# Baby Mandelbrot sets, Renormalization and MLC

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*To memory of Adrien Douady*

## 1. Preface

In the end of 1970's the dynamical systems theory witnessed two great events: discovery of the Universality phenomenon and discovery of the Mandelbrot set. At first glance, they had little to do with each other. It was the remarkable paper by Douady and Hubbard "On the dynamics of polynomial-like maps" [DH3] that linked these two events tightly and penetrated deeply into their nature.

The Universality phenomenon discovered by Feigenbaum and independently by Coullet and Tresser is concerned with rigidity of the dynamical and parameter objects of various dynamical systems, and associated self-similarity of these objects. The underlying mathematical mechanism comes from hyperbolicity of the renormalization operator that relates behavior of the system in different scales. This Renormalization Conjecture became a central theme in dynamical systems for several decades.

A striking feature of the Mandelbrot set $M$ is presence of "babies" inside of itself which are visually indistinguishable of the set $M$ itself (see Figure 1). To explain this self-similarity phenomenon, Douady and Hubbard defined a *complex* renormalization operator. This operator first allowed them to account for all babies in $M$, and then grew up into a key tool of the Renormalization theory, for complex as well as for real systems.

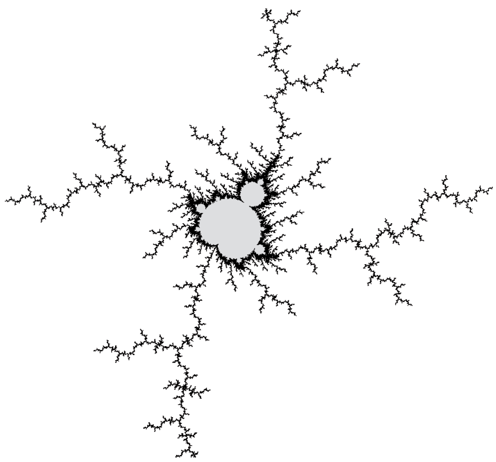

FIGURE 1. A baby Mandelbrot set.

## 2. Quadratic-like maps and complex renormalization

The renormalization theory requires a big enlargement of the one-parameter family of quadratic polynomials $P_c : z \mapsto z^2 + c$ to an infinite-dimensional space $\mathcal{Q}$ of quadratic-like maps.

Let $U \Subset U'$ be two topological disks in the complex plane. A degree two holomorphic branched covering $f : U \to U'$ is called a *quadratic-like map*. We normalize quadratic-like maps to put their critical points at the origin. The *filled Julia set* of $f$ is defined as the set of non-escaping points,

$$K(f) = \{z : \ f^n z \in U, \ n = 0, 1, \ldots\},$$

and the *Julia set* $J(f)$ is defined as the boundary of $K(f)$. These sets are either connected or Cantor depending on whether the critical point of $f$ is non-escaping or otherwise.

For instance, any quadratic polynomial can be restricted to a quadratic-like map. In fact, according to the Straightening Theorem [DH3], these restrictions represent all possible topological types of quadratic-like maps. Even better, let us say that two quadratic-like maps are *hybrid equivalent* if they are topologically conjugate by a quasiconformal map which is conformal a.e. on the filled Julia set. Then *any quadratic-like map $f$ is hybrid equivalent to some (restricted) quadratic-polynomial $P_c : z \mapsto z^2 + \chi(f)$, and in the case of connected Julia set, this polynomial is unique.* This leads to the following picture: the connectedness locus $\mathcal{C}$ (of quadratic-like maps with connected Julia set) is decomposed into hybrid classes $\mathcal{H}_c$ each of which meets the quadratic family at a single point $c$ of the Mandelbrot set (such that $c = \chi(f)$ for $f \in \mathcal{H}_c$).

Now we can define the complex renormalization. Assume there exists a natural number $p \geqslant 2$ and a topological disk $V \ni 0$ such that $g = f^p$ maps $V$ onto its image $V'$ as a quadratic-like map with connected Julia set $J'$, and such that the "little Julia sets" $f^k(J')$, $k = 0, 1, \ldots, p-1$, are "almost disjoint"[1]. Then the map $f$ is called *renormalizable* with period $p$, and the quadratic-like map $g : V \to V'$ (considered up to rescaling) is called the *(complex) renormalization $Rf$* of $f$. If the little Julia sets are actually disjoint then the renormalization is called *primitive*, otherwise it is called *satellite*.

There is some combinatorial data attached to this renormalization picture: the period $p$ of the renormalization and the "position" of the little Julia sets on the sphere[2].

It turns out that renormalizable parameters $c \in M$ with a given combinatorics form exactly one baby Mandelbrot copy ("$M$-copy"). Moreover, if the renormalization is satellite then this copy is attached to some hyperbolic component of int $M$ (being born in a bifurcation that "creates" an attracting periodic orbit of period $p$ out of an attracting periodic orbit of a smaller period $p/n$). Otherwise, the copy is born in a saddle-node bifurcation. Such a primitive copy has a distinguished cusp $c$ where $f_c$ has a parabolic periodic orbit of period $p$ with multiplier 1.

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[1] "Almost disjointness" can be defined as the property that $J' \smallsetminus f^k(J')$ is connected for any $1 \leqslant k \leqslant p-1$, compare [Mc1, §7.3].

[2] This can be defined as the homotopy class of $f$ rel to the little Julia sets collapsed to the points.

Thus, to any quadratic map $f$ we can associate a canonical sequence of periods $p_1 < p_2 < \dots$ for which $f$ is renormalizable. Depending on whether the sequence is empty, finite, or infinite, the map $f$ is called respectively *non-renormalizable*, *at most finitely renormalizable*, or *infinitely renormalizable*.

Let us now pass to the theory of quadratic-like families which links renormalization to the baby $M$-copies.

## 3. Quadratic-like families and baby Mandelbrot sets

Let us consider a holomorphic family

$$\mathbf{f} = \{f_\lambda : U_\lambda \to U'_\lambda\}, \quad \lambda \in \Lambda,$$

of quadratic-like maps over a smooth Jordan disk $\Lambda$. For such a family we can define a *Mandelbrot-like* set $M_{\mathbf{f}}$ of parameters $\lambda \in \Lambda$ for which the Julia set $J(f_\lambda)$ is connected. It turns out that if the family $\mathbf{f}$ possesses nice topological properties then the set $M_{\mathbf{f}}$ is canonically homeomorphic to the standard Mandelbrot set $M$. (In this sense, the Mandelbrot set presents a topologically universal bifurcation diagram for quadratic-like maps!) The canonical homeomorphism $M_{\mathbf{f}} \to M$ is given by the straightening $\lambda \mapsto \chi(f_\lambda)$ (and is also called straightening and is denoted by $\chi$).

Here are assumptions that make our family nice:

– The disks $U_\lambda$ and $U'_\lambda$ move holomorphically over $\Lambda$;

– The family is *proper* in the sense that $f_\lambda(0) \in \partial U'_\lambda$ for $\lambda \in \partial\Lambda$;

– The family is *unfolded* in the sense that the critical value $\lambda \mapsto f_\lambda(0)$ winds once around the critical point 0 as $\lambda$ goes once around $\partial\Lambda$.

**Theorem 3.1.** *Under the above assumptions on the family* $\mathbf{f}$*, there exists a homeomorphism* $\chi$ *from* $\Lambda$ *onto a neighborhood of* $M$ *that coincides with the canonical straightening* $M_{\mathbf{f}} \to M$ *on the Mandelbrot-like set. This homeomorphism is conformal on* $\operatorname{int} M$ *and quasiconformal on* $\Lambda \smallsetminus M$*.*

*Outline of the proof.* The proof of this fundamental result comprises four basic steps:

*Step 1: Looking from the outside.* In the very first work by Douady and Hubbard on holomorphic dynamics [DH1], they constructed an explicit uniformization of the complement of the Mandelbrot set by a "conformal position" of the critical value $c$. A similar (quasiconformal) uniformization can be constructed for any Mandelbrot-like family in question. This gives a natural correspondence between the complements of $M_{\mathbf{f}}$ and $M$.

*Step 2: Looking from the inside.* Theoretically, there are two types of components of $\operatorname{int} M$: *hyperbolic* components (for which the polynomial $f_c$ has an attracting cycle) and *queer* components (others: one of the central conjectures of the field is that these in fact do not exist).

By another early result by Douady and Hubbard [DH1], any hyperbolic component of $\operatorname{int} M$ is uniformized by the multiplier of the attracting periodic point. On the other hand, by the work of Mañé-Sad-Sullivan [MSS], queer components can be uniformized by the one-parameter family of invariant conformal structures on the Julia set with constant dilatation. These results are still valid for quadratic-like families under consideration, and they imply that the straightening is a conformal isomorphism on each component of $\operatorname{int} M_{\mathbf{f}}$.

*Step 3: Continuity.* The above insights from outside and inside match continuously on the boundary of $M$. It is shown by a simple argument making use of compactness of the space of $K$-quasiconformal maps and quasiconformal rigidity of parameters on $\partial M$. However, a similar result fails for higher degree polynomials. As Douady famously put it, "only by miracle, it is true in the quadratic case".

*Step 4: Topological Argument Principle.* What is left is to show that the map $\chi$ is one-to-one. It is done by showing that the fibers of this map are discrete (using the theory of holomorphic motions), that the map is "topologically holomorphic" (in the sense that it has positive degree at any point), and applying the Topological Argument Principle (using that the family is unfolded).

Now, let $H$ be a primitive hyperbolic component of $M$ with period $p > 1$. Then one can find a domain $\Lambda \supset H$ and topological disks $U'_\lambda \Supset U_\lambda \ni 0$ such that the renormalizations $f^p_\lambda : U_\lambda \to U'_\lambda$ form a proper unfolded quadratic-like family $\mathbf{f}$ (see [D]). According to Theorem 3.1, the Mandelbrot-like set $M_{\mathbf{f}} \subset \Lambda$ is canonically homeomorphic to $M$. This is the desired primitive Mandelbrot copy.

In the satellite case, the situation is more subtle, as there is no proper quadratic-like family that produces the satellite copy $M'$. However, there is an almost proper family: the only place where properness is lost is the root of $M'$ (where $M'$ is attached to some hyperbolic component). One can adjust the above argument to deal with such a family as well.

## 3.1. MLC Conjecture

Besides many wonderful results, Douady and Hubbard contributed to holomorphic dynamics a great conjecture widely known under the name MLC ("Mandelbrot set is Locally Connected"). There are several good reasons that make this conjecture so prominent:

– If it is true then there is an explicit topological model for the Mandelbrot set. Namely, let us consider the Riemann mapping $\varphi : \mathbb{C} \smallsetminus \bar{\mathbb{D}} \to \mathbb{C} \smallsetminus M$. If the Mandelbrot set is locally connected then by the classical Carathéodory Theorem, it extends continuously to the unit circle $\mathbb{T}$. So the boundary of $M$ becomes the quotient of $\mathbb{T}$ by a certain equivalence relation (that can be nicely visualized by means of a Thurston's geodesic lamination of the unit disk [Th]). Remarkably, this equivalence relation can be explicitly described: Douady and Hubbard, Orsay Notes [DH2].

– It was also shown in [DH2] that MLC would imply another central conjecture in the field: that hyperbolic maps are dense in the quadratic family.

– The MLC can be formulated as a Rigidity Conjecture: *two combinatorially equivalent*[3] *non-hyperbolic quadratic polynomials $P_c$ and $P_{c'}$ must coincide.* This conjecture is a dynamics counterpart of Thurston's Ending Lamination Conjecture (ELC) in hyperbolic three-dimensional geometry, which in turn is a deep extension of the classical Mostow Rigidity Theorem. The ELC has been recently proved completing a ten-years long effort by Yair Minsky with several collaborators (see [BCM] for the final shot). But MLC still resists.

[3] Two quadratic polynomials are "combinatorially equivalent" if the landing property of rational external rays (which always land at some points of the Julia set, even in the non-locally connected case) defines the same equivalence relation on the rational circle $\mathbb{Q}/\mathbb{Z}$.

A deep breakthrough in the MLC Conjecture was made in the late 80's - early 90's by J.-C. Yoccoz:

**Theorem 3.2** (see [H, M]). *If a a quadratic map $P_c$, $c \in M$, is not infinitely renormalizable then the Mandelbrot set is locally connected at $c$.*

This result reduces the MLC to infinitely renormalizable parameters, and thus (surprisingly) embeds it into the Renormalization Theory. A class of locally connected infinitely renormalizable parameters was described in [L1]. It led, in particular, to a proof of the real counterpart of the MLC with an implication that hyperbolic parameters are dense in the real quadratic family.

The MLC Conjecture shed a beautiful new light on the field of holomorphic dynamics and influenced its mainstream development for the past two decades. For recent advances in this problem see Kahn [K] and Kahn-Lyubich [KL].

## 3.2. Proof of the Renormalization Conjecture

The complex renormalization (with a given combinatorics) can be viewed as an operator $R$ acting in the space $\mathcal{Q}$ of quadratic-like maps. Then the Feigenbaum-Coullet-Tresser Renormalization Conjecture (adapted to this setting) asserted that $R$ has a unique fixed point $f_* \in \mathcal{Q}$, and it is hyperbolic with a one-dimensional unstable direction (see Figure 2).

Importance of the Douady-Hubbard theory of quadratic-like maps for the Renormalization theory was articulated by Dennis Sullivan in his address to the Berkeley ICM-1986 [S1]. Here Sullivan laid down a program of constructing the renormalization fixed point $f_*$ by the methods of Teichmüller theory. In this framework, the hybrid class of $f_*$ gets nicely interpreted as the stable manifold of the renormalization operator $R$ at $f_*$. This program was realized in [S2, S3]. Another approach to the construction (but still within the quadratic-like framework) was then suggested by C. McMullen [Mc2]. It was also proved in this work that the renormalization operator is strongly contracting on the hybrid class of $f_*$. Finally, hyperbolicity of

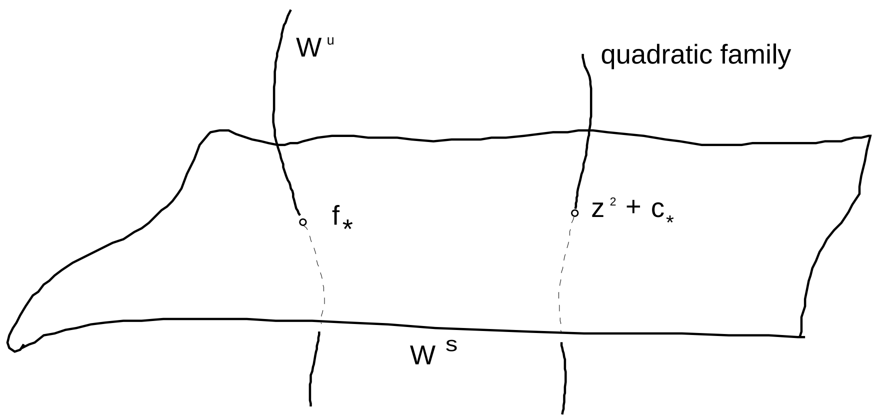


FIGURE 2. Renormalization fixed point.

$R$ at $f_*$ (in the space of quadratic-like maps) was established in [L2]. A good part of [L2] is concerned with endowing the space of quadratic-like maps (or rather, "quadratic-like germs") with a complex analytic structure that turns hybrid classes into codimension-one analytic manifolds, thus enhancing the picture described in [DH3] with a precise analytic content. Among other consequences of this structure is the fact that all primitive $M$-copies are quasiconformally equivalent to the whole Mandelbrot set.

We see that the Douady-Hubbard theory of quadratic-like maps played a prominent role in the progress made in several central themes in holomorphic and one-dimensional dynamics. By now it has become a classical background that young researchers in holomorphic dynamics learn in the graduate school along with the principles of hyperbolic metric and the Measurable Riemann Mapping Theorem

## 4. References

[BCM] J. Brook, R. Canary and Y. Minsky. The classification of Kleinian surface groups II: The Ending Lamination Conjecture, Preprint 2003.

[D] A. Douady. Chirurgie sur les applications holomorphes. In: *Proceedings of the International Congress of Mathematicians* (Berkeley 1986), 724-738.

[DH1] Douady and Hubbard. Itération des polynômes quadratiques. *C.R. de l'Académie des Sciences*, t. 294, ser. 1, 123-126.

[DH2] Douady and Hubbard. Etude dynamiques des polynômes complexes. Publications Math. d'Orsay, 84-02 (1984) and 85-04 (1985).

[DH3] Douady and Hubbard. On the dynamics of polynomial-like maps. *Ann. Sci. Éc. Norm. Sup.*, v. 18 (1985), 287-344.

[H] J. H. Hubbard, Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. In: "Topological Methods in Modern Mathematics" (Proceedings of Symposium in honor of John Milnor's 60th birthday), Publish or Perish, 1993, 467-511.

[K] J. Kahn. *A priori bounds* for some infinitely renormalizable maps, I. Bounded combinatorics. Preprint IMS Stony Brook, 2006, # 5.

[KL] J. Kahn and M. Lyubich, *A priori bounds* for some infinitely renormalizable maps, II. Decorations. Preprint IMS Stony Brook, 2006, # 6.

[L1] M. Lyubich, Dynamics of quadratic polynomials. I, II. Acta Math. 178 (1997), no. 2, 185-247, 247-297.

[L2] M. Lyubich. Feigenbaum-Coullet-Tresser Universality and Milnor's Hairiness Conjecture. *Annals of Math.*, v. 149 (1999), 319-420.

[MSS] R. Mañé, P. Sad, and D. Sullivan. On the dynamics of rational maps, *Ann. scient. Ec. Norm. Sup.*, **16** (1983), 193-217.

[Mc1] C. McMullen. Complex dynamics and renormalization. *Annals of Math. Studies*, Princeton, 1994, # 135.

[Mc2] C. McMullen. Renormalization and 3-manifolds which fiber over the circle. *Annals of Math. Studies, Princeton*, 1996, # 142.

[M] J. Milnor. Local connectivity of Julia sets: Expository lectures. In: "The Mandelbrot set: theme and variations". *London Math. Soc. Lecture Notes* (2000), v. 274 (ed. Tan Lei), p. 67-116.

[S1] D. Sullivan. Quasiconformal homeomorphisms in dynamics, topology, and geometry. In: Proceedings of the International Congress of Mathematicians (Berkeley 1986), 1216–1228.

[S2] D. Sullivan. Bounds, quadratic differentials, and renormalization conjectures. AMS Centennial Publications, v. 2: Mathematics into Twenty-first Century (1992).

[S3] D. Sullivan. Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers. In: "Topological Methods in Modern Mathematics", (Proceedings of Symposium in honor of John Milnor's 60th Birthday), Publish or Perish, 1993, 543-564.

[Th] W. Thurston. On the combinatorics and dynamics of iterated rational maps. Preprint 1982.