COLLOQUIUM MATHEMATICUM

LI DÉDIÉ À M. STANISŁAW HARTMAN 1987

LIFE AND WORK OF BRONISŁAW KNASTER (1893–1980)

BY

ROMAN DUDA (WROCŁAW)

I. Life. At the end of XIX century Warsaw was the main city of a border "Vistula Province" in the Russian Empire. The province was underdeveloped, the Russian rule oppressive and anti - Polish, the general progress slow. Born there on 22 May 1893 as a son to a popular hospital physician, Bronisław Knaster soon experienced himself difficulties of the national life: his school years were marked by the 1904–1906 "school strike", whose main aim was to achieve teaching in Polish (official language was Russian and it was not allowed to speak Polish in schools and offices). Authorities answered by closing revolting schools and young Knaster wandered around the city attending four of them before he got a secondary - school certificate in 1911.

There was a university in Warsaw, but in those times it was Russian and since the days of the "school strike" it has been boycotted by the patriotically minded Polish youth. Like many other contemporaries of his – including his later friends W. Sierpiński, Z. Janiszewski, K. Kuratowski – Knaster went abroad. Following his own father's example, he went to Paris and began to study medicine there. The three years which followed gave him such a close familiarity of the French way of life, of the language and culture, that they soon became his second nature.

The outbreak of World War I found him in Poland. In October 1914 he married Maria Morska who later made her name as a journalist. When in Summer 1915 Germans and Austrians took command over central Poland, retreating Russians evacuated Warsaw University with all its professors and most of students. And then, out of nothing and in a couple of months only, the remaining Poles have organized a new Polish university. The university started its activity in autumn 1915 and Bronisław Knaster was among its very first students. At first he studied logic under Professor J. Łukasiewicz (in 1918 L. Condorcet's *Algebra of Logic* appeared in his translation into Polish), but soon changed his mind and started studying mathematics. For a man well over twenty, the decision has been a little short of a drama. Not the least, it has wiped out several years of hard medicine studies and shattered all previous plans. Asked later for motivation, Knaster used to say that mathematics attracted him by the purity of its methods and certitude of its results, in sharp contrast to medicine.

His mathematical studies were interrupted, shortly but fatally for him, by the Polish–Soviet War of 1920. In July 1920, at the moment of greatest advance of Soviet armies towards Warsaw, Knaster volunteered and made the campaign as a corporal stretcher-bearer. He served until November 1920 and has been later awarded with the Legion Cross by Marshall Piłsudski. However, at that war time he has also become infected with a rare type of a tropical malaria. The disease has tormented him for ten years, bringing almost to the verge of a physical ruine.

Knaster's university years were a period of framing the Warsaw School of Mathematics, designed by Zygmunt Janiszewski (1888–1920), since 1915 professor of mathematics at the Warsaw University. Janiszewski proposed a program how "to win an independent place for Polish mathematics". Its main two features were: 1° concentration of all creative power upon a few selected branches of mathematics, and 2° foundation of a new journal devoted exclusively to the research in the chosen area. The choice was topology, set theory, real functions, mathematical logic, and the new journal was "Fundamenta Mathematicae". Chosen branches were of a fresh origin and so more easily accessible to the men with no sound mathematical tradition behind and no good library at disposal. And the whole program succeeded because three young men – Z. Janiszewski, S. Mazurkiewicz, W. Sierpiński – have decided to work together and soon have found followers, among the first of which were B. Knaster, K. Kuratowski and S. Saks. Also the idea of a mathematical journal which covers only some parts of mathematics, in those times quite unorthodox, proved successful.

Knaster was one of the first doctors of the Warsaw University, receiving his Ph. D. degree in 1923 on the basis of the thesis [*3*]. And three years later he received the title of an assistant professor, thus entering ranks of those who have possessed recognized professor's qualifications and have been waiting for a permanent professor's position. For the time being he could deliver lectures, receiving some remuneration but no permanent salary.

However, with the development of malaria his health quickly deteriorated. A special sanatory near Dresden did not help him and in 1924 he went to Italy, where he felt somewhat better. In Italy he lived and worked for a couple of years but even there the fits of malaria have gradually become hardly tenable. By a good chance, however, he learned a new method of treating his illness and fully recovered within a 1930 year.

After returning to Warsaw, Knaster undertook duties of a teacher, organizer, editor. In 1929 he has started his "advanced seminar in topology", in temporary cooperation with S. Mazurkiewicz, K. Kuratowski, or K. Borsuk, through which many Polish mathematicians passed, including N. Aronszajn, S. Eilenberg, S. Hartman, E. Marczewski, R. Sikorski, Z. Waraszkiewicz, M. Wojdysławski, and some others. The seminar has been successfully continued after 1945 in Wrocław, almost to the death of Knaster.

In 1931 the series "Mathematical Monographs" was founded and Knaster has become one of its editors. In particular, it was he who translated Banach's *Théorie des opérations linéaires* and Saks' *Théorie de l'intégrale* into French. With the passing of time the editorial work has preoccupied him more and more.

Till 1939 he has written nearly 100 reviews for the "Zentralblatt für Mathematik und ihre Grenzgebiete" and has been collecting reprints of all topological papers of that time. The collection was burned in the Warsaw Uprising 1944.

Since 1935 he has been also Secretary of the Polish Mathematical Society.

Knaster's favourite diversion were long evening walks. Extensive literary reading, love of music, vivid intelligence, acquaintance with foreign cultures, sharp tongue – all these have afforded a great pleasure to all who took part in those walks. He and his wife have also established closed relations with the poets of the Scamandrite group, where his wit, intelligence, ardent pursuit after an apt word have won him an admiration and friendship of Tuwim, Wittlin, Słonimski – eminent Polish poets and writers.

The life of Knaster, as of all people of his generation, has been changed abruptly and decisively by the outbreak of World War II. Knaster and his wife have fled from Germans to Lvov, soon taken by the Soviets who opened there the Ukrainian University. Some Poles have become professors in that University and among them was Knaster.

The first Soviet period was relatively short. When in June 1941 Germans came in, Knasters did not flee. During subsequent three long years of the German occupation of Lvov Knaster has nourished lice in the medical institute of Prof. Weigel, needed to produce vaccines against typhoid fever. Rather unusual job of spending several hours a day in the institute with a box of lice attached to a forearm, in those gloomy days was a rather good one: yellow "Ausweis" with a blue bar and the stamp of the "Institut für Fleckfieber" had been securing to its owner an almost perfect safety.

When Soviets came back in 1944, the Ukrainian University has been reopened and Knaster has been reaffirmed as its professor. He could stay in Lvov, but after the establishment of new borders of Poland he has decided to leave that city and in 1945 went to Cracow. There he lectured in the Jagiellonian University, simultaneously helping to reestablish the University's printing-office. It was mainly due to him that in December 1945 there appeared the XXXIIIrd volume of *Fundamenta Mathematicae*, a vivid evidence that Polish mathematics was arising from ashes and persisting to exist.

In the autumn 1945, when old universities of Cracow, Warsaw and Poznań were reopened and new ones in Wrocław, Lublin, Łódź and Toruń established, Knaster received invitations from all the latter four. After some hesitations, he decided in favour of Wrocław.

When Knaster came to Wrocław, he left behind 52 years full of events and dramas, including a cruel war which brought the painful death of his mother and tragic death of his beloved wife. A mathematician of world renown, an energetic organizer of scientific life, an experienced mathematical editor – Knaster was a man of greatest importance for the new university. He married there Regina Szpalerska, a widow of an underground Home Army soldier, and soon devoted himself completely to didactical, organizing and editorial tasks.

He was not alone in that matter. Besides him there were also E. Marczewski, H. Steinhaus, W. Ślebodziński. Those four men, "the great four", were pioneers of the Polish mathematical life in Wrocław.

In those early days in Wrocław Knaster's greatest passion was the organization of scientific editorial work. This included the printing-house which was organized in 1947. In that year resumed here its existence *Studia Mathematica* (transferred first from Lvov to Wrocław and then, after some years, to Warsaw), as well as the series "Mathematical Monographs", and a new journal *Colloquium Mathematicum*, founded by the "great four" in a kind of reminiscence of the program of Janiszewski. The journal has been intended to be a mean for integration of Wrocław mathematics. After 1949, when the newly founded State Mathematical Institute (now the Institute of Mathematics of the Polish Academy of Sciences) took over the care of all the editorial work in Polish mathematics, Knaster withdrew himself to a large extent from that area.

He was one of the founders of the Wrocław Scientific Society in 1946, remaining in its board and acting as its editor for many years to come. In particular, in the thirty years 1949–1978 he edited 131 volumes of the Series B of that Society. His editorial passion can be seen in the article [*21*] describing scientific printing-houses in Poland.

In the 1950s Knaster returns to active mathematical work, publishes over twenty original papers, and raises many open problems both in the *New Scottish Book* and in *Colloquium Mathematicum*. It is also the period of second rise of his "advanced topology seminar" which becomes then the converging point of talented mathematical youth including Jan Mycielski (now Boulder, Colorado), K. Urbanik (Wrocław), M. Reichaw (Jerusalem), A. Lelek (Houston, Texas), J. Mioduszewski (Katowice), and many others.

In spite of some face stiffness, he was a deeply good man. He loved mathematics and university life above all and it sufficed, either for a student or a beadle, to reveal a similar passion to acquire his most vivid sympathy. Intolerable to a passive attitude of mind, he strongly favoured creative work of a very concrete type. Everywhere. In garden, where he tried to get new sorts of roses, in editorial work, where he exhibited perfectionism difficult to meet, in music, which he liked to play for himself, in conversation, where he admired a sharp wit and verbal dispute, and above all in mathematics, his greatest love. Favouring creativity, he disliked history, philosophy and all kinds of reflection upon past and dead.

With all his sensitivity to a talent, he highly treasured a character in a man. He admired Bertrand Russell and Albert Einstein, although he grieved at the latter's involvement in the work on atomic bomb. Among his close friends were P. S. Aleksandrov, R. H. Bing, K. Borsuk, E. Čech, K. Menger.

On his 85th birthday's eve there came a cerebral haemorrhage. Although he later regained consciousness and, partly, ability to speak and to move around – his last year and a half were heavy to him. He died at home on 3 November 1980.

II. Work. Knaster was proud to be called a topologist and in topology, the new and much influential branch of XX-century mathematics, which he also considered the most important one, he was a master of geometric constructions. His main area of interest was the study of the connectedness and his major contributions were some peculiar constructions in making of which he revealed a special talent. Here is a list of some of them (for a more detailed account see below):

a) a *biconnected* set [*1*], i.e. a connected set which is not a union of two disjoint connected subsets,

b) a hereditarily indecomposable plane continuum [*3*], i.e. the *pseudo-arc*, as the re-inventors have called it,

c) a connected and locally connected subset of the Sierpiński triangle curve, which contains no perfect subset [*9*],

d) an irreducible continuum with a continuous decomposition into layers [*14*],

e) for each $n > 1$, a biconnected subset of n-dimensional euclidean space which divides that space [*18*],

f) an effective (i.e., without *AC*) decomposition of the square into two dense, connected, locally connected and punctiform subsets [*30*].

It has been a deep Knaster's conviction that constructions are an important form of the existential theorems in mathematics and that the less they could be expected, the more important they are.

Although his main object of interest had always been connectedness, we divide it, for the sake of clarity, into three sections: continua, general connected spaces, continuous functions. Outside topology he made contributions to foundations of mathematics, functional equations etc. and they are collected in the fourth section. The fifth and last is devoted to his problems.

1. Continua. After initial impulses from Cantor and Poincaré, the decades on the turn of XIX century saw a slow and difficult progress in establishing the basic concepts of topology. One of them was connectedness.

The first modern ideas leading to the concept of connectedness appeared in the mid-XIX century by B. Bolzano, but the first explicit definition is due to G. Cantor (for a wider covering of that story see [37]). In 1883 the latter defined [8] a subset X of a euclidean space R^n to be connected if for every two points t and t' of X and for arbitrary $\varepsilon > 0$ there exists a finite sequence of points $t_1, \ldots, t_n \in X$ such that all distances $tt_1, t_1 t_2, \ldots, t_n t'$ are smaller than ε. Cantor considered only *perfect* (i.e., closed and dense-in-itself) subsets of R^n and in this way he initiated the study of *continua* as of perfect, connected, bounded (or *general continua*, if not bounded) subsets of euclidean spaces.

In 1904 A. Schönflies started publishing a series of papers [30], an important step in the development of the theory of continua by introducing new concepts, bringing new results, and even making some fault assertions. Relying heavily on intuition, Schönflies has claimed that there do not exist three regions in the plane with the common boundary. The claim was refuted by L. E. J. Brouwer [7] who constructed continua which are the common boundary of three regions and showed that they are *indecomposable*, i.e. they are "closed curves, which cannot be divided into two proper subcurves". Besides Brouwer, indecomposable continua were announced also by A. Denjoy [10] and by Yoneyama [39], the latter describing the examples due to Wada.

More precisely, a continuum is *decomposable* if it is a union of two proper subcontinua, otherwise it is indecomposable. Although originally indecomposable continua were considered rather pathological, in later years they gained importance.

Apparently the most simple indecomposable continuum is that of Brouwer. Following Knaster's description (cf. [20], p. 209), we can define it in the following way. Take Cantor ternary set C in the segment $[0, 1]$ of the X-axis and consider its subsets P and Q, where P consists of all extremities of complementary segments, and $Q = C \setminus P$. Attach to C all semicircles lying in the upper half-plane, with the centre $1/2$ in X-axis and ends in P. Now consider C as the union of a sequence of Cantor's ternary sets $C_1, C_2, C_3, \ldots$ and of point 0, where C_1 is the "right half" of C, i.e. the subset of C lying in the segment $[2/3, 1]$, C_2 is the "right half" of $C \setminus C_1$, i.e. the subset of C lying in the segment $[2/9, 2/3]$, C_3 is the "right half" of $C \setminus \bigcup_{i=1}^{2} C_i$, and so on. And to each C_i attach all semicircles lying in the lower half-plane, with the centre in the mid-point of C_i and ends in $P \cap C_i$. The union of all thus attached semicircles (to C as a whole and to each C_i separately) is a sort of a selfwinding line. Since P is not compact, the line is not a continuum. To get Brouwer's example B, one must take the closure of that line in the plane. One can prove that B is a union of continuum many similarly looking disjoint selfwinding lines, each of which is simultaneously boundary and dense in B. If we try to decompose B into two proper subcontinua A and D, we have two possibilities: either A is a subset of one of those lines, and then $D = B$, or A meets at least two such lines and then $A = B$. Therefore continuum B is indecomposable.

The discovery of indecomposable continua inevitably led to several new problems. First came the question which Knaster posed himself: Does there exist a hereditarily indecomposable continuum, i.e. an indecomposable continuum whose every subcontinuum is also indecomposable. If it existed, it could not contain any continuum known so far, neither arc nor any continuum containing arc. And Knaster showed that it does exist! He succeeded by a tremendous work in analytic geometry which consisted in constructing a sequence of more and more complicated versions of continuum B and passing to the limit. The masterpiece later became his Ph. D. thesis [*3*].

The success was so great that when some twenty years later the continuum was rediscovered by R. H. Bing [3] and E. E. Moise [26] under the name of a pseudo-arc and with the elegant and powerful inverse limit procedure, the new inventors could hardly believe that it had been done much earlier when no such procedure was available.

Knaster continuum (as Russians called it, cf. [34]) or pseudo-arc (the name now commonly accepted) soon became a standard, in a way, answer to many questions. Does there exist a continuum which contains no arc? Yes, it does: pseudo-arc. Does there exist a continuum, distinct from an arc, which is homeomorphic to each of its proper subcontinua? Yes, it does: pseudo-arc [26]. Does there exist a continuum, distinct from a circle, which is homogeneous? Yes, it does: pseudo-arc [3]. And it turned out that it is not a rare phenomenon, since pseudo-arcs form a G_δ dense subset of the hyperspace of all subcontinua of the square [23].

One of the standard theorems in plane topology is the Janiszewski theorem [18] on cuts of the plane R^2, i.e. on continua $C \subset R^2$ such that there are two points in $R^2 \setminus C$ which cannot be joined by a continuum lying outside C. Kuratowski proved [21] that each cut of the plane into finitely many regions contains an irreducible cut. And Knaster constructed [*6*] a cut into countably many regions which contains no irreducible cut. Answering one of the earlier questions he also showed (ibidem) that there exists a common boundary of arbitrarily many (finite or countable in number) regions and that there exists a family of continuum many disjoint cuts, each of which is homeomorphic to a curve called "Warsaw circle".

G. T. Whyburn constructed [35] a hereditary cut of the plane. Knaster noticed [*15*] that Whyburn's cut is a *rational* curve, i.e. that each point of it has arbitrarily small neighbourhoods with countable boundaries.

In the theory of curves a point is called an *end* of a curve, if it has arbitrarily small neighbourhoods (in that curve) with one-point boundary. K. Menger proved ([24], p. 97) that the set of all end-points of any curve is a 0-dimensional G_δ-set. Knaster and Reichbach showed [23] that the condition is also sufficient, i.e. for each 0-dimensional G_δ-set B there exists a plane curve, the set of end-points of which is homeomorphic to B. The result was later improved by Knaster and Urbanik [25] who showed that each 0-dimensional and metrisable G_δ-set is homeomorphic to a closed subset of the Cantor set C which differs from C for some points of P only (see the definition above). As such, it is the set of end-points of a dendrite, the ramification points of which have all order 3.

In an attempt to characterize topologically the segment [0, 1] L. Zoretti introduced [40] the concept of irreducibility: a set is said to be *irreducible* between its points a and b provided that it is connected and these two points cannot be joined by any closed connected subset different from the whole set. For continua the definition sounds simpler: continuum is *irreducible* between a and b, if the points a and b cannot be joined by any proper subcontinuum. The study of general irreducible continua was initiated by Janiszewski [17] and Kuratowski proved [20] that there exists an upper semicontinuous decomposition of an irreducible continuum into subcontinua called *layers*, the hyperspace of which is an arc (layers of arc are obviously points). Knaster constructed [*14*] an irreducible continuum, the decomposition of which into layers is continuous (like in an arc) but no layer is a point. Irreducible continua like that of Knaster were later investigated and Moise proved [27] that there must be layers which are not arcs, Hamstrom proved [14] that some of them are not locally connected, and Dyer proved [12] that among them are indecomposable continua.

In an isolated paper [*13*] Hurewicz and Knaster showed that each compact metrizable space can be extended, by an addition of at most 2-dimensional set, to a locally connected and unicoherent continuum. (A continuum is called *unicoherent*, if for each decomposition of it into two subcontinua their common part is connected; unicoherence is another generalization of a property seen in simple objects: arc is unicoherent, circle is not.)

In his last paper [*31*] (published when he was 86), Knaster constructed a singular plane curve, answering a question raised by S. M. Nadler.

2. General connected spaces. As we have seen, the study of continua as of perfect (in particular, closed) and connected subsets of euclidean spaces started already in 1880s and it was only some twenty years later that there appeared first attempts at the definition of a connectedness of a set (still lying in a euclidean space) which is not necessarily closed. Cantor's definition of connectedness is obviously invalid for such sets (consider rationals) and the men who made noticeable contributions here were N. J. Lennes [22] and F. Riesz [29]. In 1914, apparently unaware of either Lennes' or Riesz' work, F. Hausdorff [15] defined connected sets as those which cannot be divided into two non-empty, closed and disjoint subsets. And it is essentially that definition that ever since became commonly accepted (for metric compact spaces, in particular for continua, it is obviously equivalent to that of Cantor).

Thus in the early 1920s there was a definition of a general connected space, there were also known some variants of that definition like local connectedness, some properties of (locally) connected spaces were recognized – but there was no systematic study of that concept alone, no theory of connected spaces. It is upon this background that one must look upon the paper [*1*] which undertook such a study, established many fundamental properties of connected spaces which have ever since become classic, and provided some extraordinary examples.

The paper starts with a foreword which reveals the authors' attitude:

The study of the concept of a connected set is important from two different points of view. First, simplicity of the very definition of that concept is remarkable from the logical point of view. Second, it is interesting to investigate relations which exist between theorems true for general connected sets and those related to continua.

Connected sets have not yet become the object of a systematic study. It is the aim of this article to provide such by a methodical examination of some fundamental problems concerning those sets, without any claim to exhaust the subject.

Chapter 1 is on general theorems, Chapter 2 on connected sets irreducible between two points, Chapter 3 on unions and common points of connected sets irreducible between two points, Chapter 4 on sets complementary to plane connected sets, and Chapter 5 provides examples.

Sets A and B are called *separated* if

$$A \cap \bar{B} \cup \bar{A} \cap B = \emptyset.$$

A set is called *connected* if it contains more than one point and cannot be decomposed into two separated subsets. The first theorem of Chapter 1 says that if a connected set is contained in a union of two separated sets, then it is contained entirely in one of them. Having it, it is easy to show the next theorem: If connected sets S_1 and S_2 are not separated, then their union is also a connected set. Follows the corollary: If a class of connected sets contains a set which is not separated with any other set of the class, then the union of all sets of that class is connected. And so, step by step, it goes.

How methodically the authors proceeded, can be seen in the following example. After having proved that each connected set is a union of two different connected sets, they asked whether a connected set can be the union of two disjoint connected sets. And answered that question negatively by providing an example of a biconnected set, i.e. of a connected set which is not a union of two disjoint connected subsets.

Let C be the Cantor ternary set in the segment [0, 1] of X-axis, and let P and Q be its subsets defined as above. Consider the segments $L(c)$ joining points $c \in C$ to the point $p = (1/2, 1/2)$, and in each such segment take all points of the second coordinate rational if $c \in P$ and all points of the second coordinate irrational if $c \in Q$. All those points form the set S.

It is not obvious that the set S is connected (in fact, the proof is rather long) but it is easy to see that each connected subset of S must contain the "explosive" point p (later on, such a point p was called a *dispersion* point of a connected set), for the set $S \setminus (p)$ clearly contains no connected subset.

The paper [*1*] stimulated a vivid research in the area of general connected spaces, in particular on biconnected sets. (For more details on biconnected sets with dispersion points, see [11].) There were also discovered biconnected sets without dispersion point [25]. Like pseudo-arc, also biconnected set of Knaster and Kuratowski has become a standard example. For instance with the help of it, Knaster answered [5] a question of R. L. Wilder [36].

Paper [7] gives an example of a G_δ biconnected set (with dispersion point), thus answering a question of S. Mazurkiewicz who proved that there can be no F_σ biconnected set. The construction relies on some properties of real functions: graph of a pointwise limit function of continuous functions is a connected set if and only if the function satisfies the Darboux property; such a graph is always a G_δ-set.

Any biconnected set with a dispersion point shows that a complement of one point in a connected set can have rather complicated structure. On the other hand, however, such a complement differs a little, for a point only, from a connected set and P. Aleksandrov put the question to characterize such complements internally, i.e. without referring to a connected set in which it lies. In [*17*] Knaster answered that question by showing that a set S can be made connected by adjoining one point only if and only if it contains a descending sequence of closed and diffused subsets, the common part of which is empty (a subset A of a space X is called *diffused* in X if, for every decomposition of X into two non-empty separated subsets, the set A meets both of them).

In still another paper [*18*] Knaster constructed, for each $n \geqslant 2$, a biconnected set with a dispersion point which cuts the euclidean space R^n.

Paper [*8*] contains the following, rather unexpected theorem: a continuum C, irreducible between the points a and b, contains a connected set, irreducible between a and b, if and only if each subcontinuum of C, which is not a continuum of condensation in C, is decomposable.

It is known that each locally connected continuum is arcwise connected. However, without the assumption of compactness it need not be so and R. L. Moore [28] was the first to show that there exists a set which is connected and locally connected but contains no arc. That result was strengthened in a short joint note [9] which shows that the Sierpiński triangle curve contains a connected and locally connected set with no perfect subset. The construction depends on the axiom of choice and is rather straightforward but examples of that kind were important. They showed the limits of the concept, in the case of connectedness, rather broad. As a result, theory of general connected sets was not pursued very strongly and much more research, at least in the Warsaw School, was concentrated upon compact connected spaces, i.e. continua.

Peculiar connected sets were always of much interest to Knaster and he returned to them also much later by showing [30] that the square can be e f f e c t i v e l y decomposed into two dense, connected, locally connected and punctiform subsets.

In the joint paper of three authors [29] it was noticed that some connected sets possess a much stronger property than that of connectedness itself: they cannot be decomposed not only in two but even into c o u n t a b l y many mutually separated subsets. A set with that property was called *σ-connected* and the three authors investigated that concept a little. By a known result of Sierpiński [31] each continuum is σ-connected, but σ-connected is also original biconnected set of Knaster and Kuratowski. Examples of sets which are not σ-connected had already existed [4], but the authors provided also new ones.

3. Continuous mappings. Scientific interests of Knaster were concentrated upon connectedness, and continuous mappings were another tool for him rather than a new area of research.

In the joint paper [11] the three authors provided a short proof of the Brouwer's fixed point theorem, elegancy and simplicity of which has not been surpassed to this day.

It is known that a continuous image of an arcwise connected continuum is again an arcwise connected continuum. A generalization of an arc is a *λ-continuum*, i.e. an irreducible continuum, each layer of which is boundary, and a generalization of an arcwise connected continuum is a *λ-connected* continuum, i.e. a continuum, any two points of which can be joined by a λ-continuum. Knaster and Mazurkiewicz showed [12] that a continuous image of a λ-continuum need not be λ-connected.

An interest in homogeneity can be seen in the joint paper [24] which shows that each homeomorphism between two boundary and closed subsets of the Cantor set C can be extended to a homeomorphism of C onto itself.

Consider a mapping $f\colon X \to R$, where X is a metric space and R is the reals. Point $x \in X$ is called a *limit-point* of f if and only if there is $a \in R$ such that $f(x_n) \to a$ for each sequence $x_n \to x$. And if $a = f(x)$, the point x is called a *point of continuity* of f. It is known that the mapping f is a limit of continuous mappings if and only if each partial mapping $f|F$, where F is a closed subset of X, has a point of continuity. Answering a question of E. Marczewski, three authors showed [*26*] that for X metric complete the condition can be weakened: it suffices if $f|F$ has a limit point.

Continuous mappings $f\colon X \to Y$ are related to upper semi-continuous decompositions of X parametrized by Y: elements of decomposition are sets $f^{-1}(y)$ and if f is continuous, then the sets $f^{-1}(y)$ are closed and $y_n \to y$ implies $\mathrm{Ls}\, f^{-1}(y_n) \subset f^{-1}(y)$, which is equivalent to upper semicontinuity. Such decompositions had become the subject of vivid research and Knaster added to it an interesting concept of fixation [*27*]. A decomposition of a metric space is called *fixable* if for each $\varepsilon > 0$ there exists a finite family of closed sets which are pairwise disjoint, have diameters $< \varepsilon$, and their union meets each element of the decomposition. He has also constructed an upper semicontinuous decomposition of a continuum into continua of large diameters which is not fixable.

Paper [*28*] is related to that of Borsuk and Molski [6], where there were considered *simple* mappings, i.e. mappings such that each counter-image of a point consists of one or two points only. Answering one of the questions stated there, Knaster and Lelek showed that the carpet of Sierpiński can be obtained from the segment by a simple mapping.

4. Other topics. Here we will discuss Knaster's work in the set theory, functional equations, and applied mathematics.

In the early years of topology, and especially in the Warsaw School, set-theoretical methods were prevailing and Knaster was one of those who had a deep understanding of the set theory. His insight was highly valued by such masters as A. Tarski and W. Sierpiński.

Knaster's first paper in that area was written jointly with W. Sierpiński [*2*]. It deals with the classes $\mathscr{L}$ of Fréchet, now abandoned and forgotten attempt at the definition of a general space. The two authors prove that there exists a class $\mathscr{L}$ with the following property: each of its elements is a limit of every uncountable subset, and show some of its features.

In the 1920s, of much interest in Warsaw and Lvov was the Cantor–Bernstein theorem on 1–1 mappings. S. Banach has generalized it and applied, together with A. Tarski, to their paradoxical decomposition of a ball [2]. In 1927 both Tarski and Knaster published short notes [32], [*10*], devoted to the subject. Knaster's idea was to consider set-valued functions. He reduced some theorems to the following lemma: if a function h, defined on subsets of A and with the values which are subsets of A, is monotone, then there is a fixpoint, i.e. a set D such that $h(D) = D$. See [*4*] and [33].

In [*16*] Knaster considered the famous problem (stated in the first volume of *Fundamenta Mathematicae*) of M. Suslin: let X be a linearly ordered set such that the order is continuous, without the first and the last elements, and satisfies the condition

(S) each family of pairwise disjoint intervals is at most countable; is then X isomorphic to the reals?

Condition (S) can be expressed also in the form: each uncountable family of intervals contains two intervals, the common part of which is not empty.

Knaster proposed to consider the following stronger condition:

(K) each uncountable family of intervals contains an uncountable family of intervals, any two of which have a non-empty intersection.

Since (K) implies separability, one gets a new characterization of the reals: linear and continuous order, without the first and the last element, condition (K). Consequently, implication (S) $\Rightarrow$ (K) is equivalent to Suslin's hypothesis. As is known, Suslin's hypothesis is independent of ZF.

One of the problems of pragmatic partition can be expressed in this way: consider a sandwich with a layer of cheese and bacon, is it possible to cut it with a knife into two parts, equivalent with respect to the amount of bread, cheese and bacon? In the paper [*19*] there is an elegant solution to that problem, and the paper [*20*] reviews this and some other pragmatic partition problems.

Paper [*22*] contains an ingenious proof that for each linearly ordered set and each monotone binary operation, the distribution-in-itself

$$x(yz) = (xy)(zx)$$

implies bisymmetry

$$(xy)(zu) = (xz)(yu).$$

5. Problems. Knaster has been well known for his habit to ask questions. Reading a paper or listening to a lecture, he was continuously asking whether this or that hypothesis were essential, whether the thesis cannot be strengthened, whether the construction cannot be simplified etc. They were often quite simple questions at the beginning, with often quite simple answers, but it could also happen that such an unexpected question led to a vivid discussion, involving all present, and afterwards to a new problem. Sometimes quite hard.

In the first years of its existence, *Fundamenta Mathematicae* used to publish open problems. They were not numerous, but included some magnificent ones like that of Suslin. In the 8th volume there is a problem of

Knaster: *Let D be a closed subset of the plane, lying in the 3-dimensional euclidean space. Is it true that each point of D is accessible in the space?* (A point d of D is *accessible* in E if there exists a continuum $C \subset E$ such that $d = C \cap D$.)

Later *Fundamenta Mathematicae* abandoned the custom but it was revived in this journal (*Colloquium Mathematicum*), where the problems department is alive to this day and the number of problems surpassed twelve hundred. It is here that most of Knaster problems have appeared in print. Let us mention some of them.

PROBLEM 4 (*Colloq. Math.* 1 (1947), p. 30). (...) *Let be given three points p_1, p_2, p_3 on a 2-dimensional sphere S^2 and a continuous mapping of S^2 into the real line E^1. Do there exist three points q_1, q_2, q_3 on S^2 which are isometric to p_1, p_2, p_3 and have the common image on E^1, i.e.*

$$f(q_1) = f(q_2) = f(q_3)?$$

(...) *More generally, is it true that for any given k points $p_1, p_2, \ldots, p_k$ on S^n and any continuous mapping of S^n into E^{n-k+2}, where $k = 2, 3, \ldots, n+1$, there exists a set of points $q_1, q_2, \ldots, q_k$ isometric to $p_1, p_2, \ldots, p_k$ which has the same image in E^{n-k+2}? And how many such sets are there?*

The problem asks for a possible generalization of Borsuk's "Antipodensatz" [5]. The case was shown to be true by H. Hopf [16] and Knaster asks for a general k. As noted in volumes 4 and 5 of *Colloquium*, the case $k = 3$, $n = 2$, had been proved by E. E. Floyd [13] and the case $k = 3$, n arbitrary, had been proved by Chung-Tao Yang [38]. And finally, the problem was answered positively in its full generality by R. P. Jerrard [19].

PROBLEMS 177 and 178 (*Colloq. Math.* 4 (1957), p. 243). *Let us say that sets lying in a space can be threaded if there exists an arc (...) which contains at least one point of each of the sets. If a compact set lying in a cube of $n > 1$ dimension is decomposed in a continuous way into disjoint continua, can they always be threaded? And is this still possible if the decomposition is only semicontinuous, in particular into components?*

The problems seem to be still open.

As we have noted before, E. Dyer proved [12] that an irreducible continuum with a continuous decomposition into layers must contain an indecomposable continuum as a layer. Knaster asks (Problem 202, *Colloq. Math.* 5 (1957), p. 118) whether it must contain a hereditarily indecomposable continuum. The problem is still open([1]).

PROBLEM 264 (*Colloq. Math.* 6 (1958), p. 334). *Let f be a function defined on a space X, with the values in a Hausdorff space Y. We say that f*

([1]) The answer is negative. See L. Mohler and L. G. Oversteegen, *On the structure of tranches in continuously irreducible continua*, this journal 54 (to appear). [Note of the Editors] *has multiplicity n at a point* $y \in Y$ *if the set* $f^{-1}(y)$ *consists of n points. Let X be the segment* [0, 1]. *Is any continuous function of a finite multiplicity, at most n at each point, a superposition of finitely many continuous functions on X of multiplicity at most* 2 *at each point*?

The problem seems to be still open.

PROBLEM 284 (*Colloq. Math.* 7 (1960), p. 108). *Does there exist a connected and hereditarily connected set of dimension greater than* 1?

The problem is still open.

PROBLEM 304 (*Colloq. Math.* 7 (1960), p. 310). *Is any locally connected subcontinuum of an n-dimensional Sierpiński carpet its retract*?

The problem is still open.

PROBLEM 323 (*Colloq. Math.* 8 (1961), p. 139). *Let us call a dendroid each continuum which is arcwise connected and hereditarily unicoherent. Characterize topologically* (*by internal properties*) *the family of all dendroids which have homeomorphic images in the plane.*

The problem, still open, contains the first definition of a dendroid, the general study of which has become so much extensive since that time. Problems 340 (*Colloq. Math.* 8 (1961), p. 278), 370 (ibidem 9 (1962), p. 169), 480 and 481 (ibidem 12 (1964), p. 294–295) also pertain to dendroids.

Problems 554 and 562 (*Colloq. Math.* 15 (1966), p. 160 and 320) ask whether a hereditarily indecomposable continuum, in particular a pseudo-arc, contains a non-trivial retract. Answering the problem J. L. Cornette showed [9] that each subcontinuum of the pseudo-arc is its retract. The general answer is not yet known.

REFERENCES

I. Papers of B. Knaster [2]

[1] (joint with K. Kuratowski) *Sur les ensembles connexes*, Fundamenta Mathematicae 2 (1921), p. 206–255.

[2] (joint with W. Sierpiński) *Sur un ensemble abstrait dont chaque élément est un élément-limite de chaque sous-ensemble non dénombrable*, ibidem 3 (1922), p. 35–40.

[3] *Un continu dont tout sous-continu est indécomposable*, ibidem 3 (1922), p. 247–286.

[4] (joint with K. Kuratowski) *Sur les continus non bornés*, ibidem 5 (1924), p. 25–38.

[5] *Sur un problème de M. R. L. Wilder*, ibidem 7 (1925), p. 191–197.

[6] *Quelques coupures singulières du plan*, ibidem 7 (1925), p. 264–289.

[7] (joint with K. Kuratowski) *Sur quelques propriétés topologiques des fonctions dérivées*, Rendiconti del Circolo Matematico di Palermo 49 (1925), p. 382–386.

[8] *Sur les ensembles connexes irréductibles entre deux points*, Fundamenta Mathematicae 10 (1927), p. 276–297.

[9] (joint with K. Kuratowski) *A connected and connected im kleinen point set which contains no perfect subset*, Bulletin of the American Mathematical Society 33 (1927), p. 106–109.

(2) The full list of Knaster's papers has been published by Wiadomości Matematyczne 25 (1984), p. 120–122.

[*10*] *Un théorème sur les fonctions d'ensemble*, Annales de la Société Polonaise de Mathématique 6 (1927), p. 133–134.

[*11*] (joint with K. Kuratowski and S. Mazurkiewicz) *Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe*, Fundamenta Mathematicae 14 (1929), p. 132–137.

[*12*] (joint with S. Mazurkiewicz) *Sur un problème concernant les transformations continues*, ibidem 21 (1933), p. 85–90.

[*13*] (joint with W. Hurewicz) *Ein Einbettungssatz der henkelfreie Kontinua*, Proceedings of the Koninklijke Akademie van Wetenschappen te Amsterdam. Section of Sciences 36 (1933), p. 557–560.

[*14*] *Un continu irréductible à décomposition continue en tranches*, Fundamenta Mathematicae 25 (1935), p. 568–577.

[*15*] *Über rationale Kurven ohne Bogen*, Monatshefte für Mathematik und Physik 42 (1935), p. 37–44.

[*16*] *Sur une propriété caractéristique de l'ensemble des nombres réels*, Matematičeskiĭ Sbornik Akademii Nauk SSSR 16 (58) (1945), p. 281–290.

[*17*] *Sur un problème de P. Alexandroff*, Fundamenta Mathematicae 33 (1945), p. 308–313.

[*18*] *Sur les coupures biconnexes des espaces euclidiens de dimension $n > 1$ arbitraire*, Matematičeskiĭ Sbornik Akademii Nauk SSSR 19 (61) (1946), p. 9–18.

[*19*] *Sur le problème du partage pragmatique de H. Steinhaus*, Annales de la Société Polonaise de Mathématique 19 (1947), p. 228–230.

[*20*] (joint with H. Steinhaus) *Sur le partage pragmatique*, Comptes Rendus de la Société des Sciences et des Lettres de Wrocław 2 (1947), p. 1–13.

[*21*] *On scientific printing houses in Poland*, Časopis pro pěstování matematiký a fysiký 74 (1949), p. 341–346 (in Polish).

[*22*] *Sur une équivalence pour les fonctions*, Colloquium Mathematicum 2 (1949), p. 1–4.

[*23*] (joint with M. Reichbach) *Sur la caractérisation topologique de l'ensemble des bouts d'une courbe*, Fundamenta Mathematicae 40 (1953), p. 13–28.

[*24*] (joint with M. Reichbach) *Notion d'homogénéité et prolongements des homéomorphies*, ibidem 40 (1953), p. 180–193.

[*25*] (joint with K. Urbanik) *Sur les espaces complets séparables de dimension* 0, ibidem 40 (1953), p. 194–202.

[*26*] (joint with J. Mioduszewski and K. Urbanik) *Points-limites et points de continuité*, Colloquium Mathematicum 3 (1955), p. 164–169.

[*27*] *Sur la fixation des décompositions*, Bulletin de l'Académie Polonaise des Sciences, Classe III 4 (1956), p. 193–196.

[*28*] (joint with A. Lelek) *Coutures et tapis*, Fundamenta Mathematicae 45 (1958), p. 186–199.

[*29*] (joint with A. Lelek and J. Mycielski) *Sur les décompositions d'ensembles connexes*, Colloquium Mathematicum 6 (1958), p. 227–246.

[*30*] *Une décomposition effective du carré en deux ensembles denses, connexes, localement connexes et ponctiformes*, Commentationes Mathematicae 14 (1970), p. 19–23.

[*31*] *A singular plane curve*, Colloquium Mathematicum 42 (1979), p. 263–271.

II. Other quoted papers

[1] J. Aczél, *Vorlesungen über Funktionalgleichungen und ihre Anwendungen*, Basel 1961.

[2] S. Banach and A. Tarski, *Sur la décomposition des ensembles de points en parties respectivement congruentes*, Fundamenta Mathematicae 6 (1924), p. 244–277.

[3] R. H. Bing, *A homogeneous indecomposable plane continuum*, Duke Mathematical Journal 15 (1948), p. 729–742.

[4] G. Birkhoff, *Lattice Theory*, New York 1948.

[5] K. Borsuk, *Drei Sätze über die d-dimensionale euklidische Sphäre*, Fundamenta Mathematicae 20 (1933), p. 177–190.

[6] K. Borsuk and R. Molski, *On a class of continuous mappings*, ibidem 45 (1958), p. 84–98.

[7] L. E. J. Brouwer, *Zur Analysis Situs*, Mathematische Annalen 68 (1910), p. 422–434.

[8] G. Cantor, *Über unendliche lineare Punktmannigfaltigkeiten, 5. Fortsetzung*, ibidem 21 (1883), 545–590; reprinted in: G. Cantor, *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, Berlin 1980.

[9] J. L. Cornette, *Retracts of the pseudo-arc*, Colloquium Mathematicum 19 (1968), p. 235–239.

[10] A. Denjoy, *Continu et discontinu*, Comptes Rendus de l'Académie des Sciences 151 (1910).

[11] R. Duda, *On biconnected sets with dispersion points*, Rozprawy Matematyczne 37, Warszawa 1964.

[12] E. Dyer, *Irreducibility of the sum of the elements of a continuous collection of continua*, Duke Mathematical Journal 20 (1953), p. 589–592.

[13] E. E. Floyd, *Real-valued mappings of spheres*, Proceedings of the American Mathematical Society 6 (1955), p. 957–959.

[14] M. E. Hamstrom, *Concerning continuous collections of continuous curves*, Bulletin of the American Mathematical Society 58 (1952), p. 204.

[15] F. Hausdorff, *Grundzuge der Mengenlehre*, Leipzig 1914.

[16] H. Hopf, *Eine Verallgemeinerung bekannter Abbildungs- und Überdeckungssätze*, Portugaliae Mathematica 4 (1943–45), p. 129–139.

[17] Z. Janiszewski, *Thesis* (Paris 1911), Journal de l'Ecole Polytechnique II, 16 (1912), reprinted in: Z. Janiszewski, *Oeuvres choisies*, Warszawa 1962.

[18] –, *Sur les coupures du plan*, Prace Matematyczno-Fizyczne 26 (1913), p. 55.

[19] R. P. Jerrard, *On Knaster's conjecture*, Transactions of the American Mathematical Society 170 (1972), p. 385–402.

[20] K. Kuratowski, *Théorie des continus irréductibles entre deux points I*, Fundamenta Mathematicae 3 (1922), p. 200–231; *part II*, ibidem 10 (1927), p. 225–275.

[21] –, *Sur les coupures irréductibles du plan*, ibidem 6 (1924), p. 130–145.

[22] N. J. Lennes, *Curves in nonmetrical Analysis Situs with an application in the calculus of variations*, American Journal of Mathematics 33 (1911), p. 287–326.

[23] S. Mazurkiewicz, *Sur l'existence des continus indécomposables*, Fundamenta Mathematicae 25 (1935), p. 327–328.

[24] K. Menger, *Kurventheorie*, Leipzig–Berlin 1932.

[25] E. W. Miller, *Concerning biconnected sets*, Fundamenta Mathematicae 29 (1937), p. 123–133.

[26] E. E. Moise, *An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua*, Transactions of the American Mathematical Society 63 (1948), p. 581–594.

[27] –, *A theorem on monotone interior transformations*, Bulletin of the American Mathematical Society 55 (1949), p. 810–811.

[28] R. L. Moore, *A connected and regular point set which contains no arc*, ibidem 32 (1926), p. 331–332.

[29] F. Riesz, *Die Genesis des Raumbegriffes*, Mathematische und Wissenschaftliche Berichte aus Ungarn 24 (1907), p. 309–353; reprinted in: F. Riesz, *Oeuvres complètes I*, Paris 1960.

[30] A. Schoenflies, *Beitrage zur Theorie der Punktmengen I*, Mathematische Annalen 58 (1904), p. 195–244; *part II*, ibidem 59 (1904), p. 129–160; *part III*, ibidem 62 (1906), p. 286–326.

[31] W. Sierpiński, *Un théorème sur les continus*, The Tôhoku Mathematical Journal 13 (1918), p. 300–303.

[32] A. Tarski, *Quelques théorèmes généraux sur les images d'ensembles*, Annales de la Société Polonaise de Mathématique 6 (1927–28), p. 132–133.

[33] –, *Cardinal algebras*, New York 1949.

[34] P. Urysohn, *Une propriété des continus de B. Knaster*, Fundamenta Mathematicae 10 (1927), p. 175–176.

[35] G. T. Whyburn, *A continuum every subcontinuum of which separates the plane*, American Journal of Mathematics 52 (1930), p. 319–330.

[36] R. L. Wilder, *On the dispersion sets of connected point sets*, Fundamenta Mathematicae 6 (1924), p. 214–228.

[37] –, *Evolution of the topological concept of "connected"*, American Mathematical Monthly 85 (1978), p. 720–726.

[38] Chung-Tao Yang, *On maps from spheres to Euclidean spaces*, American Journal of Mathematics 79 (1957), p. 725–732.

[39] Kunizô Yoneyama, *Theory of continuous sets of points*, The Tôhoku Mathematical Journal 12 (1917), p. 43–158; see p. 60.

[40] L. Zoretti, *La notion de ligne*, Annales de l'Ecole Normale Supérieure 26 (1909).

Reçu par la Rédaction le 15. 10. 1984