A Survey of Multiple Contractions

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Abstract. The AGM theory of belief contraction is extended to *multiple contraction*, i.e. to contraction by a set of sentences rather than by a single sentence. There are two major variants: In *package contraction* all the sentences must be removed from the belief set, whereas in *choice contraction* it is sufficient that at least one of them is removed. Constructions of both types of multiple contraction are offered and axiomatically characterized. Neither package nor choice contraction can in general be reduced to contractions by single sentences; in the finite case choice contraction allows for reduction.

Key words: belief revision, belief contraction, theory change.

1. BACKGROUND

This paper contains a substantive generalisation and extension of the logic of theory (or belief) change as developed by Carlos Alchourrón, Peter Gärdenfors, and David Makinson (henceforth AGM). Though from a technical point of view the present paper is self-contained, we have primarily in mind a reader who is acquainted with the key ideas and results of the AGM theory, as described e.g. in AGM (1985). The present paper is concerned with the theory of contractions; a sequel will treat revisions.

A change operation in the sense of AGM operates on a theory, say T, and a sentence, say α, to deliver a new, changed, theory, T', where the change from T to T' is constrained by the sentence α. In the AGM theory three types of changes – i.e. three ways of constraining the transition from T to T' – are considered:

- *expansions*: we just add α to T and close under logical consequence;
- *contractions*: we remove α from T, trying to save as much of T as we can;
- *revisions*: we add α to T while trying to maintain (or create) consistency.

Clearly, expansions are easily definable: they pose no problem. It is widely agreed that revisions can be reduced to contractions and expansions: first

contract T so as to open up for the consistent addition of α, i.e. retract $\neg \alpha$, then expand by α. Thus, the AGM theory focusses on the task of characterising suitable contraction operations. This task may be accomplished in two different ways. First, we may give recipes or algorithms for constructing a contracted theory $T - \alpha$ from any theory T and sentence α. Second, contractions may be characterised more indirectly by laying down a number of conditions each reasonable contraction operation should obey. In AGM (1985) and subsequent writings both paths have been taken and suitable representation results have been proven.

The plan of this paper is as follows. In the next two sections we shall argue informally that a theory of multiple contractions is a new and useful tool for analysing changes of theories or belief sets. In Section 4 we introduce some terminology and notation and reproduce, for the reader's convenience, the AGM postulates for contractions. For motivations of these postulates the reader is referred to Makinson (1985) or Gärdenfors (1988). In Section 5 we observe a fundamental distinction between two kinds of multiple contraction operations. These operations will then be studied by way of suitable postulates (Sections 5 to 7) and by looking at models of multiple contractions (Sections 8 to 11). In Section 12 we briefly return to the topic of possible reductions of multiple to singleton contractions. We close with a section on possible applications and future developments.

2. DELINEATING THE TOPIC

The AGM theory of belief change exhibits a curious asymmetry in the arguments that a change operation may take. A change operation in the sense of AGM (a contraction, expansion, or revision) may be seen as a binary operation that takes two sets of sentences, X and Y, to return a third set of sentences X':

$$(X, Y) \mapsto X'$$

Intuitively, the left-hand argument, X, is the item to be changed. The right-hand argument forces – up to uniqueness – a certain "direction of change": it represents the item to be "changed by". Asymmetry enters the AGM theory by way of three constraints:

  1. the left argument is closed under consequence;
  2. the right argument is a singleton set.

Sometimes (as in Gärdenfors (1988) but not in AGM (1985)) it is also required that

  3. the left argument is consistent.

However, the last condition plays virtually no role in the AGM theory. It may be waived without significant consequences to the theory as a whole.

The first condition marks one of the differences between theory contraction and what has come to be called base contraction (Hansson (1989), Fuhrmann (1991)). Though this condition is now usually associated with the AGM theory, it was not there in the beginning. In the first papers by Alchourrón and Makinson the condition was not required. And in their survey of partial meet contractions (1985) AGM took much care to signal where the condition is actually used.

The second condition has by and large withstood the tide of time. A first sketch of a theory of change operations that are not generally singular on the right is contained in Fuhrmann (1988). Hansson (1989), Niederée (1991), and Rott (1992) have independently pursued the idea of multiple change operations. The purpose of this paper is to systematize previous ideas and results and to probe further into the theory of multiple change operations.

The term "multiple contraction" was proposed in Fuhrmann (1988) for operations of contraction that allow for simultaneous contraction by more than one sentence. It should be distinguished from "repeated" or "iterated" contraction, i.e., the performance of two or more contractions in a sequence (not to be discussed here). Multiple contraction, as we understand it, is simultaneous contraction by a set of sentences that need not be a singleton.

Here, we shall be orthodox on the left and liberal on the right: We shall consider changes of logically closed sets of sentences by (arbitrary) sets of sentences. This asymmetry is maintained mainly for expository purposes. Ultimately we should want to be liberal both on the left-hand and on the right-hand side of the change operations; the fully general theory is planned for Fuhrmann (1994).

In this paper we shall focus on multiple contraction operations. In a sequel to this paper we will treat multiple revisions and their reduction to multiple contractions by a suitable generalisation of the Levi identity for singletons.

3. THE UBIQUITY AND NECESSITY OF MULTIPLE CONTRACTION

Before introducing the formal theory, we owe it to the reader to show, in an informal setting, that multiple contraction is a useful tool in modelling belief change, and that it can be expected to do more than singleton contraction.

Three questions need to be distinguished:

1. Do multiple contractions occur?
2. Need multiple contractions occur?
3. Should multiple contractions occur?

We shall shortly argue that, as a matter of brute fact, multiple contractions do occur in our cognitive practice – though it may require a step of abstraction to recognize their presence.

Still, multiple contractions may only occur as a matter of convenience. Perhaps they can, at least in principle, always be replaced by singleton contractions and, hence, need not occur. We shall show that none of the multiple contractions we consider here generally allow for such reduction, though under certain conditions reduction is possible. Thus, given that multiple contractions are *sui generis*, there is a sense in which they *must* occur: there are no alternatives to achieve their effects.

The last question is perhaps the most difficult of the three: given that information is precious, can it ever be rational to engage in multiple contraction? This is a question we shall not address here. Instead we proceed from the assumption that decisions to multiply contract are sometimes taken and that some ways of implementing such decisions are better than others. (Perhaps it is generally irrational to run nuclear power plants. But given that nuclear power plants exist, there are more and less rational ways of operating them.)

Although we shall not here pursue the question as to which contractions ought to be carried out, we recommend it to the reader's attention. According to Levi there are only two situations in which it is rational to contract: opening one's mind for new possibilities and retreating from inconsistency. Levi merges the second with the third question by proffering the conjecture (in correspondence) that every legitimate contraction, that is, every contraction that serves one of the two purposes, must be (equivalent to) a singleton contraction. This is not a general reduction thesis but one pertaining only to those contractions one *should* engage in, i.e. the legitimate or rational contractions. Evidently, the conjecture turns much on the requisite notion of legitimacy; it is accordingly much more difficult to assess than the general reduction theses considered, and refuted, below.

If Levi's conjecture can be shown true, it will constitute a very substantial philosophical insight into the nature of rational belief change. Such insights can happily live alongside the formal characterisations aimed at in this paper. (However, in the best of all possible worlds accessible to the authors, Levi's conjecture turns out false.)

It is generally accepted in the belief change literature that it is difficult to find cases of pure contraction. In most cases, the retraction of a belief is provoked by the acquisition of some other belief which forces the old one out. Contraction seems to occur most frequently as part of a more complex operation which involves both the removal and the addition of information. The most clear case of pure contraction that has been discussed in the literature is what may be called 'contraction for the sake of argument' or 'mind-opening

contraction': one may wish to give a belief that α a hearing although it contradicts one's present state of belief. To open up for α one should then contract by $\neg\alpha$. (Levi (1980, ch. 3), (1991, ch. 4); Gärdenfors (1988, p. 60); Fuhrmann (1991, Sec. 4).)

Contraction for the sake of argument will often involve the simultaneous removal of more than one belief. Suppose, for instance, that you wish to give each of two different views – both of which contradict your own standpoint – a fair hearing. You then need to open up your belief state to make it consistent with either of the two views. Representing the two views by the sentences, say $\neg\alpha$ and $\neg\beta$ respectively, this is equal to removing both α and β from your belief set. The result of this operation should be a belief set that implies neither α nor β and is otherwise as similar as possible to the original belief set.

It is important to distinguish the operation of completely removing a set of sentences, say, $\{\alpha, \beta\}$, from a theory from several other operations which, at first sight, seem to have similar effects:

1. contracting by $\alpha \vee \beta$,
2. intersecting the results of contracting by α and of contracting by β,
3. first contracting by α and then by β, or vice versa,
4. contracting by $\alpha \wedge \beta$.

In Section 11 below we shall investigate possible strategies for reducing contractions by sets to contractions by single sentences. For now the following informal remarks may serve to indicate that it is far from obvious that such reduction strategies can be successful.

As to the first operation, it is true that in order to remove a disjunction from a theory one needs to remove both disjuncts. But not conversely: contracting by the set $\{\alpha, \beta\}$ does not require removal of $\alpha \vee \beta$. One may open one's mind to both $\neg\alpha$ and $\neg\beta$ without opening it to $\neg(\alpha \vee \beta)$. For a rather drastic counterexample to this particular reduction thesis let $\beta = \neg\alpha$.

Still (as David Makinson has pointed out in correspondence), contractions by disjunctions may gain interest when one turns to multiple revisions. Suppose a set A is to be consistently added to a theory T, i.e. T is to be *revised* by A. A natural approach (natural, that is, from the viewpoint of partial meet constructions; see below) is to first consider the collection of all subsets of T that are maximally consistent with all of A, then to add A to what is common to all "preferred" such subsets, and finally to close the resulting set under consequence. Now, a subset X of T is (classically) consistent with A just in case no finite disjunction $\neg\alpha_1 \vee \ldots \vee \neg\alpha_n$ $(\alpha_i \in X)$ can be derived from A. This simple observation suggests that a multiple revision by a (finite) set A should be defined in terms of a singleton contraction by $\bigvee \bar{A}$ (the disjunction of all negated elements of A), followed by a singleton expansion by $\bigwedge A$. This

strategy will be explored in a sequel to the present paper. Whatever the proper approach to multiple revisions may turn out to be, we take the objection of the last paragraph to the disjunctive approach to multiple *contractions* to be decisive. Here, then, may be a point where the connection – as formulated in the Levi Identity – between contractions and revisions comes apart.

The second operation may result in a theory which is too small to represent the corresponding multiple contraction. Still, this reduction strategy seems to be the most promising of the four and we shall return to it at the end of Section 9.

The third operation would introduce asymmetry where there should be none. For, the order of contraction may make a difference (see e.g. Hansson 1992c). We cannot expect in general that the result of first contracting by α and then by β is the same as that of first contracting by β and then by α. But it is part of the very idea of a multiple contraction that all sentences to be retracted are equal. Thus, sequential contraction cannot be the same operation as multiple contraction. Multiple contraction is simultaneous: it does not discriminate between items to be removed by some assignment of priority.

Symmetry could be restored by intersecting all possible contraction sequences; in the case of a two-elements-set we could intersect the result of first retracting α, then β with the result of first retracting β, then α. But this operation would make the contracted theory even smaller than the one that results from applying the second strategy above. Given that the second operation cannot serve our purpose (as will be shown below), its symmetrizing mate must fail too.

That the fourth operation is unsuitable as a representation of a contraction by $\{\alpha, \beta\}$ is easy to see: to remove a conjunction it suffices to remove one of the conjuncts. Thus, we may have α staying in a contraction by $\alpha \wedge \beta$ but, of course, α should not stay in a multiple contraction by $\{\alpha, \beta\}$ as we have used the term so far.

The last operation points towards an alternative notion of multiple contraction: instead of removing a certain set completely from a theory one might be interested in an operation that modifies a theory such that this set is no longer contained in it. This notion of a multiple contraction is useful whenever a theory is faced with contravening evidence which, however, is not specific enough to determine exactly which sentences ought to be retracted in order to accommodate the evidence. This is a situation frequently encountered: the recalcitrant evidence may, on its own, not determine which parts of the theory ought to be given up – in this sense it may be inconclusive. In the worst case the evidence gives no guidance at all as to which part of the theory is likely to be false. More usually, however, the evidence will incriminate a proper part

of the theory from which culprits should be picked. In any case we are faced with a choice which can only be made in the context of the whole theory. That is to say, the theorist has to judge – according to some contextually fixed relevant parameters – which part of the theory she can best afford to live without.

(The theory of belief revision itself supplies a fine example of this kind of contraction. Gärdenfors' (1988, sec. 7.4) impossibility theorem concerning the generation of conditionals in systems of belief revision is commonly taken to require that, in the presence of the Ramsey test, the full set of revision postulates be retracted. But, of course, the retraction ought not to be a complete one but one that involves choice – it is far from clear, however, which choice is to be made.)

We can thus discern two kinds of multiple contraction. According to one kind of contraction all members of a set are retracted: they have to go in a package. There is another kind of contraction where one only needs to ensure that some set is no longer a subset of the theory in question. For that purpose it suffices to remove some elements of that set from the theory: one needs to choose which ones. We shall call the first type of multiple contractions *package contraction* and the second *choice contraction*. It will emerge (see Observation 17 below) that in the finite case the latter reduces to contraction by the conjunction of all sentences to be retracted.

We close this section by adducing three more clusters of situations where multiple contractions are called for. First, suppose you have inadvertently expanded a consistent belief set by some sentence α such that $\neg\alpha$ is in the belief set. As was observed by Levi (1991, sec. 4.8), in such cases it is often best to restore consistency by retracting both α and $\neg\alpha$. In general there is no reason to contract first by α and then by $\neg\alpha$, or the other way around. We therefore believe that this is a case of multiple contraction by $\{\alpha, \neg\alpha\}$. If the epistemic agent is only concerned with regaining consistency, then choice contraction is sufficient. If, on the other hand, she is more cautious (or "sceptical" to use another popular AI-term) and wishes to investigate again both the statements that led to contradiction (as Levi tends to recommend), then a package contraction is called for.

Second, multiple contraction is applicable in non-epistemic contexts as well. Consider, for instance, the dynamics of normative systems – which, by the way, was one of the main sources for the logic of theory change; see Alchourrón and Makinson (1981). Changes in legal systems often come in large pieces: they typically involve the simultaneous alteration of many parts of a legal code. Contractions, or derogations as one would say in legal contexts, are frequently of the package variety. It seems difficult, however, to find clear examples of choice derogations.

Third, most theories come in layers. For example, physics is (partly) based on a background of mathematics which in turn is (partly) based on a background of logic. This picture – a heritage from the Viennese Circle – is, perhaps, too simple but nevertheless essentially correct. The pictures does not only apply to the architectonics of science but also models ways information is structured in artificial intelligence systems, say, in distributed databases. Consider then a partial order of theories with a maximal element, T, (incorporating all theories below in the order) such that for all theories T_1 and T_2: if $T_1 \leq T_2$ and $\alpha \to \beta \in T_1$ and $\alpha \in T_2$, then $\beta \in T_2$. (Probably the first formal study of such structures is Meyer (197+).) There may be occasions when we need to study the effect of removing a whole theory T_i from the partial order – perhaps, in order to replace it by some other theory. (This is like removing a module from a distributive database.) It appears promising to represent such a change as either a package or a choice contraction of the top theory T by the subtheory T_i.

This concludes, for the time being, our informal motivation and characterisation of multiple contraction operations. At this stage we can hardly do more than broadly outline our target notions and give some indication that we are engaged in a worthwhile enterprise. Once we are equipped with a corpus of formal definitions and results, we shall return to some of the questions raised in this section.

4. FORMAL PRELIMINARIES

We assume that our theory applies to languages that have at least a Boolean structure. In particular we assume that there are operations on sentences expressing negation ($\neg$) and conjunction ($\wedge$). The set of all sentences of the language under consideration will be denoted by *'Fml'*. Small Greek letters, $\alpha, \beta, \gamma, \ldots$, range variably over sentences. Capital roman letters, A, B, C, $\ldots$, X, Y, Z stand for sets of sentences. We reserve the letter T, sometimes dashed or subscripted, to denote theories.

A *theory* is a set of sentences closed under logical consequence, Cn:

$$T = \mathrm{Cn}(T)$$

We assume that Cn includes classical consequence. For the most part, however, we only need a few general facts about (finitary) consequence operations. For the reader's convenience we recall these facts here:

$$X \subseteq \mathrm{Cn}(X) \tag{Refl}$$

$$X \subseteq Y \implies \mathrm{Cn}(X) \subseteq \mathrm{Cn}(Y) \tag{Mon}$$

$$\text{Cn}(\text{Cn}(X)) \subseteq \text{Cn}(X) \tag{Idm}$$

$$\text{Cn}(X) = \bigcup \{\text{Cn}(X_0) : X_0 \text{ is a finite subset of } X\} \tag{Fin}$$

Apart from these abstract properties of consequence operations we shall at times make use of some further elementary properties pertaining to specific connectives in the language. Prominent among these properties is the so-called deduction equivalence,

$$\alpha \to \beta \in \text{Cn}(X) \iff \beta \in \text{Cn}(X \cup \{\alpha\})$$

It will be convenient to freely switch between an operational and a relational notation. Thus, instead of $\alpha \in \text{Cn}(A)$ we sometimes write $A \vdash \alpha$. More generally, given a consequence operation, Cn, we define two *consequence relations*, $\vdash$ and $\Vdash$, as follows:

DEFINITION 1. *For sets of sentences X and Y:*
  1. $X \vdash Y \implies Y \cap \text{Cn}(X) \neq \emptyset$;
  2. $X \Vdash Y \implies Y \subseteq \text{Cn}(X)$.

Thus, the relation $\vdash$ holds between X and Y if some of Y are among the consequences of X, and the relation $\Vdash$ holds if all of Y is contained in the consequences of X. The relation $\vdash$ is familiar from Gentzen and subsequent work in multiple conclusion sequents (such as Smiley and Shoesmith (1978)).

Since we shall make frequent use of the two kinds of consequence relation, we take the time to emphasize that, according to the above definition, $X \nvdash Y$ is another way of saying that none of Y is a consequence of X and $X \nVdash Y$ means that Y is not contained in the consequences of X.

We shall economise a little in writing down elements of the consequence relations: for instance $X \cup Y \vdash Z \cup \{\alpha\}$ will be abbreviated to $X, Y \vdash Z, \alpha$, and similarly for $\Vdash$.

Note, first, that for singleton right-hand-sides the two relations coincide:

$$X \vdash \alpha \iff X \Vdash \alpha \quad (\iff \alpha \in \text{Cn}(X))$$

Note, second, that

$$X \Vdash Y \ \& \ Y \Vdash X \iff \text{Cn}(X) = \text{Cn}(Y)$$

However, from $X \vdash Y$ and $Y \vdash X$ ($X \dashv\vdash Y$) it does not in general follow that $\text{Cn}(X) = \text{Cn}(Y)$ – though the converse does of course hold as long as X and Y are non-empty.

The principal objects under investigation in this paper are certain change operations, in particular contraction operations. Formally, we consider operations f which take a theory, T, and a set of sentences, A, to a new theory, $f(T, A)$, the contraction of T by A. In the case of singleton sets to be retracted we write down the result of contracting a theory T by $\{\alpha\}$ as $T - \alpha$. Here then are

THE AGM POSTULATES FOR CONTRACTIONS

$$T = \mathrm{Cn}(T) \implies T - \alpha = \mathrm{Cn}(T - \alpha) \qquad \text{(closure)}$$

$$\emptyset \nvdash \alpha \implies \alpha \notin T - \alpha \qquad \text{(success)}$$

$$T - \alpha \subseteq T \qquad \text{(inclusion)}$$

$$\alpha \notin T \implies T \subseteq T - \alpha \qquad \text{(vacuity)}$$

$$\alpha \dashv\vdash \beta \implies T - \alpha = T - \beta \qquad \text{(extensionality)}$$

$$T \subseteq \mathrm{Cn}((T - \alpha) \cup \{\alpha\}) \qquad \text{(recovery)}$$

SUPPLEMENTARY POSTULATES

$$(T - \alpha) \cap (T - \beta) \subseteq T - (\alpha \wedge \beta) \qquad \text{(intersection)}$$

$$\alpha \notin T - (\alpha \wedge \beta) \implies T - (\alpha \wedge \beta) \subseteq T - \alpha \qquad \text{(conjunction)}$$

5. PACKAGE CONTRACTION AND CHOICE CONTRACTION

In this section, we shall search for suitable generalisations of the success postulate for contraction by singleton sets. As we shall see, such generalisations branch into two directions.

The purpose of the success postulate,

$$\alpha \notin T - \alpha, \qquad \text{unless } \emptyset \vdash \alpha \qquad \text{(success)}$$

is to ensure that the sentence to be removed from a theory should no longer be contained in the theory after the contraction has been performed. This

intuitive requirement needs some adjustment in the face of another postulate, i.e. closure:

$$T = \text{Cn}(T) \implies T - \alpha = \text{Cn}(T - \alpha) \qquad \text{(closure)}$$

– the result of contracting is not any old set of sentences but a set which is closed under logical consequence. Thus the closure of the empty set, $\text{Cn}(\emptyset)$, is a subset of any contraction result. Accordingly, consequences of the empty set (Logical Truths) are immune to effective retraction, as stated in the unless-clause of success.

In the sought generalisation of success the formula α must be replaced by a set A of formulae. Contracting a theory T by A should "remove" A from T. There is an evident ambiguity here: to remove A from T can mean either

– to let the intersection of T with A be empty, or
– to let A be no longer contained in T.

As argued above, both possibilities are interesting in their own right and need to be investigated.

To completely contract T by A is to remove all elements of A from T. We call this kind of removal operation *package* contraction and use the notation $T - [A]$. The tentative success postulate for package contractions is: $A \cap (T - [A]) = \emptyset$. (In Fuhrmann (1988) the term "meet" contraction was used instead. The new term avoids confusion with the concept of a partial meet contraction as it occurs in the writings of AGM.)

To contract T so that A is no longer contained in T it suffices to remove at least one of the elements of A from T. We shall refer to this kind of contraction as a *choice* contraction and use the notation $T - \langle A \rangle$. The tentative success postulate for choice contraction is: $A \not\subseteq T - \langle A \rangle$.

Sometimes it is convenient to remain ambiguous between package and choice contraction. We shall then speak of (multiple) contraction and use the (ambiguous) notation $T - A$. To aid browsing we shall frequently – but not always! – prefix the name of a postulate with a 'P' (for 'Package') or a 'C' (for 'Choice') even when the context leaves little doubt which kind of multiple contraction is under consideration.

Since we are concerned here with *theory* contraction, we will assume that multiple contractions, just as their singleton counterparts, should satisfy a closure condition:

$$T = \text{Cn}(T) \implies T - A = \text{Cn}(T - A)$$

This immediately calls for a qualification of the success conditions for multiple contractions. For, if A consists of theorems only, A cannot be choice-removed from any theory; and if A contains some theorem, then A cannot be

completely or package-removed from T. Hence,

$$\emptyset \not\Vdash A \implies A \not\subseteq T - \langle A \rangle \qquad\qquad \text{(Choice success)}$$

$$\emptyset \not\vdash A \implies A \cap (T - [A]) = \emptyset \qquad\qquad \text{(Package success)}$$

The condition that the set A must contain no theorems for a package contraction by A to be successful, may appear to be a very strong one. To illustrate somewhat drastically, suppose that A is very large but contains a single theorem. Then, for all we know so far, an application of package contraction may not remove any sentence from the original theory, simply because the condition under which success is guaranteed, does not obtain.

An intuitively more satisfactory variation on the theme of package contractions is a contraction operation which, as it were, tries to do its best: it removes all those sentences in A from a theory that can be removed. Let $\smile$ stand for such a contraction operation. An appropriate success postulate would be

$$(A \backslash \mathrm{Cn}(\emptyset)) \cap (T \smile A) = \emptyset$$

where $\backslash$ denotes set-theoretical subtraction. This kind of contraction appears to be well-positioned between choice and package contractions. But it is easily defined in terms of the latter by putting

$$T \smile A := T - [A \backslash \mathrm{Cn}(\emptyset)]$$

Some consideration has to be given to contraction by the empty set. The unconditional success postulate for choice contraction would be false for contractions by the empty set. But the unless-clause makes the postulate hold for $\emptyset$-contractions by falsity of the antecedent. Trivially the success condition for package contraction holds in the case of the empty set. In other words, the success conditions do allow for contractions by the empty set. However, in that limiting case, these conditions convey no information specific to the notion of contraction.

6. BASIC PROPERTIES OF MULTIPLE CONTRACTION

AGM's list of six basic and two supplementary postulates for singleton contraction has played a central rôle in studies of singleton contractions. In the last section, we generalised two of these postulates, closure and success, to multiple contraction. In this section, we shall discuss the generalisation of the remaining four basic AGM postulates and some closely related properties.

According to the *inclusion* condition, the result of a singleton contraction should be a subset of the original set:

$$T - \alpha \subseteq T$$

The basic intuition behind inclusion is that contraction is "pure" in the sense of not involving the acquisition of any new item of belief. Although it may be maintained that pure contraction in this sense is not common as an isolated phenomenon, it may occur as part of more complex changes in belief. In this sense, pure contraction is at least a useful logical abstraction – like material implication – and an important tool in the analysis of belief change. This holds true for multiple just as well as for singleton contraction.

The appropriate generalisation of inclusion is obvious and it is the same for both choice and package contraction:

$$T - A \subseteq T \qquad\qquad\qquad \text{(P/C-inclusion)}$$

According to AGM's *vacuity* postulate, the contraction by something that was not in the original belief set is an idle operation. In other words, if what should be achieved by the contraction has already been achieved, then the operation of contraction is vacuous. If you do not believe that London is the capital of France, then the contraction of your belief set by that belief involves no change at all. In general we should require that

$$\alpha \notin T \Longrightarrow T = T - \alpha$$

One half of the consequent holds already unconditionally by inclusion; the other half is the vacuity condition. For package contraction, the corresponding principle should come into force only for sets that are completely disjoint from T. Only this case completely anticipates the effect of the proposed contraction. To illustrate this, suppose that you do not believe that London is the capital of France but believe that Berlin is the capital of Germany. Then the contraction of your belief set by the set containing these two beliefs should be a real change which removes the last-mentioned belief from your set of beliefs. If, on the other hand, you entertained neither of these beliefs, then the contraction is vacuous. Thus we have the following vacuity condition for package contraction:

$$A \cap T = \emptyset \Longrightarrow T = T - [A] \qquad\qquad \text{(Package vacuity)}$$

With choice contraction, the intuition behind vacuity gives rise to a different principle. A choice contraction aims at removing at least one element of the set to be contracted. Therefore, for vacuity to come into force, it suffices that there is one element of A that is not an element of T. Thus,

$$A \nsubseteq T \Longrightarrow T = T - \langle A \rangle \qquad\qquad \text{(Choice vacuity)}$$

The next among Gärdenfors' basic postulates is *extensionality*. This postulate ensures that the contraction of a theory by two logically equivalent sentences yields the same result:

$$\alpha \dashv\vdash \beta \Longrightarrow T - \alpha = T - \beta.$$

The most immediate generalisation of this postulate to multiple contraction would be to require that a contraction by logically equivalent sets yields the same result, i.e.:

$$\mathrm{Cn}(A) = \mathrm{Cn}(B) \Longrightarrow T - A = T - B \qquad (*)$$

The principle will be shown correct for choice contractions, i.e.

$$\mathrm{Cn}(A) = \mathrm{Cn}(B) \Longrightarrow T - \langle A \rangle = T - \langle B \rangle \qquad \text{(Choice extensionality)}$$

But (*) is not the right kind of extensionality condition for package contractions. For, $\mathrm{Cn}(p \wedge q) = \mathrm{Cn}(p \wedge q, p)$. Yet, if we put $T = \mathrm{Cn}(p)$ we have $T - [p \wedge q] = T$ by vacuity whilst $T - [p \wedge q, p] \neq T$ by success. Instead of (*) we shall consider two further generalisations of singleton extensionality:

$$A \cong B \Longrightarrow T - [A] = T - [B] \qquad \text{(Package extensionality)}$$

where $A \cong B$ represents the property that for every element of A there is a logically equivalent element of B, and vice versa. Furthermore:

$$A \equiv_T B \Longleftrightarrow T - [A] = T - [B] \qquad \text{(Package uniformity)}$$

where $\equiv_T$ represents a relation of equivalence-according-to-T:

$$A \equiv_T B \Longleftrightarrow \forall X \subseteq T : X \vdash A \Longleftrightarrow X \vdash B$$

The uniformity condition has first been formulated in Hansson (1992a). In general P-extensionality entails P-uniformity. Note that as far as contraction by singletons is concerned, the converse holds also, since $\alpha \dashv\vdash \beta \Longrightarrow \alpha \equiv_T \beta$.

The most controversial among the six basic Gärdenfors postulates is that of *recovery*. According to this postulate, if a removed sentence is reinserted into the contracted belief set, then the original belief set is recreated, or, more precisely, can be recreated by logical closure:

$$T \subseteq \mathrm{Cn}((T - \alpha) \cup \{\alpha\})$$

The plausibility of recovery – especially in situations where $T \neq \mathrm{Cn}(T)$ – has been questioned by several authors (e.g. Makinson (1987), Fuhrmann (1991), Hansson (1991), Levi (1991)). However, without this postulate the remaining

AGM postulates do not suffice to achieve minimality of belief change. Whereas the postulate of inclusion precludes the addition of new sentences, and that of vacuity precludes subtractions in the vacuous limiting case, recovery is the only one among the AGM postulates that prevents unmotivated subtractions in the general case. It does so by ensuring that incisions into a theory are so small that contractions can be undone by simply adding the removed sentence. Indeed, without recovery the other five postulates are compatible with an operation such that if $\alpha \in T$, then $T - \alpha = \text{Cn}(\emptyset)$; see Hansson (1991). The following is a straightforward generalisation of singleton recovery; we formulate it for both choice and package contraction:

$$T \subseteq \text{Cn}((T - A) \cup A) \qquad\qquad \text{(P/C-recovery)}$$

The following weakening of recovery will turn out to be useful:

$$\text{If } A \text{ is finite, then } T \subseteq \text{Cn}((T - A) \cup A) \qquad \text{(finite P/C-recovery)}$$

The intuitive idea behind recovery is perhaps too strong, but nevertheless we need some postulate that imposes informational economy. Another condition that, like recovery, requires contraction to treasure information is the postulate of *relevance*. Its singleton version was introduced in Hansson (1991) to capture the intuition that in contracting one should not remove items without reason. That is to say, whatever is being removed from a belief set in the course of a contraction does in some way contribute to entailing the sentence to be retracted. Relevance for singleton contraction is defined as follows:

If $\beta \in T \backslash (T - \alpha)$, then there is some set T' such that
  (a) $T - \alpha \subseteq T' \subseteq T$,
  (b) $T' \not\vdash \alpha$ and
  (c) $T', \beta \vdash \alpha$ (relevance)

One might have hoped that relevance should characterise a different set of operations than the apparently stronger principle of recovery. However, as the following two observations will show, the singleton version of relevance and recovery are interchangeable in the presence of the other basic AGM postulates. (This result does not hold if the theory T is replaced by an arbitrary set; cf. Hansson (1991).)

OBSERVATION 2.
  1. *If the operation – satisfies closure, inclusion, vacuity, and recovery for singletons, then it satisfies relevance for singletons.*
  2. *If the operation – satisfies relevance for singletons, then it satisfies recovery for singletons.*

Proof. Ad 1. If $\alpha \notin T$, then $T = T - \alpha$ by vacuity and inclusion, so that relevance is vacuously satisfied. In the principal case, when $\alpha \in T$, let $\beta \in T\backslash(T - \alpha)$ and let $T' = (T - \alpha) \cup \{\beta \to \alpha\}$. By the logical closure of T, $\beta \to \alpha \in T$. By the inclusion postulate, $T - \alpha \subseteq T$. It follows that clause (a) of the definition of relevance is satisfied. Clearly, (c) is also satisfied. To see that (b) is satisfied, suppose to the contrary that $T' \vdash \alpha$. Then it follows by deduction equivalence that $T - \alpha \vdash (\beta \to \alpha) \to \alpha$, i.e. $T - \alpha \vdash \alpha \vee \beta$. By recovery and again deduction equivalence, $T - \alpha \vdash \alpha \to \beta$. We may conclude that $T - \alpha \vdash \beta$. However, it follows from $\beta \notin T - \alpha$ and closure that $T - \alpha \nvdash \beta$, contrary to our assumptions. This contradiction concludes the proof.

Ad 2. Let T be logically closed and let $-$ be an operation on T that satsfies relevance for singletons. Suppose that recovery is violated, so that there are α and β such that $\beta \in T$ and $\beta \notin \mathrm{Cn}((T - \alpha) \cup \{\alpha\})$. Then $\alpha \to \beta \notin T - \alpha$. By the logical closure of T, it follows from $\beta \in T$ that $\alpha \to \beta \in T$. By relevance, there is some T' such that $T - \alpha \subseteq T' \subseteq T$, $T' \nvdash \alpha$, and $T', \alpha \to \beta \vdash \alpha$. However, $T', \alpha \to \beta \vdash \alpha$ is equivalent to $T' \vdash (\alpha \to \beta) \to \alpha$, i.e., $T' \vdash \alpha$. This contradiction concludes the proof. □

The generalisation of singleton relevance to package and choice contractions is straightforward:

If $\beta \in T\backslash(T - [A])$, then there is some T' such that
(a) $T - [A] \subseteq T' \subseteq T$,
(b) $T' \nvdash A$ and
(c) $T', \beta \vdash A$ (Package relevance)

If $\beta \in T\backslash(T - \langle A \rangle)$, then there is some T' such that
(a) $T - \langle A \rangle \subseteq T' \subseteq T$,
(b) $T' \nmid\vdash A$ and
(c) $T', \beta \mid\vdash A$ (Choice relevance)

The next property, *failure*, also follows from the basic AGM postulates. It follows from closure and recovery and, hence, from closure and relevance:

$$\emptyset \vdash \alpha \implies T \subseteq T - \alpha$$

As was indicated in the previous section, if A contains at least one theorem, then A cannot be completely removed from T. Thus, failure should be generalised to package contraction as follows:

$$\emptyset \vdash A \implies T \subseteq T - [A] \qquad\qquad \text{(Package failure)}$$

For choice contraction, a generalisation of failure must be based on the observation that if A consists of theorems only, then it cannot be choice-removed. Thus,

$$\emptyset \Vdash A \implies T \subseteq T - \langle A \rangle \qquad \text{(Choice failure)}$$

Failure will play an important rôle in the representation theorems for multiple contractions to be proved below.

We close this section by recording some of the entailment relations holding between groups of postulates. These facts will be used later when proving representation results for package and choice contractions.

LEMMA 3.
1. *P-inclusion and P-relevance imply P-closure, $T - [A] = \text{Cn}(T - [A])$.*
2. *P-relevance implies P-failure, $\emptyset \vdash A \implies T \subseteq T - [A]$.*

Proof. Ad 1. Suppose (1) $T = \text{Cn}(T)$ and (2) $T - [A] \vdash \alpha$ while $\alpha \notin T - [A]$. From (2) we infer by inclusion and weakening that $T \vdash \alpha$ whence $\alpha \in T$ by (1). So $\alpha \in T \backslash T - [A]$. We may thus apply relevance to infer that there exists some set S such that

$$T - [A] \subseteq S \subseteq T \ \& \ S \not\vdash A \ \& \ S, \alpha \vdash A$$

From (2) and the third conjunct we obtain by cut that $S, T - [A] \vdash A$. It follows from the first conjunct that $S \vdash A$ – contrary to the second conjunct.

Ad 2. Assume relevance and $\emptyset \vdash A$ while $T \not\subseteq T - [A]$. Then for some sentence α, $\alpha \in T \backslash T - [A]$. It follows by relevance that there is some set S with the property $S \not\vdash A$ – contrary to $\emptyset \vdash A$. $\square$

LEMMA 4.
1. *C-inclusion and C-relevance imply C-closure, $\text{Cn}(T - \langle A \rangle) = T - \langle A \rangle$.*
2. *C-relevance implies C-failure: $\emptyset \Vdash A \implies T \subseteq T - \langle A \rangle$.*

Proof. As for Lemma 3. $\square$

7. THE SUPPLEMENTARY POSTULATES

In this section we shall briefly discuss the so-called supplementary postulates for contractions. These postulates will not be treated in the following sections where we propose models of multiple change. We mention them here only for the sake of completeness and for future reference. In a sequel to this paper, we intend to extend the modelling techniques of the following sections to encompass the supplementary postulates.

$$(T - \alpha) \cap (T - \beta) \subseteq T - (\alpha \wedge \beta) \qquad \text{(intersection)}$$

$$\alpha \notin T - (\alpha \wedge \beta) \Longrightarrow T - (\alpha \wedge \beta) \subseteq T - \alpha \qquad \text{(conjunction)}$$

These postulates are called 'supplementary' because, unlike the other ('basic') postulates they make reference to the logical structure of sentences. Abstracting for a moment from the specific structure referred to, the supplementary postulates make assertions about how the contraction by a complex item relates to contractions by its components. In the singleton case the complex and its components are a conjunction and its conjuncts respectively. To remove a conjunction from a theory, it suffices to choose and remove one of the conjuncts. The supplementary postulates give some guidance as to how a contraction operation ought to behave in such a choice situation. In this sense the postulates may be seen as an attempt at a theory of finitary multiple contractions, to be more precise: of finitary choice contractions, as we shall see in a moment. It should thus come as no surprise that in a suitable generalisation of the supplementary postulates the reference to sentence structure vanishes and, thus, these postulates lose their "special" character.

To begin with package contraction, note that contraction by some sentence α also involves the removal of $\alpha \wedge \beta$. Since $\alpha \wedge \beta$ is removed when α is removed, we may assume that package contraction by α is identical to package contraction by $\{\alpha, \alpha \wedge \beta\}$. Similarly, package contraction by β should be identical to package contraction by $\{\beta, \alpha \wedge \beta\}$. We can then rewrite intersection as follows:

$$(T - [\alpha, \alpha \wedge \beta]) \cap (T - [\beta, \alpha \wedge \beta]) \subseteq T - [\alpha \wedge \beta]$$

This may be generalised to:

$$(T - [A]) \cap (T - [B]) \subseteq T - [A \cap B]$$

Similarly, the postulate of conjunction can be rewritten as follows:

$$\alpha \notin T - [\alpha \wedge \beta] \Longrightarrow T - [\alpha \wedge \beta] \subseteq T - [\alpha, \alpha \wedge \beta]$$

This can be generalised to the following two properties. (The first of these was introduced under the name of non-deterioration in Hansson (1992b)):

$$\alpha \notin T - [B] \Longrightarrow T - [B] \subseteq T - [B \cup \{\alpha\}]$$
$$A \cap (T - [B]) = \emptyset \Longrightarrow T - [B] \subseteq T - [A \cup B]$$

For the generalisation of the supplementary postulates to choice contraction we propose the use of a quite different principle. Choice contraction by the union of two sets corresponds to singleton contraction by the conjunction of two sentences in the sense that it is sufficient for success to remove one of

the sets (one of the sentences). Therefore, the supplementary postulates seem best to be generalised to conditions on choice contractions as follows:

$$T - \langle A \rangle \cap T - \langle B \rangle \subseteq T - \langle A \cup B \rangle$$
$$A \not\subseteq T - \langle \cup B \rangle \implies T - \langle A \cup B \rangle \subseteq T - \langle A \rangle$$

We leave the subject of the supplementary postulates with these very tentative generalisations. In what follows we confine attention on the basic properties of contractions as presented in the last two sections.

8. TOOLS FOR CONSTRUCTING MULTIPLE CONTRACTION

So far we have rested content with an indirect characterisation of contraction operations on theories: we have discussed a number of conditions on such functions and we have generalised the AGM set of basic postulates to multiple contraction operations. We shall now attempt more direct characterisations: we shall give recipes for how to construct multiply contracted theories. The postulates will be used as integrity constraints on such constructions in two ways. First, the constructed functions will be required to satisfy the postulates; and, second, every operation satisfying the postulates will have to be definable by means of the proposed construction. Accordingly, the main technical results in the next sections are suitable representation theorems for multiple contractions.

In this section, we shall introduce the formal tools that will be used below to construct models of package and choice contraction.

In the AGM tradition, the most important definition of singleton contraction is that of *partial meet* (p.m.) contraction; AGM (1985). The basic idea is that one should try to lose as little information as possible when contracting a belief set. Suppose we are to contract a theory T by some sentence α. As a first approximation towards contracting without incurring loss of information beyond necessity, we may restrict attention to the *maximal* subsets of T that do not entail α. Call such subsets of T *remainders* and let $T \perp \alpha$ be the set of remainders (of T after removing α). There are many such remainders; in fact, there are too many remainders to let their intersection (so-called full meet contraction) be a viable candidate for the contraction of T by α. On the other hand, picking an arbitrary remainder brings in an element of gambling where rational choice is asked for. Besides, remainders are in a way "too large" to qualify as candidates for the contracted theory, as in so-called maxichoice contraction; see AGM (1985) for some negative results. To our logical apparatus we need to add the brute assumption that among a collection of alternative remainders we can somehow pick those that are, in some sense, the most preferred ones in that collection. Note that it is not assumed that the

choice can always be narrowed down to uniqueness: there may be more than one most valuable remainder. Given that we have revealed our preferences by choosing a set of remainders we define $T - \alpha$ to be the set of sentences that are common to all preferred remainders; i.e.

$$T - \alpha = \bigcap s(T \perp \alpha)$$

where s is a mapping (a selection function) from a non-empty class of theories (the remainders) into a non-empty subset of that class (the preferred remainders). We need to take care of the special case when $T \perp \alpha$ is empty. This happens just when α cannot be removed because it is a logical truth. In that case we simply put $s(T \perp \alpha) = \{T\}$ whence $T - \alpha = T$.

There are at least two plausible ways to generalise the remainder operation $(\ ) \perp (\ )$ to operate on two sets instead of one set and one sentence. Indeed, in the paper in which the $\perp$-notation was first introduced (Alchourrón and Makinson (1981), p. 128), it was defined as an operation on two sets, with $T \perp A$ denoting the set of maximal subsets of T that do not *overlap* with A.

DEFINITION 5 (Package remainders). $X \in T \perp A$ *if and only if*
  (a) $X \subseteq T$,
  (b) $X \not\vdash A$, *and*
  (c) $\forall Y : X \subset Y \subseteq T \Longrightarrow Y \vdash A$.

But just as the theory of multiple contractions naturally splits into a choice and a package branch, so there is a further natural generalisation of the AGM remainder operation. We may want to consider the set of maximal subsets of T that do not *contain* the set A:

DEFINITION 6 (Choice remainders). $X \in T \angle A$ *if and only if*
  (a) $X \subseteq T$,
  (b) $X \not\Vdash A$, *and*
  (c) $\forall Y : X \subset Y \subseteq T \Longrightarrow Y \Vdash A$.
    As will be seen in a moment, $\perp$ is a suitable basis for constructing package contraction and $\angle$ plays the same rôle for choice contraction.

Finally we introduce a function which chooses "preferred" remainders.

DEFINITION 7. *A selection function (for a theory T) is any function*

$$s_T : \wp(\wp(T)) \rightarrow \wp(\wp(T))$$

such that $\emptyset \subset s_T(X) \subseteq X$ for all $X \neq \emptyset$, and $s_T(X) = \{T\}$ otherwise. (In the sequel we omit subscripts to s wherever convenient.)

9. CONSTRUCTING PACKAGE CONTRACTIONS

Only a slight alteration of the AGM definition of partial meet contractions is needed to cover the case of package contraction:

DEFINITION 8. *An operation* $[-] : \wp(Fml) \times \wp(Fml) \rightarrow \wp(Fml)$ *is a* $\perp$-*based partial meet* ($\perp$-*pm*) *contraction if and only if for each theory* T *there exists a selection function* s_T *such that*

$$T - [A] = \bigcap s(T \perp A)$$

THEOREM 9. *An operation* $[-]$ *is a* $\perp$-*pm contraction if and only if it satisfies the following conditions for each theory* T *and sets* A *and* B.

$$T - [A] \subseteq T \qquad\qquad\qquad \text{(P-inclusion)}$$

$$\emptyset \not\vdash A \Longrightarrow A \cap (T - [A]) = \emptyset \qquad\qquad \text{(P-success)}$$

$$A \equiv_T B \Longrightarrow T - [A] = T - [B] \qquad\qquad \text{(P-uniformity)}$$

$$\alpha \in T - [A] \Longrightarrow \qquad\qquad\qquad \text{(P-relevance)}$$
$$\exists S : T - [A] \subseteq S \subseteq T \& S \not\vdash A \& S, \alpha \vdash A$$

Proof. ($\Longrightarrow$) inclusion: If a sentence is contained in all selected remainders, it must be contained in T.

For success assume that $\emptyset \not\vdash A$. Then $T \perp A \neq \emptyset$ whence $s(T \perp A) \subseteq T \perp A$, by the definition of s. But for all $T' \in T \perp A$ we have $T' \cap A = \emptyset$. So, since $\bigcap s(T \perp A) \subseteq T'$ for at least one $T' \in T \perp A$, $\bigcap s(T \perp A) \cap A = \emptyset$.

For uniformity assume $A \equiv_T B$, i.e.

$$\forall X \subseteq T : X \vdash A \Longleftrightarrow X \vdash B \qquad\qquad (*)$$

It will suffice to show that $T \perp A = T \perp B$. Thus, assume that $X \in T \perp A$, i.e. (a) $X \subseteq T$, (b) $X \not\vdash A$, and (c) $\forall Y : X \subset Y \subseteq T \Longrightarrow Y \vdash A$. Then, given (*), we may replace A by B in (b) and (c). Hence, $X \in T \perp B$.

For relevance assume

(1) $\alpha \in T$ and (2) $\alpha \notin T - [A]$

It follows from (2) that there is some $S \in s(T \perp A)$ with $\alpha \notin S$. Since we have assumed that $\alpha \in T$ (1) and since S is maximal, $S, \alpha \vdash A$. Moreover, since $S \in s(T \perp A), \bigcap s(T \perp A) \subseteq S \subseteq T$, i.e. $T - [A] \subseteq S \subseteq T$.

($\Longleftarrow$) For each theory T we define a function s_T such that

$$s(T \perp A) = \begin{cases} \{T' \in T \perp A : T - [A] \subseteq T'\} & \text{if } T \perp a \neq \emptyset \\ \{T\} & \text{otherwise} \end{cases}$$

We need to show
(1) s is well-defined, i.e. $T \perp A = T \perp B \Longrightarrow s(T \perp A) = s(T \perp B)$;
(2) $s(T \perp A) = \{T\}$ if $T \perp A = \emptyset$, which is immediate from the definition;
(3) $s(T \perp A) \subseteq T \perp A$ if $T \perp A \neq \emptyset$, which is likewise immediate from the definition;
(4) $s(T \perp A) \neq \emptyset$; and
(5) $T - [A] = \bigcap s(T \perp A)$.
   For (1) assume that $T \perp A = T \perp B$. We first show that

$$A \equiv_T B \tag{*}$$

Suppose $X \subseteq T$ and $X \nvdash B$. Then there exists a set $X' \supseteq X$ such that $X' \in T \perp B$ whence $X' \in T \perp A$ by our assumption. So $X' \nvdash A$ and since $X \subseteq X'$, $X \nvdash A$. We have thus shown that $\forall X \subseteq T : X \vdash A \Longrightarrow X \vdash B$. The converse follows similarly.
   Next assume the principal case $T \perp A \neq \emptyset$ – otherwise the required conclusion follows trivially. It follows that

$$X \in s(T \perp A) \Longrightarrow T - [A] \subseteq X \tag{**}$$

Suppose that $X \in s(T \perp A)$. To show that $X \in s(T \perp B)$ (in the principal case $T \perp B \neq \emptyset$, which holds by assumption) it is sufficient to show that $T - [B] \subseteq X$. We may detach the consequent from (**): $T - [A] \subseteq X$. From (*) it follows by uniformity that $T - [A] = T - [B]$ whence $T - [B] \subseteq X$ as required.
   (4) is trivial whenever $T \perp A = \emptyset$. So assume $T \perp A \neq \emptyset$. Then $\emptyset \nvdash A$. Thus, by success, $A \cap (T - [A]) = \emptyset$. By closure then $T - [A] \nvdash A$. By inclusion, $T - [A] \subseteq T$. Thus, either $T - [A] \in T \perp A$ or there is some set S such that $T - [A] \subset S \subseteq T$ which is maximal w.r.t. the property of not entailing A, i.e. $S \in T \perp A$, as required.
   As to (5), the inclusion from left to right follows immediately from the definition of s. For the converse we first consider the case that $T \perp A = \emptyset$. Then the definition puts $\bigcap s(T \perp A) = T$. So we need to show that $T \subseteq T - [A]$. From the assumption $T \perp A = \emptyset$ it follows that $\emptyset \vdash A$. Hence, by failure, which follows from relevance by Lemma 3, $T \subseteq T - [A]$ as required.
   Next we consider the principal case: $T \perp A \neq \emptyset 0$. We need to show that for all formulae α, if

$$\alpha \notin T - [A] \tag{*}$$

then $\exists T' \in T \perp A$ such that $T - [A] \subseteq T'$ while $\alpha \notin T'$.

We may assume that

$$\alpha \in T \qquad\qquad\qquad\qquad\qquad (**)$$

For, if $\alpha \notin T$, then no remainder in $T \perp A$ will contain α whence the inclusion $\bigcap s(T \perp A) \subseteq T - [A]$ holds trivially. From (*) and (**) it follows by relevance that there is some S such that $T - [A] \subseteq S \subseteq T$ and $S \not\vdash A$ while $S, \alpha \vdash A$ whence $S \not\vdash \alpha$. Thus there must be a set $S' \in T \perp (\alpha, A)$ such that $S \subseteq S' \subseteq T$; hence, $T - [A] \subseteq S' \subseteq T$ and $\alpha \notin S'$. To show that $S' \in T \perp A$ we need to verify that S' excludes A maximally. Thus consider any subset X of T such that $S' \subset X$. We know that $X \vdash \alpha, A$ since $S \subset X$ and $S \in T \perp (\alpha, A)$. Moreover, we have $S, \alpha \vdash A$ and so, since $S \subset X$, $X, \alpha \vdash A$. It follows from $X \vdash \alpha, A$ and $X, \alpha \vdash A$ by (multiple conclusion) cut that $X \vdash A$, as required. $\square$

Some of the postulates discussed in Section 6 are not mentioned in the statement of the theorem. For example P-closure, P-failure and P-vacuity all follow essentially from P-relevance. This condition thus turns out to be rather powerful. Though slightly less transparent than P-recovery it is, on reflection, an intuitively much more convincing expression of the minimality maxim that should govern contraction. P-relevance is more easily understood if one observes that it is directly equivalent (given elementary assumptions about Cn) to the condition: if $\beta \in T \backslash (T - [A])$, then there is a $T' \in T \perp A$ with $T - [A] \subseteq T'$ and $\beta \notin T'$. In words: if β gets removed from T (in the course of retracting A), then some remainder (maximal consistent subset of T excluding A) cannot contain β.

Extensionality – the condition that logically equivalent sets determine the same contraction – which is used in the AGM characterisation of singleton p.m. contractions, does not hold for (multiple) $\perp$-pm contraction. But even in the case of singleton contraction there is a sense in which uniformity rather than extensionality should be the preferred postulate. For, it just so happens that in the singleton case extensionality can step in for uniformity. Extension to the multiple case, however, reveals that uniformity would have been the more informative postulate to use all along. This suggestion is based on two observations.

First, if A and B are singleton sets, say $\{\alpha\}$ and $\{\beta\}$, then uniformity implies extensionality, $\alpha \dashv\vdash \beta \implies T - [\alpha] = T - [\beta]$. Thus, extensionality for singletons is correct for $\perp$-based p.m. contractions. Second, it is trvial from the definition of $\equiv_T$ that if α and β are in T and $\alpha \equiv_T \beta$, then $a \dashv\vdash b$. It is this result that allows to use the weaker extensionality condition rather than uniformity in completing step (1) in the above proof when singleton

contractions only are at issue; cf. Observation 2.5 of AGM (1985) where the required step (1) is, as elsewhere in the AGM-literature, omitted.

We now come back to the question (posed in Section 3) whether package contraction can be reduced to singleton contraction, using the reduction schema

$$T - [\alpha_1, \ldots, \alpha_n] = T - \alpha_1 \cap \ldots \cap T - \alpha_n \qquad (1)$$

Note that (1) implies

$$A \subseteq B \Longrightarrow T - [B] \subseteq T - [A] \qquad (2)$$

Despite the initial attraction of (2), it does not tally with the partial meet modelling adopted here. Consider a set $X \in T \perp B$. Then $X \subseteq T$, $\mathrm{Cn}(X) \cap B = \emptyset$ (whence $\mathrm{Cn}(X) \cap A = \emptyset$), and for all X', if $X \subset X' \subseteq T$ then $X' \vdash B$. However, the sentences in B that can be derived from X' may not be contained in A; so $X' \not\vdash A$ whence $X \notin T \perp A$. Thus,

$$A \subseteq B \Longrightarrow T \perp A \subseteq T \perp B \qquad (3)$$

may fail and so may (2). It is easy to see that the possibility of failure is preserved when we restrict attention to cases where $A = \{\alpha\}$ and $B = \{\alpha, \beta\}$.

We do not know whether (full) P-recovery holds for multiple $\perp$-partial meet contraction. However, the weaker postulate of finite P-recovery (recovery from contraction by a finite set) can be shown to hold. Indeed, it follows from the postulate of P-relevance alone:

OBSERVATION 10. *If the operation* $[-]$ *satisfies P-relevance, then it satisfies finite P-recovery.*

Proof. Let T be a logically closed set and let $[-]$ be an operation on T that satisfies relevance. Suppose that finite recovery is not satisfied. There is then some finite set A and some sentence β such that $\beta \in T$ and $\beta \notin \mathrm{Cn}((T - [A]) \cup A)$. Letting $\bigwedge A$ denote the conjunction of all elements of A we then have $\bigwedge A \to \beta \notin T - [A]$. It follows by relevance that there is some T' such that $T - [A] \subseteq T' \subseteq T$, $T' \not\vdash A$ and $T' \cup \{\bigwedge A \to \beta\} \vdash A$. It follows from the last of these expressions that $T' \vdash (\bigwedge A \to \beta) \to \alpha$ for some $\alpha \in A$. It follows from $\bigwedge A \vdash \alpha$ that $(\bigwedge A \to \beta) \to \alpha$ is equivalent to α, so that $T' \vdash \alpha$ and thus $T' \vdash A$ contrary to the initial conditions for T'. We can conclude from this contradiction that finite recovery holds.   □

10. PACKAGE CONTRACTION: RELEVANCE BEYOND RECOVERY

Whereas singleton relevance and singleton recovery are interchangeable in the presence of the other basic postulates, finite package recovery is a distinctly

weaker property than package relevance. Indeed, it can be used, in conjunction with the other basic postulates, to characterize a wider group of contraction operations that do not in general satisfy package relevance. This result may appear somewhat surprising, since the intuitive principle behind the relevance postulates seems weaker than that behind the recovery postulates.

The more general group of operations is defined as follows:

DEFINITION 11.

1. *A set $D \subseteq \wp T$ is B-covering if and only if $D \cap (T \perp \beta) \neq \emptyset$ for all $\beta \in B$.*
2. *$T \triangle A = \bigcup \{T \perp \alpha : \alpha \in A\}$*
3. *A function σ is a covering function for T if and only if for all $A \subseteq T$: If $T \triangle A$ is A-covering, then $\sigma(A)$ is an A-covering subset of $T \triangle A$. Otherwise, $\sigma(A) = \{T\}$.*
4. *The operator $[-]$ for T is a $\triangle$-based p.m. contraction if and only if there is some covering function σ for T such that:*
   $$T - [A] = \bigcap \sigma(A)$$

Thus, in $\triangle$-based partial meet contraction, instead of selecting among elements of $T \perp A$, we select among elements of $T \perp \alpha$ for all $\alpha \in A$. The covering condition ensures that for each $\alpha \in A$, at least one element of $T \perp \alpha$ is selected, except in the limiting case when $T \perp \alpha$ is empty for some $\alpha \in A$ (or equivalently: when $A \cap \mathrm{Cn}(\emptyset) \neq \emptyset$).

The following observation shows that $\perp$-pm contractions are a proper case of $\triangle$-pm contractions.

OBSERVATION 12.

1. *If the operator $[-]$ for T is a $\perp$-pm contraction, then it is a $\triangle$-pm contraction.*
2. *It does not hold in general that if the operator $[-]$ for T is a $\triangle$-pm contraction, then it is a $\perp$-pm contraction.*

Proof. Ad 1. Clearly, for each $X \in T \perp A$ and each $\alpha \in A$ there is some Y such that $X \subseteq Y$ and $Y \in T \perp \alpha$. Therefore, it is sufficient for us to prove that if $X \in T \perp A$, then $X = \cap\{Y : (\exists \alpha \in A)(Y \in T \perp \alpha \ \& \ X \subseteq Y)\}$. Let $X \in T \perp A$. Then $X \subseteq \cap\{Y : (\exists \alpha \in A)(Y \in T \perp A \ \& \ X \subseteq Y)\}$ is obvious. To prove the converse inclusion let β be such that

$$\forall Y : (\exists \alpha \in A : Y \in T \perp \alpha \ \& \ X \subseteq Y) \implies \beta \in Y \tag{1}$$

Suppose for reductio that $\beta \notin X$. Clearly, $\beta \in T$. Since we assume that $X \in T \perp A$ it follows that $X, \beta \vdash A$ whence there must be some $\gamma \in A$ such

that

$$X, \beta \vdash \gamma \tag{2}$$

Consider now any $Z \in T \bot \gamma$ such that $X \subseteq Z$. It follows from (1) that

$$\beta \in Z \tag{3}$$

From (2) and $X \subseteq Z$ we may infer that

$$Z, \beta \vdash \gamma \tag{4}$$

But from (3) and (4) it follows (by cut) that $Z \vdash \gamma$, contrary to our assumption that $Z \in T \bot \gamma$.

Ad 2. Let T be a theory that contains the three logically independent elements α, β, and γ. Then there are sets Z_1 and Z_2 such that:

$$\{\alpha, \gamma \rightarrow \beta\} \subseteq Z_1 \in T \bot \beta$$
$$\{\alpha \vee \beta, \alpha \rightarrow \gamma\} \subseteq Z_2 \in T \bot \gamma$$

Let $[-]$ be a $\triangle$-based p.m. contraction that is generated by a covering function σ such that $\sigma(\{\beta, \gamma\}) = \{Z_1, Z_2\}$. We show that $[-]$ is not a $\bot$-based p.m. contraction. Clearly, both Z_1 and Z_2 are closed under logical consequence.

We are first going to show that $\gamma \rightarrow \beta \in Z_2$. Suppose not. Then, since $Z_2 \in T \bot \gamma$, we have $(\gamma \rightarrow \beta) \rightarrow \gamma \in Z_2$, i.e., $\gamma \in Z_2$, contrary to the conditions. Thus, $\gamma \rightarrow \beta \in Z_2$. Since $\alpha \in Z_1$ yields $\alpha \vee \beta \in Z_1$, we may conclude that $\{\alpha \vee \beta, \gamma \rightarrow \beta\} \subseteq Z_1 \cap Z_2 = T - \{\beta, \gamma\}$.

It also follows from $\alpha \rightarrow \gamma \in Z_2$ and $Z_2 \in T \bot \gamma$ that $\alpha \notin Z_2$, thus $\alpha \notin T - \{\beta, \gamma\}$.

Now suppose that $[-]$ is also a $\bot$-pm contraction. Then, since $\{\alpha \vee \beta, \gamma \rightarrow \beta\} \subseteq T - [\beta, \gamma]$ and $\alpha \notin T - [\beta, \gamma]$, there must be some X such that $\alpha \notin X \in T \bot \{\beta, \gamma\}$, and $\{\alpha \vee \beta, \gamma \rightarrow \beta\} \subseteq X$. It follows from $\alpha \notin X \in T \bot \{\beta, \gamma\}$ that either $X, \alpha \vdash \beta$ or $X, \alpha \vdash \gamma$. In both cases, $\{\alpha \vee \beta, \gamma \rightarrow \beta\} \subseteq X$ yields $\beta \in X$, contrary to the conditions. $\square$

THEOREM 13. *An operator $[-]$ for contractions of T by finite sets is an operator of $\triangle$-based p.m. contraction if and only if it satisfies the following postulates:*

$$T - [A] = \mathrm{Cn}(T - [A]) \qquad \text{(P-closure)}$$

$$T - [A] \subseteq T \qquad \text{(P-inclusion)}$$

$$T \not\vdash A \Longrightarrow T \subseteq T - [A] \hspace{4cm} \text{(P-vacuity)}$$

$$\emptyset \not\vdash A \Longrightarrow A \cap (T - [A]) = \emptyset \hspace{3cm} \text{(P-success)}$$

$$\emptyset \vdash A \Longrightarrow T \subseteq T - [A] \hspace{4cm} \text{(P-failure)}$$

$$T \subseteq \mathrm{Cn}((T - [A]) \cup A), \text{ if } A \text{ is finite} \hspace{1.5cm} \text{(finite P-recovery)}$$

Proof. For one direction of the proof, we have to show that the postulates are satisfied.

(P-closure): The intersection of a set of closed sets is itself closed.

(P-inclusion): Directly from the definition.

(P-vacuity): If $T \not\vdash A$, then $T \perp \alpha = \{T\}$ for all $\alpha \in A$.

(P-success): If $\emptyset \not\vdash A$, then it follows for each $\alpha \in A$ that $T \perp \alpha$ is non-empty.

(P-failure): If $\emptyset \vdash A$, then there is some $\alpha \in A$ such that $T \perp \alpha = \emptyset$. It follows that $T \triangle A$ is not A-covering, and thus, $\sigma(A) = \{T\}$.

(Finite P-recovery): If $T \triangle A$ is not A-covering, then $\cap \sigma(A) = T$, from which (finite P-recovery) follows directly. For the principal case, when $T \triangle A$ is A-covering, let $\varepsilon \in T$. We show that $\varepsilon \in \mathrm{Cn}((T - [A]) \cup A)$. Let $Z \in \sigma(A)$. There is then some $\alpha \in A$ such that $Z \in T \perp \alpha$.

Let $\bigwedge A$ denote the conjunction of the elements of A. We prove that $(\bigwedge A \to \varepsilon) \in Z$. Suppose to the contrary that $(\bigwedge A \to \varepsilon) \notin Z$. By $\varepsilon \in T$ and the logical closure of T we have $(\bigwedge A \to \varepsilon) \in T$. It follows from this and $(\bigwedge A \to \varepsilon) \notin Z \in T \perp \alpha$ that $\alpha \in \mathrm{Cn}(Z \cup \{\bigwedge A \to \varepsilon\})$. It follows that $(\bigwedge A \to \varepsilon) \to \alpha \in \mathrm{Cn}(Z)$. Since $\bigwedge A \vdash \alpha$, $(\bigwedge A \to \varepsilon) \to \alpha$ is equivalent to α. We therefore have $\alpha \in \mathrm{Cn}(Z)$, contrary to $Z \in T \perp \alpha$. From this contradiction we may conclude that $(\bigwedge A \to \varepsilon) \in Z$. Since this holds for all $Z \in \sigma(A)$, we can conclude that finite P-recovery holds.

For the other direction of the proof, let $[-]$ be an operator that satisfies the postulates listed in the theorem. We use the function σ, defined as follows:

$$\sigma(B) = \begin{cases} \{Z \in T \triangle B : T - [B] \subseteq Z\} & \text{if } T \triangle B \text{ is } B\text{-covering} \\ \{T\} & \text{otherwise} \end{cases}$$

We need to show that

(1) σ is a covering function for T.

(2) $T - [B] = \bigcap \sigma(B)$.

Ad 1. It is sufficient to show that if $T \triangle B$ is B-covering and $\alpha \in B$, then there is some Z such that $T - [B] \subseteq Z \in T \perp \alpha$. By P-success and P-closure,

$T - [B] \not\vdash \alpha$ and by P-inclusion $T - [B] \subseteq T$. The desired conclusion follows directly.

Ad 2. We first show that $T - [B] \subseteq \bigcap \sigma(B)$. If $T \triangle B$ is not B-covering, then $\bigcap \sigma(B) = T$, so that the desired conclusion follows by P-inclusion. If $T \triangle B$ is B-covering, then $B \cap \text{Cn}(\emptyset) = \emptyset$. By P-success, $B \cap (T - [B]) = \emptyset$. It follows by P-closure and the construction of σ that $T - [B] \subseteq \bigcap \sigma(B)$.

For the proof of $\bigcap \sigma(B) \subseteq (T - [B])$ it should first be noted that this follows by P-vacuity if $B \cap \text{Cn}(T) = \emptyset$ and by P-failure if $B \cap \text{Cn}(\emptyset) \neq \emptyset$. In the remaining case, let $\alpha \notin T - [B]$. It follows from P-closure that $\alpha \notin \text{Cn}(T - [B])$. We need to show that $\alpha \notin \bigcap \sigma(B)$. Since this is obvious if $\alpha \notin T$ we may assume that $\alpha \in T$.

It follows from (finite P-recovery) that $\alpha \in \text{Cn}((T - [B]) \cup B)$ and thus, letting $\bigwedge B$ denote the conjunction of the elements of B, $\bigwedge B \rightarrow \alpha \in \text{Cn}(T - [B])$. It follows from this and $\alpha \notin \text{Cn}(T - [B])$ that it cannot hold for all $\beta \in B$ that $\alpha \vee \beta \in \text{Cn}(T - [B])$. Let β be an element of B such that $\alpha \vee \beta \notin \text{Cn}(T - [B])$.

It follows that there is some X such that $T - [B] \subseteq X \in T \bot (\alpha \vee \beta)$. It follows (Lemma 2.4 of AGM 1985) that $X \in T \bot \beta$ and thus $X \in T \triangle B$. By the definition of σ, $X \in \sigma(B)$. It follows from $X \in T \bot (\alpha \vee \beta)$ that $\alpha \notin \text{Cn}(X)$ and thus $\alpha \notin \bigcap \sigma(B)$. This concludes the proof. $\square$

11. CONSTRUCTING CHOICE CONTRACTIONS

Choice contractions can be characterised in a way that corresponds closely to what was obtained for the $\bot$-operation.

DEFINITION 14. *An operation* $\langle - \rangle : \wp(Fml) \times \wp(Fml) \rightarrow \wp(Fml)$ *is an* $\angle$-*based partial meet* ($\angle$-pm) *contraction if and only if for each theory* T *there exists a selection function* s_T *such that*

$$T - \langle A \rangle = \bigcap s(T \angle A)$$

THEOREM 15. *An operation* $\langle - \rangle$ *is an* $\angle$-pm *contraction if and only if it satisfies the following conditions for each theory* T *and sets* A *and* B:

$T - \langle A \rangle \subseteq T$ \hfill (C-inclusion)

$\emptyset \not\vdash A \Longrightarrow A \not\subseteq T - \langle A \rangle$ \hfill (C-success)

$\text{Cn}(A) = \text{Cn}(B) \Longrightarrow T - \langle A \rangle = T - \langle B \rangle$ \hfill (C-extensionality)

$$\alpha \in T \backslash T - \langle A \rangle \langle \Longrightarrow \qquad \text{(C-relevance)}$$
$$\exists S : T - \langle A \rangle \subseteq S \subseteq T \ \& \ S \not\Vdash A \ \& \ S, \alpha \Vdash A$$

Proof. The argument is almost completely parallel to that for $\bot$-based p.m. contraction. For the proof of sufficiency we supply the verification of C-extensionality. We assume that (*) $\mathrm{Cn}(A) = \mathrm{Cn}(B)$, and show that $T \angle A = T \angle B$. Suppose that $X \in T \angle A$, i.e. (a) $X \subseteq T$, (b) $X \not\Vdash A$, and (c) $\forall Y : X \subset Y \subseteq T \Longrightarrow Y \Vdash A$. It will suffice to show that A may be replaced by B in (b) and (c). Suppose for reductio that $X \Vdash B$, i.e. $B \subseteq \mathrm{Cn}(X)$. Then, using (*), we have $A \subseteq \mathrm{Cn}(A) \subseteq \mathrm{Cn}(B) \subseteq \mathrm{Cn}(X)$ whence $X \Vdash A$ contrary to (b); so $X \not\Vdash B$. Next consider any set Y such that $X \subset Y \subseteq T$. It follows from (c) that $A \subseteq \mathrm{Cn}(Y)$ whence $\mathrm{Cn}(A) \subseteq \mathrm{Cn}(Y)$. Using (*) we obtain the following chain of inclusions: $B \subseteq \mathrm{Cn}(B) \subseteq \mathrm{Cn}(A)$ whence $B \subseteq \mathrm{Cn}(Y)$, i.e. $Y \Vdash B$ as required.

To prove the necessity direction of the theorem we define for each theory T a function s_T such that

$$S(T \angle A) = \begin{cases} \{T' \in T \angle A : T - \langle A \rangle \subseteq T'\} & \text{if } T \angle A \neq \emptyset \\ \{T\} & \text{otherwise} \end{cases}$$

We need to show
  (1) s is well-defined, i.e. $T \angle A = T \angle B \Longrightarrow s(T \angle A) = s(T \angle B)$;
  (2) $s(T \angle A) = \{T\}$ if $T \angle A = \emptyset$, which is immediate from the definition;
  (3) $s(T \angle A) \subseteq T \angle A$ if $T \angle A \neq \emptyset$, which is likewise immediate from the definition;
  (4) $s(T \angle A) \neq \emptyset$; and
  (5) $T - \langle A \rangle = \bigcap s(T \angle A)$.

For (1) assume that $T \angle A = T \angle B$. The sought conclusion is immediate in the cases when $T \angle A = \emptyset$ or $T \angle A = \{T\}$. In the remaining case $\emptyset \neq T \angle A \neq \{T\}$ we have $A \subseteq T$ and $B \subseteq T$. We first show that

$$\mathrm{Cn}(A) = \mathrm{Cn}(B) \qquad (*)$$

Note that $\forall X \subseteq T : X \Vdash A \Longleftrightarrow X \Vdash B$ (**). For, suppose for reductio that there is some $X \subseteq T$ such that $X \Vdash A$ and yet $X \not\Vdash B$. Then there exists a set $X' \supseteq X$ such that $X' \in T \angle B$. So, by our assumption, $X' \in T \angle A$ whence $X' \not\Vdash A$. But then $X \not\Vdash A$ since X is no bigger than $X' -$ contradiction. This proves one direction of (**); the converse follows similarly. Since $A, B \subseteq T$, we may infer from (**) that $A \Vdash B$ and $B \Vdash A$ whence $\mathrm{Cn}(A) = \mathrm{Cn}(B)$. It follows by the definition of s that

$$X \in s(T \angle A) \Longrightarrow T - \langle A \rangle \subseteq X \qquad (***)$$

Suppose that $X \in s(T \angle A)$. To show that $X \in s(T \angle B)$ it is sufficient to show that $T - \langle B \rangle \subseteq X$. From our assumptions we may conclude that $X \in T \angle B$ and from (***) that $T - \langle A \rangle \subseteq X$. From (*) it follows by C-extensionality that $T - \langle A \rangle = T - \langle B \rangle$; hence, $T - \langle B \rangle \subseteq X$. This proves half of what is required to show that s is well-defined; the other half follows in an exactly similar fashion.

(4) is trivial whenever $T \angle A = \emptyset$. So assume $T \angle A \neq \emptyset$. Then $\emptyset \not\vdash A$. Thus, by C-success, $A \not\subseteq T - \langle A \rangle$. By C-closure then $T - \langle A \rangle \not\vdash A$. By C-inclusion, $T - \langle A \rangle \subseteq T$. Thus, either $T - \langle A \rangle \in T \angle A$ or there is some set S such that $T - \langle A \rangle \subset S \subseteq T$ which is maximal w.r.t. the property of not entailing all of A, i.e. $S \in T \angle A$, as required.

As to (5), the inclusion from left to right follows by C-inclusion and C-success immediately from the definition of s. For the converse we first consider the case when $T \angle A = \emptyset$. Then the definition puts $\bigcap s(T \angle A) = T$. So we need to show that $T \subseteq T - \langle A \rangle$. From the assumption $T \angle A = \emptyset$ it follows that $\emptyset \| \vdash A$. Hence, by C-failure (Lemma 4), $T \subseteq T - \langle A \rangle$ as required.

Next we consider the principal case: $T \angle A \neq \emptyset$. We need to show that for all formulae α, if $\alpha \notin T - \langle A \rangle$, then $\exists T' \in T \angle A$ such that $T - \langle A \rangle \subseteq T'$ while $\alpha \notin T'$. Excluding a trivial case we may assume that $\alpha \in T$. From $\alpha \in T$ and $\alpha \notin T - \langle A \rangle$ it follows by C-relevance that there is some S such that $T - \langle A \rangle \subseteq S \subseteq T$ and $S \not\Vdash A$ while $S, \alpha \| \vdash A$, whence $S \not\vdash \alpha$. Thus there must be a set $S' \in T \angle A$ such that $S \subseteq S' \subseteq T$ and $\alpha \notin S'$; hence, $T - \langle A \rangle \subseteq S' \subseteq T$ and $\alpha \notin S'$. Thus S' has all the properties required for T'.   □

Partial meet contractions based on the choice-remainder operation $\angle$ satisfy also (C-vacuity) and (finite C-recovery). Contrary to what was found for $\perp$-pm contractions, (finite C-recovery) determines – in the presence of the other basic postulates – the same finite contractions as (C-relevance). In this respect, choice contraction departs less dramatically from singleton contraction than package contraction. As an alternative to what has been proved in the last theorem we thus have the following representation result.

THEOREM 16. *An operation $\langle - \rangle$ for contraction by finite sets is an $\angle$-pm contraction if and only if it satisfies the following conditions for each theory T and sets A and B.*

$$\mathrm{Cn}(T - \langle A \rangle) = T - A \qquad \text{(C-closure)}$$

$$T - \langle A \rangle \subseteq T \qquad \text{(C-inclusion)}$$

$$\emptyset \not\vdash A \Longrightarrow A \not\subseteq T - \langle A \rangle \qquad\qquad \text{(C-success)}$$

$$\emptyset \vdash A \Longrightarrow T \subseteq T - A \qquad\qquad \text{(C-failure)}$$

$$A \not\subseteq T \Longrightarrow T \subset T - \langle A \rangle \qquad\qquad \text{(C-vacuity)}$$

$$\text{Cn}(A) = \text{Cn}(B) \Longrightarrow T - \langle A \rangle = T - \langle B \rangle \qquad\qquad \text{(C-extensionality)}$$

$$T \subseteq \text{Cn}((T - \langle A \rangle) \cup A), \text{ if } A \text{ is finite} \qquad\qquad \text{(finite C-recovery)}$$

Proof. We prove this theorem from Theorem 15 by showing the following five relations between the postulates:
(1) C-relevance and C-inclusion imply C-closure.
(2) C-relevance implies C-failure.
(3) C-relevance implies C-vacuity.
(4) C-relevance implies finite C-recovery
(5) C-closure, C-inclusion, C-vacuity, and finite C-recovery imply C-relevance for finite sets.

For (1) and (2) see Lemma 2.

Ad 3. Let $B \not\subseteq T$. Suppose that $T \not\subseteq T - \langle B \rangle$. There is then some ε such that $\varepsilon \in T$ and $\varepsilon \notin T - \langle B \rangle$. It follows by C-relevance that there is some T' such that $T - \langle B \rangle \subseteq T' \subseteq T$, $B \not\subseteq \text{Cn}(T')$ and $B \subseteq \text{Cn}(T' \cup \{\alpha\})$. However, since $T' \cup \{\varepsilon\}$ is a subset of T, it follows from $B \not\subseteq T$ that $B \subseteq T' \cup \{\varepsilon\}$ cannot hold. We may conclude from this contradiction that $T \subseteq T - \langle B \rangle$, and consequently that C-vacuity holds.

Ad 4. Suppose that $\beta \notin \text{Cn}((T - \langle A \rangle) \cup A)$. We are going to show that $\beta \notin T$.

Let $\bigwedge A$ be the conjunction of the elements of A. In order to show that $\bigwedge A \to \beta \notin T$, suppose to the contrary that $\bigwedge A \to \beta \in T$. It follows from $\beta \notin \text{Cn}((T - \langle A \rangle) \cup A)$ that $\bigwedge A \to \beta \notin \text{Cn}(T - \langle A \rangle)$. We can therefore use C-relevance to conclude that there is some set T' such that $T - \langle A \rangle \subseteq T' \subseteq T$, $A \not\subseteq \text{Cn}(T')$, and $A \subseteq \text{Cn}(T' \cup \{\bigwedge A \to \beta\})$. It follows from the latter expression that $\bigwedge A \in \text{Cn}(T' \cup \{\bigwedge A \to \beta\})$ and thus $(\bigwedge A \to \beta) \to \bigwedge A \in \text{Cn}(T')$, or equivalently $\bigwedge A \in \text{Cn}(T')$, contrary to $A \not\subseteq \text{Cn}(T')$. It follows from this contradiction that $\bigwedge A \to \beta \notin T$.

Since $\bigwedge A \to \beta$ follows logically from β, it follows directly from $\bigwedge A \to \beta \notin T$, by the logical closure of T, that $\beta \notin T$. This is sufficient to show that finite C-recovery holds.

Ad 5. Let $\beta \in T$ and $\beta \notin T - \langle A \rangle$. We need to show that there is some T' such that $T - \langle A \rangle \subseteq T' \subseteq T$, $T' \not\vdash A$ and $T' \cup \{\beta\} \vdash A$.

If $A \not\subseteq T$, then it follows by C-inclusion and C-vacuity that $T = T - \langle A \rangle$. Then there can be no β such that $\beta \in T$ and $\beta \notin T - \langle A \rangle$, and we are done.

For the principal case, when $A \subseteq T$, let $T' = (T - \langle A \rangle) \cup \{\beta \to \bigwedge A\}$. We need to show that (i) $T' \subseteq T$, (ii) $T' \not\models A$, and (iii) $T' \cup \{\beta\} \models A$.

(i): By C-inclusion, $T - \langle A \rangle \subseteq T$. Since $\beta \to \bigwedge A$ is a logical consequence of A, it follows from $A \subseteq T$ and the logical closure of T that $(T - \langle A \rangle) \cup \{\beta \to \bigwedge A\} \subseteq T$, i.e. $T' \subseteq T$.

(ii): Suppose to the contrary that $T' \models A$, i.e., $(T - \langle A \rangle) \cup \{\beta \to \bigwedge A\} \models A$. Then $(T - \langle A \rangle) \cup \{\beta \to \bigwedge A\} \vdash \bigwedge A$, i.e., $(T - \langle A \rangle) \vdash (\beta \to \bigwedge A) \to \bigwedge A$, i.e., $(T - \langle A \rangle) \vdash \bigwedge A \vee \beta$.

Since $\beta \in T$ it follows from finite C-recovery that $(T - \langle A \rangle) \vdash \bigwedge A \to \beta$. It follows from this and $(T - \langle A \rangle) \vdash \bigwedge A \vee \beta$ that $(T - \langle A \rangle) \vdash \beta$. However, it follows by C-closure from $\beta \notin T - \langle A \rangle$ that $(T - \langle A \rangle) \not\vdash \beta$. We can conclude from this contradiction that $T' \not\models A$.

(iii) Directly from $\{\beta \to \bigwedge A\} \subseteq T'$.

This concludes the proof. $\square$

12. EXPLORING REDUCTION STRATEGIES

In section 3, we gave some informal reasons for maintaining that both kinds of multiple contractions – package and choice – are independent of singleton contractions. In this section we shall look at reduction strategies more thoroughly. With due caution we shall conclude that such strategies are not likely to be successful. The theory of multiple contractions appears to be not only a natural and in many ways necessary but indeed a proper extension of the AGM theory of singleton contraction.

We need to specify more precisely what may be meant by reduction. There is a weak version of reduction that allows for repeated contractions. The picture is as follows. There is a space of theories. To each theory T we may associate two families of theories: those accessible from T by way of a sequence of multiple contractions and those accessible from T by way of a sequence of singleton contractions. The reduction thesis asserts that for each theory the two families coincide. (There may be a number of provisos – which need not concern us now – such as that the language must be sufficiently expressive.)

Since the framework of this paper, just like the original AGM framework, does not provide for repeated contraction, we will only be concerned with stronger versions of the reduction thesis, that refer to single steps of contraction. Such versions all derive from the following schema:

Reduction Schema: Let T be any theory. For each set $A \subseteq Fml$

there exists a formula fA such that $T - A = T - fA$.

Particular instances or classes of instances of the schema are generated by (a) specifying the kind of contraction operation at issue and (b) defining the mapping $f : \wp(Fml) \to Fml$. If nontrivial mappings f can be found for which the thesis is true, then every multiple contraction $T - A$ may be represented by a singleton contraction $T - fA$.

Mappings to be avoided are those that depend on the particular contraction at issue. Thus, the equation $T - A = T - fA$ should not be made to hold by specifying f in terms of $T - A$.

Other conditions are less indisputable but still reasonable. For example, the formula fA should be constructed only from material supplied by A, i.e.

$$\text{var}(fA) \subseteq A$$

where $\text{var}(fA)$ is the set of variables occuring in fA.

It would be nice, if matters could be kept "simple", e.g. conforming to the condition

fA is a truth function of a finite subset of A

Obviously, without some such substantial constraints the reduction schema is impossible to assess. We shall therefore focus on a reduction thesis which seems – at least to us – particularly natural and tempting. This thesis can be factored out into two components:

Finitude: For each set $A \subseteq Fml$ there exists a finite subset A' of A such that $T - A = T - A'$.

Sententiality: For each finite set $A \subseteq Fml$ there exists a formula fA such that $T - A = T - fA$.

The biggest hurdle to reduction is the finitude component: contraction by an infinite set cannot in general be simulated by retracting a finite set.

To see why, note first that for particular sets A and theories T, choice and meet contractions may coincide: $T - [A] = T - \langle A \rangle$. To take a classical – though finite – example, suppose you decide to open your mind to the possibility that Bizet and Verdi were compatriots. Then it suffices to choice-retract the set {Bizet is French, Verdi is Italian}. But if you cannot make up your mind which one of the two sentences to give up, you will have to retract both. In that case choice and package contractions are indistinguishable.

Consider now a theory T and an infinite set A such that $T - [A] = T - \langle A \rangle$. Let all sentences in A be logically independent. Thus, if $\alpha, \beta \in A$, then no logical connection between α and β settles the question whether $\alpha \in T - \beta$

or *vice versa*. Extending this thought, we may observe that for arbitrary subsets B of A, no logical considerations can settle the question whether $T - A = T - B$. Thus, A and T may be chosen such that for no – finite or infinite – proper subset B of A do we have $T - A = T - B$.

This abstract counterexample to finitude may be fleshed out in many ways. Suppose for example that one decides to retract from T the proposition that Peter hates all prime numbers. (Hans Rott sugggested this example in conversation.) Let us also assume that the universal closure of a sentence is in T just in case T contains all of its instances (ω-completeness). So retracting that Peter hates all prime numbers requires the removal of an infinite set of sentences,

$$A = \{\text{Peter hates the number } 2, 3, 5, \ldots\}$$

To make the universal sentence fail, it suffices to retract one of the sentences in A: we have a paradigm case of choice contraction at hand. Moreover, as long as there is no information as to which particular prime number(s) Peter has ceased to hate, it will be best to withdraw all of A; thus, $T - \langle A \rangle = T - [A]$. Clearly, since all elements in A are logically independent, there is no finite or even infinite subset of A whose deletion from T has the same effect as the deletion of all of A. Thus finitude fails.

Since the sententiality schema may be instantiated in a potentially infinite number of ways a general negative result seems impossible to obtain. However, in contrast to the general reduction thesis, we have a particularly simple positive result for choice contractions.

OBSERVATION 17. *For all finite sets A, $T - \langle A \rangle = T - \bigwedge A$.*

  Proof. It suffices to show that $T \angle \{\alpha, \beta\} = T \angle \{\alpha \wedge \beta\}$, for arbitrary sentences α and β, which follows immediately from the fact that $X \Vdash \alpha, \beta \iff X \Vdash \alpha \wedge \beta$. $\square$

No corresponding simple relationship holds for package contraction. In particular,

$$T - [A] = T - \bigvee A, \text{ for finite } A$$

fails, essentially because $X \vdash \alpha, \beta$ does not follow from $X \vdash \alpha \vee \beta$. As can straightforwardly be verified, it does not help to replace $\bigvee$ by some other truth function.

As we have seen, reduction of either choice or package contraction via finitude is not in general feasible. But for finite sets choice contractions may easily be represented as singleton contractions. A dual strategy fails for package contractions.

There are reduction strategies not covered by our reduction schema. For example, we may allow for representations defined in terms of intersections of sets of singleton contractions, or we may allow for sequences of singleton contractions. Variations and combinations abound: we leave their exploration to the adventurous reader.

13. CONCLUSION, FURTHER APPLICATIONS AND OUTLOOK

In the last section we have supplied further evidence that both package and choice contractions are change operations *sui generis*. Our approach to multiple contractions started from a search for suitable generalisations of the AGM postulates for singleton contraction. Not surprisingly then, in one direction the relation between the theory of multiple contractions and the AGM theory is a simple one:

- the postulates for package and choice contractions coincide for contractions by singletons, and
- for contractions by singletons the postulates for multiple contractions are equivalent to the AGM postulates.

Thus, the AGM theory emerges as a common limiting case of the two types of multiple contractions, as it should:

$$T - [\{\alpha\}] = T - \alpha = T - \langle\{\alpha\}\rangle$$

for any theory T and single sentence α.

There is a sense, however, in which the theory developed here is not *simply* an extension of the AGM theory. For, our postulates did not always result from the AGM postulates by – *cum grano salis* – substituting sets for sentences. In the case of package contraction, the postulates of uniformity and relevance depart significantly from their AGM approximations, extensionality, and recovery. These departures resulted in a group of postulates which is stronger than what we would have obtained, if we had, as it were, blindly adapted the AGM postulates to cover contraction by sets. Without this additional strength multiple contractions cannot be represented as partial meet contractions.

This paper is only a beginning in the study of multiple change operations. From a sizable agenda for future investigations we mention only four topics.

First, the status of AGM's supplementary postulates with respect to multiple contractions needs closer investigation. Presumably this is connected, as in singleton contractions, with partial meet constructions in which the preferred remainder sets are identified as the maximal elements of some preference order. This approach is also more informative than the somewhat opaque use of selection functions.

Second, we have not investigated plausible principles that mediate between package and choice contractions. Except in limiting and uninteresting cases our basic theory does not support such principles. Prima facie plausible connections such as

$$T - [A] \subseteq T - \langle A \rangle$$

are remarkably resistant to modelling attempts. Perhaps, on reflection, one will find that such connections ought not to be part of the general theory but should only hold under special conditions.

Third, though the notion of a contraction is in a certain sense purer than that of a revision, it is the latter which is of primary interest once we turn to applying our theory of change operations. We know how to obtain a theory of revisions from a theory of contractions in the singleton case, namely by means of the Levi identity. But we have made no attempt here to apply a suitable generalisation of the Levi identity to obtain a theory of multiple revisions. Just as there are two natural ways to contract by sets – i.e. package and choice – there should be corresponding modes of multiple revisions. (Cf. Fuhrmann 1988 and Hansson 1992a.)

Fourth, as Makinson and Gärdenfors have noted, there is a close connection between theory change – particularly theory revision – and nonmonotonic inference.

> "The key idea is:
> (1) See the revision of a theory T by a proposition α, forming a theory $T * \alpha$, as a form of nonmonotonic inference *from* α;
> (2) Conversely, see a nonmonotonic inference of a proposition β from a proposition α as a discovery that β is contained in the result of revising a *fixed background theory* T so as to integrate α.
> In this way, the nonmontonic relation $\alpha \mathrel{|\!\sim} \beta$ serves as a shorthand for $\alpha \mathrel{|\!\sim_T} \beta$ which indicates that the nonomonotonic inference is dependent on the background theory T."
> (Makinson and Gärdenfors (1989))

This idea can be turned into a translation from theory revision to nonmonotonic inference and vice versa. Under these translations core principles of nonmonotonic inference and the AGM postulates for theory revision exhibit a close correspondence. However, the AGM theory imposes a rather serious stricture on inference relations thus generated: they must be singular on the left-hand-side. This is in stark contrast to other studies of nonmonotonic inference in which inferences are usually drawn from sets rather than single sentences. Indeed, that we may extract conclusions from multiple premisses

is usually considered one of the hallmarks of inference or consequence relations as opposed to implication connectives that are frozen at the first degree. A theory of multiple revisions would not issue in such strictures and could produce genuine inference relations. But, as just remarked, such a theory still awaits formulation.

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