Topology Comprehensive Exam
Summer 2005

Below are twelve problems. Do as many parts of as many problems as you can, giving complete details. Partial credit will be given for parts successfully completed. Each problem is worth of 10 points. You must get 70 points to pass.

The symbol $C_p(X)$ denotes the space of all continuous real-valued functions on X with the topology of pointwise convergence (that is, the topology inherited from $\mathbf{R}^X$). If X is a Tikhonov space, the symbol βX denotes the Čech – Stone compactification of X. In particular, $\beta\omega$ is the Čech – Stone compactification of the discrete countable space $\omega = \{0, 1, 2, \dots\}$. All compact spaces are assumed to be Hausdorff.

1. A topological space X has the *Suslin property* if there is no uncountable family of pairwise disjoint non-empty open subsets of X. Prove that every paracompact space with the Suslin property is Lindelöf.

2. Is the space of irrational numbers homeomorphic to an F_σ subset of the real line? ('F_σ' means 'the union of countably many closed sets'.)

3. Let X be a non-compact metric space. Prove that βX is not metrizable.

4. Let X be the metric space of all bounded sequences $(a_n)_(n = 1)^\infty$ of real numbers, with the metric defined by $d((a_n), (b_n)) = \sup\{|a_n - b_n| : n = 1, 2 \dots\}$. Let $Y \subset X$ be the subspace of all sequences that converge to zero. Determine whether X and Y are separable or not.

5. Let X be compact, $x \in X$, and suppose that the singleton $\{x\}$ is a G_δ-subset of X (that is, there exists a countable family γ of open subsets of X such that $\{x\} = \cap\gamma$). Prove that X has a countable base at x.

5. Let X be a separable Tikhonov space. Prove that every compact subspace of $C_p(X)$ is metrizable.

6. Consider the space $X = C_p(\mathbf{R})$, where $\mathbf{R}$ is the real line. For each of the following statements, determine whether it is true or false. Explain your answers.

a) X is metrizable;
b) X has a countable base;
c) X has a countable network;
d) X is first-countable;
b) X is Lindelöf;
c) X is separable;
d) X is normal.

7. Prove that the square of the Sorgenfrey line is not paracompact.

8. Prove that a compact space X is metrizable if and only if the Banach space $C(X)$ is separable. *Note.* $C(X)$ is the space of all continuous real-valued functions on X. The norm on $C(X)$ is defined by $\|f\| = \sup\{|f(x)| : x \in X\}$.

9. Is the property of being collectionwise normal preserved by continuous closed onto mappings?

10. Let X be a first-countable separable Hausdorff space. Prove that the cardinality of X does not exceed 2^ω. Is it true that the cardinality of every regular separable space does not exceed 2^ω?

11. Prove that the Boolean algebra of all clopen subsets of $\beta\omega$ is isomorphic to the Boolean algebra of all subsets of ω.

12. Prove that every compact subspace of the Sorgenfrey line is countable.