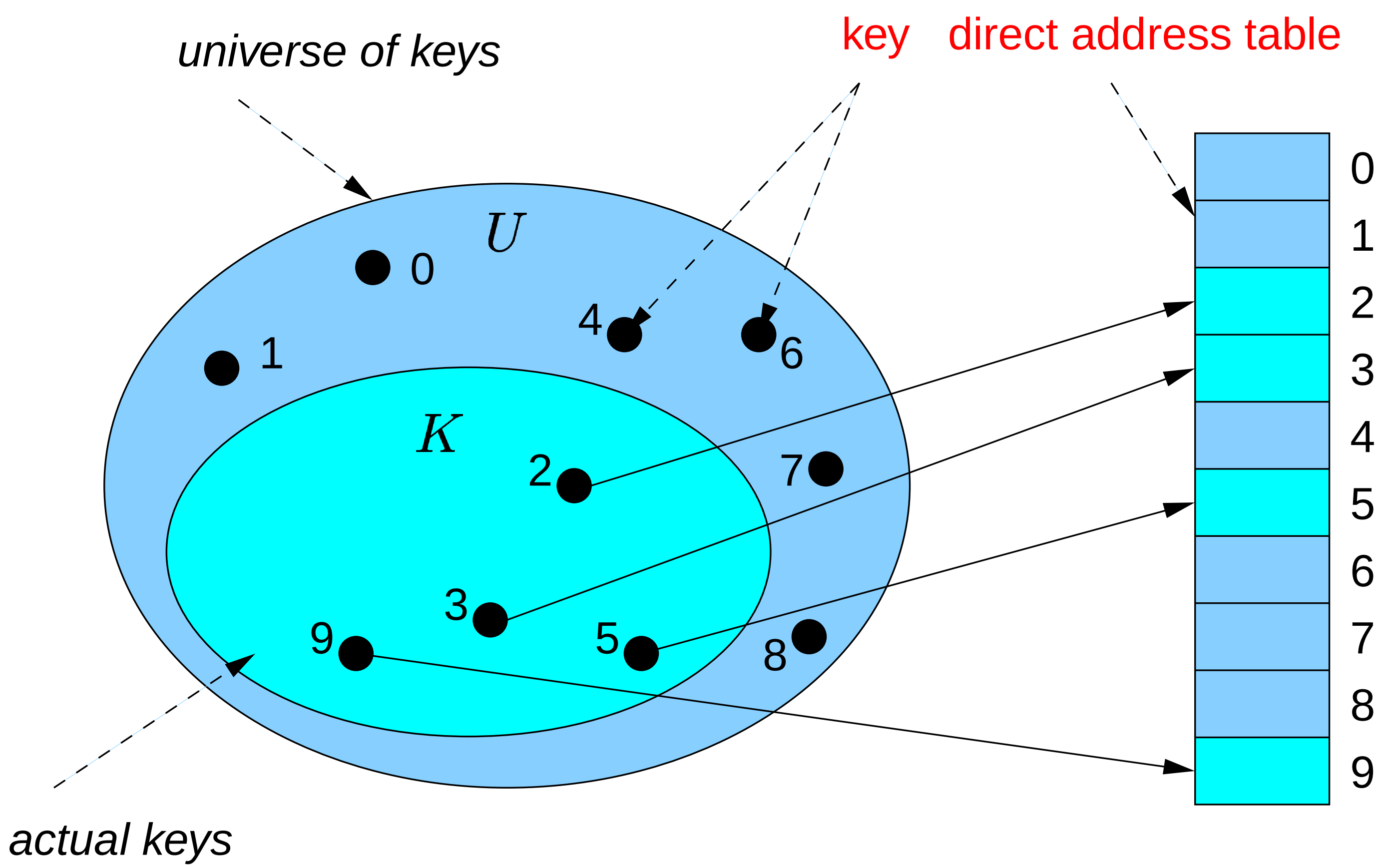
Direct addressing

- Allocate an array that has a position for each possible key
- Each array element contains a pointer to the stored information
  - for simplicity we omit the information associated to keys in this lecture

  $\Rightarrow$ the techniques and analysis results remain valid
- For universe $U = \{\, 0, 1, \ldots, n{-}1 \,\}$ of keys we have:
  - a direct-address table $T[0 \ldots n{-}1]$ with $T[k]$ corresponding to key k
  - *search*(k): return $T[k]$
  - *insert*(e): boils down to $T[key[e]] = e$
  - *delete*(e): simply means $T[key[e]] =$ *nil*
- Runtime for each of the operations is $\Theta(1)$ in worst case

Direct addressing



Check for duplicates in linear time

assume all elements are positive integers of at most k

```
bool checkDuplicates(int [1..n] E) {
    int [1..k] Count;                              // direct-address table for E[i]
    for (i = 1; i ⩽ k; i++) Count[i] = 0;          // initialize Count
    for (i = 1; i ⩽ n; i++) {
        if (Count[E[i]] > 0) return true;          // duplicate found
        else Count[E[i]]++; }                      // count occurrence of E[i]
    return false;                                  // no duplicate found
}
```

Counting sort

assume all elements are positive integers of at most k

void *countSort*(**int** $[1..n]$ E) $\{$

  int $[1..k]$ *Count*, **int** $i, j, l = 0;$

  for $(i = 1; i \leqslant k; i{+}{+})$ *Count*$[i] = 0;$

  for $(i = 1; i \leqslant n; i{+}{+})$ *Count*$[E[i]]{+}{+};$

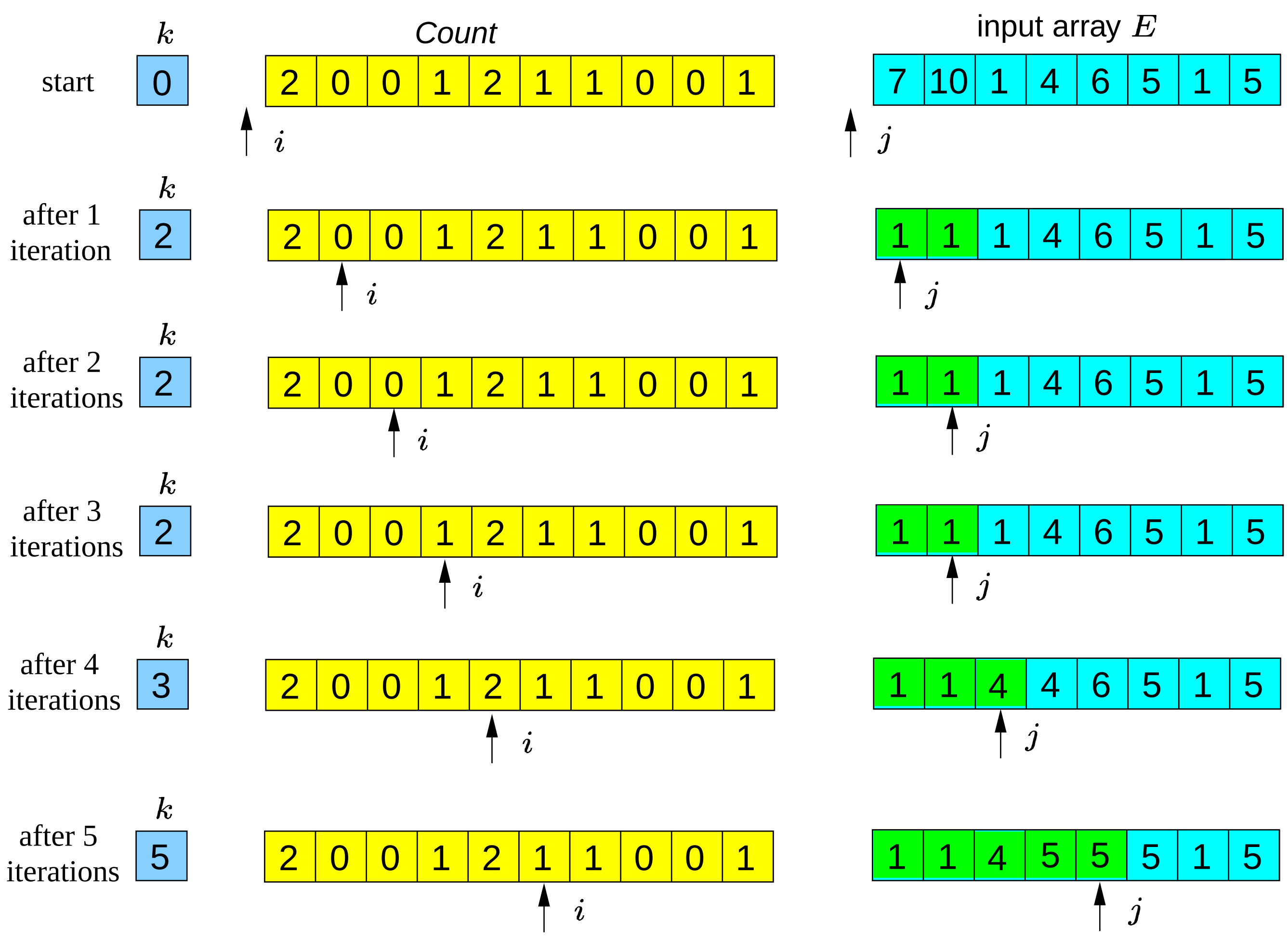
  for $(i = 1; i \leqslant n; i{+}{+})$ $\{$

    for $(j =$ *Count*$[i] + l; j > l; j{-}{-})$ $E[j] = i;$

    $l =$ *Count*$[i] + l;$ $\}$

$\}$

Counting sort: example



Counting sort

- Note that we now sort with worst-case complexity $\Theta(n)$
  - compare this to the lower-bound of $\Theta(n \cdot \log n)$ that we obtained earlier
  - but this algorithm is incomparable to quicksort, heapsort and the like

  $\Rightarrow$ it is not based on element-wise comparisons, but counts occurrences
- Why does this trick work: exploit direct addressing
- Insertion, deletion and searching takes $\Theta(1)$ in worst case
- Main complication: excessive space consumption (size of array = $|U|$)
  - e.g., if keys are strings of 20 symbols, we need about 2^{100} array entries

can we avoid this huge memory consumption while remaining efficient?

yes! by using hashing

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Hashing

- In practice only a small fraction of keys is used, i.e., $|K| \ll |U|$

  $\Rightarrow$ with direct addressing most of the direct address table T is wasted

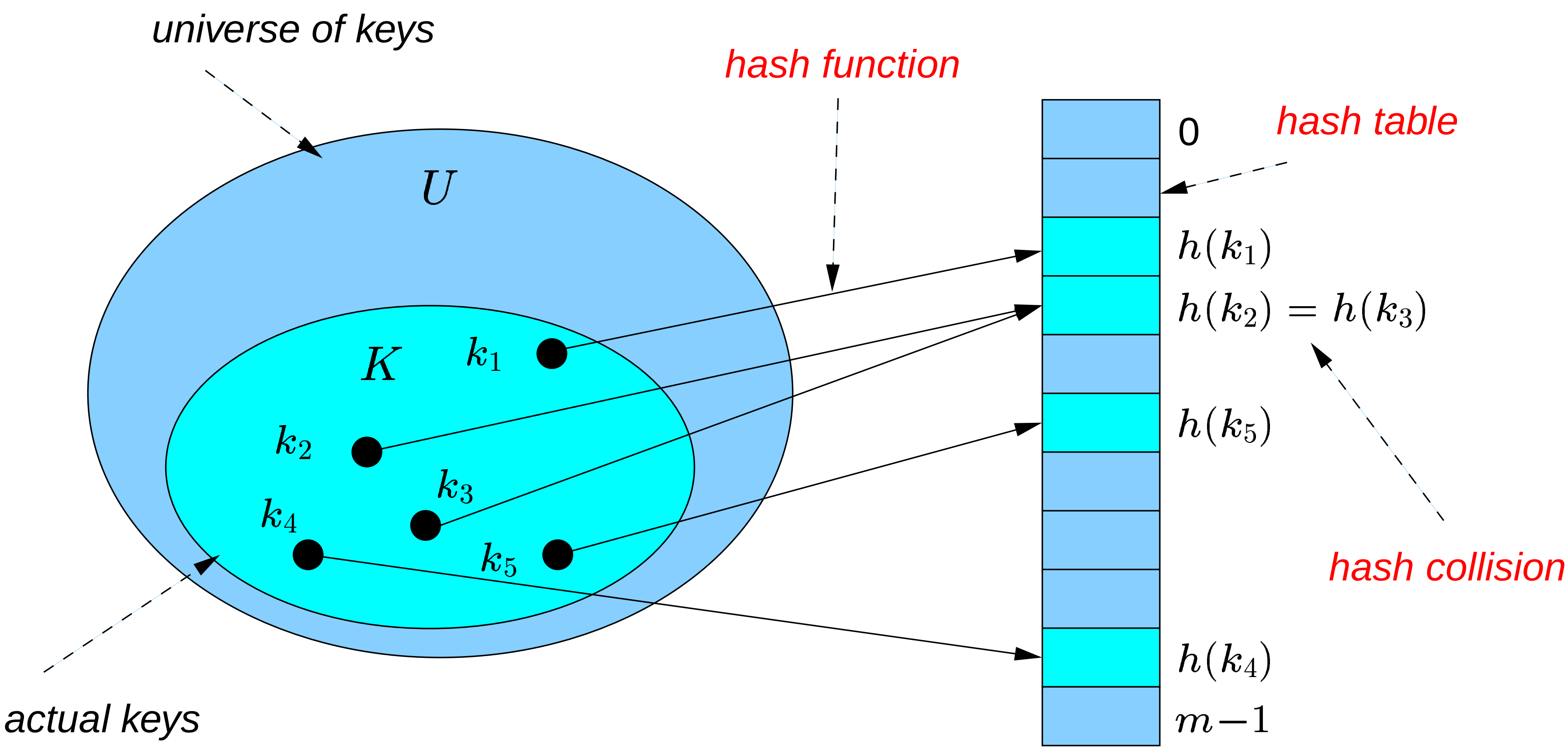
- The aim of hashing is:
  - map an extremely large key space onto a reasonable small range (of integers)
  - such that it is unlikely that two keys are mapped onto the same integer

- A hash function maps a key onto an index in the hash table T:

  $h : U \longrightarrow \{\, 0, 1, \ldots, m{-}1 \,\}$ where m is the table-size and $|U| = n$

- Hash collisions, i.e., $h(k) = h(k')$ for $k \neq k'$, raise the issues:
  - how to obtain a hash function that is cheap to evaluate and minimizes collisions?
  - how to treat hash collisions when they occur?

Hashing


universe of keys
hash function
hash table
U
K
k_1
k_2
k_3
k_4
k_5
0
$h(k_1)$
$h(k_2) = h(k_3)$
$h(k_5)$
$h(k_4)$
$m-1$
hash collision
actual keys

Hash collisions: the birthday paradox

No matter how good our hash function is, we better be prepared for collisions

- This is due to the birthday paradox:
  - the probability that your neighbor has the same birthday is $\frac{1}{365} \approx 0.027$
  - if you ask 23 people, this probability raises to $\frac{23}{365} \approx 0.063$
  - but, if there are 23 people in a room, two of them have the same birthday

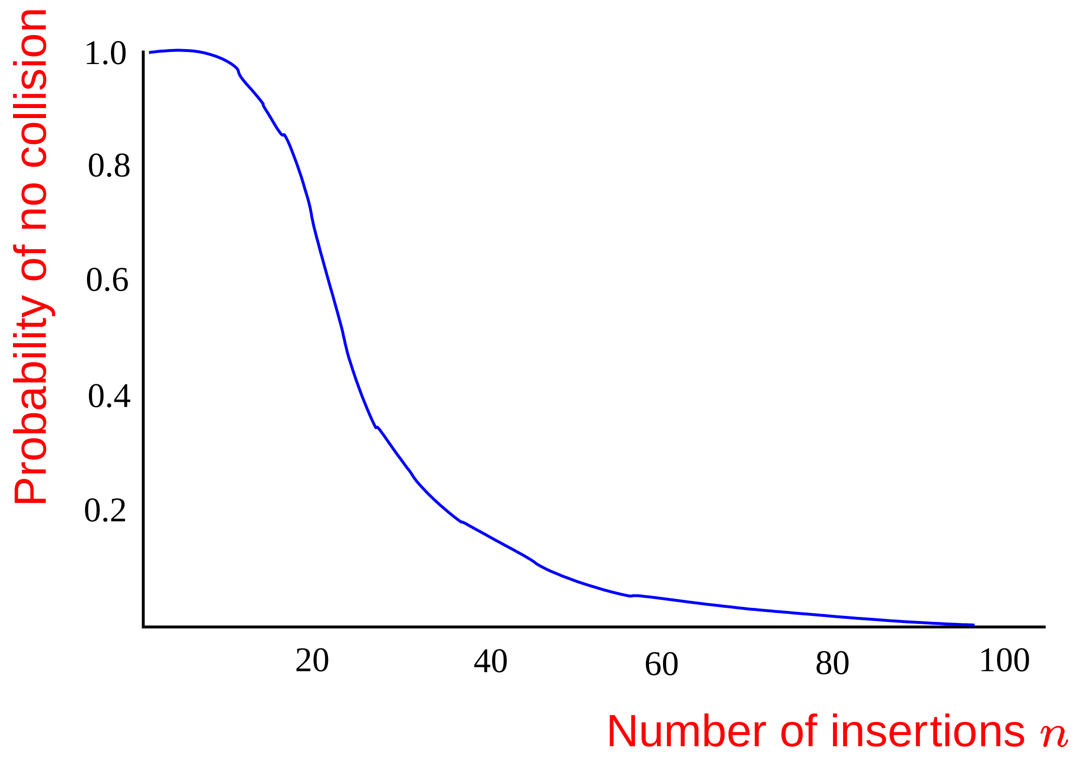
with probability: $$1 - \left(\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \ldots \cdot \frac{343}{365} \right) \approx 0.5$$

- Applying this to hashing yields:
  - the probability of no collisions after k insertions into an m-element table:

$$\frac{m}{m} \cdot \frac{m-1}{m} \cdot \ldots \cdot \frac{m-k+1}{m} = \prod_{i=0}^{k-1} \frac{m-i}{m}$$

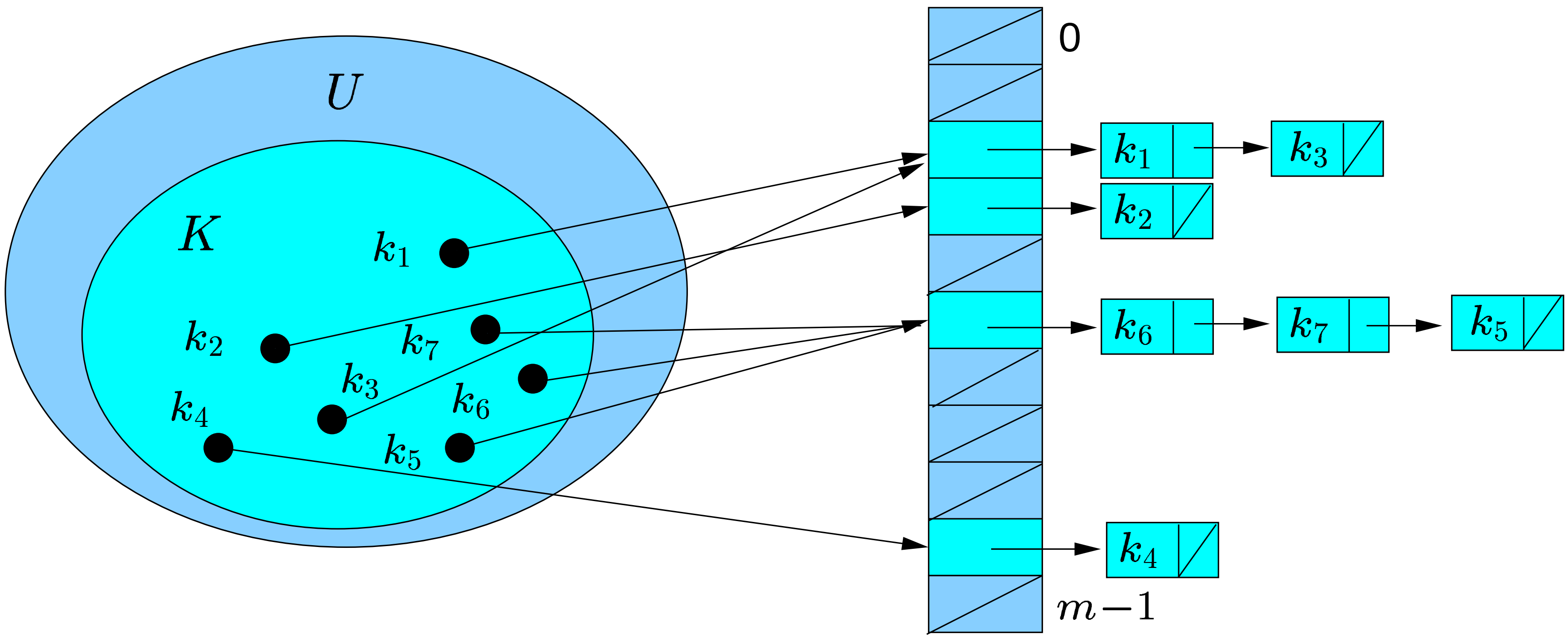
  - for $m = 365$ and $k \geqslant 50$ this probability goes to 0

Hash collisions: the birthday paradox



Collision resolution by chaining

concept: put all keys that hash to the same integer in a linked list [Luhn 1953]



Collision resolution by chaining

- Dictionary operations when using chaining:
  - *search*(k): search for an element with key k in the list $T[h(k)]$
  - *insert*(e): put element e at the front of list $T[h(key[e])]$
  - *delete*(e: delete element e from list $T[h(key[e])]$
- Worst-case complexity of these operations:
  - assuming computing $h(k)$ is rather efficient, say $\Theta(1)$
  - searching: proportional to the length of the list $T[h(k)]$
  - insertion: in constant time (note: no check whether element e is already present)
  - deletion: proportional to the length of the list $T[h(k)]$
- In worst case all keys are hashed onto the same slot
  - searching and deletion have same complexity as for lists! $\Theta(n)$

The average case complexity of hashing with chaining is efficient, though

Average case analysis of chaining (I)

- Assumptions:
  - we have n possible keys and m hash-table entries $n \gg m$
  - uniform hashing: each key is equally likely hashed to any integer
  - the hash value $h(k)$ can be computed in constant time
- The filling degree of hash table T is $\alpha(n, m) = \frac{n}{m}$
  - note that the average length of list $T[j]$ is also α
- What is the expected # elts examined in $T[h(k)]$ to search key k?
  - distinguish between unsuccessful and successful search (like in lecture #1)
- Technical point:
  - extend definition of O, Θ and Ω for functions with two parameters (like α)
  - e.g., $g \in O(f)$ if $\exists c > 0, n_0, m_0$ such that

$$\forall n \geqslant n_0, m \geqslant m_0 : 0 \leqslant g(n, m) \leqslant c \cdot f(n, m)$$

Average case analysis of chaining (II)

- An unsuccessful search takes $\Theta(1+\alpha)$ time on average
  - expected time to search for key k = expected time to search list $T[h(k)]$
  - this list has expected length α
  - the computation of $h(k)$ takes a single time unit
  - $\Rightarrow$ together this yields $1+\alpha$ time units on average
- A successful search also takes $\Theta(1+\alpha)$ time on average
  - let k_i be the i-th inserted key and $A(k_i)$ be the expected time to search k_i:

$$A(k_i) = 1 + \text{ average \# of keys inserted in } T[h(k_i)] \text{ after } k_i \text{ was inserted}$$

  - using the uniform hashing assumption this reduces to: $A(k_i) = 1 + \sum_{j=i+1}^{n} \frac{1}{m}$
  - take the average over all n insertions into the hash-table $\frac{1}{n} \sum_{i=1}^{n} A(k_i)$

Average case analysis of chaining (III)

The expected number of elements examined in a successful search is

$$\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\frac{1}{m}\right)$$

$$= \quad \text{(* calculus *)}$$

$$\frac{1}{n}\sum_{i=1}^{n}1+\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=i+1}^{n}1$$

$$= \quad \text{(* calculus *)}$$

$$1+\frac{1}{nm}\sum_{i=1}^{n}(n-i)$$

$$= \quad \text{(* calculus *)}$$

$$1+\frac{1}{nm}\left(n^2-\frac{n(n+1)}{2}\right)$$

$$= \quad \text{(* calculus *)}$$

$1+\dfrac{n-1}{2m}=1+\dfrac{\alpha}{2}-\dfrac{\alpha}{2n}$ and thus in $\Theta(1+\alpha)$

Complexity of dictionary operations using chaining

- Assume the number m of entries is (at least) proportional to n
- Then filling degree $\alpha(n,m) = \frac{n}{m} \in \frac{O(m)}{m} = O(1)$
- Then all dictionary operations take $O(1)$ time on average
- This includes searching, so we can sort in $O(n)$ on average!

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  - Collision resolution using chaining
  - Complexity analysis of chaining

$\Rightarrow$ *Open addressing*
  - Probing strategies
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- *Hash functions*

Collision resolution by open addressing

- Unlike chaining all elements are stored in the hash table itself

$\Rightarrow$ at most n keys can be stored, i.e., $\alpha(n, m) = \frac{n}{m} \leqslant 1$ [Amdahl 1954]

- Since no memory is used for pointers, more data can be stored

$\Rightarrow$ this helps to reduce the number of hash collisions

- Insertion of a key k:
  - *probe* the entries of the hash table until an empty slot is found
  - sequence of slots probed depends on key k to be inserted
  - the hash function depends on the key k and the probe number:

$$h : U \times \{\, 0, 1, \ldots m{-}1 \,\} \longrightarrow \{\, 0, 1, \ldots m{-}1 \,\}$$

  - hash function h should eventually consider every entry in the hash table

Insertion using open addressing

```
void hashInsert(int T, key k) {
    int i = 0, j;                          // i is probe number
    repeat
        j = h(k, i);                       // compute (i+1-st probe
        if T[j] == nil {                   // free entry found
            T[j] = k; return ; }           // store key k and stop
        else i = i+1;
    until (i == T.length);                 // check entire table
    return hash table overflow;            // no free entry left
}
```

Searching using open addressing

```
int hashSearch(int T, key k) {
    int i = 0, j;                          // i is probe number
    repeat
        j = h(k, i);              // compute (i + 1)-st probe
        if T[j] == k return j;                // key k found
        else i = i+1;
    until (i == T.length || T[j] == nil);
                 // check entire table or find an empty slot
    return nil;                 // key k has not been found
}
```

Deletion using open addressing

- Deleting key k from slot i by $T[i] =$ *nil* is inappropriate

  $\Rightarrow$ if at insertion of k slot i was occupied we cannot retrieve k anymore

- Solution: mark $T[i]$ as special value DELETED (or “obsolete”)

  $\Rightarrow$ *hashInsert* needs to be adapted to treat such slots as empty
  $\Rightarrow$ *hashSearch* remains unchanged as DELETED slots are ignored

- Search times now no longer depend on filling degree α only

$\Rightarrow$ If keys are to be deleted, chaining is more commonly used

How to select the next probe?

- How to generate the probing sequence for a given key k:

$$\langle h(k,0), h(k,1), \ldots, h(k,m-1)\rangle$$

  - which is a permutation of $\langle 0, \ldots m-1\rangle$ for each key k

  $\Rightarrow$ this guarantees that all slots are eventually considered

- Ideally we have uniform hashing
  - i.e. each of the $m!$ permutations is equally likely as probing sequence
  - only used for analysis, in practice too expensive and approximated

- Different policies exist to select the next probe
  - we consider linear probing, quadratic probing and double hashing
  - quality is indicated by the number of distinct probing sequences generated

Linear probing

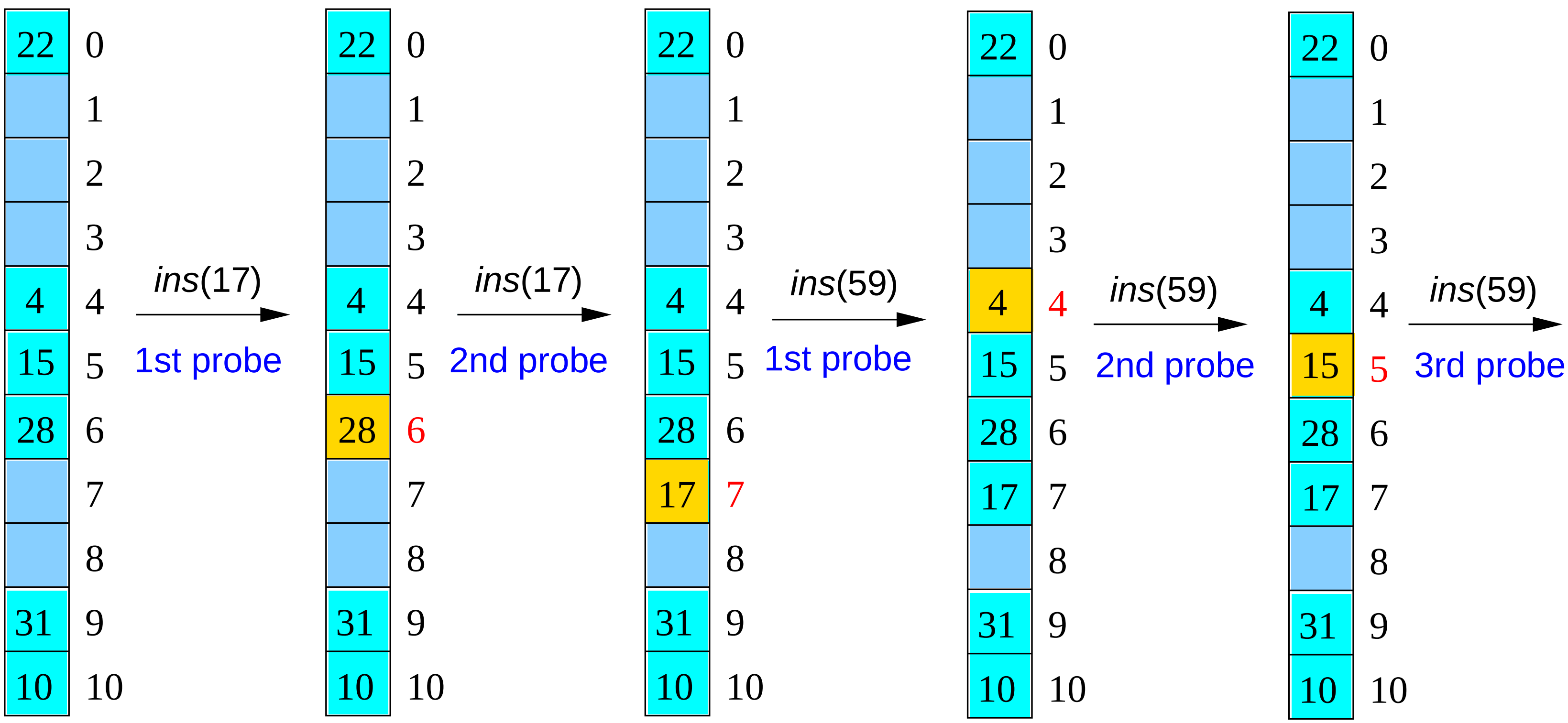
- Uses the hash function $h(k,i) = (h'(k) + i)$ **mod** m (for $i < m$)
  - where h' is an auxiliary hash function
- Subsequent probed slots are offset by a linear dependence on i
- Initial probe determines the entire probe sequence

$\Rightarrow$ m distinct probe sequences can be generated

- Suffers from clustering, i.e., long sequences of occupied slots
  - an empty slot preceded by i full slots gets filled next with probability $\frac{i+1}{m}$

$\Rightarrow$ long sequences of occupied slots tend to get longer

Linear probing: example



$$h'(k) = k \textbf{ mod } 11$$

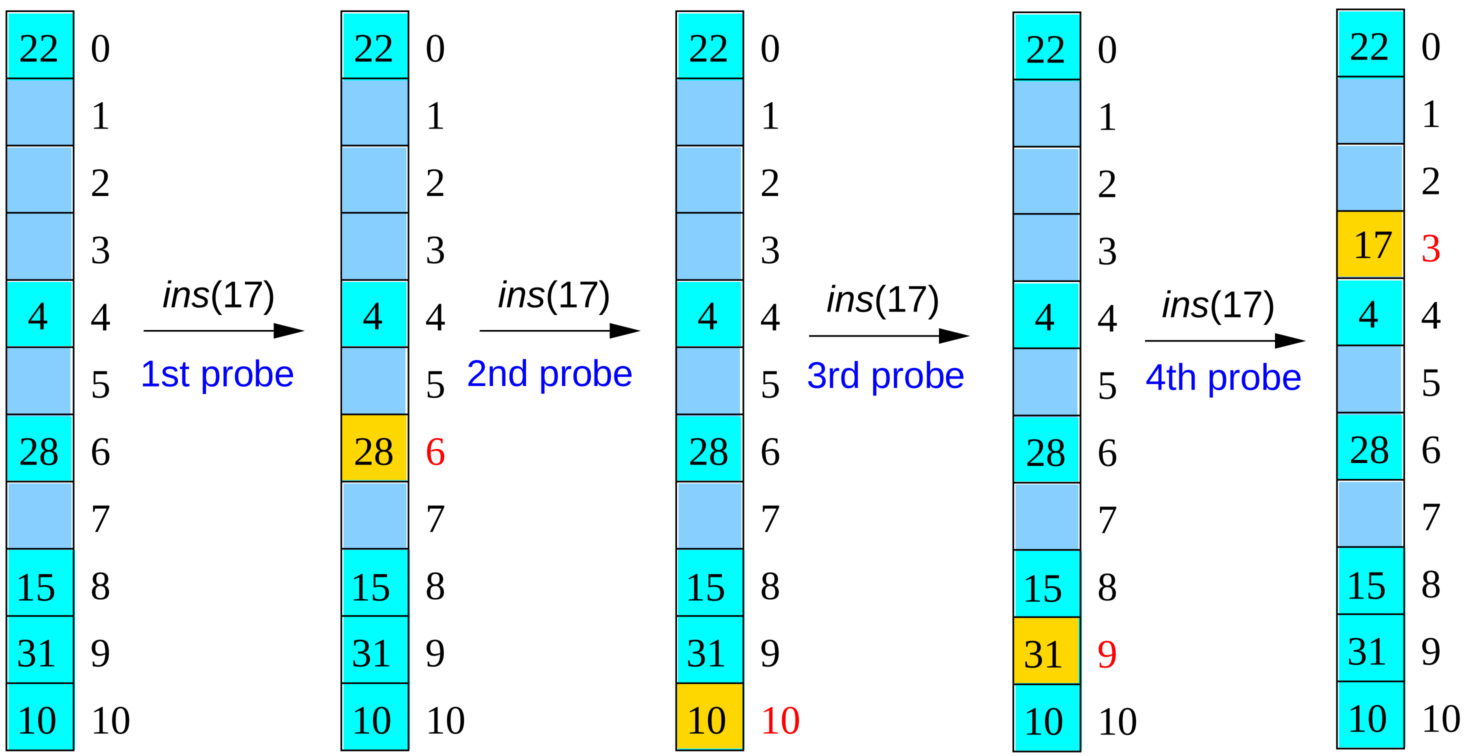
$$h(k,i) = (h'(k) + i) \textbf{ mod } 11$$

Quadratic probing

- Uses the hash function $h(k,i) = (h'(k) + c_1{\cdot}i + c_2{\cdot}i^2)$ **mod** m (for $i < m$)
  - where h' is an auxiliary hash function and non-zero constants c_1, c_2
- Subsequent probed slots are offset by a quadratic dependence on i
- Initial probe determines the entire probe sequence

$\Rightarrow$ m distinct probe sequences can be generated (like for linear probing)
  - ...... provided the values of m and constants c_1 and c_2 are appropriately chosen
- Suffers from *secondary* clustering
  - $h(k,0) = h(k',0)$ implies $h(k,i) = h(k',i)$ for all i
  - but avoids the clustering appearing with linear probing

Quadratic probing: example



$$h'(k) = k \textbf{ mod } 11$$

$$h(k,i) = (h'(k) + i + 3i^2) \textbf{ mod } 11$$

Double hashing

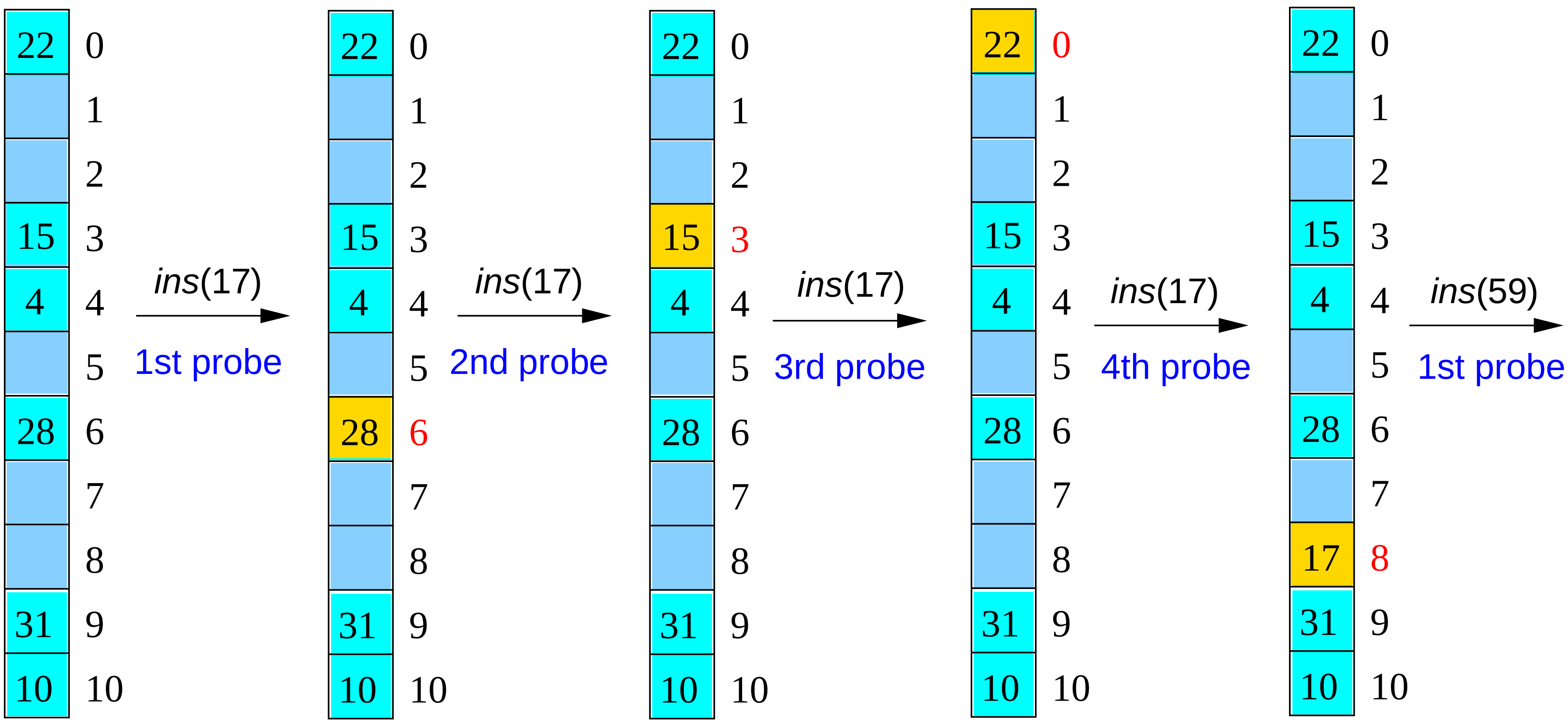
- Uses the hash function $h(k) = (h_1(k) + i \cdot h_2(k))$ **mod** m (for $i < m$)
  - where h_1 and h_2 are auxiliary hash functions
- Subsequent probed slots are offset by the amount $h_2(k)$

$\Rightarrow$ the initial probe does not determine the probe sequence
$\Rightarrow$ this yields a better distribution of keys in the hash table
$\Rightarrow$ approximates the uniform hashing strategy

- If $h_2(k)$ and m are relatively prime, the entire hash table is searched
  - e.g., choose $m = 2^k$ and h_2 such that it produces an odd number
- Each possible pair $h_1(k)$ and $h_2(k)$ yields a distinct probe sequence

$\Rightarrow$ double hashing generates m^2 distinct permutations

Double hashing: example



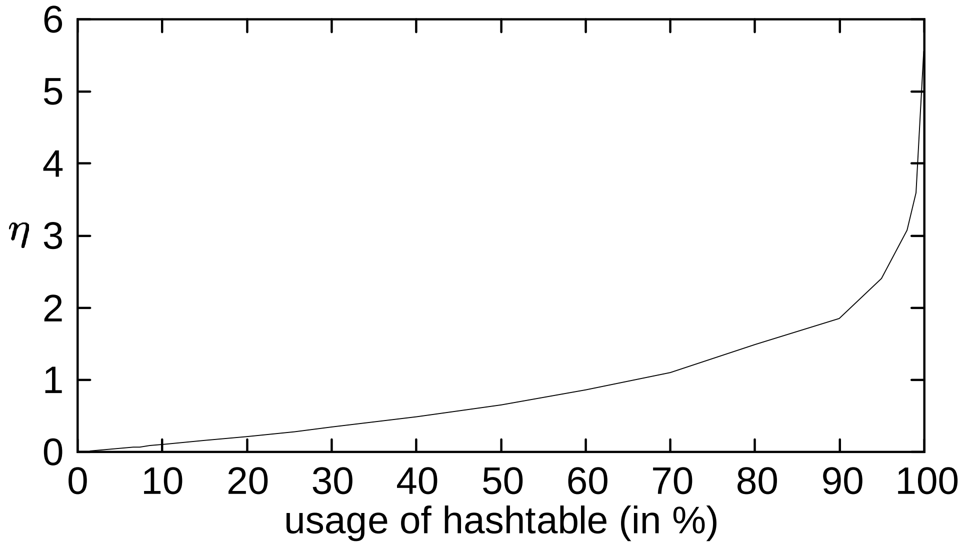
$$h_1(k) = k \textbf{ mod } 11$$
$$h_2(k) = 1 + k \textbf{ mod } 10$$

$$\boxed{h(k,i) = (h_1(k) + i \cdot h_2(k)) \textbf{ mod } 11}$$

Practical efficiency of double hashing

- Hash table with $538\,051$ entries (final filling 99.95%)

- *Mean* number of collisions per insertion into hash table:



Efficiency of open addressing

Under the assumption of uniform hashing we have:

- An unsuccessful search takes $O\left(\frac{1}{1-\alpha}\right)$ time on average
  - if hash table is half full, 2 probes are necessary on average
  - if hash table is 90% full, 10 probes are necessary on average
- A successful search takes $O\left(\frac{1}{\alpha} \cdot \ln \frac{1}{1-\alpha}\right)$ time on average
  - if hash table is half full, about 1.39 probes are necessary on average
  - if hash table is 90% full, about 2.56 probes are necessary on average
- Recall that for chaining this was $\Theta(1+\alpha)$ for both cases

Analyzing unsuccessful search (I)

$$\Pr\{\text{\# probes } \geqslant i\}$$

$=$ (* A_i is the event that there is an i-th probe and it is to an occupied slot *)

$$\Pr\{A_1 \cap A_2 \cap \ldots \cap A_{i-1}\}$$

$=$ (* probability theory *)

$$\Pr\{A_1\} \cdot \Pr\{A_2 \mid A_1\} \cdot \Pr\{A_3 \mid A_1 \cap A_2\} \ldots \Pr\{A_i \mid A_1 \cap \ldots \cap A_{i-1}\}$$

$=$ (* there are n elements and m slots *)

$$\frac{n}{m} \cdot \frac{n-1}{m-1} \cdot \ldots \cdot \frac{n-i+2}{m-i+2}$$

$\leqslant$ (* bound to above *)

$$\left(\frac{n}{m}\right)^{i-1}$$

$=$ (* definition of α *)

$$\alpha^{i-1}$$

Analyzing unsuccessful search (II)

the expected number of probes

$=$ (* property of E *)

$$\sum_{i=1}^{\infty} \Pr\{\#\text{ probes } \geqslant i\}$$

$\leqslant$ (* use previous derivation on $\Pr\{\#\text{ probes } \geqslant i\}$ *)

$$\sum_{i=1}^{\infty} \alpha^{i-1}$$

$=$ (* rewrite slightly *)

$$\sum_{i=0}^{\infty} \alpha^{i}$$

$=$ (* geometric series *)

$$\frac{1}{1-\alpha}$$

Analyzing successful search (I)

average number of probes in a successful search

$=$ (* definition of average *)

$$\frac{1}{n} \cdot \sum_{i=0}^{n-1} \text{average number of probes for } (i+1)\text{-st inserted key}$$

$\leqslant$ (* average number of probes for $(i+1)$-st inserted key is at most $\frac{m}{m-i}$ *)

$$\frac{1}{n} \cdot \sum_{i=0}^{n-1} \frac{m}{m-i}$$

$=$ (* calculus *)

$$\frac{m}{n} \cdot \sum_{i=0}^{n-1} \frac{1}{m-i}$$

Analyzing successful search (II)

$$\frac{m}{n} \cdot \sum_{i=0}^{n-1} \frac{1}{m-i}$$

$=$ (* calculus *)

$$\frac{1}{\alpha} \cdot \left(\sum_{k=m-n+1}^{m} \frac{1}{k} \right)$$

$\leqslant$ (* approximate summation by integral (cf. Example 1.7) *)

$$\frac{1}{\alpha} \cdot \int_{m-n}^{m} \frac{1}{x}\, dx$$

$=$ (* integral calculus *)

$$\frac{1}{\alpha} \ln \left(\frac{m}{m-n} \right)$$

$=$ (* definition of α *)

$$\frac{1}{\alpha} \ln \left(\frac{1}{1-\alpha} \right)$$

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Hash functions

- A hash function maps a key onto an integer (i.e., an index)
  - the hash function $h(k)$ should be cheap to evaluate
  - it should be surjective on the range $0 \ldots m-1$
  - it should tend to use all indexes with uniform frequency
  - it should tend to put similar keys in different parts of the hash table
- Three major techniques to obtain a “good” hash function:
  - the division method
  - the multiplication method
  - universal hashing

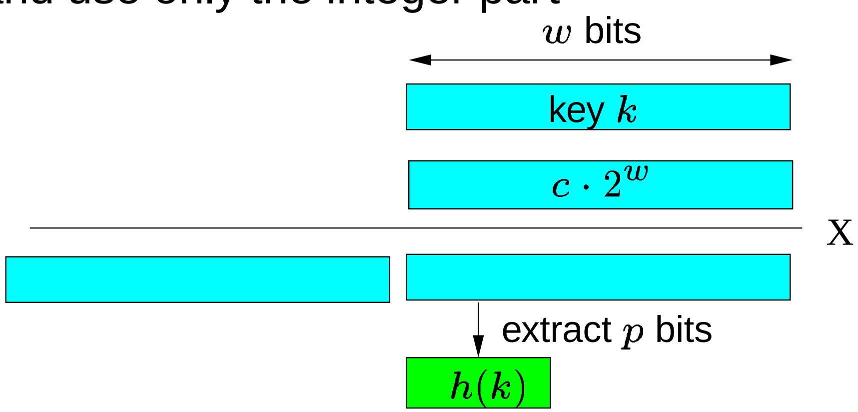
Division method

- Uses the hash scheme $h(k) = k$ **mod** m (for $i < m$)
- Using this method, the value of m should be chosen with care
  - if $m = 2^p$, then k **mod** m amounts to select the p least significant bits of k
- Practical good choice: m is prime and not too close to power of 2
  - example: consider 2,000 character strings
  - allow on average about 3 probes for an unsuccessful search
  - choose $m = 2000/3 \longrightarrow 701$

Multiplication method

- Uses the hash scheme $h(k) = \lfloor m \cdot (k{\cdot}c \textbf{ mod } 1) \rfloor$ (for $i < m$)
  - with constant $0 < c < 1$ (Knuth suggests $c \approx (\sqrt{5} - 1)/2 \approx 0.62$)
  - note that $k{\cdot}c$ **mod** 1 is the fractional part of $k{\cdot}c$

  $\Rightarrow$ the value of m is not critical here
- Usual scheme take $m = 2^p$ and $c = \frac{s}{2^w}$ where $0 < s < 2^w$ and then:
  - first compute $k \cdot s$ (= $k \cdot c \cdot 2^w$)
  - divide by 2^w, use only the fractional part
  - multiply by 2^p and use only the integer part



Universal hashing

- Greatest problem with hashing:
  - there is always an adversarial sequence of keys all mapped onto the same slot
- Choose randomly a hash function from a given small set H
  - that is independent of the keys which are going to be used
- For k, k' the fraction of functions in H such that k and k' collide is $\frac{|H|}{m}$
  - probability that k, k' collide is $\frac{1}{|H|}.\frac{|H|}{m} = \frac{1}{m}$
- Example: define the elements of the class of hash functions by:

$$h_{a,b}(k) = ((a \cdot k + b) \textbf{ mod } p) \textbf{ mod } m$$

  - where p is a prime number such that $p > m$ and $p >$ largest key
  - integers a $(1 \leqslant a < p)$ and b $(0 \leqslant b < p)$ are chosen at execution time