## THE UNIVERSITY OF AKRON

## Mathematics and Computer Science

## Lesson 5: Expansion

## Directory

- Table of Contents
- Begin Lesson 5 I 1 Om

$$
\begin{gathered}
\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C} \\
a^{3} a^{4}=a^{7}(a b)^{10}=a^{10} b^{10} \\
-(a b-(3 a b-4))=2 a b-4 \\
(a b)^{3}\left(a^{-1}+b^{-1}\right)=(a b)^{2}(a+b) \\
(a-b)^{3}=a^{3}-3 a^{2} b+3 a b^{2}-b^{3} \\
2 x^{2}-3 x-2=(2 x+1)(x-2) \\
\frac{1}{2} x+13=0 \Longrightarrow x=-26 \\
G=\{(x, y) \mid y=f(x)\} \\
\end{gathered}
$$

Copyright © 1995-2000 D.P/ Story
Last Revision Date: 8/19/2000

## Lesson 5: Expansion Table of Contents

5. Expansion
5.1. General Methods of Expansion

- Special Products • Radicals Revisited
5.2. The Binomial Formula
- The Binomial Expansion Algorithm


## 5. Expansion

In this lesson we discuss and illustrate methods of expanding a product of algebraic expressions. Initially, we discuss General Methods of Expansion. Following that, we consider the problem of computing powers of a binomial. Such expansion can be carried out quickly and quietly by using the Binomial Formula.

The student will find these techniques to be invaluable tools in their study of algebra, calculus, and beyond ... perhaps into an engineering discipline.

### 5.1. General Methods of Expansion

In Lesson 4, in the section entitled The Distributive Law, we looked at products of the form $a(b+c)$ and saw that

$$
\begin{equation*}
a(b+c)=a b+a c \tag{1}
\end{equation*}
$$

When read from left-to-right, the formula is a rule for expanding a product. When read from right-to-left, the above formula can be used for simple factoring or combining of similar terms.

In this lesson we take up the problem of expanding more complicated expressions than we considered in Lesson 4. These more complicated products are products in which both factors are the sum of several terms: such as $\left(x^{2}+2\right)(3 x-4),(a+b)(a+b+c)$, or $(x+1)^{2}(x+2)^{3}$.

An algebraic expression consisting of exactly two terms is called a binomial. Consider the problem of computing the product of two binomials: $(a+b)(c+d)$. We can and do expand this product by the Distributive Law.

$$
\begin{aligned}
(a+b)(c+d) & =a \cdot(c+d)+b \cdot(c+d) & & \triangleleft \text { by }(1) \\
& =a c+a d+b c+b d & & \triangleleft \text { by }(1)
\end{aligned}
$$

Thus,

General Multiplication Rule:

$$
\begin{equation*}
(a+b)(c+d)=a c+a d+b c+b d \tag{2}
\end{equation*}
$$

Key Point. If you study the formula (2), you can see that the product of two binomials is the sum of all possible products obtained by taking one term from the first factor and one term from the second factor. This observation is valid even when there are an arbitrary number of factors in the product and an arbitrary number of terms in each factor.

The above observation then eliminates the need to memorize formula (2)! Let's go the examples.

Example 5.1. Expand and combine each of the products.
(a) $(x+1)(x+2)$
(b) $(2 w-3 s)(5 w+2 s)$
(c) $(2 x-3)\left(x^{2}-2\right)(\mathrm{d})(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$

Study the above examples. Try to "see" the pattern and get a "feel" for multiplying out binomials.

Exercise 5.1. Expand and combine each of the following.
(a) $(4 x-5)(3 x+2)$
(b) $(x-2 y)^{2}$
(c) $\left(x^{3 / 2}+1\right)\left(x^{1 / 2}+2\right)$

Section 5: Expansion

Now let's look at multiplying binomials and trinomials (three terms).

$$
\begin{aligned}
(a+b)(x+y+z) & =a(x+y+z)+b(x+y+z) \\
& =a x+a y+a z+b x+b y+b z
\end{aligned}
$$

Thus,

$$
(a+b)(x+y+z)=a x+a y+a z+b x+b y+b z
$$

Notice that every term of the first factor is multiplied by every term of the second factor. This is key to understanding how to expand arbitrary complex expressions.

Example 5.2. Expand and combine each of the following.
(a) $(x+y)\left(x^{2}-x y+y^{2}\right)$
(b) $\left(2 x^{2}-4 y^{3}\right)\left(6 x^{3}+3 x y+2 y^{2}\right)$

Practice the process of expanding on the next set of problems.
Exercise 5.2. Expand and combine each of the following.
(a) $(2 x-y)\left(x^{2}-2 y^{2}+x y\right)(b)(a b-c)(a-b+c)$

Section 5: Expansion
There is a perhaps a more convenient method for multiplying out sums. The following example illustrates the technique.

Example 5.3. Expand and combine $(2 x-3 y)\left(x^{2}-y^{2}+2 x y\right)$.
Exercise 5.3. Use the techniques in Example 5.3 to expand and combine the expression $(4 x-7 y)\left(2 x^{2}-3 y^{2}-4 x y\right)$.

## - Special Products

There are many special products that I could present to you at this time. Here are a few of the more important ones.

Some Special Products:

$$
\begin{array}{ll}
\triangleright & (x-y)(x+y)=x^{2}-y^{2} \\
\triangleright & (x+y)^{2}=x^{2}+2 x y+y^{2} \\
\triangleright & (x-y)^{2}=x^{2}-2 x y+y^{2} \\
\triangleright & (x+a)(x+b)=x^{2}+(a+b) x+a b \tag{6}
\end{array}
$$

Section 5: Expansion

Each of these equations is just a special case of the General Multiplication Rule; even so, these formulae speed up the expansion of these common types of products and are, therefore, quite useful to know.

Each of the product rules has a verbalization: Just click on the green triangle to jump there.

In the next series of examples we illustrate each of the above special product rules. Use their verbalizations to help you understand how to use the formula - the verbalizations are independent of the letters being used in the expansions and are very useful for that reason.

Illustration 1. Expand each of the following.
(a) Equation (3) tells us that the product of the sum and difference of two expression is the difference of the squares of the two expressions. This formula has a verbalization which you should recite every time you use the formula. This will help you remember these simple expansion formulae.

1. $\left(x^{2}-y^{3}\right)\left(x^{2}+y^{3}\right)=\left(x^{2}\right)^{2}-\left(y^{3}\right)^{2}=x^{4}-y^{6}$.
2. $(\sqrt{2}-1)(\sqrt{2}+1)=(\sqrt{2})^{2}-1=2-1=1$.

Section 5: Expansion
3. $\left(\frac{x^{2}}{2}+\frac{1}{\sqrt{3}}\right) \cdot\left(\frac{x^{2}}{2}-\frac{1}{\sqrt{3}}\right)=\frac{x^{4}}{4}-\frac{1}{3}$.
4. $(x-y)^{2}(x+y)^{2}=[(x-y)(x+y)]^{2}=\left[x^{2}-y^{2}\right]^{2}$.
(b) Equations (4) and (5) tells us how to square a binomial. Each have verbalizations and when you square you simply verbalize as you expand. The verbalizations are: (4) and (5).

1. $(2 x+y)^{2}=4 x^{2}+4 x y+y^{2}$.
2. $(x-y)^{2}(x+y)^{2}=[(x-y)(x+y)]^{2}=\left[x^{2}-y^{2}\right]^{2}=x^{4}-$ $2 x^{2} y^{2}+y^{4}$.
3. $\left(\frac{x}{3}-\frac{3}{x}\right)^{2}=\frac{x^{2}}{9}-2\left(\frac{x}{3}\right)\left(\frac{3}{x}\right)+\frac{9}{x^{2}}=\frac{x^{2}}{9}-2+\frac{9}{x^{2}}$.
(c) The previous formulae are just special cases of the general expansion formula for binomials. Here are some examples of (6)-it too has a verbalization.
4. $(x+1)(x+3)=x^{2}+(1+3) x+3=x^{2}+4 x+3$.
5. $(x+3)(x-4)=x^{2}+(3-4) x-12=x^{2}-x-12$.

Section 5: Expansion
3. $(y-1)(y+2)=y^{2}+(-1+2) y-2=y^{2}+y-2$. More general products such as $(4 x-3)(9 x+2)$ can be carried out by the General Multiplication Rule.
Illustration Notes: Notice how in example (b), part 2, the expansion of the more complicated expression was easily accomplished by first multiplying the two bases together, as is permitted by Law \#2, then the special product formula (3) was applied followed by (5). A lot of ideas were used in making this little calculation.
Here is a couple of sets of problems for your consideration. Work them out first before you look at the solutions. If you missed some of them, study the solutions to understand what went wrong.

Exercise 5.4. Expand and combine each of the following.
(a) $\left(3 x^{2}-8 y^{4}\right)\left(3 x^{2}+8 y^{4}\right)$
(b) $(x y+\sqrt{x})(x y-\sqrt{x})$
(c) $\left(x^{-3}-2 x^{-2}\right)\left(x^{-3}+2 x^{-2}\right)$

Exercise 5.5. Expand and combine each of the following.
(a) $(3 x-2)^{2}$
(b) $\left(6 y^{2}+1\right)^{2}$
(c) $(4-\sqrt{2})^{2}$
(d) $\left(2 x^{2} y^{3}-3\right)^{2}$

Those were so easy (and important), let's have more. If you erred, study the solutions and test you understanding again.

Exercise 5.6. Expand and combine each of the following.
(a) $\left(5-3 x^{4}\right)^{2}$
(b) $(\sqrt{5}+\sqrt{6})^{2}$
(c) $\left(x^{-3 / 2}-x^{-1 / 2}\right)^{2}$
(d) $\left(4 x^{6} y^{4}+5\right)^{2}$

## - Radicals Revisited

We can exploit multiplication formula (3) in order to Rationalize the Denominator the denominators in certain situations.

If we are working with a ratios of the form

$$
\begin{equation*}
\frac{3}{3+\sqrt{2}} \quad \text { or } \quad \frac{2}{\sqrt{5}-\sqrt{2}} \tag{7}
\end{equation*}
$$

(or some other variation on the same theme), we can rid ourselves of the dastardly radicals in the denominator by multiplying by the conjugate of the denominator.

What is a conjugate? The conjugate of $\sqrt{5}-\sqrt{3}$ is $\sqrt{5}+\sqrt{3}$. The conjugate of $3+\sqrt{2}$ is $3-\sqrt{2}$. In general, if we have a sum or difference of two terms, change the sign of the second term to get the conjugate.

Example 5.4. Rationalize the following two expressions.
(a) $\frac{3}{3+\sqrt{2}}$
(b) $\frac{2 x}{\sqrt{5}-\sqrt{3}}$
(c) $\frac{3-\sqrt{2}}{3+\sqrt{2}}$

Try a few simple one's yourself please. Study the level of simplification of Example 5.4 and strive to attain the same level of simplicity. When solving all these problems, be neat and organized.

Exercise 5.7. Rationalize the denominator.
(a) $\frac{7}{4-\sqrt{2}}$ (b) $\frac{6}{\sqrt{3}(\sqrt{3}-1)}$ (c) $\frac{2+\sqrt{5}}{2-\sqrt{5}}$

### 5.2. The Binomial Formula

Let us turn now the problem of computing higher powers of binomials. In this section, we are interested in learning how to compute arbitrary

Section 5: Expansion
powers of a binomial:
Problem: Compute $(a+b)^{n}$, for $n=1,2,3,5, \ldots$
Let's list out the first few cases:

- $(a+b)^{2}=a^{2}+2 a b+b^{2}$
- $(a+b)^{3}=a^{3}+3 a^{2} b+3 a b^{2}+b^{3}$
- $(a+b)^{4}=a^{4}+4 a^{3} b+6 a^{2} b^{2}+4 a b^{3}+b^{4}$.
- $(a+b)^{5}=a^{5}+5 a^{4} b+10 a^{3} b^{2}+10 a^{2} b^{3}+5 a b^{4} b^{5}$.
- $(a+b)^{6}=a^{6}+6 a^{5} b+15 a^{4} b^{2}+20 a^{3} b^{3}+15 a^{2} b^{4}+6 a b^{5}+b^{6}$.

The symbolic factors follow a definite pattern. In the expansion of $(a+b)^{n}$ the symbolic factors are

$$
a^{n} b^{0}, a^{n-1} b, a^{n-2} b^{2}, a^{n-3} b^{3}, \ldots, a^{2} b^{n-2}, a b^{n-1}, a^{0} b^{n}
$$

Pattern of the Symbolic Factors: Notice the power of $a$ begin at $n$ and decrease to power 0 ; the powers of $b$ begins at 0 and increases to power $n$.

Pattern of the Numerical Coefficients: The pattern of the coefficients is a little harder to see. There is a general formula for these coefficients, called binomial coefficients, but that formula will not be presented here. Instead, an algorithm for expanding a binomial will be emphasized.

## - The Binomial Expansion Algorithm

Here is a series of steps that will enable you to expand without memorizing formulas.

Algorithm for expanding $(a+b)^{n}$.
(a) The first term is $a^{n}$. Call this the current term.
(b) plus...

1. the product of the coefficient of the current term and the current exponent of $a$, divided by the term number of the current term, times ...
2. a raised to one less power, times ...
3. $b$ raised to one greater power.

Section 5: Expansion
(c) If the exponent of $a$ just computed is zero then you are done. If the exponent of $a$ is not zero, then call the term you just computed the current term and go to step (b).

The next example is the binomial algoritm in action. Study this example very closely! The same reasoning can be used to expand any binomial.

Example 5.5. Illustrate this algorithm by using it to expand $(a+b)^{3}$.
If you survived the reading of Example 5.5, and you have a basic "feel" for the algorithm, you know it is easier to expand out the term than it is to type out a detailed explanation how to do it!

It is important to realize that the binomial algorithm is set up for expanding basic symbols like $a$ and $b$. When you want to expand something like $(2 x+3 y)^{3}$, you expand using the algorithm with $a=2 x$ and $b=3 y$. Thus,

$$
\begin{equation*}
(2 x+3 y)^{3}=(2 x)^{3}+3(2 x)^{2}(3 y)+3(2 x)(3 y)^{2}+(3 y)^{3} . \tag{8}
\end{equation*}
$$

The powers of the first term decrease and the powers of the second term increase. The coefficients are computed by multiplying the current coefficient by the current exponent of the ' $a$ ' and dividing by the term number.

Further simplification of (8) is necessary:

$$
(2 x+3 y)^{3}=8 x^{3}+36 x^{2} y+54 x y^{2}+27
$$

Example 5.6. Expand $\left(2 x^{2}-y^{3}\right)^{4}$.
The binomials can be expanded by the algorithm, by memorizing the first so many expansions-as listed above, by using Pascal's Triangle to compute the coefficients or by applying the Binomial coefficient formula. The latter two are not presented here. :-( $\mathfrak{P S}$

The expansion techniques is quite general and can be applied with fractional exponents or negative exponents.

Example 5.7. Expand $\left(x^{1 / 2}-1\right)^{3}$.

Section 5: Expansion
Let's finish off this section with some exercises.
Exercise 5.8. Expand each of the following.
(a) $(4 x-3 y)^{2}$
(b) $\left(x^{1 / 2}-x^{-1 / 2}\right)^{2}$
(c) $(\sin (x)-\cos (x))^{2}$

Exercise 5.9. Expand each of the following.
(a) $(x-2 y)^{3}$
(b) $\left(x^{3}+y^{5}\right)^{4}$
(c) $\left(x^{1 / 2}-1\right)^{5}$

Tip. Perhaps you may have noticed that when expanding a difference $(a-b)^{n}$ the signs alternate. This observation accelerates the process of expansion.

Illustration 2. Expand $\left(2 x^{2}-x^{-3}\right)^{3}$.

$$
\begin{aligned}
\left(2 x^{2}-x^{-3}\right)^{3} & =\left(2 x^{2}\right)^{3}-3\left(2 x^{2}\right)^{2}\left(x^{-3}\right)+3\left(2 x^{2}\right)\left(x^{-3}\right)^{2}-\left(x^{-3}\right)^{3} \\
& =8 x^{6}-12 x^{4} x^{-3}+6 x^{2} x^{-6}-x^{-9} \\
& =8 x^{6}-12 x+6 x^{-4}-x^{-9}
\end{aligned}
$$

Additional simplifications are possible, but I'll call it quits.
And now for the last exercise of this lesson!

Section 5: Expansion

Exercise 5.10. Expand each using alternating signs as a short-cut. Passing grade is $100 \%$.
(a) $\left(3 x^{4}-2\right)^{3}$
(b) $\left(x^{2} y^{3}-1\right)^{4}$

This is the end of Lesson 5. Click on Lesson 6 to continue.

The Product of a sum and difference:

$$
(x-y)(x+y)=x^{2}-y^{2}
$$

The product of the sum and difference of $x$ and $y$ is the square of the first minus the square of the second.

Squaring a Binomial:

$$
(x+y)^{2}=x^{2}+2 x y+y^{2}
$$

The square of a binomial is the square of the first, plus twice the product of the first and second, plus the square of the second.

Squaring a Binomial:

$$
(x-y)^{2}=x^{2}-2 x y+y^{2}
$$

The square of a binomial is the square of the first, minus twice the product of the first and second, plus the square of the second.

The Product of Two Binomials:

$$
(x+a)(x+b)=x^{2}+(a+b) x+a b
$$

The product of two binomials is the product of the two first terms, plus the sum of the two cross-product terms, plus the product of the second terms.

## Solutions to Exercises

5.1. Solutions:
(a) Expand and combine $(4 x-5)(3 x+2)$.

$$
\begin{aligned}
(4 x- & 5)(3 x+2) \\
\quad & =(4 x)(3 x)+(4 x)(2)+(-5)(3 x)+(-5)(2) \\
& =12 x^{2}+8 x-15 x-10 \\
& =12 x^{2}-7 x-10
\end{aligned}
$$

(b) Expand and combine $(x-2 y)^{2}$.

$$
\begin{aligned}
(x-2 y)^{2} & =(x-2 y)(x-2 y) \\
& =x^{2}+x(-2 y)+(-2 y)(x)+(-2 y)(-2 y) \\
& =x^{2}-2 x y-2 x y+4 y^{2} \\
& =x^{2}-4 x y+4 y^{2}
\end{aligned}
$$

Solutions to Exercises (continued)
(c) Expand and combine $\left(x^{3 / 2}+1\right)\left(x^{1 / 2}+2\right)$.

$$
\begin{aligned}
\left(x^{3 / 2}\right. & +1)\left(x^{1 / 2}+2\right) \\
& =\left(x^{3 / 1}\right)\left(x^{1 / 2}\right)+\left(x^{3 / 2}\right)(2)+(1)\left(x^{1 / 2}\right)+2 \\
& =x+2 x^{3 / 2}+x^{1 / 2}+2
\end{aligned}
$$

Optionally, the result can be left in radical notation:

$$
\left(x^{3 / 2}+1\right)\left(x^{1 / 2}+2\right)=x+2 x \sqrt{x}+\sqrt{x}+2
$$

And, if we wanted to be true to our algebraic roots, we could write

$$
\left(x^{3 / 2}+1\right)\left(x^{1 / 2}+2\right)=x+(2 x+1) \sqrt{x}+2
$$

Solutions to Exercises (continued)
5.2. Solutions:
(a) Expand and combine $(2 x-y)\left(x^{2}-2 y^{2}+x y\right)$.

$$
\begin{aligned}
&(2 x-y)\left(x^{2}-2 y^{2}+x y\right) \\
&=2 x\left(x^{2}\right)-2 x\left(2 y^{2}\right)+2 x(x y)-y\left(x^{2}\right)+y\left(2 y^{2}\right)-y(x y) \\
& \quad=2 x^{3}-4 x y^{2}+2 x^{2} y-x^{2} y+2 y^{3}-x y^{2} \\
&=2 x^{3}+\left(2 x^{2} y-x^{2} y\right)+\left(-4 x y^{2}-x y^{2}\right)+2 y^{3} \\
&=2 x^{3}+x^{2} y-5 x y^{2}+2 y^{3}
\end{aligned}
$$

(b) Expand and combine $(a b-c)(a-b+c)$.

$$
\begin{aligned}
(a b-c) & (a-b+c) \\
& =a b(a)+a b(-b)+a b(c)-c(a)-c(-b)-c(c) \\
& =a^{2} b-a b^{2}+a b c-a c+b c-c^{2}
\end{aligned}
$$

In this last problem, no additional simplification is necessary.

Solutions to Exercises (continued)
5.3. Solution: Put the trinomial on top and the binomial on bottom.

$$
\begin{aligned}
& 2 x^{2}-3 y^{2}-4 x y \\
& 4 x-7 y \\
& \hline 8 x^{3}-12 x y^{2}-16 x^{2} y \\
& \quad+28 x y^{2}-14 x^{2} y+21 y^{3} \\
& \hline 8 x^{3}+16 x y^{2}-30 x^{2} y+21 y^{3}
\end{aligned}
$$

Solutions to Exercises (continued)
5.4. Solutions:
(a) $\left(3 x^{2}-8 y^{4}\right)\left(3 x^{2}+8 y^{4}\right)=9 x^{4}-64 y^{8}$.
(b) $(x y+\sqrt{x})(x y-\sqrt{x})=x^{2} y^{2}-x$.
(c) $\left(x^{-3}-2 x^{-2}\right)\left(x^{-3}+2 x^{-2}\right)=x^{-6}-4 x^{-4}=x^{-6}\left(1-4 x^{2}\right)=$
$\quad \frac{1-4 x^{2}}{x^{6}}$.

Study the last solution ... several facts about factoring out and negative exponents were used.

## Solutions to Exercises (continued)

5.5. Solutions: The square of a binomial is the square of the first (term) plus/minus twice the product of the first and second (terms), plus the square of the second (term). Apply that to each.

$$
\begin{aligned}
& \text { (a) }(3 x-2)^{2}=9 x^{2}-12 x+4 . \\
& \text { (b) }\left(6 y^{2}+1\right)^{2}=36 y^{4}+12 y^{2}+1 . \\
& \text { (c) }(4-\sqrt{2})^{2}=16-8 \sqrt{2}+2=18-8 \sqrt{2} . \\
& \text { (d) }\left(2 x^{2} y^{3}-3\right)^{2}=4 x^{4} y^{6}-12 x^{2} y^{3}+9 .
\end{aligned}
$$

Short and sweet! $\mathfrak{P S}$

Solutions to Exercises (continued)
5.6. Solutions:
(a) $\left(5-3 x^{4}\right)^{2}=25-30 x^{4}+9 x^{8}=9 x^{8}-30 x^{4}+25$.
(b) $(\sqrt{5}+\sqrt{6})^{2}=5+2 \sqrt{5} \sqrt{6}+6=11+2 \sqrt{30}$.
(c) Expand $\left(x^{-3 / 2}-x^{-1 / 2}\right)^{2}$.

$$
\begin{aligned}
\left(x^{-3 / 2}-x^{-1 / 2}\right)^{2} & =x^{-3}-2 x^{-3 / 2} x^{-1 / 2}+x^{-1} \\
& =x^{-3}-2 x^{-2}+x^{-1} \\
& =x^{-3}\left(1-2 x+x^{2}\right) \\
& =\frac{x^{2}-2 x+1}{x^{3}} \quad \triangleleft \text { get rid of neg. exp. }
\end{aligned}
$$

- The other option to expanding this particular expression is the rid yourself of the negative exponents and then square. Try solving this problem this way as well. Check out the solution by clicking on the green dot.
(d) $\left(4 x^{6} y^{4}+5\right)^{2}=16 x^{12} y^{8}+40 x^{6} y^{4}+25$.

Solutions to Exercises (continued)
5.7. Solutions:
(a) Rationalize $\frac{7}{4-\sqrt{2}}$.

$$
\begin{array}{rlrl}
\frac{7}{4-\sqrt{2}} & =\frac{7}{4-\sqrt{2}} \cdot \frac{4+\sqrt{2}}{4+\sqrt{2}} & & \triangleleft \text { mul. by conjugate } \\
& =\frac{7(4+\sqrt{2})}{(4-\sqrt{2})(4+\sqrt{2})} & & \triangleleft \text { sum times difference } \\
& =\frac{7(4+\sqrt{2})}{16-2} & & \triangleleft \text { by }(3) \\
& =\frac{7(4+\sqrt{2})}{14} & & \\
& =\frac{1}{2}(4+\sqrt{2}) &
\end{array}
$$

Solutions to Exercises (continued)
(b) Rationalize $\frac{6}{\sqrt{3}(\sqrt{3}-1)}$.

$$
\begin{align*}
\frac{6}{\sqrt{3}(\sqrt{3}-1)} & =\frac{6}{3-\sqrt{3}} & & \\
& =\frac{6}{3-\sqrt{3}} \cdot \frac{3+\sqrt{3}}{3+\sqrt{3}} & & \triangleleft \text { mul. by conjugate } \\
& =\frac{6(3+\sqrt{3})}{(3-\sqrt{3})(3+\sqrt{3})} & & \triangleleft \text { sum times diff. } \\
& =\frac{6(3+\sqrt{3})}{(9-3)} & & \triangleleft \text { by }(3)  \tag{3}\\
& =\frac{6(3+\sqrt{3})}{6} & & \triangleleft \text { cancel the } 6 ’ \mathrm{~s}! \\
& =3+\sqrt{3} & & \triangleleft \text { interesting! }
\end{align*}
$$

Solutions to Exercises (continued)
(c) Rationalize $\frac{2+\sqrt{5}}{2-\sqrt{5}}$.

$$
\begin{aligned}
\frac{2+\sqrt{5}}{2-\sqrt{5}} & =\frac{2+\sqrt{5}}{2-\sqrt{5} \cdot \frac{2+\sqrt{5}}{2+\sqrt{5}}} & & \triangleleft \text { mul. by conjugate } \\
& =\frac{(2+\sqrt{5})(2+\sqrt{5})}{(2-\sqrt{5})(2+\sqrt{5})} & & \triangleleft \text { sum times difference } \\
& =\frac{(2+\sqrt{5})^{2}}{4-5} & & \triangleleft \text { by }(3) \\
& =\frac{4+4 \sqrt{5}+5}{-1} & & \triangleleft \text { from }(5) \\
& =-(9+4 \sqrt{5}) & &
\end{aligned}
$$

Solutions to Exercises (continued)
5.8. Solutions:
(a) $(4 x-3 y)^{2}=16 x^{2}+2(4 x)(-3 y)+9 y^{2}=16 x^{2}-24 x y+9 y^{2}$.
(b) $\left(x^{1 / 2}-x^{-1 / 2}\right)^{2}=x-2\left(x^{1 / 2}\right)\left(x^{-1 / 2}\right)+x^{-1}=x-2+x^{-1}$.
(c) Expand $(\sin (x)-\cos (x))^{2}$.

$$
\begin{aligned}
(\sin (x)-\cos (x))^{2} & =\sin ^{2}(x)-2 \sin (x) \cos (x)+\cos ^{2}(x) \\
& =\left(\sin ^{2}(x)+\cos ^{2}(x)\right)-2 \sin (x) \cos (x) \\
& =1-2 \sin (2 x)
\end{aligned}
$$

I tossed in this problem to shake you up a little. Here, I have recalled the identities: $\sin ^{2}(x)+\cos ^{2}(x)=1$ and $2 \sin (x) \cos (x)=\sin (2 x)$.
5.9. Solutions: Just expand each by the binomial algorithm. Note that as I type out these solutions, I am expanding by the algorithmI'm not using a list of formulas.
(a) Expand $(x-2 y)^{3}$

$$
\begin{aligned}
(x-2 y)^{3} & =x^{3}+3 x^{2}(-2 y)+3 x(-2 y)^{2}+(-2 y)^{3} \\
& =x^{3}-6 x^{2} y+12 x y^{2}-8 y^{3}
\end{aligned}
$$

(b) Expand $\left(x^{3}+y^{5}\right)^{4}$.

$$
\begin{aligned}
\left(x^{3}+y^{5}\right)^{4}= & \left(x^{3}\right)^{4}+4\left(x^{3}\right)^{3}\left(y^{5}\right)+6\left(x^{3}\right)^{2}\left(y^{5}\right)^{2} \\
& \quad+4\left(x^{3}\right)\left(y^{5}\right)^{3}+\left(y^{5}\right)^{4} \\
= & x^{12}+4 x^{9} y^{5}+6 x^{6} y^{10}+4 x^{3} y^{15}+y^{20}
\end{aligned}
$$

Solutions to Exercises (continued)
(c) Expand $\left(x^{1 / 2}-1\right)^{5}$

$$
\begin{aligned}
\left(x^{1 / 2}-1\right)^{5}= & \left(x^{1 / 2}\right)^{5}+5\left(x^{1 / 2}\right)^{4}(-1)+10\left(x^{1 / 2}\right)^{3}(-1)^{2} \\
& \quad+10\left(x^{1 / 2}\right)^{2}(-1)^{3}+5\left(x^{1 / 2}\right)(-1)^{4}+(-1)^{5} \\
= & x^{5 / 2}-5 x^{2}+10 x^{3 / 2}-10 x+5 x^{1 / 2}-1
\end{aligned}
$$

These terms can be converted to radical notation. Make it so!

Solutions to Exercises (continued)
5.10. Solutions:
(a) $\left(3 x^{4}-2\right)^{3}$.

$$
\begin{aligned}
\left(3 x^{4}-2\right)^{3} & =\left(3 x^{4}\right)^{3}-3\left(3 x^{4}\right)^{2}(2)+3\left(3 x^{4}\right)(2)^{2}-2^{3} \\
& =27 x^{12}-54 x^{8}+36 x^{4}-8
\end{aligned}
$$

(b) $\left(x^{2} y^{3}-1\right)^{4}$.

$$
\begin{aligned}
\left(x^{2} y^{3}-1\right)^{4}= & \left(x^{2} y^{3}\right)^{4}-4\left(x^{2} y^{3}\right)^{3}+6\left(x^{2} y^{3}\right)^{2} \\
& -4\left(x^{2} y^{3}\right)+1 \\
= & x^{8} y^{12}-4 x^{6} y^{9}+6 x^{4} y^{6}-4 x^{2} y^{3}+1
\end{aligned}
$$

## Solutions to Examples

5.1. Solutions: In each case, we add up all possible products of terms from the first factor with terms in the second factor.
(a) Expand and combine $(x+1)(x+2)$.

$$
\begin{aligned}
(x+ & 1 & (x+2) & \\
& =(x)(x)+(x)(2)+(1)(x)+(1)(2) & & \triangleleft \text { Expand by }(2) \\
& =x^{2}+2 x+x+2 & & \\
& =x^{2}+3 x+2 & & \triangleleft \text { and combine! }
\end{aligned}
$$

(b) Expand and combine $(2 w-3 s)(5 w+2 s)$.

$$
\begin{aligned}
(2 w- & 3 s)(5 w+2 s) \\
& =(2 w)(5 w)+(2 w)(2 s)+(-3 s)(5 w)+(-3 s)(2 s) \\
& =10 w^{2}+4 s w-15 s w-6 s^{2} \\
& =10 w^{2}-11 s w-6 s^{2}
\end{aligned}
$$

(c) Expand and combine $(2 x-3)\left(x^{2}-2\right)$.

$$
\begin{aligned}
(2 x- & 3)\left(x^{2}-2\right) \\
& =(2 x)\left(x^{2}\right)+(2 x)(-2)+(-3)\left(x^{2}\right)+(-3)(-2) \\
& =2 x^{3}-4 x-3 x^{2}+6 \\
& =2 x^{3}-3 x^{2}-4 x+6
\end{aligned}
$$

Notice how the negative terms are handled during the expansion of this product. This would be a good style for you.
(d) Expand and combine $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$.

$$
\begin{aligned}
(\sqrt{a} & +\sqrt{b})(\sqrt{a}-\sqrt{b}) \\
& =\sqrt{a}(\sqrt{a})+\sqrt{a}(-\sqrt{b})+\sqrt{b}(\sqrt{a})+\sqrt{b}(-\sqrt{b}) \\
& =a-\sqrt{a} \sqrt{b}+\sqrt{b} \sqrt{a}-b \\
& =a-b
\end{aligned}
$$

In this calculation, the so-called cross-product terms eliminate each other.

Solutions to Examples (continued)
5.2. Solutions: We multiply each term of the first factor by each term of the second factor, and add together the resultant calculations.
(a) Expand and combine $(x+y)\left(x^{2}-x y+y^{2}\right)$.

$$
\begin{aligned}
(x+ & y) \\
& \left(x^{2}-x y+y^{2}\right) \\
& =x\left(x^{2}\right)+x(-x y)+x\left(y^{2}\right)+y\left(x^{2}\right)+y(-x y)+y\left(y^{2}\right) \\
& =x^{3}-x^{2} y+x y^{2}+x^{2} y-x y^{2}+y^{3} \\
& =x^{3}+y^{3}
\end{aligned}
$$

(b) Expand and combine $\left(2 x^{2}-4 y^{3}\right)\left(6 x^{3}+3 x y+2 y^{2}\right)$.

$$
\begin{aligned}
\left(2 x^{2}-\right. & \left.4 y^{3}\right)\left(6 x^{3}+3 x y+2 y^{2}\right) \\
= & 2 x^{2}\left(6 x^{3}\right)+2 x^{2}(3 x y)+2 x^{2}\left(2 y^{2}\right) \\
& \quad-4 y^{3}\left(6 x^{3}\right)-4 y^{3}(3 x y)-4 y^{3}\left(2 y^{2}\right) \\
= & 12 x^{5}+6 x^{3} y+4 x^{2} y^{2}-24 x^{3} y^{3}-12 x y^{4}-8 y^{5}
\end{aligned}
$$

5.3. This technique is similar to multiplying out numbers by pencil and paper.

$$
\begin{aligned}
& \begin{array}{c}
x^{2}-y^{2}+2 x y \\
2 x-3 y
\end{array} \\
& \hline 2 x^{3}-2 x y^{2}+4 x^{2} y \\
& \frac{-6 x y^{2}-3 x^{2} y+3 y^{3}}{2 x^{3}-8 x y^{2}+x^{2} y+3 y^{3}}
\end{aligned}
$$

The third row is obtained by taking the first term in the second row and multiplying it by each term in the first row.

The fourth row is obtained by taking the second term in the second row and multiplying it by each term in the first row, being sure to place similar terms in the same column.

Solutions to Examples (continued)
5.4. Solutions:
(a) $\frac{3}{3+\sqrt{2}}$.

$$
\begin{aligned}
\frac{3}{3+\sqrt{2}} & =\frac{3}{3+\sqrt{2}} \cdot \frac{3-\sqrt{2}}{3-\sqrt{2}} & & \triangleleft \text { mul. by conjugate } \\
& =\frac{3(3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})} & & \triangleleft \text { sum times difference } \\
& =\frac{3(3-\sqrt{2})}{9-2} & & \triangleleft \text { by }(3) \\
& =\frac{3(3-\sqrt{2})}{7}=\frac{3}{7}(3-\sqrt{2}) & &
\end{aligned}
$$

Solutions to Examples (continued)
(b) $\frac{2 x}{\sqrt{5}-\sqrt{3}}$.

$$
\begin{array}{rlrl}
\frac{2 x}{\sqrt{5}-\sqrt{3}} & =\frac{2 x}{\sqrt{5}-\sqrt{3}} \cdot \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}} & & \triangleleft \text { mul. by conjugate } \\
& =\frac{2 x(\sqrt{5}+\sqrt{3})}{(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})} & \triangleleft \text { sum times difference } \\
& =\frac{2 x(\sqrt{5}+\sqrt{3})}{5-3} & & \triangleleft \text { by }(3) \\
& =\frac{2 x(\sqrt{5}+\sqrt{3})}{2} & & \\
& =x(\sqrt{5}+\sqrt{3}) &
\end{array}
$$

Solutions to Examples (continued)
(c) $\frac{3-\sqrt{2}}{3+\sqrt{2}}$.

$$
\begin{align*}
\frac{3-\sqrt{2}}{3+\sqrt{2}} & =\frac{3-\sqrt{2}}{3+\sqrt{2}} \cdot \frac{3-\sqrt{2}}{3-\sqrt{2}} & & \triangleleft \text { mul. by conjugate } \\
& =\frac{(3-\sqrt{2})(3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})} & & \triangleleft \text { sum times difference } \\
& =\frac{(3-\sqrt{2})^{2}}{9-2} & & \triangleleft \text { by }(3)  \tag{3}\\
& =\frac{9-6 \sqrt{2}+2}{7} & & \triangleleft \text { from }(5) \\
& =\frac{11-6 \sqrt{2}}{7} & &
\end{align*}
$$

5.5. Expand $(a+b)^{n}$ using the binomial algorithm.

1. The first term is $a^{3}$. This is the current term.
$\triangleright(a+b)^{3}=a^{3}+\ldots$
2. plus...
a. the product of the coefficient of the current term (1) and the current exponent of $a(3)$, divided by the current term number (1): (1)(3)/1 = 3 times ...
b. a raised to one less power: $3 a^{2}$ times ...
c. $b$ raised to one greater power: $3 a^{2} b$
3. Thus, the second term is $3 a^{2} b$. This is our new current term. The expansion so far is ...
$\triangleright(a+b)^{3}=a^{3}+3 a^{2} b+\ldots$
4. The exponent of $a$ is nonzero so we repeat step (b).
5. plus ...
a. the product of the coefficient of the current term (3) and the current exponent of $a$ (2), divided by the current term number (2): $(3)(2) /(2)=3$ times $\ldots$
b. $a$ raised to one less power: $3 a$ times ...
c. $b$ raised to one greater power: $3 a b^{2}$.
6. Thus, the third term is $3 a b^{2}$. This is our new current term. The expansion so far is ...
$\triangleright(a+b)^{3}=a^{3}+3 a^{2} b+3 a b^{2}+\ldots$
7. The exponent of $a$ is nonzero so we repeat step (b).
8. plus ...
a. the product of the coefficient of the current term (3) and the current exponent of $a$ (1), divided by the current term number (3): $(3)(1) /(3)=1$ times $\ldots$
b. a raised to one less power: (1) $a^{0}$ times ...
c. $b$ raised to one greater power: (1) $a^{0} b^{3}=b^{3}$.
9. Thus, the fourth term is $b^{3}$. This is our current term.
$\triangleright(a+b)^{3}=a^{3}+3 a^{2} b+3 a b^{2}+b^{3}$
10. The exponent of $a$ is down to zero! Finished!

We have expanded $(a+b)^{3}$ as

$$
(a+b)^{3}=a^{3}+3 a^{2} b+3 a b^{2}+b^{3} .
$$

And that's all there is to it!

Solutions to Examples (continued)
5.6. We first use the binomial algorithm to obtain the form of the expansion. Note that $a=2 x^{2}$ and $b=-y^{3}$.

$$
\begin{aligned}
\left(2 x^{2}-y^{3}\right)^{4}= & \left(2 x^{2}+\left(-y^{3}\right)\right)^{4} \\
= & \left(2 x^{2}\right)^{4}+4\left(2 x^{2}\right)^{3}\left(-y^{3}\right)+6\left(2 x^{2}\right)^{2}\left(-y^{3}\right)^{2} \\
& \quad+4\left(2 x^{2}\right)\left(-y^{3}\right)^{3}+\left(-y^{3}\right)^{4}
\end{aligned}
$$

Now we simplify from there:

$$
\left(2 x^{2}-y^{3}\right)^{4}=16 x^{8}-32 x^{6} y^{3}+24 x^{4} y^{6}-8 x^{2} y^{9}+y^{12}
$$

The student, that's you, should verify the above simplification.
Example 5.6.

Solutions to Examples (continued)
5.7. Expand the cubic using the algorithm with $a=x^{1 / 2}$ and $b=1$ :

$$
\begin{aligned}
\left(x^{1 / 2}-1\right)^{3} & =\left(x^{1 / 2}\right)^{3}+3\left(x^{1 / 2}\right)^{2}(-1)+3\left(x^{1 / 2}\right)(-1)^{2}+(-1)^{3} \\
& =x^{3 / 2}-3 x+3 x^{1 / 2}-1
\end{aligned}
$$

Presentation of Answer:

$$
\left(x^{1 / 2}-1\right)^{3}=x^{3 / 2}-3 x+3 x^{1 / 2}-1
$$

Or we can take a radical approach to presentation the answer.

$$
(\sqrt{x}-1)^{3}=x \sqrt{x}-3 x+3 \sqrt{x}-1
$$

## Important Points

Important Points (continued)
Alternate Solution: Expand $\left(x^{-3 / 2}-x^{-1 / 2}\right)^{2}$.

$$
\begin{aligned}
\left(x^{-3 / 2}-x^{-1 / 2}\right)^{2} & =\left(x^{-3 / 2}(1-x)\right)^{2} & \triangleleft \text { factor out lowest power } \\
& =\left(x^{-3 / 2}\right)^{2}(1-x)^{2} & \triangleleft \text { Law \#2 } \\
& =x^{-3}\left(1-2 x^{2}+x^{2}\right) & \triangleleft \text { Law \#3 \& (5) } \\
& =\frac{x^{2}-2 x+1}{x^{3}} &
\end{aligned}
$$

Same expansion as before! No surprise.

