Crib Notes on Campbell-Baker-Hausdorff expansions

Cosmas Zachos

High Energy Physics Division, Argonne National Laboratory, Argonne, IL 60439-4815, USA
zachos@anl.gov

Consider/introduce auxiliary scaling parameter t, for counting powers, ultimately set to $t = 1$. For matrices, operators, etc, A and B,

$$e^{tA}e^{tB} \equiv e^{Z(t,A,B)}.$$

Based on Lemmata 1,2, and 3, below,

$$Z \equiv \ln(e^{tA}e^{tB}) = (e^{tA}e^{tB} - 1) - \frac{1}{2}(e^{tA}e^{tB} - 1)^2 + \frac{1}{3}(e^{tA}e^{tB} - 1)^3 + \cdots$$

is evaluated recursively [1] through algorithms in Applications 1,2,3,...,

$$Z = t(A+B) + \frac{t^2}{2}[A,B] + \frac{t^3}{12}\Big([[A,B],B] + [A,[A,B]]\Big) + \frac{t^4}{24}[[[B,A],A],B]$$

$$-\frac{t^5}{720}([[[[A,B],B],B],B] + [[[[B,A],A],A],A]) + \frac{t^5}{360}([[[[A,B],B],B],A] + [[[[B,A],A],A],B])$$

$$-\frac{t^5}{120}([[[[A,B],B],A],B] + [[[[B,A],A],B],A]) + \cdots$$

• Powers of t higher than the first have coefficients which are *always commutators: Lie polynomials*—they are in the Lie Algebra. (Campbell, 1897; Poincaré, 1899. Structures Lie's converse (third fundamental) theorem: exponentiation of the algebra yields the simply connected group.) A concise proof of this fact was discovered by Eichler (1968).

• $Z(t,A,B) = -Z(-t,B,A)$, whence even powers of t are $A-B$ antisymmetric, while odd ones are symmetric.

• Thompson representation: $Z(t,A,B) = t\Big(e^{-tW}Ae^{tW} + e^{t\tilde{W}}Be^{-t\tilde{W}}\Big)$, where W and $\tilde{W}$ are in the Lie Algebra (A, B, and commutators), and $\tilde{W}(t,A,B) = W(-t,B,A) = \frac{A+B}{4} + \cdots$.

• Zassenhaus expansion:

$$e^{t(A+B)} = e^{tA}\ e^{tB}\ e^{-\frac{t^2}{2}[A,B]}\ e^{\frac{t^3}{3!}(2[B,[A,B]]+[A,[A,B]])}\ e^{-\frac{t^4}{4!}([[[A,B],A],A]+3[[[A,B],A],B]+3[[[A,B],B],B])}\cdots$$

Example: $e^{t(\partial+f'(x))} = e^{-f(x)}e^{t\partial}e^{f(x)} = e^{t\partial}\ e^{f(x)-f(x-t)} = e^{t\partial}e^{tf'(x)}e^{-\frac{t^2}{2!}f''(x)}e^{\frac{t^3}{3!}f'''(x)}e^{-\frac{t^4}{4!}f''''(x)}\cdots.$

• Triple formula:

$$e^{tV(t)} = e^{tA}e^{tB}e^{tA},$$

so that $V(t,A,B)$ is an even function of t, $V(t) = V(-t)$. Evaluated by, e.g., Application 1:

$$V = 2A + B + \frac{t^2}{6}[(A+B),[B,A]] + \cdots\ .$$

Note the duality between B and $-V$.

• Lie group commutator: $e^{tA}e^{tB}e^{-tA}e^{-tB} = e^{U}$.

$$U(t,A,B) = -U(t,B,A)$$

$$= t^2[A,B] + \frac{t^3}{2}[(A+B),[A,B]] + \frac{t^4}{3!}\left([[[B,A],A],B]/2 + [(A+B),[(A+B),[A,B]]]\right)\cdots.$$

Lemma 1: (Duhamel's formula)

$$\delta e^Z = \int_0^1 ds\; e^{(1-s)Z}\;\; \delta Z\;\; e^{sZ},$$

for any matrix (noncommutative) Z. E.g., $\delta = \frac{\partial}{\partial t}$.

Provable directly from the finite N definition of the exponential, and then $N \to \infty$.

Alternative proof: Consider Δ as the operator δ acting on everything to its right.

$$\frac{d}{ds}\left(e^{-sZ}\;\Delta\; e^{sZ}\right) = e^{-sZ}\;[\Delta, Z]\; e^{sZ}\;;$$

then integrate by $\int_0^1 ds$ to obtain

$$e^{-sZ}\;\Delta\; e^{sZ} - \Delta = \int_0^1 ds\; e^{-sZ}\;[\Delta,Z]\; e^{sZ},$$

hence

$$[\Delta, e^Z] = \delta e^Z = \int_0^1 ds\; e^{(1-s)Z}\;\; \delta Z\;\; e^{sZ}.$$

Lemma 2 (Hadamard formula): $e^A B\, e^{-A} = e^{[A}\; B \equiv B + [A,B] + \frac{1}{2!}[A,[A,B]] + \cdots.$

Proof: Note left commutation $[A,B] \equiv Ad(A)\; B$ acts on B like a derivative operator—obeys operator Leibniz' chain rule. For a parameter s,

$$\frac{d}{ds}\left(e^{sA}\; B\; e^{-sA}\right) = \left[A\;,\; e^{sA}\; B\; e^{-sA}\right],$$

so $f \equiv e^{sA}\; B\; e^{-sA}$ satisfies

$$\frac{df}{ds} = [A, f],$$

with B.C. $f(0) = B$. In turn, this is formally solved by the series in s, $f(s) = e^{s[A}\; f(0)$.

Alternatively, it can be proved by induction in powers of s,

$$\frac{d^n f}{ds^n} = [A,[A,[A,\cdots,f]\cdots]]].$$

A direct consequence follows, $e^A e^B e^{-A} = e^{B+[A,B]+\frac{1}{2!}[A,[A,B]]+\cdots}$, and hence the braiding relation,

$$e^A e^B = e^{B+[A,B]+\frac{1}{2!}[A,[A,B]]+\cdots}\;\; e^A\;.$$

These two lemmata lead to

Lemma 3 (Campbell-Poincaré fundamental identity):

$$(\delta e^{Z})\ e^{-Z} = \frac{e^{[Z} - 1}{[Z}\ \delta Z\ ,$$

or equivalently,

$$\delta Z = \frac{[Z}{e^{[Z} - 1}\ \left((\delta e^{Z})\ e^{-Z} \right),$$

where the fraction is the celebrated generating function of the Bernoulli numbers.
Other equivalent forms are,

$$e^{-Z} \delta e^{Z} = \frac{1 - e^{-[Z}}{[Z}\ \delta Z = \delta Z\ \frac{e^{Z]} - 1}{Z]}\ ,$$

etc. So suitable choices for simple δe^{Z}s lead to determination of δZ and hence Z.

Application Algorithm 1. (Poincaré. Readily exhibits Z construction out of nested commutators, and applied mathematicians and Lie Group textbooks like it, cf. [2], but clumsy computationally).

Set now $e^{Z(t)} = e^{A} e^{tB}$. For $\delta Z = \partial_t Z \equiv Z'$ in Lemma 3, note that

$$B = e^{-Z} \delta e^{Z} = \frac{1 - e^{-[Z}}{[Z}\ Z',$$

hence

$$Z' = \frac{[Z}{1 - e^{-[Z}}\ B = \psi \left(e^{[Z} \right)\ B,$$

where

$$\psi(x) \equiv \frac{x \ln x}{x - 1} = 1 - \sum_{n=1}^{\infty} \frac{(1-x)^n}{n(n+1)}\ .$$

(NB. $\psi(e^{-y}) = \sum_{n=0}^{\infty} B_n \frac{y^n}{n!}$, for the Bernoulli numbers $B_n(1)$: $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, ...)

Since, from Lemma 2, $e^{[Z} = e^{[A} e^{t[B}$, this reads

$$Z'(t) = \psi \left(e^{[A} e^{t[B} \right)\ B,$$

and so one may integrate over t from $Z(0) = A$, to finally obtain

$$Z(1) = A + \left(\int_0^1 dt\ \psi \left(e^{[A} e^{t[B} \right) \right)\ B.$$

It is manifest that all subleading terms to $A + B$ are commutators, so Z is in the Lie Algebra.

This is essentially an algorithm to produce the series by judicious power expansion of ψ and its arguments. For example, if only the term linear in B is sought (e.g.[3]), trivially, then,

$$Z = A + \psi(e^{[A})\ B + O(B^2) = A + \frac{[A}{1 - e^{-[A}}\ B + O(B^2),$$

where all expansion coefficients are simply related to the Bernoulli numbers as above.

Corollary: If $O(B^2)$ terms vanish, e.g. by virtue of special relations such as $[A,B]=sB$, it follows that $Z=A+\frac{s}{1-\exp(-s)}B=A+\psi(e^s)B$.

Taking inverses and rescaling yields a braiding relation, $e^Ae^B=e^{(\exp s)B}\,e^A$. Thus, the group commutator amounts to just $e^Ae^Be^{-A}e^{-B}=e^{(e^s-1)B}$.

Similarly, for $A=t\partial$, $B=f(x)-f(x-t)=(1-\exp(-[A))f(x)$, so that $[B,[A,B]]=0=[B,[A,[A,B]]]=$ ..., this exact expansion collapses to just $Z=A+[A,f(x)]=t(\partial+f'(x))$, the example employed for the above Zassenhaus expansion.

Application Algorithm 2. By virtue of its mechanical recursiveness, this one is favored by phycisists, e.g. [4]. Set $e^{Z(t)}\equiv e^{tA}e^{tB}$. Operate both sides by $\delta=\partial_t$, and multiply by e^{-Z} on the right. By Lemmata 3, and 2,

$$\frac{e^{[Z}-1}{[Z}\,Z'=A+e^{t[A}\,B.$$

Now, setting

$$Z\equiv\sum_{n=1}^{\infty}t^nZ_n\ ,$$

the Z_ns can be solved for recursively in the power of t^{n-1} components of this eqn, so

$$Z_1=A+B\quad\Longrightarrow\quad Z_2\quad\Longrightarrow\quad Z_3\quad\Longrightarrow\quad\cdots$$

Manifestly, again, for $n>1$, each Z_n is a function of commutators only.

• $Z(t,A,B)=-Z(-t,B,A)$, $\Longrightarrow$ even powers of t are $A-B$ antisymmetric, while odd ones are symmetric.

Application Algorithm 3. (Hausdorff, cf [5]. Most systematic as a power expansion in A or B.)

Consider the *replacement operators*

$$\delta_A\equiv\left(\delta A\frac{\partial}{\partial A}\right),\qquad\qquad \delta_B\equiv\left(\delta B\frac{\partial}{\partial B}\right),$$

which act on functions of A and B to successively replace each occurrence of A by δA, to first order, preserving the orderings, in accord with Leibniz's rule. Seek a symmetry of $Z(A,B)$, upon infinitesimal dilation of B, $\delta B=\epsilon B$, i.e. find $\delta A=-\epsilon D(A,B)$ s.t., to $O(\epsilon^2)$,

$$Z(A,B)=Z(A-\epsilon D,B+\epsilon B)+O(\epsilon^2),$$

so that

$$e^Ae^B=e^{A-\epsilon D}e^{B+\epsilon B}=e^A(1-\epsilon e^{-A}\delta_Ae^A)(1+\epsilon B)e^B+O(\epsilon^2).$$

So, evaluating δ_Ae^A by Lemma 3, one has to $O(\epsilon^2)$,

$$\frac{1-e^{-[A}}{[A}\,D=B,$$

whence

$$D = \frac{[A}{1 - e^{-[A}} \; B.$$

Consequently, $\delta Z = \delta_A Z + \delta_B Z = 0$ amounts to

$$\left(\left(\frac{[A}{1 - e^{-[A}} \; B \right) \frac{\partial}{\partial A} \right) Z = \left(B \frac{\partial}{\partial B} \right) Z.$$

The l.h.side raises the power of B, so the eqn may be solved recursively in each term Z_n of $O(B^n)$ in Z,

$$Z_n = \frac{1}{n} \left(\left(\frac{[A}{1 - e^{-[A}} \; B \right) \frac{\partial}{\partial A} \right) Z_{n-1},$$

that is,

$$Z_0 = A, \qquad Z_1 = \frac{[A}{1 - e^{-[A}} \; B = D, \qquad \cdots$$

etc, as in Application 1.

References

[1] J Campbell, Proc Lond Math Soc 28 (1897) 381-390; ibid, 29 (1898) 14-32;
H Poincaré, Compt Rend Acad Sci Paris 128 (1899) 1065-1069; Camb Philos Trans 18 (1899) 220-255;
H Baker, Proc Lond Math Soc (1) 34 (1902) 347-360; ibid (1) 35 (1903) 333-374; ibid (Ser 2) 3 (1905) 24-47;
F Hausdorff, Ber Verh Sächs Akad Wiss Leipzig 58 (1906) 19-48.
M Eichler, J Math Soc Jp 20 (1968) 23-25; http://projecteuclid.org/euclid.jmsj/1260463291

[2] W Miller, *Symmetry Groups and their Applications*, Academic Press, NY 1972, pp 159-161;
V S Varadarajan, *Lie groups, Lie algebras, and their representations*, Springer, NY, 1974 (expansion of V Bargmann's Princeton University notes on CBH).

[3] J Wess & J Bagger, *Supersymmetry and Supergravity*, Princeton Univ Press, 1983. Cf eqn (7.19).

[4] P Fishbane et al, Phys Rev D24 (1981) 2324;
R Wilcox, JMP 8 (1967) 962-982;
M Suzuki, JMP 26 (1985) 26;
M Veltman, *Diagrammatica*, Cambridge University Press, 1994, cf Appendix A.4;
....

[5] W Magnus, Comm Pur Appl Math VII (1954) 649-673;
W Magnus, A Karass, and D Solitar, *Combinatorial Group Theory*, John Wiley & Sons, 1966, New York.