

Minimum-Length Polygons of First-Class Simple Cube-Curves

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Abstract. We consider simple cube-curves in the orthogonal 3D grid. The union of all cells contained in such a curve (also called the tube of this curve) is a polyhedrally bounded set. The curve's length is defined to be that of the minimum-length polygonal curve (MLP) fully contained and complete in the tube of the curve. So far only one general algorithm called rubber-band algorithm was known for the approximative calculation of such an MLP. A proof that this algorithm always converges to the correct curve, is still an open problem. This paper proves that the rubber-band algorithm is correct for the family of first-class simple cube-curves.

1 Introduction

The analysis of cube-curves is related to 3D image data analysis. A cube-curve is, for example, the result of a digitization process which maps a curve-like object into a union S of face-connected closed cubes. The definition of length of a simple cube-curve in 3D Euclidean space can be based on the calculation of the minimal length polygonal curve (MLP) in a polyhedrally bounded compact set [3, 4].

The computation of the length of a simple cube-curve in 3D Euclidean space was a subject in [5]. But the method may fail for specific curves. [1] presents an algorithm (rubber-band algorithm) for computing the approximating MLP in S with measured time $O(n)$, where n is the number of grid cubes of the given cube-curve.

The difficulty of the computation of the MLP in 3D may be illustrated by the fact that the Euclidean shortest path problem (i.e., find a shortest obstacle-avoiding path from source point to target point, for a given finite collection of polyhedral obstacles in 3D space and a given source and a target point) is known to be NP-complete [8]. However, there are some algorithms solving the approximate Euclidean shortest path problem in 3D with polynomial-time, see [9]. The rubber-band algorithm is not yet proved to be always convergent to the correct 3D-MLP.

Recently, [6] developed an algorithm for calculation of the correct MLP (with proof) for a special class of cube-curves. The main idea is to decompose a cube-curve into arcs by finding “end angles” (see Definition 3 below).

More recently, [7] constructed an example of a (special - see title of reference) simple cube-curve, and generalized this by characterizing the class of all of those

cube-curves. In particular, it is true that these cube-curves do not have any end angle; and this means that we cannot use the MLP algorithm proposed in [6] which is provable correct. This was the basic importance of the result in [7]: we showed the existence of cube-curves which require further algorithmic studies.

Both [6] and [7] focus on a special class of simple cube-curves which are called first-class simple cube-curves (defined below). This paper proves that the rubber-band algorithm is correct for first-class simple cube-curves.

The paper is organized as follows: Section 2 defines the notations used in this paper. Section 3 describes theoretical proofs of our results. Section 3 discusses the computational complexity. Section 4 gives the conclusions.

2 Definitions

Following [1], a grid point $(i, j, k) \in \mathbb{Z}^3$ is assumed to be the center point of a *grid cube* with *faces* parallel to the coordinate planes, with *edges* of length 1, and *vertices* as its corners. *Cells* are either cubes, faces, edges, or vertices. The intersection of two cells is either empty or a joint *side* of both cells. A *cube-curve* is an alternating sequence $g = (f_0, c_0, f_1, c_1, \dots, f_n, c_n)$ of faces f_i and cubes c_i , for $0 \leq i \leq n$, such that faces f_i and f_{i+1} are sides of cube c_i , for $0 \leq i \leq n$ and $f_{n+1} = f_0$. It is *simple* iff $n \geq 4$ and for any two cubes $c_i, c_k \in g$ with $|i - k| \geq 2 \pmod{n+1}$, if $c_i \cap c_k \neq \emptyset$ then either $|i - k| = 2 \pmod{n+1}$ and $c_i \cap c_k$ is an edge, or $|i - k| \geq 3 \pmod{n+1}$ and $c_i \cap c_k$ is a vertex.

A *tube* \mathbf{g} is the union of all cubes contained in a cube-curve g . A tube is a compact set in \mathbb{R}^3 , its frontier defines a polyhedron, and it is homeomorphic with a torus in case of a simple cube-curve. A curve in \mathbb{R}^3 is *complete* in \mathbf{g} iff it has a nonempty intersection with every cube contained in g . Following [3, 4], we define:

Definition 1. A minimum-length polygon (MLP) of a simple cube-curve g is a shortest simple curve P which is contained and complete in tube \mathbf{g} . The length of a simple cube-curve g is defined to be the length $l(P)$ of an MLP P of g .

It turns out that such a shortest simple curve P is always a polygonal curve, and it is uniquely defined if the cube-curve is not only contained in a single layer of cubes of the 3D grid (see [3, 4]). If it is contained in one layer, then the MLP is uniquely defined up to a translation orthogonal to that layer. We speak about *the* MLP of a simple cube-curve.

A *critical edge* of a cube-curve g is such a grid edge which is incident with exactly three different cubes contained in g .

Definition 2. If e is a critical edge of g and l is a straight line such that $e \subset l$, then l is called a *critical line* of e in g or *critical line for short*.

Definition 3. Assume a simple cube-curve g and a triple of consecutive critical edges e_1, e_2 , and e_3 such that $e_i \perp e_j$, for all $i, j = 1, 2, 3$ with $i \neq j$. If e_2 is parallel to the x -axis (y -axis, or z -axis) implies that the x -coordinates (y -coordinates, or z -coordinates) of two vertices (i.e., end points) of e_1 and e_3 are

equal, then we say that e_1 , e_2 and e_3 form an end angle, and g has an end angle, denoted by $\angle(e_1, e_2, e_3)$; otherwise we say that e_1 , e_2 and e_3 form a middle angle, and g has a middle angle.

Definition 4. A simple cube-curve g is called first-class iff each critical edge of g contains exactly one vertex of the MLP of g .

Figure 1 shows a first-class simple cube-curve (left) and a non-first-class simple cube-curve (right). Because the vertices of the MLP must be in e_0 , e_1 , e_3 , e_4 , e_5 , e_6 and e_7 . In other words, the critical edge e_2 does not contain any vertex of the MLP of this simple cube-curve.

The rubber-band algorithm is published in [1, 2].

Definition 5. One iteration of the rubber-band algorithm is a complete pass through the main loop of the algorithm.

Let g be a simple cube-curve. Let $AML P_n(g)$ be an n -polygon of g , where $n = 1, 2, \dots$. Let $AML P = \lim_{n \rightarrow \infty} AML P_n(g)$. Let $p_i(t_{i_0})$ be the i -th vertex of $AML P$, where $i = 0, 1, \dots, m+1$. Let $d_i = d_e(p_{i-1}, p_i) + d_e(p_i, p_{i+1})$, where $i = 1, 2, \dots, m$. Let $d(t_0, t_1, \dots, t_m, t_{m+1}) = \sum_1^m d_i$.

Definition 6. Let $e_0, e_1, e_2, \dots, e_m$ and e_{m+1} be all consecutive critical edges of g and $p_i \in e_i$, where $i = 0, 1, 2, \dots, m$ or $m+1$. We call the $m+2$ tuple $(p_0, p_1, p_2, \dots, p_m, p_{m+1})$ a critical point tuple of g . We call it an $AML P$ critical point tuple of g if it is the set of the vertices of an $AML P$ of g .

Definition 7. Let $P = (p_0, p_1, p_2, \dots, p_m, p_{m+1})$ be a critical point tuple of g . Using P as an initial point set, and n iterations of the rubber-band algorithm,

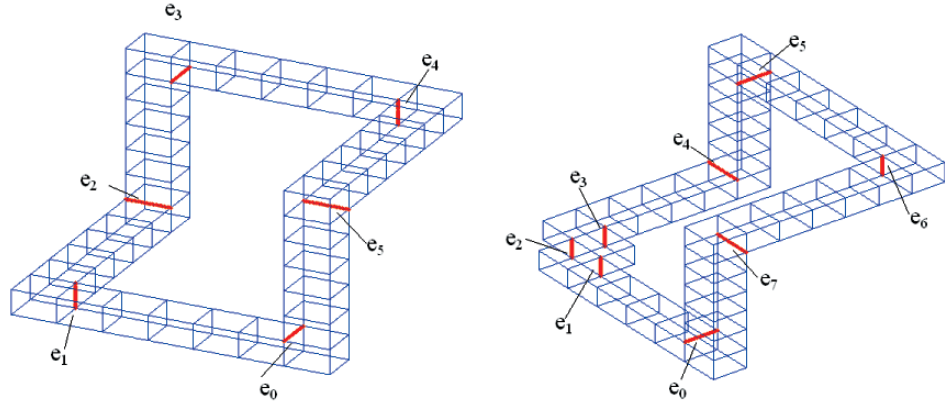


Fig. 1. (1) A first-class simple cube-curve. (2) A non-first-class simple cube-curve.

we get another critical point tuple of g , say $P' = (p'_0, p'_1, p'_2, \dots, p'_m, p'_{m+1})$. The polygon with vertex set $\{p'_0, p'_1, p'_2, \dots, p'_m, p'_{m+1}\}$ is called an n -polygon of g , denoted by $AMLP_n(g)$, or $AMLP_n$ for short, where $n = 1, 2, \dots$

Definition 8. Let $\frac{\partial d(t_0, t_1, \dots, t_m, t_{m+1})}{\partial t_i} \big|_{t_{i0}} = 0$, where $i = 0, 1, \dots, \text{or } m+1$. Then we say that $(t_{00}, t_{10}, \dots, t_{m0}, t_{m+10})$ is a critical point of $d(t_0, t_1, \dots, t_m, t_{m+1})$.

Definition 9. Let $P = (p_0, p_1, p_2, \dots, p_m, p_{m+1})$ be a critical point tuple of g . Using P as an initial point set, n iterations of the rubber-band algorithm, we calculate an n -rubber-band transform of P , denoted by $P \xrightarrow{(r-b)_n} Q$, or $P \rightarrow Q$ for short, where Q is the resulting critical point tuple of g , and n is an positive integer.

Definition 10. Let $P = (p_0, p_1, p_2, \dots, p_m, p_{m+1})$ be a critical point tuple of g . For sufficiently small real $\epsilon > 0$, the set

$\{ (p'_0, p'_1, p'_2, \dots, p'_m, p'_{m+1}) : x'_i \in (x_i - \epsilon, x_i + \epsilon) \text{ and } y'_i \in (y_i - \epsilon, y_i + \epsilon) \text{ and } z'_i \in (z_i - \epsilon, z_i + \epsilon) \text{ and } p'_i = (x'_i, y'_i, z'_i) \text{ and } p_i = (x_i, y_i, z_i), \text{ where } i = 0, 1, 2, \dots, m, m+1 \}$

is called P 's ϵ -neighborhood, denoted by $U(P, \epsilon)$.

Definition 11. Let n be a positive integer. Let $x = (x_1, x_2, \dots, x_n)$. Let T be the family of subsets of \mathbb{R}^n defined by open intervals, i.e., a subset K of \mathbb{R}^n belongs to T iff for each $r = (r_1, r_2, \dots, r_n)$ in K there are real numbers a_i, b_i such that $a_i < r_i < b_i$ and

$$\{x : x \in \mathbb{R}^n, a_i < x_i < b_i, i = 1, \dots, n\} \subset K$$

The topological space (\mathbb{R}^n, T) is called the n -dimensional usual topology.

Definition 12. ([12], Definition 4.1) Let $Y \subset X$, where (X, T) is a topological space. Let T' be the family of sets defined as follows: A set W belongs to T' iff there is a member U in T such that $W = Y \cap U$. The family T' is called the relativization of T to Y , denoted by $T|_Y$.

3 Proofs

We provide mathematical fundamentals to prove that the rubber-band algorithm is correct for any first-class simple cube-curve. We start with citing a basic theorem from [1]:

Theorem 1. Let g be a simple cube-curve. Critical edges are the only possible locations of vertices of the MLP of g .

Let $d_e(p, q)$ be the Euclidean distance between points p and q .

Let $e_0, e_1, e_2, \dots, e_m$ and e_{m+1} be $m+2$ consecutive critical edges in a simple cube-curve, and let $l_0, l_1, l_2, \dots, l_m$ and l_{m+1} be the corresponding critical lines.

We express a point $p_i(t_i) = (x_i + k_{x_i}t_i, y_i + k_{y_i}t_i, z_i + k_{z_i}t_i)$ on l_i in general form, with $t_i \in \mathbb{R}$, where $i = 0, 1, \dots$, or $m + 1$.

In the following, $p_i(t_i)$ will be denoted by p_i for short, where $i = 0, 1, \dots$, or $m + 1$.

Theorem 2. ([10], Theorem 8.8.1) Let $f = f(t_1, t_2, \dots, t_k)$ be a real-valued function defined on an open set U in \mathbb{R}^k . Let $C = (t_{10}, t_{20}, \dots, t_{k0})$ be a point of U . Suppose that f is differentiable at C . If f has a local extremum at C , then $\frac{\partial f}{\partial t_i} = 0$, where $i = 1, 2, \dots, k$.

Lemma 1. $(t_{00}, t_{10}, \dots, t_{m0}, t_{m+10})$ is a critical point of $d(t_0, t_1, \dots, t_m, t_{m+1})$.

Proof. $d(t_0, t_1, \dots, t_m, t_{m+1})$ is differentiable at each point $(t_0, t_1, \dots, t_m, t_{m+1}) \in [0, 1]^{m+2}$. Because $AMLP_n(g)$ is a n -polygon of g , where $n = 1, 2, \dots$ and $AMLP = \lim_{n \rightarrow \infty} AMP_n(g)$, so $d(t_{00}, t_{10}, \dots, t_{m0}, t_{m+10})$ is a local minimum of $d(t_0, t_1, \dots, t_m, t_{m+1})$. By Theorem 2, $\frac{\partial d}{\partial t_i} = 0$, where $i = 0, 1, 2, \dots, m + 1$. \square

Theorem 3. ([7], Theorem 2) If $e_i \perp e_j$, where $i, j = 1, 2, 3$ and $i \neq j$, then e_1, e_2 and e_3 form an end angle iff the equation $\frac{\partial(d_e(p_1, p_2) + d_e(p_2, p_3))}{\partial t_2} = 0$ has a unique root 0 or 1.

Theorem 4. ([7], Theorem 3) If $e_i \perp e_j$, where $i, j = 1, 2, 3$ and $i \neq j$, then e_1, e_2 and e_3 form a middle angle iff the equation $\frac{\partial(d_e(p_1, p_2) + d_e(p_2, p_3))}{\partial t_2} = 0$ has a root t_{20} such that $0 < t_{20} < 1$.

Theorem 5. ([7], Theorem 4) e_0 and e_{m+1} are on different grid plane iff $0 < t_{10} < t_{20} < \dots < t_{m0} < 1$.

Let $p_i(t_{i0})$ be i -th vertex of an $AMLP$, where $i = 0, 1, \dots$, or $m + 1$.

By Lemma 1 and Theorems 3, 4 and 5, we immediately prove the following theorem.

Theorem 6. If e_{i-1}, e_i and e_{i+1} form an end angle, then $t_{i0} = 0$ or 1; otherwise, $0 < t_{i0} < 1$, where $i = 1, 2, \dots$, or m .

By the proofs of the two lemmas (Lemmas 1 and 2) of [7], we have

Lemma 2. If $e_1 \perp e_2$, then $\frac{\partial d_e(p_1, p_2)}{\partial t_2}$ can be written as $\frac{t_2 - \alpha}{\sqrt{(t_2 - \alpha)^2 + (t_1 - \beta)^2 + \gamma}}$, where α, β , and γ are reals.

Lemma 3. If $e_1 \parallel e_2$, then $\frac{\partial d_e(p_1, p_2)}{\partial t_2}$ can be written as $\frac{t_2 - t_1}{\sqrt{(t_2 - t_1)^2 + \alpha}}$, where α is a real.

Theorem 7. ([6], Theorem 4) $\frac{\partial d_2}{\partial t_2} = 0$ implies that we have one of the following representations for t_3 : we can have

$$t_3 = \frac{-c_2 t_1 + (c_1 + c_2) t_2}{c_1}$$

if $c_1 > 0$; we can also have

$$t_3 = 1 - \sqrt{\frac{c_1^2(t_2 - a_2)^2}{(t_2 - t_1)^2} - c_2^2}$$

or

$$t_3 = \sqrt{\frac{c_1^2(t_2 - a_2)^2}{(t_2 - t_1)^2} - c_2^2}$$

if a_2 is either 0 or 1, and c_1 and c_2 are positive; and we can also have

$$t_3 = 1 - \sqrt{\frac{(t_2 - a_2)^2[(t_1 - a_1)^2 + c_1^2]}{(t_2 - b_1)^2} - c_2^2}$$

or

$$t_3 = \sqrt{\frac{(t_2 - a_2)^2[(t_1 - a_1)^2 + c_1^2]}{(t_2 - b_1)^2} - c_2^2}$$

if a_1 , a_2 , and b_1 are either 0 or 1, and c_1 and c_2 are positive reals.

Lemma 4. *The number of critical points of $d(t_0, t_1, \dots, t_m, t_{m+1})$ in $[0, 1]^{m+2}$ is finite.*

Proof. Let $d = d(t_0, t_1, \dots, t_m, t_{m+1})$.

Case 1. The simple cube-curve g has some end angles.

Assume that e_i, e_{i+1} , and e_{i+2} form an end angle, and also e_j, e_{j+1} , and e_{j+2} , and no other three consecutive critical edges between e_{i+2} and e_j form an end angle, where $i \leq j$ and $i, j = 0, 1, 2, \dots, m-2$. By Theorem 6 we have $t_{i+3} = f_{i+3}(t_{i+1}, t_{i+2})$, $t_{i+4} = f_{i+4}(t_{i+2}, t_{i+3})$, $t_{i+5} = f_{i+5}(t_{i+3}, t_{i+4})$, \dots , t_j , and $t_{j+1} = f_{j+1}(t_{j-1}, t_j)$. This shows that $t_{i+3}, t_{i+4}, t_{i+5}, \dots, t_j$, and t_{j+1} can be represented by t_{i+1} , and t_{i+2} . In particular, we obtain an equation $t_{j+1} = f(t_{i+1}, t_{i+2})$, or

$$g(t_{j+1}, t_{i+1}, t_{i+2}) = 0,$$

where t_{j+1} , and t_{i+1} are already known, or

$$g_1(t_{i+2}) = 0. \tag{1}$$

By Lemmas 2 and 3, function $g_1(t_{i+2})$ can be decomposed into finitely many monotonous functions. Therefore, Equation (1) has finite solutions. This implies that the system formed by $\frac{\partial d}{\partial t_i} = 0$ (where $i = 0, 1, \dots$, and $m+1$.) has finite solutions.

Case 2. The simple cube-curve g does not have any end angle.

Analogous to Case 1, the system formed by $\frac{\partial d}{\partial t_i} = 0$ (where $i = 0, 1, \dots$, and $m+1$.) implies a two variables system formed by

$$h_1(t_0, t_1) = 0 \quad (2)$$

$$h_2(t_0, t_1) = 0 \quad (3)$$

Again by Lemmas 2 and 3, Equations (2) and (3) can be decomposed into finite monotonous functions, so the system formed by Equations (2) and (3) has finite solutions. This implies that the system formed by $\frac{\partial d}{\partial t_i} = 0$ (where $i = 0, 1, \dots, \text{and } m + 1$.) has finitely many solutions. \square

By Lemmas 4 and 1, we have

Lemma 5. *g has only a finite number of AMLP critical point tuples.*

Let e_0, e_1 and e_2 be three consecutive critical edges. Let $p_i(p_{i_1}, p_{i_2}, p_{i_3}) \in e_i$, where $i = 0, 1, 2$. Let the two endpoints of e_i be $a_i(a_{i_1}, a_{i_2}, a_{i_3})$ and $b_i(b_{i_1}, b_{i_2}, b_{i_3})$, where $i = 0, 1, 2$.

Lemma 6. *There is an algorithm such that its computing complexity of finding a point $p_1 \in e_1$ with $d_e(p_1, p_0) + d_e(p_1, p_2) = \min\{p_1 | d_e(p_1, p_0) + d_e(p_1, p_2), p_1 \in e_2\}$ is $O(1)$.*

Proof. p_1 can be written as $(a_{1_1} + (b_{1_1} - a_{1_1})t, a_{1_2} + (b_{1_2} - a_{1_2})t, a_{1_3} + (b_{1_3} - a_{1_3})t)$. Note that

$$d_e(p_1, p_0) = \sqrt{\sum_{i=1}^3 ((a_{1_i} - p_{1_i}) + (b_{1_i} - a_{1_i})t)^2}$$

can be simplified. In fact, the straight line a_1b_1 is parallel to one coordinate axis (x, y or z axis) So, only one element of the set $\{b_{1_i} - a_{1_i} : i = 1, 2, 3\}$ is 1 and the other two should be 0. Without loss of generality, we can assume that $d_e(p_1, p_0) = \sqrt{(t + A_1)^2 + B_1}$, where A_1 and B_1 are functions of a_{1_i}, b_{1_i} and p_{1_i} , where $i = 0, 1, 2$. Analogously, $d_e(p_1, p_2) = \sqrt{(t + A_2)^2 + B_2}$, where A_2 and B_2 are functions of a_{1_i}, b_{1_i} and p_{2_i} , where $i = 0, 1, 2$. In order to find a point $p_1 \in e_1$ such that $d_e(p_1, p_0) + d_e(p_1, p_2) = \min\{p_1 | d_e(p_1, p_0) + d_e(p_1, p_2), p_1 \in e_1\}$, we can solve the equation $\frac{\partial(d_e(p_1, p_0) + d_e(p_1, p_2))}{\partial t} = 0$: the unique solution is $t = -(A_1B_2 + A_2B_1)/(B_2 + B_1)$. \square

By the proof of Lemma 6, and if we represent p_i as $(a_{i_1} + (b_{i_1} - a_{i_1})t_i, a_{i_2} + (b_{i_2} - a_{i_2})t_i, a_{i_3} + (b_{i_3} - a_{i_3})t_i)$, then we have

Lemma 7. *$t_2 = t_2(t_1, t_3)$ is a continous function at each tuple $(t_1, t_3) \in [0, 1]^2$.*

Lemma 8. *If $\overrightarrow{P(r-b)_1}Q$, then for every sufficient small real $\epsilon > 0$, there is a sufficient small real $\delta > 0$ such that $P' \in U(P, \delta)$ and $\overrightarrow{P'(r-b)_1}Q'$ implies $Q' \in U(Q, \epsilon)$.*

Proof. By Lemma 6 and note that g has $m + 2$ critical edges, so by using Lemmas 1 repeatedly $m + 2$ times we prove this lemma. \square

By Lemma 8, we have

Lemma 9. *If $\overrightarrow{P(r-b)_n}Q$, then for every sufficiently small real $\epsilon > 0$, there is a sufficiently small real $\delta_\epsilon > 0$ and a sufficiently large integer N_ϵ such that $P' \in U(P, \delta_\epsilon)$ and $\overrightarrow{P'(r-b)_{n'}}Q'$ implies $Q' \in U(Q, \epsilon)$, where n' is an integer and $n' > N_\epsilon$.*

By Lemma 5, let Q_1, Q_2, \dots, Q_N with $N \geq 1$ be the set of all AMLP critical point tuples of g . Let ϵ be a sufficiently small positive real such that $U(Q_i, \epsilon) \cap U(Q_j, \epsilon) = \emptyset$, where $i, j = 1, 2, \dots, N$ and $i \neq j$. Let $D_i = \{P : P \rightarrow Q', Q' \in U(Q_i, \epsilon), P \in [0, 1]^{m+2}\}$, where $i = 1, 2, \dots, N$.

The following two lemmas are straightforward.

Lemma 10. *If $N > 1$ then $D_i \cap D_j = \emptyset$, where $i, j = 1, 2, \dots, N$ and $i \neq j$.*

Lemma 11. $\cup_{i=1}^N D_i = [0, 1]^{m+2}$.

We consider the usual topology $T = R^{m+2}|_{[0,1]^{m+2}}$.

Lemma 12. D_i is an open set of T , where $i = 1, 2, \dots, N$ with $N \geq 1$.

Proof. By Lemma 9, for each $P \in D_i$, there is a sufficiently small real $\delta_P > 0$ such that $U(P, \delta_P) \subseteq D_i$. So we have $\cup_{P \in D_i} U(P, \delta_P) \subseteq D_i$.

On the other hand, for $P \in U(P, \delta_P)$, we have $D_i = \cup P \subseteq \cup_{P \in D_i} U(P, \delta_P)$. Note that $U(P, \delta_P)$ is an open set of T . So $D_i = \cup_{P \in D_i} U(P, \delta_P)$ is an open set of T . □

Lemma 13. ([11], Proposition 5.1.4) *Let $U \subset R$ be an arbitrary open set. Then there are countably many pairwise disjoint open intervals U_n such that $U = \cup U_n$.*

Lemma 14. g has a unique AMLP critical point tuple.

Proof. By contradiction. Suppose that Q_1, Q_2, \dots, Q_N with $N > 1$ are all AMLP critical point tuples of g . Then there exists $i \in \{1, 2, \dots, N\}$ such that $D_i|_{e_j} \subset [0, 1]$, where e_j is a critical edge of g , $i, j = 1, 2, \dots, N$. Otherwise we have $D_1 = D_2 = \dots = D_N$. This is a contradiction to Lemma 10.

Let $E = \{e_j | D_i|_{e_j} \subset [0, 1]\}$, where e_j is a critical edge of g . We can select a critical point tuple of g as follows: go through each $e \in \{e_0, e_1, \dots, e_m, e_{m+1}\}$. If $e \in E$, by Lemmas 12 and 13, select the minimum left endpoint of the open intervals whose union is $D_i|_e$. Otherwise select the midpoint of e . We denote the resulting critical point tuple as $P = (p_0, p_1, p_2, \dots, p_{m+1})$. By the selection of P , we know that P is not in D_i . By Lemma 11 there is $j \in \{1, 2, \dots, N\} - \{i\}$ such that $P \in D_j$. Therefore there is a sufficiently small real $\delta > 0$ such that $U(P, \delta) \subset D_j$. Again by the selection of P , there is a sufficiently small real $\delta' > 0$ such that $U(P, \delta') \cap D_i \neq \emptyset$. Let $\delta'' = \min\{\delta, \delta'\}$. Then we have $U(P, \delta'') \subset D_j$ and $U(P, \delta'') \cap D_i \neq \emptyset$. This implies that $D_i \cap D_j \neq \emptyset$, and it is a contradiction to Lemma 10. □

Let g be a simple cube-curve. Let $AMLP_n(g)$ be an n -polygon of g , where $n = 1, 2, \dots$. $AMLP = \lim_{n \rightarrow \infty} AMLP_n(g)$.

Theorem 8. *The $AMLP$ of g is the MLP of g .*

Proof. By Lemma 14 and the proof of Lemma 1, $d(t_0, t_1, \dots, t_m, t_{m+1})$ has a unique local minimal value. This implies that the $AMLP$ of g is the MLP of g . \square

4 Computational Complexity

Assume that a simple cube-curve g has m critical edges. By Lemma 6, the computational complexity of each iteration of running the rubber-band algorithm is $O(m)$. Let $AMLP_n(g)$ be an n -polygon of g , where $n = 1, 2, \dots$. Then the computational complexity of finding $AMLP_n(g)$ is $nO(m)$. Suppose $\lim_{n \rightarrow \infty} AMLP_n(g) = AMLP$. By Theorem 8, we can use $AMLP_{N(\epsilon)}(g)$ as an approximate MLP of g , where ϵ is the error between the length of $AMLP_{N(\epsilon)}(g)$ and that of MLP . The computational complexity is $N(\epsilon)O(m)$.

5 Conclusions

We have proved that the rubber-band algorithm is correct for the family of first-class simple cube-curves and that the algorithm's computational complexity of finding an approximate MLP of a simple cube-curve is linear for this family of curves.

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