Connections of K-Theory to Geometry and Topology

Maxim Stykow

July 25, 2013

University of British Columbia, Vancouver, BC
E-mail address: `maxim@math.ubc.ca`

Abstract. Recent research in algebraic K-theory focusses on Waldhausen's construction of the K-theory spectrum and computations in homotopy theory using trace methods. The purpose of this article is to survey the more classical foundations always keeping in mind the connections to geometry and topology.

Chapter 1 reviews the construction and properties of topological K-theory and can be skipped by a reader familiar with this material. Chapter 2 then begins the exploration of geometric connections by asking for which dimensions $\mathbb{R}^n$ admits the structure of a division algebra, a question known as the Hopf invariant one problem. Related to this is the question of how many independent vector fields fit on the spheres S^n which we answer in chapter 3. Wall's finiteness obstruction, Whitehead torsion, the K-theory of schemes, and the geometric motivation for higher K-theory will be discussed in chapter 4 for which we will introduce algebraic K-theory of rings and categories. The last chapter discusses the equivariant story.

This monograph is work in progress. Please feel free to email me with feedback, suggestions or corrections.

Contents

CHAPTER 1

Basics of Topological K-Theory

1.1. K-Theory as a Cohomology Theory

In this section all spaces are assumed to be compact and Hausdorff and we will by default be dealing with complex vector bundles unless mentioned otherwise. Our first take on K-theory will be to make the direct sum operation on vector bundles into an addition operation in a group.

Let $\mathbf{Vect}(X)$ be the set of isomorphism classes of vector bundles over a space X. The trivial n-dimensional vector bundle we write as $\epsilon^n \to X$ or $n\epsilon$ or even just n to avoid confusion with the notation E^n which we will use to denote the n^{th} tensor power of a bundle E. We denote the space of sections of a bundle $E \to X$ by ΓE. We can form new bundles from old ones by operations from linear algebra such as direct sum, tensor product, and Hom. A morphism between bundles $p : E \to X$ and $q : F \to X$ in $\mathbf{Vect}(X)$ is a map $\phi : E \to F$ such that $q \circ \phi = p$, and the restriction $\phi_x : E_x \to F_x$ is a vector space homomorphism. Morphisms between E and F form a vector space isomorphic to $\Gamma \operatorname{Hom}(E, F)$.

Whitney sum of bundles gives $\mathbf{Vect}(X)$ the structure of an abelian monoid with zero element ϵ^0 and we apply the *Grothendieck construction* or *group completion* to $\mathbf{Vect}(X)$ to obtain an abelian group $K(X)$, called the *(complex) K-theory* of X.

Recall that the Grothendieck group M^+ of an abelian monoid M is the group of formal differences $m - n$ of elements of M, where $m - n \simeq m' - n'$ if and only if there is some $p \in M$ such that $m + n' + p = m' + n + p$ in M. Denote by $[m-n]$ the equivalence class of $m-n$. There is a natural inclusion $M \to M^+$ sending m to $[m - 0] =: [m]$ and addition in M^+ is defined in the obvious way $[m-n]+[m'-n'] = [(m+m')-(n+n')]$. The inverse of $[m-n]$ is then $[n - m]$ and M^+ is an abelian group (since M is abelian). M^+ also has the universal property that any monoid homomorphism $\psi : M \to G$ to a group G factors uniquely through the inclusion $m \mapsto [m]$.

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[d, hook] & G \\ M^+ \arrow[ur, dashed, "\exists!\phi"'] & \end{tikzcd}$$

The homomorphism ϕ is defined by $\phi([m-n]) = \psi(m) - \psi(n)$.

Alternatively, one can describe the Grothendieck group M^+ of an abelian monoid M as the free abelian group on generators $[m]$, $m \in M$, subject to the relations $[m+m'] = [m]+[m']$. This construction has the same universal property by defining $\phi([m]) = \psi(m)$ and extending by linearity. Applying the universal property of each of these descriptions to each other we conclude that the resulting group completions are isomorphic.

Example 1.1.1. Vector bundles over a point are trivial so $\mathbf{Vect}(x_0) = \mathbb{N}$ and $K(x_0) = \mathbb{Z}$.

Just like in singular cohomology, along with $K(X)$ we have a reduced version $\widetilde{K}(X)$ which is roughly $K(X)$ modulo trivial bundles. Let x_0 be a basepoint of X and define *reduced* K-theory

$$\widetilde{K}(X) = \ker(i^* : K(X) \to K(x_0))$$

where i^* is restriction of vector bundles to the basepoint. Let $c : X \to x_0$ be the constant map, then $i^* \circ c^* = \mathrm{id}$ so the exact sequence of abelian groups

$$0 \to \widetilde{K}(X) \to K(X) \xrightarrow{i^*} K(x_0) \cong \mathbb{Z} \to 0$$

splits and $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$. Note that this splitting is non-canonical unless X is a pointed space.

There is another interpretation for $\widetilde{K}(X)$ which we will need: say that two bundles E and E' are *stably isomorphic* if there exist trivial bundles ϵ^n and ϵ^m such that $E \oplus \epsilon^n \cong E \oplus \epsilon^m$. This is an equivalence relation, and we denote by $\mathcal{S}(X)$ the set of stable classes $\{E\}$ of bundles over X. We can give $\mathcal{S}(X)$ the structure of an abelian monoid by defining $\{E\}+\{E'\} = \{E+E'\}$ with zero element $\{\epsilon^n\}$ for any n. Even more is true:

Fact 1.1.2. *For each vector bundle $E \to X$ with X compact Hausdorff there exists a vector bundle $E' \to X$ such that $E \oplus E' \cong \epsilon^n$ for some n.* [**Hat09**, Proposition 1.4]

It follows that $\mathcal{S}(X)$ is an abelian group and one can show

Proposition 1.1.3. *Let X be a pointed compact space. Then $\widetilde{K}(X) \cong \mathcal{S}(X)$.* [**AGP02**, Theorem 9.3.8]

Note that by (1.1.2) every element in $K(X)$ can be represented by $[E-\epsilon^n]$ for some n since $[E-E'] = [(E \oplus E'') - (E' \oplus E'')] = [(E \oplus E'') - \epsilon^n]$ for some appropriate E''.

To ease notation, let us now drop the brackets from $[E]$ and just write E. Consider the ring structure of $K(X)$ induced by tensor product of vector bundles with identity the trivial bundle 1. We will write $(E_1 - E_1')(E_2 - E_2') = E_1 \otimes E_2 - E_1 \otimes E_2' + E_1' \otimes E_2' - E_1' \otimes E_2$. If X is a pointed space $\widetilde{K}(X)$ being the kernel of the ring homomorphism $i^* : K(X) \to K(x_0)$ is an ideal and thus also a ring in its own right.

Since pullback preserves direct sums and tensor product, $K(-)$ and $\widetilde{K}(-)$ become contravariant functors from the category of (pointed) compact spaces to commutative rings. That they are also functors on the homotopy category of (pointed) compact spaces follows from the following string of propositions.

Lemma 1.1.4. *Let Y be a closed subspace of a compact space X and let $E \to X$ be a vector bundle over X. Then any section of the restriction E_Y extends to a section of E.*

Proof. Apply Tietze's extension theorem to extend the given section locally and use compactness and partitions of unity to glue the local pieces to a global section. Details in [**AB64**, Lemma 1.1]. $\square$

Lemma 1.1.5. *Let Y be a closed subspace of a compact space X and let $E \to X$ and $F \to X$ be two vector bundles over X. Then any isomorphism $s : E_Y \to F_Y$ extends to an isomorphism $E_U \to F_U$ for some open U containing Y.*

Proof. We have mentioned before that morphisms between E_Y and F_Y are in one-to-one correspondence with sections of $\mathrm{Hom}(E,F)_Y$. Seeing s as a section of $\mathrm{Hom}(E,F)_Y$ we extend it to a section t of $\mathrm{Hom}(E,F)$ by the above lemma. Let U be the subset of X of points x such that t_x is an isomorphism. Then $Y \subseteq U$ and U is open because $\mathrm{GL}_n(\mathbb{C})$ is open in $\mathrm{End}(\mathbb{C}^n)$. $\square$

Proposition 1.1.6. *Let Y be a compact space, $f : Y \times I \to X$ be a homotopy and E a vector bundle over X. Then $f_0^*E \cong f_1^*E$.*

Proof. Apply the previous lemma to the bundles f^*E and $\pi^* f_t^* E$ and the subspace $Y \times t \subset Y \times I$ where $\pi : Y \times I \to Y$ is the projection. Clearly, the two bundles are isomorphic on this subspace. Hence they are also isomorphic on some strip $Y \times \delta t$ where δt is a neighborhood of t in I. But this means that the isomorphism class of f_t^*E is a locally constant function of t. Since I is connected it must in fact be constant and $f_0^*E \cong f_1^*E$. $\square$

Corollary 1.1.7. *A homotopy equivalence $f : A \to B$ of paracompact spaces induces a bijection $f^* : \mathbf{Vect}^n(B) \to \mathbf{Vect}^n(A)$. In particular, every vector bundle over a contractible paracompact base is trivial.*

We can also define an *external product* $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ by $a * b := \mu(a \otimes b) = p_1^*(a)p_2^*(b)$ where p_1 and p_2 are the projections of $X \times Y$ onto X respectively Y. One quickly checks that this is indeed a ring homomorphism.

Let us now begin the calculation of $K(X)$ in nontrivial cases. Of particular importance are the rings $K(S^n)$ for bundles over spheres.

Proposition 1.1.8. *There is a bijection between $\mathbf{Vect}^n(S^k)$ and the set $[S^{k-1}, \mathrm{GL}_n(\mathbb{C})]$ of homotopy classes of maps $S^{k-1} \to \mathrm{GL}_n(\mathbb{C})$.*

Proof. Given such a *clutching function* $f : S^{k-1} \to \mathrm{GL}_n(\mathbb{C})$, we construct a vector bundle E_f the usual way by glueing two copies of $D^k \times \mathbb{C}^n$ (the upper and lower hemisphere of S^k) along the equator $S^{k-1} \times \mathbb{C}^n = \partial D^k \times \mathbb{C}^n$ according to $(x, v) \to (x, f(x)v)$.

Going the other way, if $E \to S^k$ is any rank n vector bundle, the restriction to the upper respectively lower hemisphere $E_\pm$ is trivial by the previous fact. Let $h_\pm : E_\pm \to D^k \times \mathbb{C}^n$ be trivializations, then $h_+h_-^{-1}$ defines a map $S^{k-1} \to \mathrm{GL}_n(\mathbb{C})$ which yields a homotopy class $[h_+h_-^{-1}] \in [S^{k-1}, \mathrm{GL}_n(\mathbb{C})]$.

These constructions are inverses of each other. Moreover, E_f depends up to isomorphism only on the homotopy class of f, and h_+ and h_- are unique up to homotopy so that these constructions are indeed well-defined [**Hat09**, Proposition 1.11]. $\square$

Since $\mathrm{GL}_n(\mathbb{C})$ is connected, we get an immediate

Corollary 1.1.9. $\mathbf{Vect}^n(S^1) \cong \{\epsilon^n\}$ *so that $K(S^1) \cong \mathbb{Z}$ and $\widetilde{K}(S^1) = 0$.*

Example 1.1.10. Over $S^2 = \mathbb{C}P^1$ we have the canonical line bundle H. It satisfies $(H \otimes H) \oplus \epsilon^1 \cong H \oplus H$. So see this, let $f : S^1 \to \mathrm{GL}_1(\mathbb{C})$ be the clutching function of H given by $z \mapsto z$ and consider the clutching functions for both sides of the claimed relation. They are the maps $S^1 \to \mathrm{GL}_2(\mathbb{C})$ given by

$$(f \otimes f) \oplus \mathrm{id} : z \mapsto \begin{pmatrix} z^2 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad f \oplus f : z \mapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$$

It is now not difficult to construct a homotopy between $(f \otimes f) \oplus \mathrm{id}$ and $f \oplus f$: let $\alpha_t \in \mathrm{GL}_2(\mathbb{C})$ be a path from the identity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ to the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ of the transformation which swaps the two factors of $\mathbb{C} \times \mathbb{C}$. Then the matrix product $(f \oplus \mathrm{id})\alpha_t(\mathrm{id} \oplus f)\alpha_t$ gives a homotopy from $f \oplus f$ to $(f \otimes f) \oplus \mathrm{id}$.

In $K(S^2)$ this relation implies $H^2 + 1 = 2H$, or $(H-1)^2 = 0$, so we have a natural ring homomorphism $\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$. Tensoring with $K(X)$ and composing with the external product μ from above yields:

Theorem 1.1.11 (Product Theorem)**.** *The map $K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X \times S^2)$ is an isomorphism of rings for all compact Hausdorff spaces X.* [**Hat09**, Theorem 2.2]

Taking X to be a point we obtain:

Corollary 1.1.12. *$K(S^2) \cong \mathbb{Z}[H]/(H-1)^2$ as rings.*

Since $\widetilde{K}(S^2) = \ker(K(S^2) \to K(x_0))$, we see that $\widetilde{K}(S^2) \cong \langle L - 1 \rangle$ as an abelian group. Moreover, since $(H-1)^2 = 0$ the multiplication in $\widetilde{K}(S^2)$ is completely trivial.

We proceed to higher dimensional spheres. To do so, we need some more computational tools. In particular, we construct long exact sequences in K-theory. Let (X, A) be a pair of spaces. Define $\widetilde{K}(X, A)$ as $\widetilde{K}(X/A)$ taking A as the basepoint. Inclusion and quotient give the exact sequence

$$A \to X \to X/A$$

and applying $\widetilde{K}$ we get the sequence

$$\widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A).$$

Fact 1.1.13. *The above sequence in $\widetilde{K}$ is exact.* [**Hat09**, Proposition 2.9]

There is a nice way to extend the short exact sequence from above to the left: let C and S denote cone and suspension respectively and consider the following diagram.

$$\begin{tikzcd}[column sep=small]
A \arrow[r] & X \arrow[r] & X \cup CA \arrow[r] \arrow[d] & (X \cup CA) \cup CX \arrow[r] \arrow[d] & ((X \cup CA) \cup CX) \cup C(X \cup CA)) \arrow[d] \\
 & & X/A & SA & SX
\end{tikzcd}$$

The pattern is simple: each space in the first row is obtained from its predecessor by attaching a cone on the subspace two steps back in the sequence. The vertical maps are the quotient maps collapsing the newly attached cone. It is often true that collapsing a contractible subspace is a homotopy equivalence which would yield an isomorphism in $\widetilde{K}$. This is in fact true:

Fact 1.1.14. *If A is contractible, the quotient map $q : X \to X/A$ induces a bijection $q^* : \mathbf{Vect}^n(X/A) \to \mathbf{Vect}^n(X)$ for all n.* [**Hat09**, Lemma 2.10]

By (1.1.14) and repeated application of (1.1.13) we obtain a long exact sequence of $\widetilde{K}$ groups

$$\cdots \to \widetilde{K}(SX) \to \widetilde{K}(SA) \to \widetilde{K}(X/A) \to \widetilde{K}(X) \to \widetilde{K}(A). \tag{1}$$

Example 1.1.15. Let $X = A \vee B$ be the one-point union of A and B, then $X/A = B$ and the sequence breaks up into split short exact sequences so that $\widetilde{K}(A \vee B) \cong \widetilde{K}(A) \oplus \widetilde{K}(B)$.

Next, we would like to understand the K-theory of the suspension of a space. Recall that $\Sigma X = S \wedge X$ where Σ is reduced suspension and $X \wedge Y = X \times Y / X \vee Y$ is wedge product. Let x_0 be the basepoint of X. Since ΣX is the quotient space of SX obtained by collapsing $\{x_0\} \times I$ to a point, we have $\widetilde{K}(SX) \cong \widetilde{K}(\Sigma X)$ by (1.1.14). We are thus led to consider the $\widetilde{K}$ long exact sequence associated to the pair $(X \times Y, X \vee Y)$:

$$\begin{tikzcd}
\widetilde{K}(S(X\times Y)) \arrow[r] & \widetilde{K}(S(X\vee Y)) \arrow[r] \arrow[d, "\cong"] & \widetilde{K}(X\wedge Y) \arrow[r] & \widetilde{K}(X\times Y) \arrow[r] & \widetilde{K}(X\vee Y) \arrow[d, "\cong"] \\
& \widetilde{K}(SX) \oplus \widetilde{K}(SY) & & & \widetilde{K}(X) \oplus \widetilde{K}(Y)
\end{tikzcd}$$

The first vertical isomorphism follows from $\Sigma(X \vee Y) \approx \Sigma X \vee \Sigma Y$. The last horizontal map is a split surjection with splitting $(a, b) \mapsto p_1^*(a) + p_2^*(b)$ where p_1 and p_2 are the projections as per usual. We thus get a splitting $\widetilde{K}(X \times Y) \cong \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y)$.

Now consider the external product on $\widetilde{K}$. Let $a \in \widetilde{K}(X)$ and $b \in \widetilde{K}(Y)$. Then $a * b = p_1^*(a)p_2^*(b) \in K(X \times Y)$. By definition $p_1^*(a)$ restricts to zero over Y and $p_2^*(b)$ restricts to zero over X so that $a * b \in \widetilde{K}(X \times Y)$ and it restricts to zero in $\widetilde{K}(X) \oplus \widetilde{K}(Y)$. Thus $a * b$ can be seen as an element in $\widetilde{K}(X \wedge Y)$ and this means reducing the external product to $\widetilde{K}$ gives a ring homomorphism $\widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$. In fact, more is true: every statement about this reduced external product is equivalent to the same statement about the unreduced external product. This follows from the splitting we already mentioned: we have

$$K(X) \otimes K(Y) \cong (\widetilde{K}(X) \otimes \widetilde{K}(Y)) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z}$$

and

$$K(X \times Y) \cong \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z}.$$

So ring homomorphisms $K(X) \otimes K(Y) \to K(X \times Y)$ are determined by ring homomorphisms $\widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$ and vice versa.

Now taking Y to be S^2 we can state the Bott Periodicity Theorem:

Theorem 1.1.16 (Bott Periodicity)**.** *The morphism* $\beta : \widetilde{K}(X) \to \widetilde{K}(S^2X)$*,* $\beta(x) = (H-1) * x$ *where* H *is the canonical line bundle over* $S^2 = \mathbb{C}P^1$ *is a ring isomorphism for all compact Hausdorff spaces* X*.*

Proof. Recall that $H-1$ is the generator of $\widetilde{K}(S^2) \cong \mathbb{Z}$ so that β is the composition

$$\widetilde{K}(X) \xrightarrow{\cong} \widetilde{K}(S^2) \otimes \widetilde{K}(X) \xrightarrow{*} \widetilde{K}(S^2X).$$

Since the reduced external product corresponds to the unreduced external product, this is equivalent to the Product Theorem by the remarks immediately preceeding this theorem. □

Example 1.1.17. We have seen earlier that $\widetilde{K}(S^1) = 0$ and $\widetilde{K}(S^2) = \mathbb{Z}$. It follows by Bott periodicity that $\widetilde{K}(S^n)$ is $\mathbb{Z}$ for n even and 0 for n odd. In particular, we see that a generator of $\widetilde{K}(S^{2k})$ is $(H-1) * \cdots * (H-1)$ and that multiplication in $\widetilde{K}(S^{2k})$ is trivial since multiplication in $\widetilde{K}(S^2)$ is trivial.

Example 1.1.18. $\widetilde{K}(S^{2k} \wedge X) \cong \widetilde{K}(S^{2k}) \otimes \widetilde{K}(X)$ as rings. This follows from iterated Bott periodicity.

Example 1.1.19. $K(S^{2k} \times X) \cong K(S^{2k}) \otimes K(X)$ as rings. This follows from the previous example by the same argument that showed the equivalence of reduced and unreduced Bott periodicity. In particular, since $K(S^{2k}) \cong \mathbb{Z}[\alpha]/(\alpha^2)$, we have $K(S^{2k} \times S^{2l}) \cong \mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$.

Bott Periodicity allows us to turn $\widetilde{K}$-theory into a reduced cohomology theory in the sense of Eilenberg and MacLane as follows. Looking at the long exact sequence (1), we define $\widetilde{K}^{-n}(X) := \widetilde{K}(S^nX)$ and $\widetilde{K}^{-n}(X, A) = \widetilde{K}(S^n(X/A))$. Negative indices are chosen so that the "coboundary maps" increase dimension just as in ordinary cohomology. We also extend to positive degrees using Bott Periodicity by setting $\widetilde{K}^{2i}(X) = \widetilde{K}^0(X) = \widetilde{K}(X)$ and $\widetilde{K}^{2i+1}(X) = \widetilde{K}^1(X) = \widetilde{K}(SX)$. Then the long exact sequence rolls up into a six-term exact sequence.

$$\begin{tikzcd}
\widetilde{K}^0(X,A) \arrow[r] & \widetilde{K}^0(X) \arrow[r] & \widetilde{K}^0(A) \arrow[d] \\
\widetilde{K}^1(A) \arrow[u] & \widetilde{K}^1(X) \arrow[l] & \widetilde{K}^1(X,A) \arrow[l]
\end{tikzcd}$$

Let $\widetilde{K}^*(X) = \widetilde{K}^0(X) \oplus \widetilde{K}^1(X)$, then we define a product on this group as follows. First notice that a product $\widetilde{K}^i(X) \otimes \widetilde{K}^j(Y) \to \widetilde{K}^{i+j}(X \wedge Y)$ is obtained from the reduced external product by replacing X and Y by S^iX and S^jY respectively. Thus we get a product $\widetilde{K}^*(X)\otimes\widetilde{K}^*(X) \to \widetilde{K}^*(X\wedge X)$. We compose this with the map $\widetilde{K}^*(X\wedge X) \to \widetilde{K}^*(X)$ induced by the diagonal map $X \to X \wedge X$, $x \mapsto (x,x)$.

While multiplication in $\widetilde{K}(X)$ is commutative (tensor product of bundles), this is not the case in $\widetilde{K}^*(X)$:

Fact 1.1.20. *Multiplication in $\widetilde{K}^*(X)$ is graded commutative, i.e. $\alpha\beta = (-1)^{ij}\beta\alpha$ for $\alpha \in \widetilde{K}^i(X)$ and $\beta \in \widetilde{K}^j(X)$.* [**Hat09**, Proposition 2.14]

Use one-point compactification to extend the definition of K-theory to locally compact spaces without basepoints $K^n(X) = \widetilde{K}^n(X_+)$. The new $K^0(X)$ and the original $K(X)$ agree when X is already compact, in which case $X_+ := X \coprod *$ is the disjoint union with a point: extend a vector bundle on X by giving it the fiber zero at the point $*$; and conversely assign to a bundle $E \to X \coprod *$ the element $(E|_X) - (E_* \times X)$ in $K(X)$, where E_* is the fiber over the disjoint basepoint.

For $n = 1$ our definition yields $K^1(X) = \widetilde{K}^1(X_+) = \widetilde{K}(S(X_+)) \cong \widetilde{K}(SX \vee S^1) \cong \widetilde{K}(SX) \oplus \widetilde{K}(S^1) \cong \widetilde{K}(SX) = \widetilde{K}^1(X)$.

Finally, since $X_+ \wedge Y_+ = (X \times Y)_+$, the external product $\widetilde{K}^*(X) \otimes \widetilde{K}^*(Y) \to \widetilde{K}^*(X \wedge Y)$ gives a product $K^*(X) \otimes K^*(Y) \to K^*(X \times Y)$ and the ring structure on $K^*(X)$ is obtained similarly to that on $\widetilde{K}^*(X)$.

1.2. K-Theory as a Homotopy Theory

The definitions from the previous section have a homotopical interpretation. To avoid confusion, let Y be a pointed space and recall that unpointed maps from a space X to Y are the same as pointed maps between X_+ and Y, in symbols $[X, Y] = [X_+, Y]_*$. For the remainder of this section we will then use $[-,-]$ to denote homotopy classes of *pointed* maps of spaces adjoining a disjoint basepoint $+$ if needed.

Again let X be a compact Hausdorff space. By the classification theorem of vector bundles $\mathbf{Vect}^n(X)$ is naturally isomorphic to $[X_+, B\mathrm{U}(n)]$ where $B\mathrm{U}(n) \simeq G_n(\mathbb{C}^\infty)$. Note that $B\mathrm{U}(n)$ is connected and comes with a natural basepoint 1. Let $\gamma_n \to B\mathrm{U}(n)$ be the universal bundle over $B\mathrm{U}(n)$. Then the bundle $\gamma_n \oplus \epsilon^1 \to B\mathrm{U}(n)$ induces a classifying map $i_n : B\mathrm{U}(n) \to B\mathrm{U}(n+1)$. This map is an inclusion. More specifically, we can think of $B\mathrm{U}(n+1)$ as $G_{n+1}(\mathbb{C}^\infty \oplus \mathbb{C})$. Then i_n sends an n-plane p in $\mathbb{C}^\infty$ to the $(n+1)$-plane $p \oplus \mathbb{C}$. We can thus define $B\mathrm{U}$ to be the colimit of $B\mathrm{U}(n)$.

Theorem 1.2.1. *Give $\mathbb{Z}$ the discrete topology. For X compact and Hausdorff, there is a natural isomorphism $K(X) \cong [X_+, BU \times \mathbb{Z}]$, and for X a pointed, compact, and Hausdorff space, there is a natural isomorphism $\widetilde{K}(X) \cong [X, BU \times \mathbb{Z}]$.*

Proof. Since both functors send disjoint unions to cartesian products, we may assume X is connected. By the discussion after (1.1.2) we can see elements of $K(X)$ as formal differences $E - \epsilon^n$. The first isomorphism sends such an element of $K(X)$ to $(f, \operatorname{rank} E - n)$ where $f : X \to B\mathrm{U}(\operatorname{rank} E) \subset B\mathrm{U}$ is the classifying map.

Now let X be a pointed space. The second isomorphism follows from the first since $i^* : K(X) \to K(x_0) \cong \mathbb{Z}$ can be identified with the map $[X_+, B\mathrm{U} \times \mathbb{Z}] \to [S^0, B\mathrm{U} \times \mathbb{Z}]$ induced by the inclusion $S^0 \hookrightarrow X_+$ since $B\mathrm{U}$ is connected. This identified map has kernel $[X/S^0, B\mathrm{U} \times \mathbb{Z}] = [X, B\mathrm{U} \times \mathbb{Z}]$. $\square$

This theorem thus enables us to represent K-theory in the sense of Brown representability [**AGP02**, Theorem 12.2.22]. We can also use this interpretation to define K-theory for non-compact spaces. For X a space of the homotopy type of a CW complex, we define $K(X) = [X_+, B\mathrm{U} \times \mathbb{Z}]$ and if X is moreover a pointed space then $\widetilde{K}(X) = [X, B\mathrm{U} \times \mathbb{Z}]$.

We would hope that these spaces have a ring structure just like in the compact case. That this is indeed the case follows from the fact that $B\mathrm{U} \times \mathbb{Z}$ is a ring space up to homotopy [**May99**, p.201].

In this context, the Bott periodicity theorem (1.1.16) says that

$$[X, B\mathrm{U} \times \mathbb{Z}] \cong \widetilde{K}(X) \to \widetilde{K}(\Sigma^2 X) \cong [X, \Omega^2(B\mathrm{U} \times \mathbb{Z})]$$

is an isomorphism. Letting $X = B\mathrm{U} \times \mathbb{Z}$, this means that we have a homotopy equivalence

$$B\mathrm{U} \times \mathbb{Z} \simeq \Omega^2(B\mathrm{U} \times \mathbb{Z}).$$

Example 1.2.2. We can use this result to calculate the homotopy groups of $B\mathrm{U}$: first note that for $i \geq 0$

$$\begin{aligned}\pi_{i+2}(B\mathrm{U}) \cong \pi_{i+2}(B\mathrm{U} \times \mathbb{Z}) \cong \pi_i(\Omega^2(B\mathrm{U} \times \mathbb{Z})) &\cong \pi_i(B\mathrm{U} \times \mathbb{Z}) \\ &\cong \begin{cases} \mathbb{Z} & \text{if } i = 0; \\ \pi_i(B\mathrm{U}) & \text{if } i \geq 1. \end{cases}\end{aligned}$$

This means the homotopy groups of $B\mathrm{U}$ repeat with period two. Since $B\mathrm{U}$ is connected we get $\pi_0(B\mathrm{U}) = 0$ and from the above we get $\pi_{2n}(B\mathrm{U}) = \pi_2(B\mathrm{U}) \cong \mathbb{Z}$. We also have $\pi_1(B\mathrm{U}) = [S^1, B\mathrm{U}] = [S^0, \Omega\, B\mathrm{U}] = [S^0, U] =$

$\pi_0(U) = 0$ since U is connected and $\Omega\, B\mathrm{U} \simeq \mathrm{U}$. By periodicity, $\pi_{2n+1}(B\mathrm{U}) = 0$. Thus,

$$\pi_i(B\mathrm{U}) = \begin{cases} 0 & \text{if } i = 0; \\ \mathbb{Z} & \text{if } i > 0 \text{ is even}; \\ 0 & \text{if } i > 0 \text{ is odd}. \end{cases}$$

As before, we can extend the definition of these $\widetilde{K}$-groups to negative integers by defining $\widetilde{K}^{-n} = \widetilde{K}(\Sigma^n X)$ and then extending to positive integers using Bott periodicity. We can show that $\widetilde{K}^*$ thus defined satisfies the axioms of a reduced cohomology theory.

Recall that an Ω-*spectrum* consists of a collection of pointed spaces $\{P_n\}_{n\in\mathbb{Z}}$ and weak homotopy equivalences $P_n \xrightarrow{\sim} \Omega P_{n+1}$ called *structure maps*. Moreover, every Ω-spectrum gives rise to a reduced generalized cohomology theory defined by $\widetilde{k}^n(X) = [X, P_n]_*$. See [**AGP02**, Theorem 12.3.3] for details.

Example 1.2.3. Using what we have just discussed, we see that the family of spaces $P_{2n} = B\mathrm{U} \times \mathbb{Z}$ and $P_{2n+1} = \Omega(B\mathrm{U} \times \mathbb{Z}) \simeq \mathrm{U}$ for $n \in \mathbb{Z}$ forms an Ω-spectrum, namely the one giving rise to $\widetilde{K}$-theory.

CHAPTER 2

The Hopf Invariant One Problem

2.1. Division Algebras, Parallelizable Spheres, and H-Spaces

As a first application, we will use K-theory to prove Adams' theorem on the Hopf invariant which shows for which dimensions $\mathbb{R}^n$ admits the structure of a division algebra.

Recall that a *division algebra* is an algebra A over $\mathbb{R}$ without zero divisors. Here are four examples:

(1) $A = \mathbb{R}$ with the usual multiplication.

(2) $A = \mathbb{R}^2 = \mathbb{C}$ with the multiplication of complex numbers. Note that if we were to define a multiplication on $\mathbb{R}^2$ by $(a,b)(c,d) = (ac, bd)$ we would get zero divisors.

(3) $A = \mathbb{R}^4 = \mathbb{H}$ with the multiplication of Hamilton *quaternions*, i.e. if $1, i, j, k$ are the four basis vectors define $ij = k$, $jk = i$, $ki = j$, $i^2 = j^2 = k^2 = -1$. Another way to obtain these rules is via the *Cayley-Dickson construction* applied to ordered pairs of complex numbers: let $a + bj = (a,b) \in \mathbb{C} \times \mathbb{C}$ and define $(a,b)(c,b) = (ac - \bar{d}b, da + b\bar{c})$. Then for instance $ij = (i,0)(0,1) = (0,i)$ and $j(0,i) = (0,1)(0,i) = (0 - \bar{i}, 0) = (i,0) = i$. So by declaring $(0,i) =: k$ we have recovered the usual rules $ij = k$ and $jk = i$. While $\mathbb{C}$ was an associative and commutative algebra, $\mathbb{H}$ is only associative.

(4) $A = \mathbb{R}^8 = \mathbb{O}$ with the multiplication of Cayley *octonians*. This multiplication is defined via the Cayley-Dickson construction applied to pairs of quaternions. $\mathbb{O}$ is a nonassociative algebra.

Note how at each stage of applying the Cayley-Dickson construction we lose more and more nice properties. First commutativity, then associativity. One may ask whether we could apply the Cayley-Dickson construction ad infinitum to come up with more examples of division algebras. This is not the case. Applying the Cayley-Dickson construction to pairs of octonians, i.e. applying it to $\mathbb{R}^{16}$, we produce an algebra called the *sedonians*, $\mathbb{S}$, which contains zero divisors. Denoting the basis vectors of $\mathbb{R}^{16}$ by $1, e_1, \ldots, e_{15}$, the reader may wish to check that $(e_3 + e_{10})(e_6 - e_{15}) = 0$. That the above four examples are in fact the only four examples of division algebras coming from $\mathbb{R}^n$ is the content of the theorem we wish to prove in this chapter.

To get there, we begin with the following result:

Proposition 2.1.1. *If $\mathbb{R}^n$ has the structure of a division algebra, then S^{n-1} is parallelizable.*

Recall that this means that $T(S^{n-1}) = \{(x, y) \in S^{n-1} \times \mathbb{R}^n : \langle x, y\rangle = 0\} \to S^{n-1}$ is trivial.

Proof. We construct $n-1$ linearly independent sections of $T(S^{n-1})$. Choose a basis $\{1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$. Take $x \in S^{n-1}$ and define $v_i(x) = xe_i - \langle x, xe_i\rangle x$ for $i \geq 2$. Then $\langle x, v_i(x)\rangle = 0$, and so $(x, v_i(x)) \in T(S^{n-1})$. Since $1, e_2, \ldots, e_n$ are linearly independent, so are $x, xe_2, \ldots, xe_n$. Thus $v_2(x), \ldots, v_n(x)$ are also linearly independent. $\square$

From here, Bott and Milnor in [**BM58**], and independently Kervaire in [**Ker58**] proved that $n = 2, 4$, or 8 by using earlier work of Bott on the orthogonal groups O_n. However, we will use a different route by observing that parallelizable spheres have an additional structure, namely that of an H-space. Recall that an *H-space* is a topological space with a continuous multiplication map having a two-sided identity element. This is weaker than a topological group since we are neither assuming associativity nor inverses. From the above four examples, we see that S^1, S^3, and S^7 are H-spaces by restricting the respective multiplications to the respective unit spheres. Note how S^7 is not a topological group since it is not associative.

Proposition 2.1.2. *If S^{n-1} is parallelizable, then S^{n-1} is an H-space.*

Proof. Let $v_1, \ldots, v_{n-1}$ be linearly independent sections of the tangent bundle. By Gram-Schmidt, we may assume that they are orthonormal for all $x \in S^{n-1}$. For e_1 the first standard basis vector, we may also assume that $v_1(e_1), \ldots, v_{n-1}(e_1)$ are the standard basis vectors $e_2, \ldots, e_n$ by changing the sign of v_{n-1} if necessary to get the orientations right and then deforming the vector fields near e_1. Let $\alpha_x \in \mathrm{SO}(n)$ send the standard basis to $x, v_1(x), \ldots, v_{n-1}(x)$. Then the map $(x, y) \mapsto \alpha_x(y)$ defines an H-space structure since $(x, e_1) = \alpha_x(e_1) = x$ and $(e_1, x) = \alpha_{e_1}(x) = x$ since α_{e_1} is the identity map. $\square$

Next, we will define an invariant that is equal to ± 1 if a sphere admits an H-space structure and show that this invariant can take on the value ± 1 only if $n = 1, 2$, or 4 thus closing the circle of implications and showing that the only $\mathbb{R}^n$ division algebras are the ones we exposed with our four examples above.

We begin by showing that n has to be even:

Proposition 2.1.3. *S^{n-1} cannot be an H-space when $n > 1$ is odd.*

PROOF. Suppose $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$ is an H-space multiplication. Since $n-1$ is even, by example (1.1.19) we get an induced ring homomorphism $\mu^* : \mathbb{Z}[\gamma]/(\gamma^2) \to \mathbb{Z}[\alpha,\beta]/(\alpha^2,\beta^2)$. $\mu^*(\gamma)$ is of the form $r + p\alpha + q\beta + m\alpha\beta$ for $r,p,q,m \in \mathbb{Z}$. We know $0 = \mu^*(\gamma^2) = (\mu^*(\gamma))^2$. This leads to

$$r^2 + 2rp\alpha + 2rq\beta + 2(rm + pq)\alpha\beta = 0$$

so $r = 0$ and $pq = 0$. However, $p = q = 1$. This can be seen by considering the inclusions $i_k : S^{n-1} \to S^{n-1} \times S^{n-1}$ for $k = 1,2$ onto either of the subspaces $S^{n-1} \times \{e\}$ or $\{e\} \times S^{n-1}$. i_1^* sends α to γ and β to zero, and the other way around for i_2^*. But the composition $i_k^* \circ \mu^* = \mathrm{id}^*$ for $k = 1,2$ since μ is an H-space structure. Hitting $\mu^*(\gamma)$ with i_k^* we conclude $p = q = 1$ as claimed which is a contradiction. □

Next, suppose we are given a map $g : S^{n-1} \times S^{n-1} \to S^{n-1}$ such as an H-structure. We can then define an associated map $H(g) : S^{2n-1} \to S^n$ called the *Hopf construction* as follows: regard S^{2n-1} as $\partial(D^n \times D^n) = S^{n-1} \times D^n \cup D^n \times S^{n-1}$, and S^n as the union of two disks D^n_+ and D^n_-. The former is also known as the *join* $S^{n-1} * S^{n-1}$ and the latter as the reduced suspension ΣS^{n-1}. Then $H(g)$ is defined on $S^{n-1} \times D^n$ as $|y|g(x, y/|y|) \in D^n_+$ and on $D^n \times S^{n-1}$ as $|x|g(x/|x|, y) \in D^n_-$, or, if you like the join/suspension point of view, this is the same as $[x,t,y] \mapsto [g(x,y),t]$.

We now specialize to spheres S^{n-1} which admit an H-space structure g. We've seen that this means n must be even. So replace n by $2n$. Let $f = H(g) : S^{4n-1} \to S^{2n}$ and consider the mapping cone C_f. This is just S^{2n} with a $4n$-cell attached via f. The quotient C_f/S^{2n} is S^{4n} and we consider the rolled up six-term exact sequence for the pair (C_f, S^{2n}).

$$\begin{tikzcd}
\widetilde{K}^0(S^{4n}) \arrow[r] & \widetilde{K}^0(C_f) \arrow[r] & \widetilde{K}^0(S^{2n}) \arrow[d] \\
\widetilde{K}^1(S^{2n}) \arrow[u] & \widetilde{K}^1(C_f) \arrow[l] & \widetilde{K}^1(S^{4n}) \arrow[l]
\end{tikzcd}$$

Since $\widetilde{K}^1(S^{2n}) = \widetilde{K}^1(S^{4n}) = 0$ this reduces to the short exact sequence

$$0 \to \widetilde{K}(S^{4n}) \xrightarrow{p^*} \widetilde{K}(C_f) \xrightarrow{i^*} \widetilde{K}(S^{2n}) \to 0.$$

Let $b_{2k} = (H-1) * \cdots * (H-1)$ denote the generator of $\widetilde{K}(S^{2k})$. We let $\alpha = p^*(b_{4n}) \in \widetilde{K}(C_f)$ be the image of the generator of $\widetilde{K}(S^{4n})$ and we let $\beta \in \widetilde{K}(C_f)$ map to the generator of $\widetilde{K}(S^{2n})$, i.e. $i^*(\beta) = b_{2n}$. Since multiplication in $\widetilde{K}(S^{2n})$ is trivial $\beta^2 = \beta \otimes \beta$ maps to zero. Thus $\beta^2 = h(f)\alpha$ where $h(f) \in \mathbb{Z}$ is called the *Hopf invariant* of f. We need to show that it is independent of the choice of β. So suppose $i^*(\beta') = b_{2n}$. Then $i^*(\beta' - \beta) = 0$, and so $\beta' - \beta = p^*(mb_{4n}) = m\alpha$ for some $m \in \mathbb{Z}$ and $(\beta')^2 = \beta^2 + 2m\alpha\beta + m^2\alpha^2 = \beta^2 + 2m\alpha\beta$ since $\alpha^2 = p^*(b_{4n}^2) = 0$. So it suffices to show that $\alpha\beta = 0$. Since $i^*(\alpha) = 0$, $i^*(\alpha\beta) = 0$ and so $\alpha\beta = k\alpha$ for some $k \in \mathbb{Z}$. Multiply this equation by β to get $k\alpha\beta = \alpha\beta^2 = m\alpha^2 = 0$ so that $\alpha\beta = 0$ as required since $\alpha\beta$ lies in the torsion free subgroup $\widetilde{K}(S^{4n}) \subseteq \widetilde{K}(C_f)$.

An alternative definition of the Hopf invariant goes via cohomology: the mapping cone of a map $f : S^{2n-1} \to S^n$ where $n \geq 2$ has a single n-cell i and a single $2n$-cell j so that the differential in the cellular chain complex of C_f is zero for dimensional reasons. Hence $H^n(C_f; \mathbb{Z})$ is free abelian on $x = [i]$ and $H^{2n}(C_f; \mathbb{Z})$ is free abelian on $y = [j]$. Then $x \cup x = ky$ for some $k \in \mathbb{Z}$. In fact, $k = h(f)$. This can be shown by using the Chern character as we will explain later in these notes (see the discussion around (3.3.5) for details).

Example 2.1.4. As an example, we calculate the Hopf invariant of the Hopf fibration $f : S^3 \to S^2$ defined by $(z_1, z_2) \mapsto [z_1, z_2]$ under the identification $S^3 \subset \mathbb{C}^2$ and $S^1 \cong \mathbb{C}P^1$. This is precisely the attaching map of a 4-cell as in the construction of $\mathbb{C}P^2$. Since $H^*(\mathbb{C}P^2; \mathbb{Z}) = \mathbb{Z}[t]/t^3$ where t is the generator of $H^2(\mathbb{C}P^2, \mathbb{Z})$ it follows that $h(f) = 1$. From the higher Hopf bundles, we also get maps of Hopf invariant one from the attaching maps of the 8-cell and 16-cell of $\mathbb{H}P^2$ and $\mathbb{O}P^2$, respectively.

We are finally ready to connect H-space structures with a particular Hopf invariant:

Proposition 2.1.5. *If S^{2n-1} admits and H-space structure g, then the Hopf construction $f := H(g)$ has Hopf invariant ± 1.*

Proof. Let e be the identity element for the H-space structure, and let $\Phi : (D^{2n} \times D^{2n}, \partial(D^{2n} \times D^{2n})) \to (C_f, S^{2n})$ be the characteristic map of the $4n$-cell of C_f. Restricting Φ to $\{e\} \times D^{2n}$ respectively $D^{2n} \times \{e\}$ is precisely the attaching map $f = H(g)$ restricted to $\{e\} \times D^{2n}$ respectively $D^{2n} \times \{e\}$. But f restricted to these sets is the identity by the fact that g is an H-space structure (see the construction of $H(g)$ above). Thus these restrictions of Φ induce homeomorphisms of $\{e\} \times D^{2n}$ to D^{2n}_+ and $D^{2n} \times \{e\}$

to D^{2n}_- respectively. We thus obtain the following commutative diagram:

$$\begin{tikzcd}
\widetilde{K}(C_f) \otimes \widetilde{K}(C_f) \arrow[rr, "\otimes"] & & \widetilde{K}(C_f) \\
\widetilde{K}(C_f, D^{2n}_+) \otimes \widetilde{K}(C_f, D^{2n}_-) \arrow[u, "(1.1.14)", "\cong"'] \arrow[rr, "\otimes"] \arrow[d, "\Phi^*\otimes\Phi^*"', "\cong"] & & \widetilde{K}(C_f, S^{2n}) \arrow[u, "p^*"'] \arrow[d, "\cong"] \\
\widetilde{K}(D^{2n}\times\{e\}, \partial D^{2n}\times\{e\}) \otimes \widetilde{K}(\{e\}\times D^{2n}, \{e\}\times\partial D^{2n}) \arrow[rr, "\cong", "(1.1.18)"'] & & \widetilde{K}(D^{2n}\times D^{2n}, \partial(D^{2n}\times D^{2n}))
\end{tikzcd}$$

We now chase the diagram: starting with $\beta^2 = \beta \otimes \beta$ in the upper left ring, we can map this element to a generator of the ring in the bottom row of the diagram since β is an element mapping to the generator of $\widetilde{K}(S^{2n})$ by definition. This generator in turn gets mapped via p^* to $\pm\alpha$ again by definition, since α was defined to be the image of a generator of $\widetilde{K}(C_f, S^{2n})$. Thus by commutativity $\beta^2 = \pm\alpha$, which means that $H(f) = \pm 1$. $\square$

So for which n does there exist a map of Hopf invariant ± 1? This is the famous Hopf invariant one problem and here is the answer:

Theorem 2.1.6 (Adams' Theorem). *There exists a map $f : S^{4n-1} \to S^{2n}$ of Hopf invariant ± 1 only when $n = 1, 2,$ or 4.*

The proof of this theorem will occupy the rest of this chapter. Meanwhile, putting the four propositions and the final theorem of this section together we obtain:

Corollary 2.1.7. *The only values of n for which $\mathbb{R}^n$ is a division algebra are $n = 1, 2, 4,$ and 8. These cases are realized by $\mathbb{R}, \mathbb{C}, \mathbb{H},$ and $\mathbb{O}$ respectively.*

On a historical note, as mentioned above this result was known by the work of Bott and Milnor respectively Kervaire in 1958 before Adams solved the Hopf invariant one problem in 1960.

2.2. Adams Operations and the Splitting Principle

To proceed with the proof of Adams' theorem we need some tools that we will introduce in this section. We begin with the analog of *Steenrod operations* in K-theory. Here are their basic properties.

Theorem 2.2.1 (Adams Operations). *There exist ring homomorphisms $\psi^k : K(X) \to K(X)$, defined for all compact Hausdorff spaces X and all integers $k \geq 0$, and satisfying:*

(1) $\psi^k f^* = f^* \psi^k$ for all maps $f : X \to Y$ (naturality),
(2) $\psi^k(L) = L^k$ if L is a line bundle,

(3) $\psi^k\psi^l = \psi^{kl}$,
(4) $\psi^p(\alpha) \equiv \alpha^p \mod p$ *for* p *a prime,*
(5) $\psi^k(\alpha) = k^n\alpha$ *for* $\alpha \in \widetilde{K}(S^{2n})$ *a generator.*

Note that since ψ^k are ring homomorphisms, $\psi^k(L_1 \oplus \cdots \oplus L_n) = L_1^k + \cdots + L_n^k$. So property (2) characterizes the operations whenever E is a sum of line bundles. We would thus like a general definition for $\psi^k(E)$ that specializes to this formula when E is a sum of line bundles. That every vector bundle can be pulled back to a sum of line bundles is the content of the *splitting principle* which we shall discuss now.

Theorem 2.2.2 (Splitting Principle)**.** *Given a vector bundle $E \to X$ with X compact Hausdorff, there is a compact Hausdorff space $F(E)$ and a map $p : F(E) \to X$ such that the induced map $p^* : K^*(X) \to K^*(F(E))$ is injective and $p^*(E)$ splits as a sum of line bundles.*

Thus by the injectivity, if a statement is true for sums of line bundles, it is true for all bundles. In particular this means that properties (1) and (2) completely characterize the Adams operations. The following Leray-Hirsch type theorem for K-theory will be used in the proof.

Fact 2.2.3. *Let $E \to X$ be a rank n vector bundle and let H be the canonical line bundle over the projectivization $p : P(E) \to X$. Then $K^*(P(E))$ is the free $K^*(X)$-module with basis $\{1, H, \ldots, H^{n-1}\}$ and module structure induced by pullback p^*. Moreover,*

$$\sum_{i=0}^{n} (-1)^i \Lambda^i(E) H^i = 0$$

where $\Lambda^i(E)$ is the i^{th} exterior power bundle constructed from E. [**May99**, p. 206]

Proof of the Splitting Principle. If E has rank 1, there is nothing to prove. So suppose E has rank $n \geq 2$ and consider the *projectivization* $P(E) \xrightarrow{p} X$ of E. This is the bundle whose fiber at a point $p \in X$ is $P(E_p)$, the projectivization of the vector space E_p. Equivalently, this bundle is described by the transition functions $\hat{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{PGL}_n(\mathbb{C})$ induced from $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}_n(\mathbb{C})$. Thus, a point of $P(E)$ is a pair (p, l) where $p \in X$ and l is a line through the origin in E_p. Consider the pullback $p^*(E) \to P(E)$. The fiber over a point $(p, l) \in P(E)$ is E_p. $p^*(E)$ contains the canonical line bundle $H \to P(E)$ whose fiber at (p, l) is the collection of vectors in E_p that lie on the line l. Thus $p^*(E)$ splits as $H \oplus E'$ for

$E' \to P(E)$ the subbundle of $p^*(E)$ orthogonal to H with respect to some choice of inner product.

By the above fact, $K^*(X)$ is included in $K^*(P(E))$ as the part generated by $1 \in K^*(P(E))$. If E' is a line bundle we are thus done. If not, repeat the process and consider $P(E')$, splitting off another line bundle. A point of $P(E')$ over (p, l_1) in $P(E)$ is a triple (p, l_1, l_2) where l_2 is a line in the orthogonal complement of l_1 in E_p. After a finite number of repetitions we obtain the *flag bundle* $F(E) \to X$, whose points are n-tuples of orthogonal lines through the origin in fibers of E and the pullback of E over $F(E)$ splits as a sum of line bundles. $F(E) \to X$ induces an injection on K^* since it is a composition of maps with this property. The whole process may be visualized as follows:

$$\begin{tikzcd} E \arrow[d] & H_1 \oplus E_1 \arrow[d] & H_1 \oplus H_2 \oplus E_2 \arrow[d] & \cdots & H_1 \oplus H_2 \oplus \cdots \oplus E_{n-1} \arrow[d] \\ X & P(E) \arrow[l] & P(E_1) \arrow[l] & \cdots \arrow[l] & P(E_{n-2}) = F(E) \arrow[l] \end{tikzcd}$$

□

Note that this procedure also works for sums of vector bundles by pulling back to the flag bundle of one summand at a time and then composing the pullbacks.

Returning to the Adams operations, the idea is to use the exterior powers $\Lambda^k(E)$ which already satisfy many desirable properties:

(i) $\Lambda^k(E_1 \oplus E_2) \cong \bigoplus_{i+j=k} (\Lambda^i E_1 \otimes \Lambda^j E_2)$,
(ii) $\Lambda^0(E) = \epsilon^1$,
(iii) $\Lambda^1(E) = E$,
(iv) $\Lambda^k(E) = 0$ for $k > \operatorname{rank} E$,
(v) $f^*(\Lambda^i(E)) = \Lambda^i(f^*(E))$ for $f : X \to Y$.

Define $\lambda_t(E) = \sum_i \Lambda^i(E) t^i \in K(X)[t]$. This sum is finite by property (iv), and we can rewrite property (i) as $\lambda_t(E_1 \oplus E_2) = \lambda_t(E_1) \otimes \lambda_t(E_2)$. When E is a sum of line bundles L_i, then $\lambda_t(E) = \prod_i \lambda_t(L_i) = \prod_i (1 + L_i t)$ by properties (ii), (iii), and (iv). But $\prod_i (1 + L_i t) = \sum_i \sigma_i(L_1, \ldots, L_n) t^i$ where σ_i is the i^{th} elementary symmetric polynomial in the L_j's. Thus $\Lambda^i(E) = \sigma_i(L_1, \ldots, L_n)$ whenever $E = L_1 \oplus \cdots \oplus L_n$.

By the fundamental theorem on symmetric polynomials, every degree k symmetric polynomial can be expressed as a unique polynomial in $\sigma_1, \ldots, \sigma_k$. In particular, $\psi^k(E) = L_1^k + \cdots + L_n^k = s_k(\sigma_1(L_1, \ldots, L_n), \ldots, \sigma_k(L_1, \ldots, L_n))$, for some s_k called a *Newton polynomial*. For a general bundle E (not necessarily a sum of line bundles), we now set

$$\psi^k(E) := s_k(\Lambda^1(E), \dots, \Lambda^k(E)),$$

then this definition extends our observation for sums of line bundles.

So what are these Newton polynomials? First of all, they are independent of n as can be seen by setting $L_n = 0$ to go from n to $n-1$. To get a recursive formula for s_k, let $n = k$ and consider $(x+t_1)\cdots(x+t_k) = x^k + \sigma_1 x^{k-1} + \cdots + \sigma_k$. Now let $x = -t_i$, then $(-1)^{k-1}t_i^k = (-1)^{k-1}\sigma_1 t_i^{k-1} + (-1)^{k-2}\sigma_2 t_i^{k-2} + \cdots + \sigma_k$. Or equivalently, $t_i^k = \sigma_1 t_i^{k-1} - \sigma_2 t_i^{k-2} + \cdots + (-1)^{k-1}\sigma_k$. Summing over i we get

$$t_1^k + \cdots + t_k^k = \sigma_1 s_{k-1} - \sigma_2 s_{k-2} + \cdots + (-1)^{k-2}\sigma_{k-1}s_1 + (-1)^{k-1}k\sigma_k.$$

Here are the first few Newton polynomials:

$$\begin{aligned}
s_1 &= \sigma_1 \\
s_2 &= \sigma_1^2 - 2\sigma_2 \\
s_3 &= \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3 \\
s_4 &= \sigma_1^4 - 4\sigma_1^2\sigma_2 + 4\sigma_1\sigma_3 + 2\sigma_2^2 - 4\sigma_4
\end{aligned}$$

Armed with this definition, we now proceed to show that the Adams operations satisfy the claimed properties.

Proof of (2.2.1). Working in $\mathbf{Vect}(X)$, property (1) is a consequence of property (v) of exterior powers. $\psi^k(E_1 \oplus E_2) = \psi^k(E_1) + \psi^k(E_2)$ follows from the defining property of the Newton polynomial and the splitting principle by the remark right after its proof.

To see that ψ^k are also multiplicative, note that if E is the sum of L_i and E' the sum of L'_j, then $E \otimes E'$ is the sum of $L_i \otimes L'_j$. So by the splitting principle the following computation suffices: $\psi^k(E \otimes E') = \sum \psi^k(L_i \otimes L'_j) = \sum (L_i \otimes L'_j)^k = \sum L_i^k \otimes {L'_j}^k = (\sum L_i^k) \otimes (\sum {L'_j}^k) = \psi^k(E)\psi^k(E')$.

For property (3), the splitting principle and additivity reduce us to the case $\psi^k\psi^l(L) = L^{kl} = \psi^{kl}(L)$. Similarly for (4), $\psi^p(E) = L_1^p + \cdots + L_n^p \equiv (L_1 + \cdots + L_n)^p = E^p \mod p$.

Since ψ^k are additive, they descend to $K^(X)$ by the universal property. All other properties descend similarly. Since $\widetilde{K}(X)$ is the kernel of $i^* : K(X) \to K(x_0)$, ψ^k restricts to an operation on $\widetilde{K}(X)$ by naturality. ψ^k also behave well with respect to the external product since $\alpha * \beta$ was defined as $p_1^*(\alpha)p_2^*(\beta)$ and so once again by naturality we get $\psi^k(\alpha*\beta) = \psi^k(\alpha)*\psi^k(\beta)$.

We use this observation to prove property (v). First, consider the case $n = 1$. It suffices to show $\psi^k(\alpha) = k\alpha$ for α a generator of $\widetilde{K}(S^2)$. One such generator is $\alpha = H - 1$ as seen just after (1.1.12). Then $\psi^k(\alpha) = \psi^k(H - 1) = H^k - 1 = (1 + \alpha)^k - 1 = 1 + k\alpha - 1 = k\alpha$ where we have used property (2) and the fact that multiplication in $\widetilde{K}(S^2)$ is trivial. When $n > 1$, assume the desired formula holds in $\widetilde{K}(S^{2n-2})$. By Bott periodicity (external product) we have $\widetilde{K}(S^{2n}) \cong \widetilde{K}(S^2) \otimes \widetilde{K}(S^{2n-2})$. Thus it suffices to check the formula on the external product of the two generators $\alpha * \beta$. $\psi^k(\alpha * \beta) = \psi^k(\alpha) * \psi^k(\beta) = k\alpha * k^{n-1}\beta = k^n(\alpha * \beta)$. $\square$

We are now ready to prove Adams' theorem.

2.3. Proof of Adams' Theorem

Recall the setup. We have a map $f : S^{2n-1} \to S^n$ with Hopf invariant ± 1. That means we have the short exact sequence

$$\begin{array}{ccccc} \widetilde{K}(S^{4n}) & \xrightarrow{p^*} & \widetilde{K}(C_f) & \xrightarrow{i^*} & \widetilde{K}(S^{2n}) \\ b_{4n} & \longmapsto & \alpha, \beta & \longmapsto & b_{2n} \end{array}$$

such that $\beta^2 = \pm\alpha$.

Proof of Adams' Theorem (2.1.6). The proof boils down to a computation using Adams operations. We have $\psi^k(\alpha) = k^{2n}\alpha$ by naturality and property (5) of the Adams operations. We also have $i^*(\psi^k(\beta)) = k^n b_{2n}$ so that $\psi^k(\beta) - k^n\beta = \mu_k\alpha$ for some $\mu_k \in \mathbb{Z}$ since $\psi^k(\beta) - k^n\beta \in \ker i^*$. Thus

$$\psi^k\psi^l(\beta) = \psi^k(l^n\beta + \mu_l\alpha) = k^n l^n\beta + (k^{2n}\mu_l + l^n\mu_k)\alpha.$$

But $\psi^k\psi^l = \psi^{kl} = \psi^l\psi^k$. This means swapping k and l in the above line gives the same expression which can only be if the coefficient of α is the same under this swap, i.e. $k^{2n}\mu_l + l^n\mu_k = l^{2n}\mu_k + k^n\mu_l$ or equivalently

$$k^n(k^n - 1)\mu_l = l^n(l^n - 1)\mu_k. \tag{2}$$

Next, by property (4) we have $\psi^2(\beta) \equiv \beta^2 = h(f)\alpha \mod 2$. But we also just computed $\psi^2(\beta) = 2^n\beta + \mu_2\alpha$. So $\mu_2 \equiv h(f) \mod 2$. By assumption $h(f) = \pm 1$ so μ_2 must be odd (in fact this is true for $h(f)$ any odd number). Setting $k = 2$ and $l = 3$ in (2) we obtain $2^n(2^n - 1)\mu_3 = 3^n(3^n - 1)\mu_2$. Thus 2^n divides $3^n - 1$ since μ_2 is odd. Applying the fact from number theory below finishes the proof. $\square$

Fact 2.3.1. *If 2^n divides $3^n - 1$ then $n = 1, 2,$ or 4.* [**Hat09**, Lemma 2.22]

There is nothing mysterious about this fact. Writing $n = 2^l m$ with m odd, one shows by induction that the highest power of 2 dividing $3^n - 1$ is 2 for $l = 0$ and 2^{l+2} for $l > 0$. Then from this we have $n \leq l + 2$, so that $2^l \leq 2^l m = n \leq l + 2$, which means $l \leq 2$ and $n \leq 4$. The cases $n = 1, 2, 3, 4$ can be checked by hand.

CHAPTER 3

Vector Fields on Spheres

3.1. From Vector Fields to Stiefel Manifolds

Related to the parallelizability of spheres is the following question: what is the maximal number k of vector fields $X_1, \ldots, X_k$ on the n-dimensional sphere S^n such that $X_1(p), \ldots, X_k(p) \in T_pS^n$ are linearly independent for each $p \in S^n$? The goal of this chapter is to answer this question.

Example 3.1.1. We claim that we can find at least one nonvanishing vector field on all odd spheres. So suppose $n = 2k - 1$ is odd. Then we can regard S^n as $\{z \in \mathbb{C}^k : |z| = 1\}$ and notice that $iz \perp z$ since, intuitively, i corresponds to a 90 degree rotation. So $X(z) = iz$ is a nonvanishing vector field on S^n. To make this precise, we use the inner product (dot product) induced from $\mathbb{R}^{2k}$, i.e. writing $z = a + bi = (a, b)$ where $a, b \in \mathbb{R}^k$, we have $\langle z_1, z_2\rangle = a_1.a_2 + b_1.b_2$. Then notice that $i(a, b) = (-b, a)$. So $\langle X(z), z\rangle = \langle iz, z\rangle = \langle(-b, a), (a, b)\rangle = -(b.a) + a.b = 0$ so that $X(z) \perp z$ and so X is a nonvanishing vector field on S^n as claimed.

Example 3.1.2 (Hairy Ball Theorem)**.** On the contrary, suppose now that n is even and suppose that we have a nonvanishing vector field X on S^n. We may assume $X(p)$ is of unit length by dividing $X(p)$ by $|X(p)|$ if necessary which we can do since X is nonvanishing. Now consider the homotopy $h : I \times S^n \to S^n$ defined by $h_t(p) = p\cos(\pi t) + X(p)\sin(\pi t)$. That this is well-defined follows from $\langle h_t(p), h_t(p)\rangle = p.p\cos^2(\pi t) + X(p).X(p)\sin^2(\pi t) = \cos^2(\pi t) + \sin^2(\pi t) = 1$. Moreover, $h_0(p) = p$ and $h_1(p) = -p$. So h is a homotopy between the identity and the antipodal map. Thus the Brouwer degree (a homotopy invariant) of the antipodal map is 1. However, it is well known (see for instance [**Vic94**, Corollary 1.22]) that the antipodal map of an n-sphere has Brouwer degree $(-1)^{n+1}$ and since n is even this produces a contradiction. Thus, there are no nonvanishing vector fields on even spheres.

We continue with the discussion. By applying Gram-Schmidt, any k-tuple of everywhere linearly independent vector fields can be converted into a k-tuple of everywhere orthonormal vector fields. An orthonormal k-tuple of vectors $v_1, \ldots, v_k \in T_pS^n$ together with the point $p \in S^n$ constitute an orthonormal $(k+1)$-tuple $(p, v_1, \ldots, v_k) \in \mathbb{R}^{n+1}$ also known as a $(k+1)$-*frame* in $\mathbb{R}^{n+1}$. The set of all orthonormal $(k+1)$-frames in $\mathbb{R}^{n+1}$, denoted by $V_{k+1}(\mathbb{R}^{n+1})$, is also known as a *Stiefel manifold*. It is a manifold in the following way: every $(k+1)$-frame $(p, v_1, \ldots, v_k)$ can be completed to an orthonormal basis $(p, v_1, \ldots, v_k, w_1 \ldots, w_m)$ of $\mathbb{R}^{n+1}$ where $m = n - k$. The vectors in such an orthonormal basis constitute the column vectors of a matrix in $\mathrm{O}(n+1)$, the Lie group of $(n+1) \times (n+1)$ orthogonal matrices. The different choices of completing vectors $(w_1, \ldots, w_m)$ correspond to an orbit for the free action of the subgroup $\mathrm{O}(m) \subset \mathrm{O}(n+1)$ placed in the lower right hand corner. $V_{k+1}(\mathbb{R}^{n+1})$ is therefore the homogeneous space $\mathrm{O}(n+1)/\,\mathrm{O}(m) = \mathrm{O}(n+1)/\,\mathrm{O}(n-k)$.

Example 3.1.3. We have $\mathrm{O}(n+1)/\,\mathrm{O}(n) \approx V_1(\mathbb{R}^{n+1}) = S^n$ and $\mathrm{O}(n+1)/\,\mathrm{O}(n-1) \approx V_2(\mathbb{R}^{n+1}) = \{(p, v) \in S^n \times T_pS^n : |v| = 1\}$ is the subspace of unit tangent vectors of TS^n. Also, $V_n(\mathbb{R}^{n+1}) = \mathrm{O}(n+1)/\,\mathrm{O}(1) = \mathrm{SO}(n+1)$ since we can complete an n-frame to an $n+1$-frame in such a way that the resulting matrix has positive determinant. In fact, by the same argument $V_{k+1}(\mathbb{R}^{n+1}) = \mathrm{SO}(n+1)/\,\mathrm{SO}(n-k)$ whenever $k < n+1$.

Define $\pi : V_{k+1}(\mathbb{R}^{n+1}) \to S^n$ by $(p, v_1, \ldots, v_k) \mapsto p$. This map corresponds to $\mathrm{O}(n+1)/\,\mathrm{O}(m) \to \mathrm{O}(n+1)/\,\mathrm{O}(n)$ induced by the inclusion $\mathrm{O}(m) \subseteq \mathrm{O}(n)$. Thus π is a fiber bundle and a k-tuple of everywhere orthonormal vector fields $X_1, \ldots, X_k$ on S^n defines a section $\sigma : S^n \to V_{k+1}(\mathbb{R}^{n+1})$ taking p to the $(k+1)$-frame $(p, X_1(p), \ldots, X_k(p))$. So now the original question has become: what is the largest k for which the bundle $\pi : V_{k+1}(\mathbb{R}^{n+1}) \to S^n$ has a section?

3.2. Clifford Algebras and the Lower Bound

Let us from now on change our indexing slightly and ask to find k linearly independent vector fields on S^{n-1}. By the discussion above this is equivalent to asking for a section of $V_{k+1}(\mathbb{R}^n) \to S^{n-1}$. We continue with answering this question by constructing as many vector fields as possible using linear algebra.

Fix $k \geq 0$, the *Clifford algebra* C_k is the free associative algebra over $\mathbb{R}$ with generators $1, e_1, \ldots, e_k$ subject to the relations $e_ie_j + e_je_i = 0$ for $i \neq j$ and $e_i^2 = -1$.

Example 3.2.1. $C_0 = \mathbb{R}$, $C_1 \cong \mathbb{C}$ by identifying e_1 with $\pm i$, and $C_2 \cong \mathbb{H}$ with for instance $e_1 \mapsto i, e_2 \mapsto j, e_1e_2 \mapsto k$. Note that none of these isomorphisms are canonical.

A basis for C_k is given by the set of words $\{e_{i_1} \cdots e_{i_m} : m \geq 0, i_1 < \cdots < i_m\}$ made up of ordered nonrepeating sequences of generators. It follows that $\dim C_k = 1 + k + \binom{k}{2} + \cdots + \binom{k}{k} = \sum_{i=0}^{k} \binom{k}{i} = 2^k$. We also need the observation that the set $G_k := \{\pm e_{i_1} \cdots e_{i_m} : m \geq 0, i_1 < \cdots < i_m\}$ is a multiplicative subgroup of C_k. Then, given an algebra representation of C_k, i.e. a C_k-module structure on some n-dimensional vector space V, we can produce a G_k-invariant inner product $\sum_{g \in G_k} \langle g-, g- \rangle$ on V by taking any inner product $\langle -, - \rangle$ on V and averaging over the G_k-action. With this inner product we can define a sphere $S(V) \approx S^{n-1}$ and obtain

Proposition 3.2.2. *Let V be a faithful C_k-module. Then there is a G_k-invariant inner product on V such that the assignments $p \mapsto e_i p$ for $1 \leq i \leq k$ define a k-tuple of orthonormal vector fields on $S(V) \approx S^{n-1}$.*

Proof. We have $(p, e_i p) = (e_i p, -p) = -(p, e_i p)$ by symmetry of the inner product. But then $(p, e_i p) = 0$ and so $e_i p \in T_p S(V)$. Moreover, $(e_i p, e_i p) = 1$ by the G_k-invariance of the inner product and $(e_i p, e_j p) = (e_i e_j e_i p, e_i e_j e_j p) = (e_j p, -e_i p) = -(e_i p, e_j p)$ so that $(e_i p, e_j p) = 0$ and our vector fields are orthonormal. Hence $(p, e_1 p, \ldots, e_k p) \in V_{k+1}(V)$. □

We will thus set out to determine representations of C_k. We start this process by computing the algebras C_k. This can be done inductively by defining related Clifford algebras C'_k with generators $1, e'_1, \ldots, e'_k$ and relations $e'_i e'_j + e'_j e'_i = 0$ for $i \neq j$ and $e'^2_i = +1$.

Example 3.2.3. C'_1 has one generator whose square is 1. So $C'_1 \cong \mathbb{R}^2$ via for instance $e'_1 \mapsto (1, -1)$ where multiplication is defined by $(a, b)(c, d) = (ac, bd)$ (not a divison algebra). We can also show that $C'_2 \cong M_2(\mathbb{R})$, the group of (2×2)-matrices over $\mathbb{R}$. One isomorphism is given by $e'_1 \mapsto A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $e'_2 \mapsto B = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, i.e. reflection through two lines that are separated by 45 degrees. That this is an isomorphism follows from the fact that any real (2×2)-matrix can be written as a linear combination of $1, A, B$, and AB.

Examples (3.2.1) and (3.2.3) are all we need to compute the remaining Clifford algebras by the following

Lemma 3.2.4. *$C_{k+2} \cong C'_k \otimes_{\mathbb{R}} C_2$ and $C'_{k+2} \cong C_k \otimes_{\mathbb{R}} C'_2$.*

Proof. The first isomorphism is given by

$$e_i \mapsto \begin{cases} 1 \otimes e_i & \text{if } i = 1, 2; \\ e'_{i-2} \otimes e_1 e_2 & \text{if } i > 2. \end{cases}$$

This map is surjective since its image generates $C'_k \otimes C_2$. Moreover, range and domain have the same dimension so that we are dealing with an isomorphism. The second isomorphism is similar. □

We then compute the C_k inductively using the following standard isomorphisms of real algebras:

(1) $M_n(\mathbb{R}) \otimes A \cong M_n(A)$ where A is any $\mathbb{R}$-algebra;
(2) $M_n(\mathbb{R}) \otimes M_m(\mathbb{R}) \cong M_{nm}(\mathbb{R})$ induced by the isomorphism $\mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{nm}$;
(3) $\mathbb{H} \otimes \mathbb{C} \cong M_2(\mathbb{C})$ where we see $\mathbb{H}$ as a subalgebra of $M_2(\mathbb{C})$ by sending $(a, b) \in \mathbb{H}$ to $\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right)$. We now have three ways of seeing elements of $\mathbb{H}$. For example, $i = (i, 0) = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $j = (0, 1) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and $k = (0, i) = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$. We then send $\left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right) \otimes z \in \mathbb{H} \otimes \mathbb{C}$ to $\left(\begin{smallmatrix} za & zb \\ -z\bar{b} & z\bar{a} \end{smallmatrix}\right)$. That this is an isomorphism follows from the fact that the generators $1, A, B$, and AB of $M_2(\mathbb{C})$ as in (3.2.3) are mapped to by $(1, 0) \otimes 1, (i, 0) \otimes -i, (0, i) \otimes -i$, and $(0, 1) \otimes 1$ respectively;
(4) $\mathbb{H} \otimes \mathbb{H} \cong M_4(\mathbb{R})$ given by $\phi(z_1 \otimes z_2)z = z_1 z \bar{z}_2$ for $z \in \mathbb{R}^4 \cong \mathbb{H}$. To show that this is an isomorphism, it suffices to show that ϕ is surjective since the dimensions of source and target agree. That this is so follows from the fact that every real matrix with just one nonzero entry, the collection of which generate $M_4(\mathbb{R})$, is in the image of ϕ. For instance, $\phi(1 \otimes 1) = 1, \phi(i \otimes i)i = i, \phi(i \otimes i)j = -j, \phi(i \otimes i)k = -k$ and similar relations hold for $\phi(j \otimes j)$ and $\phi(k \otimes k)$. Then $\phi((1 \otimes 1 + i \otimes i + j \otimes j + k \otimes k)/4)$ maps 1 to 1 and i, j, k to zero. More computations for $\phi(i \otimes j), \phi(i \otimes k)$, and $\phi(j \otimes k)$ and linear combinations thereof can be used to construct the remaining required matrices.

The result is the following table:

k	C_k	C'_k
0	$\mathbb{R}$	$\mathbb{R}$
1	$\mathbb{C}$	$\mathbb{R}^2$
2	$\mathbb{H}$	$M_2(\mathbb{R})$
3	$\mathbb{H}^2$	$M_2(\mathbb{C})$
4	$M_2(\mathbb{H})$	$M_2(\mathbb{H})$
5	$M_4(\mathbb{C})$	$M_2(\mathbb{H})^2$
6	$M_8(\mathbb{R})$	$M_4(\mathbb{H})$
7	$M_8(\mathbb{R})^2$	$M_8(\mathbb{C})$
8	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})$
$k+8$	$M_{16}(\mathbb{R}) \otimes C_k$	$M_{16}(\mathbb{R}) \otimes C'_k$

The last line, which we will refer to as *periodicity* of the Clifford algebra, follows from $C_{k+8} \cong C_2 \otimes C_2' \otimes C_2 \otimes C_2' \otimes C_k \cong M_{16}(\mathbb{R}) \otimes C_k$.

We are now ready to look at the representations. Starting with $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$, we see that these are all skew fields and thus satisfy complete reducibility, i.e. it suffices to look at irreducible representations. Up to isomorphism there is only one irreducible representation of each of these algebras, namely the action of each algebra on itself. Moreover, the category of R-modules is equivalent to the category of $M_n(R)$-modules for any ring R where a given representation V of R induces a representation of $M_n(R)$ on V^n in the obvious way. Thus, looking at the above table and using periodicity there is precisely one irreducible representation of C_k for $k = 0, 1, 2, 4, 5, 6 \mod 8$.

Finally, the category of $(R \times S)$-modules is equivalent to the category of R-modules times the category of S-modules by defining an R-module $U := (1, 0) \cdot M$ and an S-module $V := (0, 1) \cdot M$ from an $(R \times S)$-module M. Thus, by looking again at our table we find that there are precisely two irreducible representations of C_k for $k = 3, 7 \mod 8$.

Writing $a_k = \min(\dim V : V \text{ is a representation of } C_k)$ and $\phi(k) = \log_2(a_k)$ we thus obtain the following table:

k	C_k	a_k	$\phi(k)$
0	$\mathbb{R}$	1	0
1	$\mathbb{C}$	2	1
2	$\mathbb{H}$	4	2
3	$\mathbb{H}^2$	4	2
4	$M_2(\mathbb{H})$	8	3
5	$M_4(\mathbb{C})$	8	3
6	$M_8(\mathbb{R})$	8	3
7	$M_8(\mathbb{R})^2$	8	3
8	$M_{16}(\mathbb{R})$	16	4
$k+8$	$M_{16}(\mathbb{R}) \otimes C_k$	$16a_k$	$\phi(k)+4$

Let's apply this to the sphere S^{n-1}. By (3.2.2) we know that there are k linearly independent vector fields on S^{a_k-1}. Likewise, if a_k divides n then there are k linearly independent vector fields on S^{ca_k-1} which corresponds to the direct sum of c copies of the smallest dimensional irreducible representation associated to C_k. We hence wish to find the largest k for which a_k divides n.

Let's write $n = m2^{c+4d}$ where m is odd and $0 \le c \le 3$. By the above table, $a_k = 2^{\phi(k)}$ so a_k divides n if and only if $\phi(k) \le c + 4d = \phi(b) + 4d$ for some $0 \le b \le 7$. But $\phi(b) + 4d = \phi(b + 8d)$ again by the above table. So we're looking for the largest k such that $\phi(k) \le \phi(b+8d)$. We can obviously achieve equality and maximize k for a given c by taking $b = 2^c - 1$ which we can again see from the above table. Let us write $\rho(n) = 2^c + 8d$ then the largest k such that a_k divides n is $k_{\max} = \rho(n) - 1$.

Thus, using Clifford algebras, we have succeeded in constructing $\rho(n)-1$ linearly independent vector fields on S^{n-1}. Here is a table for the first few cases.

$c + 4d$	0	1	2	3	4	5	6	7
$\rho(n) - 1$	0	1	3	7	8	9	11	15

We have proved

Theorem 3.2.5 (Hurwitz-Radon-Eckmann)**.** *There exist $\rho(n) - 1$ independent vector fields on S^{n-1}.*

Adams showed that this is in fact the best we can do. The rest of this chapter will be dedicated to proving that $\rho(n)-1$ is indeed an upper bound.

3.3. K-Theory of Projective Spaces

We will from now on need to distinguish between complex and real K-theory and use the notation K_F where F is $\mathbb{C}$ or $\mathbb{R}$. Also denote by H respectively L the complex respectively real canonical line bundle.

The purpose of this section is to prove the following

Proposition 3.3.1. *Let $\sigma(k)$ be the number of integers i such that $0 < i \le k$ and $i \equiv 0, 1, 2$ or $4 \mod 8$. Then $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^k) = \mathbb{Z}/2^{\sigma(k)}$ and is generated by $\lambda = L - 1$ subject to the two relations $\lambda^2 = -2\lambda$ and $\lambda^{\sigma(k)+1} = 0$.*

We will use this fact repeatedly to prove Adams' theorem but the calculations involved are interesting in their own rights. First, here is a table of the values $\sigma(k)$ can take.

k	0	1	2, 3	4, 5, 6, 7	8	$k + 8$
$\sigma(k)$	0	1	2	3	4	$\sigma(k) + 4$

Comparing this to the table for $\phi(k)$ from the section on Clifford algebras, we see that $\sigma(k) = \phi(k)$ and so the order of $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^k)$ is in fact a_k as defined in that section.

The cohomology of real projective space is well known. Here is a reminder.

$$H^p(\mathbb{R}P^{2k+1};\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0, 2k+1; \\ \mathbb{Z}/2 & \text{if } p \text{ is even, } 0 < p < 2k+1; \\ 0 & \text{ otherwise.} \end{cases}$$

$$H^p(\mathbb{R}P^{2k};\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0; \\ \mathbb{Z}/2 & \text{if } p \text{ is even, } 0 < p \leq 2k; \\ 0 & \text{ otherwise.} \end{cases}$$

$$H^p(\mathbb{R}P^{k};\mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } 0 \leq p \leq k; \\ 0 & \text{ otherwise.} \end{cases}$$

Also recall the following

Fact 3.3.2. *Let X have the homotopy type of a CW complex. Then the first Stiefel-Whitney class $w_1 : \mathbf{Vect}^1_{\mathbb{R}}(X) \to H^1(X;\mathbb{Z}/2)$ and the first Chern class $c_1 : \mathbf{Vect}^1_{\mathbb{C}}(X) \to H^2(X;\mathbb{Z})$ define ring isomorphisms, i.e. real respectively complex line bundles are characterized by them.* [**Hat09**, Proposition 3.10]

We begin with the complex case.

Proposition 3.3.3. *$K_{\mathbb{C}}(\mathbb{C}P^k) = \mathbb{Z}[H]/(H-1)^{k+1}$ where H is the canonical line bundle.*

Proof. Recall from the construction of the Adams operations that $\Lambda^i(E) = \sigma_i(L_1, \dots, L_n)$ whenever $E = L_1 \oplus \cdots \oplus L_n$ is a sum of line bundles. So $\sum_{i=0}^n (-1)^i \Lambda^i(E) H^i = \sum_{i=0}^n (-1)^i \sigma_i(L_1, \dots, L_n) H^i = \prod_{i=1}^n (H - L_i)$. Applying the Leray-Hirsch Theorem of K-theory (2.2.3) to the trivial bundle of rank $k+1$ over a point, we obtain that $K_{\mathbb{C}}(\mathbb{C}P^k)$ is generated as a $\mathbb{Z}$-module by H subject to the relation $\prod_{i=1}^{k+1} (H-1) = (H-1)^{k+1} = 0$ as required. □

Next we want to compute $\widetilde{K}_{\mathbb{C}}(\mathbb{R}P^k)$. For this we need three tools. First, we need a result connecting the complex and real canonical line bundles. Let $c : K_{\mathbb{R}}(X) \to K_{\mathbb{C}}(X)$ be complexification of vector bundles and let $\pi : \mathbb{R}P^{2k+1} \to \mathbb{C}P^k$ be the standard projection given by sending a real line to the complex line on which it lies. Then

Lemma 3.3.4. *Over $\mathbb{R}P^{2k+1}$, $cL = \pi^* H$ and this common element is nontrivial if $k > 0$.*

PROOF. The case $k = 0$ being trivial, suppose $k > 0$. By (3.3.2) complex line bundles over $\mathbb{R}P^{2k+1}$ are classified by their first Chern class $c_1 \in H^2(\mathbb{R}P^{2k+1};\mathbb{Z}) = \mathbb{Z}/2$. Since $c_1(\pi^* H) = \pi^*(c_1(H)) \neq 0$, it suffices to show that cL is nontrivial. Let $r : K_{\mathbb{C}}(X) \to K_{\mathbb{R}}(X)$ be induced by the map forgetting the complex structure, then we have $rc = 2$. But $rcL = 2L$ has nontrivial total Stiefel-Whitney class $1 + x^2$ where x is the generator of $H^*(\mathbb{R}P^{2k+1};\mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{2k+2})$ and so cL must be nontrivial as well. $\square$

The second tool establishes a connection between K-theory and cohomology. We'll define a ring homomorphism $ch : K_F^*(X) \to H^*(X;\mathbb{Q})$ called the *Chern character* and describe this for $K^*(X) = K_{\mathbb{C}}^*(X)$. The real case is similar.

For a line bundle $L \to X$, define

$$ch(L) = e^{c_1(L)} = 1 + c_1(L) + c_1(L)^2/2! + \cdots \in H^*(X;\mathbb{Q})$$

and so for a product of line bundles $ch(L_1 \otimes L_2) = e^{c_1(L_1 \otimes L_2)} = e^{c_1(L_1)+c_1(L_2)} = ch(L_1)ch(L_2)$ by (3.3.2). For the Chern character to land in the direct sum rather than direct product, we impose from now on that X be a finite CW complex or slightly more generally a finite cell complex. For a direct sum of line bundles $E = L_1 \oplus \cdots \oplus L_n$ we would like $ch(E) = \sum_i ch(L_i) = \sum_i e^{\alpha_i} = n + (\alpha_1 + \cdots + \alpha_n) + \cdots + (\alpha_1^k + \cdots + \alpha_n^k)/k! + \cdots$ where $\alpha_i = c_1(L_i)$ and are called the *Chern roots*. This looks reminiscent of the Newton polynomials we saw during the construction of the Adams operations (2.2.1). Indeed, there we saw that $\alpha_1^k + \cdots + \alpha_n^k = s_k(\sigma_1(\alpha_1, \ldots, \alpha_n), \ldots, \sigma_k(\alpha_1, \ldots, \alpha_n))$. But now $c_j(E) = \sigma_j(\alpha_1, \ldots, \alpha_n)$ since the total Chern class of E is $(1 + \alpha_1) \cdots (1 + \alpha_n) = 1 + \sigma_1(\alpha_1, \ldots, \alpha_n) + \cdots + \sigma_n(\alpha_1, \ldots, \alpha_n)$. Hence the preceeding formula can be rewritten as

$$ch(E) = \operatorname{rank} E + \sum_{k>0} s_k(c_1(E), \ldots, c_k(E))/k!.$$

All these results only hold for E a sum of line bundles. Since, however, this last formula makes sense for arbitrary vector bundles, we take this as the general definition extending the special case.

Note that the definition of ch is natural with respect to pullback of bundles. We can thus apply the splitting principle to check in exactly the same way as with the Adams operations that $ch : \mathbf{Vect}(X) \to H^{\text{even}}(X;\mathbb{Q})$ is also multiplicative and thus extends to a ring homomorphism $ch : K(X) \to H^{\text{even}}(X;\mathbb{Q})$.

Naturality also implies that ch behaves well with respect to external product, and that there is a reduced form $ch : \widetilde{K}(X) \to \widetilde{H}^{\text{even}}(X;\mathbb{Q})$

since these reduced rings are kernels of restriction maps. We extend to $ch : K^*(X) \to H^*(X;\mathbb{Q})$ by the following commutative diagram

$$\begin{tikzcd}
\widetilde{K}^0(SX) \arrow[r, "ch"] \arrow[d, "\cong"] & \widetilde{H}^{\mathrm{even}}(SX;\mathbb{Q}) \arrow[d, "\cong"] \\
K^1(X) \arrow[r, "ch"] & H^{\mathrm{odd}}(X;\mathbb{Q}).
\end{tikzcd}$$

Theorem 3.3.5. *Let X be a finite cell complex. The map $K^*(X)\otimes\mathbb{Q} \to H^*(X;\mathbb{Q})$ induced by the Chern character is an isomorphism.*

Proof. Recall (1.1.16). Since $ch((H-1)*x) = ch(H-1)ch(x)$ we have the following commutative diagram

$$\begin{tikzcd}
\widetilde{K}(X) \arrow[r, "\cong"] \arrow[d, "ch"] & \widetilde{K}(S^2X) \arrow[d, "ch"] \\
\widetilde{H}^*(X;\mathbb{Q}) \arrow[r, "\cong"] & \widetilde{H}^*(S^2X;\mathbb{Q}).
\end{tikzcd}$$

The bottom row is cup product with $ch(H-1) = ch(H)-1 = 1+c_1(H)-1 = c_1(H)$, a generator of $H^2(S^2;\mathbb{Z})$ and so by the Künneth formula (see for instance [**Hat02**, Theorem 3.16]) this map is an isomorphism.

Observe, then, that if we take $X = S^{2n}$, we get an isomorphism $\widetilde{K}(S^{2n}) \to H^{2n}(S^{2n};\mathbb{Z})$ by induction on n. Recalling (1.1.17), this isomorphism means $K^*(S^{2n})\otimes\mathbb{Q} \cong H^*(S^{2n};\mathbb{Q})$ and we have proved the result for spheres.

We now proceed by induction on the number of cells of X. The result is trivial for just one cell, a 0-cell. For the induction step, let X be obtained from a subcomplex A by attaching a cell. Apply the rationalized Chern character to the long exact sequence in K-theory associated to the pair (X,A) to obtain the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K^*(SA)\otimes\mathbb{Q} \arrow[r] \arrow[d] & K^*(X/A)\otimes\mathbb{Q} \arrow[r] \arrow[d] & K^*(X)\otimes\mathbb{Q} \arrow[r] \arrow[d] & K^*(A)\otimes\mathbb{Q} \arrow[r] \arrow[d] & K^*(SX/SA)\otimes\mathbb{Q} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^*(SA;\mathbb{Q}) \arrow[r] & H^*(X/A;\mathbb{Q}) \arrow[r] & H^*(X;\mathbb{Q}) \arrow[r] & H^*(A;\mathbb{Q}) \arrow[r] & H^*(SX/SA;\mathbb{Q}) \arrow[r] & \cdots .
\end{tikzcd}$$

The rows are exact since tensoring with $\mathbb{Q}$ preserves exactness. Recall that the "boundary map" in K-theory was defined via pullbacks (1.1.14) so that all squares commute by naturality of the Chern character. X/A and SX/SA are spheres, and SA is homotopy equivalent to a cell complex with the same number of cells as A by collapsing the suspension of a 0-cell. Thus by induction and having proved the case of spheres, we can apply the five-lemma to get that $K^*(X)\otimes\mathbb{Q} \to H^*(X;\mathbb{Q})$ is an isomorphism as well. $\square$

The third and final tool we need for the computation of $\widetilde{K}_{\mathbb{C}}(\mathbb{R}P^k)$ is a spectral sequence in K-theory.

Theorem 3.3.6 (Atiyah-Hirzebruch Spectral Sequence)**.** *Let X be a finite cell complex and let X^p be its p-skeleton. Let $K^n_F(X)$ be filtered by the groups $K^n_{F,p}(X) = \ker(K^n_F(X) \to K^n_F(X^{p-1}))$. Then there exists a right half-plane multiplicative spectral sequence $E_2^{p,q} = H^p(X, K^q_F(*)) \Rightarrow K^{p+q}_F(X)$ with $E_\infty^{p,q} = G_p K^{p+q}_F(X) = K^{p+q}_{F,p}(X)/K^{p+q}_{F,p+1}(X)$ the p^{th} graded piece. The differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ shifts degree by $(r, -r+1)$. The multiplication on the E_2 page is given by cohomology cup product.*

By Bott periodicity, the rings $K^q_F(*)$ are periodic with period 2 for $F = \mathbb{C}$ and period 8 for $F = \mathbb{R}$ and are given as follows.

q	0	1	2	3	4	5	6	7
$K^{-q}_{\mathbb{C}}(*)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$K^{-q}_{\mathbb{R}}(*)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0

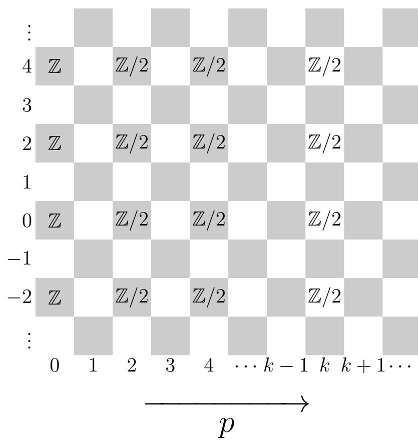
Recall (3.3.4). For odd real projective space let $\nu = c(L-1) = \pi^*(H-1) \in K_{\mathbb{C}}(\mathbb{R}P^{2k+1})$ and extend that definition to the even case by letting $\nu = i^*\nu \in K_{\mathbb{C}}(\mathbb{R}P^{2k})$ where $i : \mathbb{R}P^{2k} \to \mathbb{R}P^{2k+1}$ is the inclusion.

Proposition 3.3.7. *Let $f = \lfloor k/2 \rfloor$. Then $\widetilde{K}_{\mathbb{C}}(\mathbb{R}P^k) = \mathbb{Z}/2^f$ and is generated by ν subject to the relations $\nu^2 = -2\nu$ and $\nu^{f+1} = 0$.*

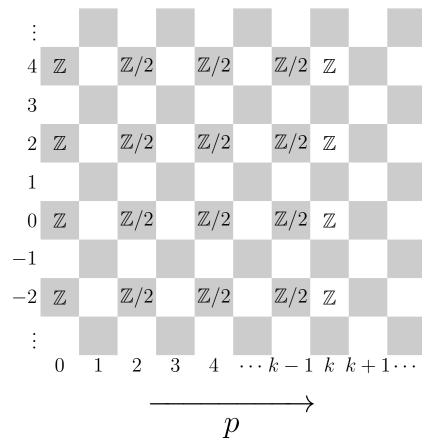
Proof. The case $k = 1$ being trivial, suppose $k > 1$ so that $\nu \neq 0$ by (3.3.4). Since tensor product commutes with pullback it suffices to show that the two relations hold for k odd, i.e. $\nu = \pi^*(H-1)$. Now $\nu^2 = -2\nu$ is equivalent to $(1+\nu)^2 = (cL)^2 = 1$, so it suffices to show that $L^2 = 1$. We either have $L^2 = 1$ or $L^2 = L$ since real line bundles are classified by their first Stiefel-Whitney class in $H^1(\mathbb{R}P^{2k+1}; \mathbb{Z}/2) = \mathbb{Z}/2$. The latter would imply the contradiction $L = 1$ since all line bundles are invertible. The other relation follows from $(H-1)^{f+1} = 0$ in $K_{\mathbb{C}}(\mathbb{C}P^f)$ (3.3.3).

Let's look at the E_2-page of the spectral sequence in complex K-theory for $\mathbb{R}P^k$. The entries are given by $H^p(\mathbb{R}P^k; K^q_{\mathbb{C}}(*))$.

$E_2^{p,q}$, k even



$E_2^{p,q}$, k odd



Recall that the differential d_r shifts degree by $(r, -r+1)$. For the even case, the checkerboard pattern forces all differentials to be zero since one of the integers $(r, -r+1)$ must be odd. Therefore $E_2^{p,q} = E_\infty^{p,q}$. The only possible difference in the odd case is that there may be some differentials (for example d_3) mapping from $\mathbb{Z}/2$ to $\mathbb{Z}$ in the last nonzero column. However, any such map is trivial and so the spectral sequence is trivial in the odd case also. Hence the associated graded ring to $\widetilde{K}_{\mathbb{C}}(\mathbb{R}P^k)$, $\bigoplus_{p\geq 1} E_\infty^{p,-p}$, is a direct sum of the $f = \lfloor k/2 \rfloor$ copies of $\mathbb{Z}/2$ on the diagonal of the E_2-page.

Since $ch(\nu) = \pi^*(ch(H-1)) = \pi^*(c_1(H))$, by the Chern character isomorphism (3.3.5) ν can be seen as the generator of $E_2^{2,-2} = H^2(\mathbb{R}P^{2k+1}; \mathbb{Z}) = \mathbb{Z}/2$. By naturality of the Chern character this is also true for $\mathbb{R}P^{2k}$ where $\nu = i^*\nu$. Hence the powers ν^i generate the successive $E_2^{2i,-2i}$ terms since multiplication on the E_2-page is given by cohomology cup product.

So now that we know all the quotients of the filtration $K_{\mathbb{C},p}(\mathbb{R}P^k) = \ker(K_{\mathbb{C}}(\mathbb{R}P^k) \to K_{\mathbb{C}}(\mathbb{R}P^{p-1}))$, it remains to inductively work our way back up to $K_{\mathbb{C},1}(\mathbb{R}P^k) = \widetilde{K}_{\mathbb{C}}(\mathbb{R}P^k)$

$$0 = K_{\mathbb{C},k+1}(\mathbb{R}P^k) \subseteq K_{\mathbb{C},k}(\mathbb{R}P^k) \subseteq \cdots \subseteq K_{\mathbb{C},1}(\mathbb{R}P^k) \subseteq K_{\mathbb{C}}(\mathbb{R}P^k).$$

We begin with the last nonzero term on the diagonal which is $E_\infty^{k,-k}$ in the even case and $E_\infty^{k-1,-(k-1)}$ in the odd case. Up to an index shift by 1, the argument is the same for the two cases from here on so let us focus on k even. Now $\mathbb{Z}/2 = E_\infty^{k,-k} = K_{\mathbb{C},k}(\mathbb{R}P^k)/K_{\mathbb{C},k+1}(\mathbb{R}P^k) = K_{\mathbb{C},k}(\mathbb{R}P^k)$. Moreover, $E_\infty^{k-1,-(k-1)} = K_{\mathbb{C},k-1}(\mathbb{R}P^k)/K_{\mathbb{C},k}(\mathbb{R}P^k) = 0$ so that $K_{\mathbb{C},k-1}(\mathbb{R}P^k) \cong K_{\mathbb{C},k}(\mathbb{R}P^k) = \mathbb{Z}/2$. This provides the base case $j = 0, 1$ of an induction that the group extensions

$$0 \to K_{\mathbb{C},k-(j-1)}(\mathbb{R}P^k) \hookrightarrow K_{\mathbb{C},k-j}(\mathbb{R}P^k) \twoheadrightarrow E_\infty^{k-j,-(k-j)} \to 0$$

yield isomorphisms

$$\begin{cases} K_{\mathbb{C},k-j}(\mathbb{R}P^k) \cong \mathbb{Z}/2^{j/2+1} & \text{when } j \text{ is even;} \\ K_{\mathbb{C},k-j}(\mathbb{R}P^k) \cong K_{\mathbb{C},k-(j-1)}(\mathbb{R}P^k) \cong \mathbb{Z}/2^{(j+1)/2} & \text{when } j \text{ is odd.} \end{cases}$$

The odd case is clear since $E_\infty^{k-j,-(k-j)} = 0$ whenever j is odd. For the even case, we have $K_{\mathbb{C},k-(j-1)}(\mathbb{R}P^k) \cong K_{\mathbb{C},k-(j-2)}(\mathbb{R}P^k) \cong \mathbb{Z}/2^{(j-2)/2+1} = \mathbb{Z}/2^{j/2}$ where the first isomorphism follows since $j-1$ is odd and the second isomorphism follows by the inductive hypothesis. Thus the group extension now becomes

$$0 \to \mathbb{Z}/2^{j/2} \hookrightarrow K_{\mathbb{C},k-j}(\mathbb{R}P^k) \twoheadrightarrow \mathbb{Z}/2 \to 0.$$

This means $K_{\mathbb{C},k-j}(\mathbb{R}P^k)$ can only be either $\mathbb{Z}/2^{j/2+1}$ or $\mathbb{Z}/2^{j/2} \oplus \mathbb{Z}/2$. However, as stated before $E_\infty^{k-j,-(k-j)} = \mathbb{Z}/2$ is generated by $\nu^{(k-j)/2}$ and $\mathbb{Z}/2^{j/2} = K_{\mathbb{C},k-(j-2)}(\mathbb{R}P^k)$ coming from the term $E_\infty^{k-(j-2)}$ is generated by $\nu^{(k-(j-2))/2} = \nu^{(k-j)/2+1}$. But now $\nu^{i+1} = -2\nu^i$ as follows from the relation $\nu^2 = -2\nu$. Hence, there is only one generator involved and so $K_{\mathbb{C},k-j}(\mathbb{R}P^k)$ must be isomorphic to $\mathbb{Z}/2^{j/2+1}$. This finishes the induction.

We then obtain $\widetilde{K}_{\mathbb{C}}(\mathbb{R}P^k)$ which is $K_{\mathbb{C},1}(\mathbb{R}P^k) = K_{\mathbb{C},k-(k-1)}(\mathbb{R}P^k) \cong K_{\mathbb{C},k-(k-2)}(\mathbb{R}P^k) \cong \mathbb{Z}/2^{(k-2)/2+1} = \mathbb{Z}/2^{k/2} = \mathbb{Z}/2^f$ since $f = \lfloor k/2 \rfloor$ and k is even. This finishes the proof. □

From here, our goal for this section to calculate $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^k)$ is finally within reach.

Proof of (3.3.1). We examine the diagonal $E_2^{p,-p} = H^p(\mathbb{R}P^k; K_{\mathbb{R}}^p(*))$ for $p \geq 1$ on the E_2-page of the spectral sequence in real K-theory to obtain information about the associated graded ring to $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^k)$. By Bott periodicity, we see that the only nonzero terms on the diagonal occur for $p \equiv 0, 1, 2, \text{ or } 4 \mod 8$ and that all of them are $\mathbb{Z}/2$. It follows that there are $\sigma(k)$ copies of $\mathbb{Z}/2$ on this diagonal and hence there are at most $2^{\sigma(k)}$ elements in the group $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^k)$.

Consider the complexification homomorphism $c : \widetilde{K}_{\mathbb{R}}(\mathbb{R}P^k) \to \widetilde{K}_{\mathbb{C}}(\mathbb{R}P^k)$. By (3.3.7) $\widetilde{K}_{\mathbb{C}}(\mathbb{R}P^k)$ is generated by $\nu = c\lambda$ and so c is an epimorphism for all k. Additionally, for $k \equiv 6, 7, 8 \mod 8$ we have $f = \lfloor k/2 \rfloor = \sigma(k)$. Indeed, if $k = 6 + 8d$ or $k = 7 + 8d$ then $\sigma(k) = 3 + 4d = f$ as we saw right after the statement of (3.3.1). Similarly, if $k = 8d$ then $\sigma(k) = 4d = f$. Hence in those cases $\widetilde{K}_{\mathbb{C}}(\mathbb{R}P^k)$ contains $2^f = 2^{\sigma(k)}$ elements and so all the nonzero E_2 terms on the diagonal survive to the E_∞-page and c must be an isomorphism and λ a generator for $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^k) = \mathbb{Z}/2^{\sigma(k)}$.

For the other cases, there is always some $K > k$ such that $K \equiv 6, 7$, or $8 \mod 8$. The inclusion $\mathbb{R}P^k \hookrightarrow \mathbb{R}P^K$ induces a map of spectral sequences and it follows that the E_2 terms survive in those cases as well and that $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^k) = \mathbb{Z}/2^{\sigma(k)}$ with generator λ for all k.

Finally, $\lambda^2 = -2\lambda$ follows from $L^2 = 1$ which we proved in (3.3.7) and $\lambda^{\sigma(k)+1}$ follows from this first relation and the fact that $2^{\sigma(k)}\lambda = 0$. $\square$

3.4. The Upper Bound

We begin with a definition: two vector bundles E and E' over a common base space X are said to be *fiber homotopy equivalent* if there exists a bundle map $\theta : E \to E'$ such that the restriction $S(E) \to S(E')$ is a homotopy equivalence over X, i.e. the homotopies in question are through maps that send fibers to fibers. This implies in particular the weaker condition that for each point $p \in X$, the map $\theta_p : S(E)_p \to S(E')_p$ is a homotopy equivalence. It is a theorem of Dold [**Jam76**, Theorem 4.2] that a bundle map inducing a homotopy equivalence on each fiber is a fiber homotopy equivalence so long as E and E' have the homotopy type of CW complexes and X is path-connected.

Recall that we're trying to find an upper bound for the number of vector fields on a sphere and reduced the problem to finding the largest k for which $V_{k+1}(\mathbb{R}^n) \to S^{n-1}$ has a section. Suppose a section s exists. Then we can define a map $\hat{s} : S^k \times S^{n-1} \to S^k \times S^{n-1}$ by sending (v, p) to $(v, s(p)v)$. Indeed, we regard S^k as a subspace of $\mathbb{R}^{k+1}$ and recall that $V_{k+1}(\mathbb{R}^n) = \mathrm{O}(n)/\mathrm{O}(n-k-1)$ so that $s(p)$ acts on v as an $(n \times (k+1))$-matrix acting on a $(k+1)$-vector. Moreover, since $s(p)$ is an orthogonal matrix, it is true that $|s(p)v| = 1$ whenever $|v| = 1$. But even more is true: taking the $\mathbb{Z}/2$-Borel quotient on the target of this map we determine that $\hat{s}(-v, p) = (-v, -s(p)v) = (v, s(p)v) = \hat{s}(v, p)$ so that $\hat{s}$ descends to a map $\mathbb{R}P^k \times S^{n-1} \to S^k \times_{\mathbb{Z}/2} S^{n-1}$.

Note that $\mathbb{R}P^k \times S^{n-1}$ is just the sphere bundle of the real trivial bundle ϵ^n (also written $n\epsilon$ or simply n) over $\mathbb{R}P^k$ and that $S^k \times_{\mathbb{Z}/2} S^{n-1}$ is the sphere bundle of nL (direct sum of n copies of L) where $L = \{(l, x) \in \mathbb{R}P^k \times \mathbb{R}^{k+1} : x \in l\} = S^k \times_{\mathbb{Z}/2} \mathbb{R}$ is the real canonical line bundle over $\mathbb{R}P^k$. This proves the first part of

Proposition 3.4.1. *Suppose S^{n-1} admits k linearly independent vector fields. Then there is a bundle map $n \to nL$ over $\mathbb{R}P^k$, which is a fiber homotopy equivalence.*

Proof. We've seen earlier that the question of finding vector fields on spheres is equivalent to finding a section of the Stiefel manifold. Given a section s, we just showed above how to construct a bundle map $\hat{s} : S(n) \to S(nL)$ and we can extend this to a bundle map $n \to nL$ by *radial extension* $(v, p) \mapsto (v, |p|\hat{s}(p/|p|))$. It remains to show that this extension is a fiber homotopy equivalence. By Dold's theorem, it suffices to show that $\hat{s}_v : S^{n-1} \to S^{n-1}$, $p \mapsto s(p)v$ is a homotopy equivalence for each $v \in \mathbb{R}P^k$. For this in turn to be true, it suffices to show that $\hat{s}_v$ is homotopic to the identity. First note that $\hat{s}_v \simeq \hat{s}_{e_1}$ where e_1 is the first basis vector in $\mathbb{R}^{k+1}$ since S^k is path-connected. Also note that $s(p)e_1 = p$ by the way we constructed the homeomorphism $V_{k+1}(\mathbb{R}^n) \approx \mathrm{O}(n)/\,\mathrm{O}(n-k-1)$. Thus $\hat{s}_{e_1} = \mathrm{id}$ and so $\hat{s}_v \simeq \mathrm{id}$ as required. □

In this case, nL is said to be fiber homotopy trivial. In fact, we can define an equivalence relation on vector bundles by saying that bundles E and E' are *J-equivalent*, written $E \simeq_J E'$, if and only if there exist integers n, m such that $S(E \oplus \epsilon^n)$ has the same fiber homotopy type as $S(E' \oplus \epsilon^m)$. That is, the idea is similar to that of $\widetilde{K}$ just that now we are dealing with sphere bundles instead of vector bundles and fiber homotopy equivalences instead of bundle isomorphisms. The set of J-equivalence classes of real vector bundles over a base space X is denoted by $J(X)$. There is a natural functor $J : \widetilde{K}_{\mathbb{R}}(X) \to J(X)$.

If E and E' are two bundles over X, then $S(E \oplus E') = S(E) * S(E')$ where $*$ denotes the *fiber join* of sphere bundles. This is the equivalent notion of Whitney sum of vector bundles in the category of sphere bundles. The join of two topological spaces X and Y is defined as $X * Y = X \times I \times Y/\sim$ where $(x, 0, y) \sim (x, 0, y')$ and $(x, 1, y) \sim (x', 1, y)$. The product of two spheres isn't another sphere, the join of two spheres, however, is. This can best be seen by the description $X * Y \approx \partial(CX \times CY)$. The fiber join is then obtained by taking the join $S(E)_p * S(E')_p$ on each fiber and then pulling back along the diagonal map $X \to X \times X$.

Now the join of two homotopy equivalences is again a homotopy equivalence since $(f * g)[x, t, y] = [f(x), t, g(y)]$. It follows that if $E_1 \simeq_J E_2$ and $E_1' \simeq_J E_2'$ then $E_1 \oplus E_1' \simeq_J E_2 \oplus E_2'$ so that direct sum grants $J(X)$ the structure of an abelian group with zero element $J(\epsilon^0)$. It also follows that J is a surjective group homomorphism. The celebrated result is

Theorem 3.4.2 (Adams). *$J : \widetilde{K}_{\mathbb{R}}(\mathbb{R}P^k) \to J(\mathbb{R}P^k)$ is an isomorphism.*

From this we finally get the answer to our initial question.

Corollary 3.4.3. *There are at most $\rho(n) - 1$ linearly independent vector fields on S^{n-1}.*

Proof. We have shown that $nL \to \mathbb{R}P^k$ is fiber homotopy trivial, i.e. $nL \simeq_J n$, when S^{n-1} admits k linearly independent vector fields. By Adams' theorem this implies $nL \simeq n$ in $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^k)$. Equivalenty, $n(L-1) = 0$ in the formal difference notation of unreduced K-theory. But by (3.3.1) $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^k)$ is of order a_k and generated by $(L-1)$. Hence a_k divides n. Hence, the largest k for which S^{n-1} admits k linearly independent vector fields is the largest k for which a_k divides n which we have seen to be $\rho(n) - 1$ when we proved theorem (3.2.5). □

By combining (3.2.5) and (3.4.3), we get that there are precisely $\rho(n)-1$ linearly independent vector fields on S^{n-1}. Notice that it thus follows that $\rho(n) = n$ if and only if S^{n-1} is parallelizable. A quick check then reveals

Corollary 3.4.4 (Bott, Milnor; Kervaire). *The only parallelizable spheres are S^0, S^1, S^3, and S^7.*

We already saw another proof of this result in (2.1.6) from Chapter 2. As mentioned there, this result was known before either of Adams' theorems.

There are two different ways in the literature in which Adams' theorem is proved. We follow [**Jam76**, Chapter 9], [**Kar78**, Chapter V], and [**Sha11**]. For the other proof see [**Ada62**], [**Gor70**] or [**Nor01**].

Proof Sketch of (3.4.2). The idea is to define characteristic classes $\rho_F^m(E) \in K_F(E)$ of F-vector bundles E where F is $\mathbb{R}$ or $\mathbb{C}$. The construction of these classes is similar to that of the Stiefel-Whitney classes except that one uses the Adams operations in lieu of Steenrod squares and the Thom isomorphism in K-theory rather than cohomology.

Recall that the Thom space of an n-vector bundle $E \to X$ is defined as $T(E) = Sph(E)/X$ where $\xi : Sph(E) \to X$ is the bundle obtained from E by taking one-point compactification on each fiber. We also have a projection map $\pi : Sph(E) \to T(E)$. By composing the product of ξ and π with the diagonal map on $Sph(E)$ we obtain a map $Sph(E) \to Sph(E) \times Sph(E) \to X \times T(E)$. Note that this map sends all points at ∞ to $X \times \{\infty\}$. Thus, it factors through a map $\Delta : T(E) \to X_+ \wedge T(E)$ called the *Thom diagonal.* In reduced complex K-theory this induces a map $\widetilde{K}(X_+ \wedge$

$T(E)) \to \widetilde{K}(T(E))$ and by external product in reduced complex K-theory a map $K(X) \otimes \widetilde{K}(T(E)) \to \widetilde{K}(T(E))$.

There is a special element $\lambda_E \in \widetilde{K}(T(E))$ termed the *Thom class* whose characterizing property is that it restricts to a generator of $\widetilde{K}(T(E_p)) = \widetilde{K}(S^n_p)$ for each $p \in B$.

Theorem 3.4.5 (Thom Isomorphism Theorem)**.** *The Thom diagonal induces a map $\Phi : K(X) \to \widetilde{K}(T(E))$, $\Phi(x) = x * \lambda_E$ which is a $K(X)$-module isomorphism.*

We see that $\lambda_E = \Phi(1)$. The same theorem holds true for real K-theory when one restricts to $\mathrm{Spin}(8d)$-bundles, that is, bundles with vanishing first and second Stiefel-Whitney classes whose rank is a multiple of 8.

We define $\rho^m_F : \mathbf{Vect}_F(X) \to K_F(X)$ by $\rho^m_F(E) = \Phi^{-1}\psi^m_F(\lambda_E)$, implicitly restricting the domain of definition to $\mathrm{Spin}(8d)$-bundles for $F = \mathbb{R}$. In honor of their discoverer, these classes are called *Bott classes*. Sometimes they are also referred to as *cannibalistic classes* as their input is a characteristic class itself (the Thom class).

Let L be the canonical line bundle over $\mathbb{R}P^k$. Then as an example:

Lemma 3.4.6. *For m odd, $\rho^m_{\mathbb{R}}(4dL \oplus 4d) = m^{4d}(1 + \frac{m^{2d}-1}{2m^{2d}}\lambda)$ where $\lambda = L - 1$ is the generator of $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^k)$.*

Lemma 3.4.7. *Let E be a $\mathrm{Spin}(8d)$-bundle over $\mathbb{R}P^k$, such that the bundles $8d$ and E are fiber homotopy equivalent. Then for m odd, $\rho^m_{\mathbb{R}}(E) = m^{4d}$.*

These lemmata enable us to prove Adams theorem. By (3.3.1) $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^k)$ is generated by $L-1$. So suppose $J(n(L-1)) = 0$, i.e. nL is fiber homotopy trivial. We wish to show that a_k, the order of $\widetilde{K}_{\mathbb{R}}(\mathbb{R}P^k)$, divides n. We begin with the observation that fiber homotopy equivalent bundles have the same Stiefel-Whitney classes as can be seen from their definition in terms of the Thom isomorphism and Steenrod squares. As discussed before, the total Stiefel-Whitney class of L, $w(L)$, is $1 + x$ where x is the generator of $H^*(\mathbb{R}P^{2k+1}; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^{2k+2})$. By fiber homotopy triviality we require that $w(nL) = (1 + x)^n \equiv 1 = w(n) \mod 2$. But $(1 + x)^n = 1 + nx + n(n-1)/2x^2 + \cdots$. Thus if $k = 0$, then n must be a multiple of $2 = a_1$. If $k > 0$, then n must be a multiple of 4, say $n = 4d$. Note that by (3.4.1) $4dL \oplus 4d$ is a $\mathrm{Spin}(8d)$-bundle fiber homotopy equivalent to $8d$. By the previous lemmata, this implies $\frac{m^{2d}}{2}(m^{2d} - 1)\lambda = 0$ for all odd m. But then $\frac{m^{2d}}{2}(m^{2d} - 1) \equiv 0 \mod a_k$. We've seen that $a_k = 2^{\phi(k)}$ so this means $(m^{2d} - 1) \equiv 0 \mod 2^{\phi(k)+1}$ since m is odd. Now let $m = 3$ and write $n = 4d = 2 \cdot 2d$ and $2d = 2^l p$

with p odd. Since $l \geq 1$, we have seen in the discussion after (2.3.1) that the largest power of 2 dividing $3^{2d} - 1$ is $l + 2$. This means $l + 2 \geq \phi(k) + 1$ and so $n = 4d = 2(2^l p) = 2^{l+1}p \geq 2^{\phi(k)}p \geq 2^{\phi(k)} = a_k$. That is, a_k divides n as required.

$\square$

CHAPTER 4

Geometry and Topology in Algebraic K-Theory

4.1. K_0 of Rings, Swan's Theorem, Wall's Finiteness Obstruction

Now that we have seen some of the power of the K-functor applied to the category of topological spaces, let us see how we can generalize it to other categories and draw applications from it. Historically, Grothendieck first defined K-theory on the category of schemes to study algebraic vector bundles which led to what is now called *algebraic* K-theory. The content of Swan's theorem, our first application, is that topological K-theory is just a special case of algebraic K-theory.

Unless otherwise stated, in this chapter let R be an associative ring with unit. Recall that an R-module P is said to be *projective* if there exists another R-module Q such that $P \oplus Q$ is a free R-module. If additionally we require that P be finitely generated then $P \oplus Q \cong R^n$ for some n. The set $\mathbf{P}(R)$ of isomorphism classes of finitely generated projective R-modules forms an abelian monoid under direct sum. We define the zeroth *algebraic K-theory* of a ring R, $K_0(R)$, as the Grothendieck group (§1.1) of $\mathbf{P}(R)$.

The reason we use a subscript 0 is that K_0 is a covariant functor from rings to abelian groups. To see this, start with a ring homomorphism $\phi : R \to S$. We then have an induced homomorphism $\phi_* : \mathbf{P}(R) \to \mathbf{P}(S)$, $P \mapsto S \otimes_R P$. Indeed, if P is a finitely generated projective R-module, then $(S \otimes_R P) \oplus (S \otimes_R Q) \cong S \otimes_R (P \oplus Q) \cong S \otimes_R (R^n) \cong S^n$ and so $S \otimes_R P$ is a finitely generated projective S-module. ϕ_* is a homomorphism since tensor product commutes with direct sum and so ϕ_* descends to the group completion to yield a homomorphism $K_0(R) \to K_0(S)$.

Example 4.1.1. If $R = F$ is a field, then a finitely generated projective F-module is just a finite dimensional F-vector space and so $\mathbf{P}(F) \cong \mathbb{N}$ and $K_0(F) \cong \mathbb{Z}$. Similarly, over a principal ideal domain every projective module is free by the structure theorem for modules over principal ideal domains. Hence $K_0(\mathbb{Z}) \cong \mathbb{Z}$ and $K_0(F[x]) \cong \mathbb{Z}$ where F is a field.

Just like in the topological case, we can define the zeroth *reduced* algebraic K-theory of a ring R. For any ring R with unit, there is a unique ring homomorphism $\iota : \mathbb{Z} \to R$ sending 1 to the unit of R. We define

$$\widetilde{K}_0(R) = \operatorname{coker}(\iota_* : K_0(\mathbb{Z}) \cong \mathbb{Z} \to K_0(R)).$$

As before, $\widetilde{K}_0(R)$ measures the non-obvious part of $K_0(R)$, i.e. $K_0(R)$ modulo free modules. Since projective modules satisfy the equivalent of fact (1.1.2) by definition, the proof of (1.1.3) goes through as before to show that $\widetilde{K}_0(R)$ are stable classes of finitely generated projective R-modules. In particular, if $[P] = 0 \in \widetilde{K}_0(R)$ then P is *stably free*, i.e. its direct sum with some free module is free.

Let $F = \mathbb{R}$ or $\mathbb{C}$. To state Swan's theorem, recall that the *sheaf of sections* of an F-vector bundle $p : E \to X$ is $\Gamma(U, E) := \{s : U \to p^{-1}(U) \text{ continuous} : p \circ s = \mathrm{id}_U\}$ for every open $U \subseteq X$. To avoid confusion, recall that the sections of a sheaf $\mathcal{F}$ over an open subset U are the elements of $\mathcal{F}(U) =: \Gamma(U, \mathcal{F})$. Once we identify a vector bundle E with its sheaf of sections there will be no ambiguity in using the same letter E for both of these and the notation $\Gamma(U, E)$ will be consistent.

Example 4.1.2. The sheaf of sections of the trivial line bundle $X \times F \to X$ is the sheaf $\mathcal{O}_X$ of continuous functions of X. That is, for every open $U \subseteq X$ one has $\mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_X) = \{f : U \to F : f \text{ is continuous}\}$.

Note that the sheaf of sections $U \mapsto \Gamma(U, E)$ is moreover an $\mathcal{O}_X$-module (i.e. an abelian sheaf $\mathcal{F}$ with a pairing $\mathcal{O}_X \otimes \mathcal{F} \to \mathcal{F}$ so that $\mathcal{F}(U)$ has the structure of an $\mathcal{O}_X(U)$-module). The module structure is given by $(f \cdot s)(x) = f(x)s(x)$.

Theorem 4.1.3 (Swan)**.** *Let X be a compact Hausdorff space. Then there is an equivalence of categories between $\mathbf{Vect}_F(X)$ and $\mathbf{P}(\Gamma(X, \mathcal{O}_X))$ which descends to an isomorphism $K^0_F(X) \to K_0(\Gamma(X, \mathcal{O}_X))$.*

Proof. Given a bundle E, we consider the global sections $\Gamma(X, E)$. We need to show that this is a finitely generated projective $\Gamma(X, \mathcal{O}_X)$-module and that every such module arises this way. Finite generation is given locally over some U by the basis vectors $e_1, \dots, e_n$ of F^n since E is locally trivial. By compactness, we can cover X by finitely many such open sets U_i and choose a partition of unity $\{f_i\}$ subordinate to this covering. Then $e_{ij} := f_i e_j$ is supported on U_i and extends to all of X by setting $e_{ij}(x) = 0$ for $x \notin U_i$. By construction the finitely many e_{ij} generate $\Gamma(X, E)$ as a $\Gamma(X, \mathcal{O}_X)$-module.

To see that $\Gamma(X, E)$ is projective, recall that by (1.1.2) there is a bundle E' such that $E \oplus E' \cong X \times F^k$ for some k. Hence $\Gamma(X, E) \oplus \Gamma(X, E') \cong \Gamma(X, E \oplus E') \cong \Gamma(X, \mathcal{O}_X)^k$.

Conversely, we start with a finitely generated projective module P. Then there exists a module Q such that $P \oplus Q \cong \Gamma(X, \mathcal{O}_X)^n$ for some n. Thus, we can regard P as a collection of functions $X \to F^n$ and define

$$E := \{(x, v_1, \ldots, v_n) \in X \times F^n : \exists s \in P \text{ with } s(x) = (v_1, \ldots, v_n)\}.$$

Let $p : E \to X$ be projection onto the first factor. The fibers are F-vector spaces since P is a $\Gamma(X, \mathcal{O}_X)$-module. It only remains to check local triviality. We do this by constructing local sections. Given $x \in X$, choose $f^1, \ldots, f^k \in P$ forming a basis at x of the subspace $E_x = p^{-1}(x)$ of F^n. Express $f^i = (f^i_1, \ldots, f^i_n)$ in terms of the standard basis vectors of F^n. Then linear independence of $f^1(x), \ldots, f^k(x)$ is equivalent to being able to find integers $1 \leq j_1 < \cdots < j_r \leq n$ such that the determinant of

$$\begin{pmatrix} f^1_{j_1} & f^2_{j_1} & \cdots & f^r_{j_1} \\ f^1_{j_2} & f^2_{j_2} & \cdots & f^r_{j_2} \\ \vdots & \vdots & & \vdots \\ f^1_{j_r} & f^2_{j_r} & \cdots & f^r_{j_r} \end{pmatrix}$$

is nonzero at x. Similarly, we can follow the above procedure for the complementary module Q and find $g^1, \ldots, g^{n-k} \in Q$ forming a basis at x of the complementary vector subspace of F^n such that the determinant of the corresponding matrix is also nonzero at x. Since determinants are continuous, there is some neighborhood U of x where neither determinant vanishes. For each y in this U $f^1(y), \ldots, f^k(y)$ are linearly independent and generate a k-dimensional subspace of F. We then have $E|_U \cong U \times F^k$ since the complement contains the $(n-k)$-dimensional subspace generated by $g^1(y), \ldots, g^{n-k}(y)$.

These constructions are inverses of each other and so we have an equivalence of categories which descends to an isomorphism in the respective K-theories since both constructions are additive. $\square$

Our next application addresses the following question: when is a space homotopy equivalent to a finite CW complex? It is known that all compact manifolds are. This is clear for piecewise linear, and smooth manifolds since they admit triangulations by finite simplicial complexes. More generally we consider *finitely dominated spaces*, that is, spaces which are deformation retracts of finite CW complexes. In fact, every finitely dominated space is homotopy equivalent to a CW complex but not necessarily a finite one (see [**Hat02**, Proposition A.11] or [**Mil59**] for the original). Consequently, we ask when is a finitely dominated CW complex homotopy equivalent to a finite one?

Examples 4.1.4. Here are some finitely dominated spaces:

(1) Every *absolute neighborhood retract* (ANR) and in particular every *euclidean neighborhood retract* (ENR) is a deformation retract of a finite simplicial complex [**Hat02**, Corollary A.8].

(2) Every compact topological manifold X is an ENR [**Hat02**, Corollary A.9]. Thus, every compact topological manifold is finitely dominated. Alternatively, one can also first show using Morse theory that such X has the homotopy type of a CW complex K. Let $f : X \to K$, $g : K \to X$ be such a homotopy equivalence. Then $f(X)$ is contained in a finite subcomplex $Q \subseteq K$, since X is compact [**Hat02**, Corollary A.1], and X is finitely dominated by Q via f and $g|_Q$.

So suppose that X is a path-connected CW complex finitely dominated by K. We are thus given maps $K \underset{i}{\overset{r}{\rightleftarrows}} X$ such that $r \circ i \simeq \mathrm{id}_X$. Starting from K and r, we are going to construct another finite complex $\bar{K}$ and a weak homotopy equivalence $\bar{r} : \bar{K} \to X$. By Whitehead's theorem $\bar{r}$ will be a homotopy equivalence.

We begin by producing an isomorphism of fundamental groups.

Proposition 4.1.5. *For $r : K \to X$ a finite domination of CW complexes, we may attach finitely many 2-cells to K to form $\bar{K}$ and extend r to $\bar{r} : \bar{K} \to X$ such that $\bar{r}$ induces an isomorphism of fundamental groups.*

Proof. The map $r_* : \pi_1(K) \to \pi_1(X)$ is surjective since $r \circ i \simeq \mathrm{id}_X$. We will succeed if we can attach finitely many 2-cells to K to kill $\ker(r_*)$. For this to work we need $\ker(r_*)$ to be finitely generated.

This is indeed so: let $\{g_i\}$ be a finite set of generators for $\pi_1(K)$ ($\pi_1(K)$ is generated by the 1-skeleton of K which is finite), and let $\alpha = i_* \circ r_* : K \to K$. We claim that the normal closure $P^{\pi_1(K)}$ of $P = \{g_i\alpha(g_i^{-1})\}$ generates $\ker r_*$. To see this, first note that

$$r_*(g_i\alpha(g_i^{-1})) = r_*(g_i)r_*d_*r_*(g_i^{-1}) = r_*(g_i)r_*(g_i^{-1}) = 1$$

since $r_* \circ d_* = \mathrm{id}_*$. Thus $P^{\pi_1(K)} \subseteq \ker(r_*)$. To show the reverse inclusion, we first note that every $g\alpha(g^{-1}) \in P^{\pi_1(K)}$. This can be done by induction on word length. For instance,

$$g_1g_2\alpha((g_1g_2)^{-1}) = g_1g_2\alpha(g_2^{-1})\alpha(g_1^{-1}) = g_1g_2\alpha(g_2^{-1})g_1^{-1}g_1\alpha(g_1^{-1}) \in P^{\pi_1(K)}.$$

Then if $g \in \ker(r_*)$, we have $g = g\alpha(g^{-1}) \in P^{\pi_1(K)}$ since $\alpha(g^{-1}) = i_* \circ r_*(g^{-1}) = 1$.

So now let $\{[\gamma_i : S^1 \to K]\}$ be a finite set of generators for $\ker(r_*)$. Attach 2-cells using the γ_i to kill $\ker(r_*)$. r extends over the new cells to a map $\bar{r}$ because the images $d_*([\gamma_i]) = [d \circ \gamma_i : S^1 \to X]$ are null-homotopic, i.e. extend over the disk D^2. Also note that $\bar{r} \circ i = r \circ i \simeq \mathrm{id}_X$, since the new cells are not in the image of i. □

Let us replace our new symbols $\bar{r}$ and $\bar{K}$ by the old r and K to avoid notational build-up. We may assume without loss of generality that r is an inclusion of the subcomplex K into X by replacing X with the mapping cylinder M_f which is homotopic to X. Consider the long exact homotopy sequence of the pair (K, X)

$$\cdots \to \pi_{k+1}(r) \to \pi_k(K) \xrightarrow{r_*} \pi_k(X) \to \pi_k(r) \to \cdots,$$

where $\pi_k(r) := \pi_k(X, K)$. An isomorphism of fundamental groups thus implies $\pi_1(r) = 0$.

We then take universal covers and lifts such that $\widetilde{r} \circ \widetilde{i} \simeq \mathrm{id}_{\widetilde{X}}$. Having turned r into an inclusion, we now have at our disposal a long exact sequence in homology:

$$\cdots \to H_3(\widetilde{X}, \widetilde{K}) \to H_2(\widetilde{K}) \xrightarrow{\widetilde{d}_*} H_2(\widetilde{X}) \to H_2(\widetilde{X}, \widetilde{K}) \to 0$$

(the terms on the right are zero since universal covers are simply-connected). $\widetilde{i}_* : H_2(\widetilde{X}) \to H_2(\widetilde{K})$ splits the long exact sequence into short exact sequences

$$\cdots \to H_{k+1}(\widetilde{X}, \widetilde{K}) \to H_k(\widetilde{K}) \to H_k(\widetilde{X}) \to 0, \qquad k \geq 2.$$

Thus $H_2(\widetilde{X}, \widetilde{K}) = 0$. By the relative Hurewicz theorem $\pi_2(\widetilde{r}) = \pi_2(\widetilde{X}, \widetilde{K}) \cong H_2(\widetilde{X}, \widetilde{K}) = 0$. But since the fiber over any point of a covering map is discrete, it follows from the long exact homotopy sequence of the fibration $\widetilde{X} \to X$ that $\pi_i(\widetilde{r}) \cong \pi_i(r)$ for $i > 1$. In particular, $\pi_2(r) = 0$. r is thus 2-connected.

From here on, the process of attaching cells to kill even higher homotopy groups is a construction due to Milnor. To review it, recall that an element of $\pi_n(r)$ is the homotopy class of a map $D^n \to X$ which carries the boundary S^{n-1} into K.

Construction 4.1.6. Let $K \xrightarrow{r} X$ be an $(n-1)$-connected map with $n \geq 3$. Then $r_* : \pi_1(K) \cong \pi_1(X)$ and we let $\widetilde{r} : \widetilde{K} \to \widetilde{X}$ denote a lift of r to the universal covers. $\pi_n(\widetilde{r}) \cong \pi_n(r)$ is a module over $\mathbb{Z}[\pi]$ where $\pi := \pi_1(X)$, and we let $\{g_j\}_{j\in J}$ denote a set of generators. Let

$$\bar{K} = K \bigcup_{g_j|_{S^{n-1}}} \{e^n_j\}_{j\in J}.$$

Then there is an extension $\bar{r} : \bar{K} \to X$ of r, where $\bar{r}|_{e^n_j}$ is defined by g_j. In case J is finite, $\bar{K}$ is obtained from K by attaching a finite number of n-cells and it isn't hard to show that $\bar{r}$ is n-connected [**Var89**, §6 Lemma 1.3].

By iterating the above construction we obtain a homotopy equivalence between X and a CW complex built from K. The key to building a *finite* CW complex is for the $\mathbb{Z}[\pi]$-modules $\pi_n(r)$ to be finitely generated at every step. How might one go about this? Given an $(n-1)$-connected map $r : K \to X$ with $n \geq 3$, we have already seen how to use the relative Hurewicz theorem to get an isomorphism $\pi_n(r) \cong H_n(\widetilde{X}, \widetilde{K})$ of $\mathbb{Z}[\pi]$-modules. It thus suffices to show that $H_n(\widetilde{X}, \widetilde{K})$ is finitely generated as a $\mathbb{Z}[\pi]$-module.

Note that the cellular chain complex $C_*(\widetilde{X})$ is generally not finitely generated so that a priori there is no reason to believe that $H_n(\widetilde{X}, \widetilde{K})$ should be. Suppose, however, that our boldest hopes were true and we could find a chain complex A_* of finitely generated, free $\mathbb{Z}[\pi]$-modules that is chain-homotopy equivalent to $C_*(\widetilde{X})$ so that we could compute $H_n(\widetilde{X}, \widetilde{K})$ as $H_n(A_*)$. We would then solve the problem:

Proposition 4.1.7. *If A_* is a positive chain complex of finitely generated, free $\mathbb{Z}[\pi]$-modules with $H_k(A_*) = 0$ for $k < n$, then $H_n(A_*)$ is a finitely generated, projective $\mathbb{Z}[\pi]$-module.*

PROOF. Let

$$\begin{aligned} Z_k &:= \ker(\partial_k : A_k \to A_{k-1}), \\ B_k &:= \operatorname{im}(\partial_{k+1} : A_{k+1} \to A_k). \end{aligned}$$

Then $Z_0 = A_0$ and $Z_k = B_k$ for $k < n$ since $H_k(A_*) = 0$ in this range. We thus have the following exact sequences of $\mathbb{Z}[\pi]$-modules:

$$\begin{aligned} &0 \to Z_1 \to A_1 \to A_0 \to 0 \\ &0 \to Z_2 \to A_2 \to Z_1 \to 0 \\ &\quad\cdots \\ &0 \to Z_n \to A_n \to Z_{n-1} \to 0. \end{aligned}$$

Since A_i are finitely generated and free by hypothesis, the first sequence splits to show that Z_1 is finitely generated, projective over $\mathbb{Z}[\pi]$. This implies that the second sequence splits and hence Z_2 is finitely generated, projective over $\mathbb{Z}[\pi]$. Proceeding thus we finally see that Z_n is finitely generated, projective. Since $H_n(A_*)$ is a quotient of Z_n, this finishes the proof. □

Alas, there is an obstruction to finding such a chain complex. Wall showed that it is always possible to find a bounded chain complex A_* of finitely generated, projective $\mathbb{Z}[\pi]$-modules chain-homotopy equivalent to $C_*(\widetilde{X}, \widetilde{K})$ if X is finitely dominated. However, this complex may or may not be free. In fact, it would suffice that the chain complex be stably free, for we will show in the proof below that a finitely generated, stably free chain complex is chain-homotopy equivalent to a finitely generated, free one. This motivates the definition of the *Wall finiteness obstruction* of X, an element of $\widetilde{K}_0(\mathbb{Z}[\pi])$, in the next theorem.

Recall that a chain complex C_* is of *finite type* if there exists an N such that $C_n = 0$ for $|n| \geq N$ and each C_k is finitely generated. For instance, the cellular chain complex of a finite CW complex is of finite type.

Theorem 4.1.8 (Wall's Finiteness Obstruction). *Let X be a path-connected, finitely dominated CW complex. Then $\pi := \pi_1(X)$ is finitely presented and $C_*(\widetilde{X})$ is chain-homotopy equivalent to a chain complex A_* of finite type of projective $\mathbb{Z}[\pi]$-modules. Moreover, the Wall finiteness obstruction $w(X)$ of X which is the Euler characteristic*

$$\chi(A_*) := \sum_i (-1)^i [A_i] \in \widetilde{K}_0(\mathbb{Z}[\pi])$$

of A_* is well-defined, and $w(X) = 0$ if and only if X is homotopy equivalent to a finite CW complex. [**Wal65, Wal66**]

Proof Sketch. We only prove the very last assertion that vanishing of the finiteness obstruction is equivalent to X having the homotopy type to a finite CW complex. For other parts of the proof see Wall's original papers cited above or Rosenberg's partial exposition in [**Ros94**, §1.7].

First suppose that X is homotopy equivalent to a finite CW complex Z. Then $C_*(\widetilde{X})$ is chain-homotopy equivalent to $C_*(\widetilde{Z})$ which is a complex of finite type of *free* $\mathbb{Z}[\pi]$-modules. Thus $w(X) = \chi(C_*(\widetilde{Z})) = 0$ since free modules vanish in $\widetilde{K}_0(\mathbb{Z}[\pi])$.

Conversely suppose that $w(X) = 0$ and that we have already found a chain-homotopy equivalent chain complex A_* of finite type of projective $\mathbb{Z}[\pi]$-modules. We show that A_* is chain-homotopy equivalent to a chain complex of finite type of free $\mathbb{Z}[\pi]$-modules which completes the proof by Milnor's construction and the discussion before the theorem.

Suppose $A_j = 0$ for j outside of an interval $\{k, k+1, \dots, k+n\}$. Choose projective modules $Q_n, \dots, Q_0$ such that $A_{k+n} \oplus Q_n$ is free, $A_{k+n-1} \oplus Q_n \oplus Q_{n-1}$ is free, and in general such that $A_{k+j} \oplus Q_{j+1} \oplus Q_j$ is free for $0 \le j < n$. If T_* is chain-contractible, then replacing A_* by $A_* \oplus T_*$ doesn't change the chain-homotopy type. So define (T_*^j, d^{T^j}) by

$$T_i^j = \begin{cases} 0 & i \neq k+j, k+j-1; \\ Q_j & i = k+j, k+j-1 \end{cases}$$

with $d_{k+j}^{T^j} : Q_j \to Q_j$ the identity map. This is clearly contractible and

$$B_* := A_* \oplus \bigoplus_{j=0}^{n} T_*^j$$

has free modules in all degrees except perhaps in degree $k-1$ where $B_{k-1} = Q_0$. Thus

$$0 = w(X) = \chi(A_*) = \chi(B_*) = (-1)^{k-1}[B_{k-1}] \in \widetilde{K}_0(\mathbb{Z}[\pi]).$$

Hence Q_0 is stably free, i.e. there are free modules F, F' such that $Q_0 \oplus F \cong F'$. Let (S_*, d^S) be defined by

$$S_j = \begin{cases} 0 & j \neq k-1, k-2; \\ F & j = k-1, k-2 \end{cases}$$

with $d_{k-1}^S : F \to F$ the identity map. Then $D_* := D_* \oplus S_*$ is of finite type and has free modules in all degrees and is chain-homotopy equivalent to A_*. $\square$

It follows that any simply-connected, finitely dominated space is homotopy equivalent to a finite CW complex. The same could be true for any finitely dominated space with torsion-free fundamental group as it is conjectured that $\widetilde{K}_0(\mathbb{Z}[\pi]) = 0$ for any torsion-free group π.

4.2. K_0 of Schemes

To avoid pathologies, we will from now on assume all rings and schemes Noetherian until further notice. We will remind the reader of this hypothesis in some crucial situations by bracketing (Noetherian).

Our goal in this section is to define a K-theory of schemes that generalizes that of just topological spaces. Since topological K-theory deals with topological vector bundles, we would like to come up with a generalized notion of "vector bundle" that we can apply to schemes. Swan's theorem showed that the category of vector bundles over a topological space X is equivalent to that of finitely generated projective $\Gamma(X, \mathcal{O}_X)$-modules. A scheme is a locally ringed space with a sheaf of commutative rings $\mathcal{O}_X$, termed the *structure sheaf*, which generalizes the sheaf of continous functions on a topological space X. We could thus define an algebraic vector bundle over a scheme X to be a finitely generated projective $\Gamma(X, \mathcal{O}_X)$-module extending the topological definition. This will be the right definition when X is an affine scheme. However, to encode the information contained in a general scheme, we need to frame the discussion in terms of sheaves rather than modules.

To do so, we first show that the category of topological vector bundles is also equivalent to that of locally free $\mathcal{O}_X$-modules, i.e. sheaves, of finite rank. Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is *locally free of rank* n if there exists a (Zariski) open covering $X = \bigcup_i U_i$ such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$.

Example 4.2.1. The sheaf of sections of a topological vector bundle of rank n defined in the previous section is a locally free $\mathcal{O}_X$-module of rank n since vector bundles are locally trivial.

Conversely, given a locally free $\mathcal{O}_X$-module $\mathcal{F}$ of rank n, we can find trivializations $\psi_i : \mathcal{F}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n}$ so that the transition maps $\psi_{ij} := (\psi_i \circ \psi_j^{-1})|_{U_i \cap U_j} : \mathcal{O}_{U_i \cap U_j}^{\oplus n} \cong \mathcal{O}_{U_i \cap U_j}^{\oplus n}$ can be seen as elements of $\mathrm{GL}_n(\mathcal{O}_{U_i \cap U_j})$. By definition, $\psi_{ij}|_{U_i \cap U_j \cap U_k} \circ \psi_{jk}|_{U_i \cap U_j \cap U_k} = \psi_{ik}|_{U_i \cap U_j \cap U_k}$ so that the ψ_{ij} are cocycles that we can use to construct a vector bundle E of rank n. Since these two constructions are inverse to each other we obtain

Proposition 4.2.2. *Associating to a vector bundle its sheaf of sections defines an equivalence of categories between vector bundles over a topological space X and locally free $\mathcal{O}_X$-modules of finite rank.*

Extending this interpretation to schemes, we define an *algebraic vector bundle* over a scheme X to be a locally free $\mathcal{O}_X$-module of finite rank. We also denote by $\mathbf{Vect}(X)$ the category of algebraic vector bundles over a scheme X.

Fact 4.2.3. *When $X = \mathrm{Spec}(R)$ is an affine scheme, then $\mathbf{Vect}(X)$ is equivalent to $\mathbf{P}(R)$.* [**Wei12**, Example I.5.1.2]

However, over a general scheme this is not necessarily true. For example, the projective lifting property fails for every vector bundle over the projective line $\mathbb{P}_R^1 = \mathrm{Proj}(R[x, y])$ [**Wei12**, Example I.5.4]. Thus, in general we cannot hope to define the K-theory of a scheme as K_0 of some ring as we did in the topological setting.

In fact, historically the K-theory of rings was only a special case of the K-theory of schemes as Grothendieck was led to define the K-group as the free abelian group with generators the coherent sheaves on a scheme subject to a relation that identifies any extension of two sheaves with their sum. By the above fact, in the affine case any extension of sheaves splits and so many authors now define the K-theory of a ring R as $K(R) := K(\mathrm{Spec}(R)) = K(\mathbf{P}(R))$.

Let us make these notions precise. First of all, there is a natural way to take the group completion of a small *symmetric monoidal category*, i.e. a small category $\mathcal{C}$ equipped with a functor $\Box : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a distinguished object e and the following three natural isomorphisms: $x \Box y \cong y \Box x$, $e \Box x \cong x$, and $x \Box (y \Box z) \cong (x \Box y) \Box z$. The isomorphism classes of objects of $\mathcal{C}$, $\mathcal{C}^{\mathrm{iso}}$, form an abelian monoid with respect to $\Box$. We thus define $K_0^{\Box}(\mathcal{C}) := K_0(\mathcal{C}^{\mathrm{iso}})$.

Examples 4.2.4. Some group completions of symmetric monoidal categories:

(1) Any category with a direct sum $\oplus$ is symmetric monoidal. This includes $\mathbf{P}(R)$ and $\mathbf{Vect}(X)$ for a topological space X. Since the above definition is precisely how we constructed the Grothendieck group in (1.1), we see that $K_0(R) = K_0^{\oplus}(\mathbf{P}(R))$ and $K^0(X) = K_0^{\oplus}(\mathbf{Vect}(X))$.

(2) Let $\mathbf{Sets}_{\mathrm{fin}}$ denote the category of finite sets. It has a product $\times$ and a coproduct, the disjoint sum $\coprod$. Then $K_0^{\coprod}(\mathbf{Sets}_{\mathrm{fin}}) = \mathbb{Z}$ while $K_0^{\times}(\mathbf{Sets}_{\mathrm{fin}}) = 0$ since the empty set satisfies $\emptyset = \emptyset \times X$ for all finite sets X. However, the set of isomorphism classes of nonempty finite sets is $\mathbb{N}_{>0}$ and the product of finite sets corresponds to multiplication. Since the group completion of the abelian monoid $\mathbb{N}_{>0}^{\times}$ is the group $\mathbb{Q}_{>0}^{\times}$ we find that $K_0^{\times}(\mathbf{Sets}_{\mathrm{fin}} \setminus \emptyset) = \mathbb{Q}_{>0}^{\times}$.

(3) Let G be a finite group and denote by $\mathbf{Rep}_{\mathbb{C}}(G)$ the category of finite-dimensional complex representations of G. It is symmetric monoidal under $\oplus$. We denote $K_0^{\oplus}(\mathbf{Rep}_{\mathbb{C}}(G))$ by $R(G)$. By Maschke's Theorem [**Ser77**, Theorem 1], all representations of G are completely reducible so that $\mathbf{Rep}_{\mathbb{C}}(G) \cong \mathbb{N}^k$, a basis being the k irreducible representations $[V_1], \ldots, [V_k]$ of G. By character theory [**Ser77**, Theorem 7], k is equal to the number of conjugacy classes of G. As an abelian group $R(G) \cong \mathbb{Z}^k$. The tensor product of two representations is also a representation and so $R(G)$ admits the structure of a ring called the *representation ring* of G. This

example is related to a variant of topological K-theory which will be discussed in chapter 5.

For a scheme X, $\mathbf{Vect}(X)$ is symmetric monoidal and we could define K-theory of a scheme with respect to this structure. However, this turns out not to be right idea. The reason is that we would like to talk about kernels, cokernels and do homological algebra but $\mathbf{Vect}(X)$ is not an abelian category. To see this, take for instance the trivial vector bundle ϵ^1 on $X = \mathbb{R}$. We can define a bundle map f from ϵ^1 to itself by $f(x, y) = (x, xy)$. If $\mathbf{Vect}(X)$ were abelian, then the kernel of f should be a bundle and so the rank of the bundle $\ker(f)$ over each point of X should be the same since X is connected. However, the rank of $\ker(f)$ is 0 everywhere except at $x = 0$ where it is 1. We can also see this algebraically: in $\mathbf{P}(\mathbb{Z})$ consider the multiplication homomorphism $n : \mathbb{Z} \to \mathbb{Z}$. Clearly, the cokernel cannot be projective since it has torsion.

Instead, we embed $\mathbf{Vect}(X)$ in the smallest abelian category containing the algebraic vector bundles and define a K-theory respecting this embedding. This is the category $\mathbf{Coh}(X)$ of coherent $\mathcal{O}(X)$-modules. Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is *coherent* if there is an open covering $X = \bigcup U_i$ such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ for some finitely generated $\mathcal{O}_X(U_i)$-module M_i where $\widetilde{M_i}$ is the $\mathcal{O}_{U_i}$-module defined on distinguished open sets by $\widetilde{M_i}(D(f)) = (M_i)_f$. In particular, every algebraic vector bundle $\mathcal{E}$ is coherent since for an open $U = \operatorname{Spec}(R)$, $\widetilde{R} = \mathcal{O}_U$ and so $\mathcal{E}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus n} \cong \widetilde{R_i}^n \cong \widetilde{R_i^n}$.

Fact 4.2.5. *Let X be a (Noetherian) scheme. The category* $\mathbf{Coh}(X)$ *is abelian.* [**Har06**, II.5.7]

Given a small abelian category $\mathcal{A}$, we define its Grothendieck group $K_0(\mathcal{A})$ as the free abelian group with generators $[A]$ for each object A of $\mathcal{A}$ and with one relation $[A] = [A'] + [A'']$ for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$. Here are some identities which hold in $K_0(\mathcal{A})$:

(1) $[0] = 0$,
(2) if $A \cong A'$, then $[A] = [A']$ (take $A'' = 0$),
(3) $[A' \oplus A''] = [A'] + [A'']$ (take $A = A' \oplus A''$).

If two abelian categories are equivalent then their Grothendieck groups are naturally isomorphic as (2) implies that they have the same presentation. By (3) the group $K_0(\mathcal{A})$ is a quotient of the group $K_0^{\oplus}(\mathcal{A})$ by considering $\mathcal{A}$ as a symmetric monoidal category. This means that $K_0(\mathcal{A})$ is often easier to compute as it is smaller.

In the same way we define K-theory for an *exact category*, which is an additive subcategory of an abelian category which is closed under extensions (any extension in the abelian category of two objects in the subcategory is isomorphic to an object in the subcategory). The function $\mathcal{A} \to K_0(\mathcal{A})$ defined by $A \mapsto [A]$ is a universal *additive function* ($f(A) = f(A') + f(A'')$ for each exact sequence $0 \to A' \to A \to A'' \to 0$) in the sense that every additive function factors through it.

Examples 4.2.6. (1) For a (Noetherian) ring R, $\mathbf{P}(R)$ is an additive subcategory of the abelian category of finitely generated R-modules. Since every exact sequence of projective modules splits, we have $K_0(\mathbf{P}(R)) = K_0(R)$. A category with this property is called *split exact*.

(2) For a topological space X, $\mathbf{Vect}(X)$ can be embedded in the abelian category of families of vector spaces over X. $\mathbf{Vect}(X)$ is also split exact by (1.1.2). Hence $K_0(\mathbf{Vect}(X)) = K_0^{\oplus}(\mathbf{Vect}(X)) = K^0(X)$.

These constructions give rise to the K-theory of a scheme X: $\mathbf{Vect}(X)$ is an additive subcategory of the abelian category $\mathbf{Coh}(X)$. We can thus associate to X the two K-groups

$$K_0(X) := K_0(\mathbf{Vect}(X)), \text{ and } K_0'(X) := K_0(\mathbf{Coh}(X)).$$

The latter is sometimes also denoted by $G_0(X)$ and called *G-theory*. The inclusion $\mathbf{Vect}(X) \subseteq \mathbf{Coh}(X)$ is an exact functor (sends exact sequences to exact sequences) and thus yields a *Cartan homomorphism* $K_0(X) \to K_0'(X)$.

Fact 4.2.7. *If X is smooth, then the Cartan homomorphism is an isomorphism.* [**Fri07**, Theorem 4.19]

[**Wei12**, Example I.5.4] shows that exact sequences in $\mathbf{Vect}(X)$ do not necessarily split. In general, the K-theory of the exact category $\mathbf{Vect}(X)$ is thus distinctly different from the K-theory of the symmetric monoidal category $\mathbf{Vect}(X)$.

Fact 4.2.8. *Let $f : X \to Y$ be a morphism of (Noetherian) schemes. Then f induces a pullback functor* [**Har06**, II.5.8]

$$f^* : \mathbf{Vect}(Y) \to \mathbf{Vect}(X), \; f^*(\mathcal{E}) = f^{-1}\mathcal{E} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

If f is flat, then f^* is exact. Moreover, if f is proper then for every coherent sheaf $\mathcal{F}$, $R^i f_*(\mathcal{F})$ is also coherent [**Vak12**, Theorem 20.8.1] *where $R^i f_*$ are the right derived functors of the direct image $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$, and f*

induces a pushforward functor

$$f_! : \mathbf{Coh}(X) \to \mathbf{Coh}(Y),\ f_!(\mathcal{F}) = \sum_i (-1)^i R^i f_*(\mathcal{F})$$

which is exact. [**Wei12**, Lemma II.6.2.6]

Thus seen as functors from **Schemes** to **Ab**, K_0' is both covariant and contravariant, and K_0 is contravariant with respect to appropriate morphisms. For smooth schemes, it follows from (4.2.7) that K_0 is also covariant.

Finally, the tensor product of vector bundles defines a biexact functor $\mathbf{Vect}(X) \times \mathbf{Vect}(X) \to \mathbf{Vect}(X)$ [**Wei12**, I.5.3] inducing a bilinear map $K_0(X) \otimes_{\mathbb{Z}} K_0(X) \to K_0(X)$. Thus $K_0(X)$ has a commutative, associative product $[\mathcal{E}] \cdot [\mathcal{E}'] = [\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}']$ with unit $[\mathcal{O}_X]$. It can be shown that the pullback and pushforward functors above preserve tensor products [**Vak12**, Theorem 17.3.7]. Thus, K_0 and K_0' are functors from **Schemes** to **CRings**.

4.3. Riemann-Roch

Having defined K-theory of a scheme, our next goal is to state Grothendieck's Riemann-Roch theorem. To do so, we need to develop an analog of the Chern character defined in singular cohomology. In all instances, the term "Chern character" refers to map from K-theory to a (co)homology theory. Since we are working with schemes, a possible (co)homology theory would be that of Chow rings. We recall the necessary notions.

An *algebraic k-cycle* on a scheme X is a finite formal sum of k-dimensional subschemes with integer coefficients. For example, on an integral scheme of dimension d, a Weil divisor is a $(d-1)$-cycle. The group $Z_k X$ of k-cycles is very large and we introduce an equivalence relation to slim it down. We say that a k-cycle Z on X is *rationally equivalent to zero* if and only if there exist $(k+1)$-dimensional subschemes $V_1, \ldots, V_n$ of $X \times \mathbb{P}^1$ with dominant projections $f_i : V_i \to \mathbb{P}^1$ such that $Z = \sum_i [V_i(0)] - [V_i(\infty)]$ where $[V_i(p)]$ denotes the cycle associated to the scheme-theoretic fibre $f_i^{-1}(p)$. If we regard $\mathbb{P}^1$ as a line, we can think of this notion as an algebro-geometric analog of cobordism. See [**Ful98**, §1.6] for more details on rational equivalence.

We define the *Chow group* of a scheme X, $A_k(X)$, as the group of k-cycles modulo rational equivalence. We write $A_*(X)$ for the direct sum of the Chow groups. A ring structure on $A_*(X)$ is given by the "Moving Lemma" [**Fri07**, Theorem 4.20] which could be seen as an algebro-geometric analog of Thom's transversality theorem. It asserts that a cycle of codimension r and a cycle of codimension s can be moved within their rational equivalence class so that their intersection is generically transverse meaning that the intersection of any two irreducible components is either empty or of codimension $r+s$. Writing $A^k(X) = A_{d-k}(X)$ where $d = \dim(X)$, this gives an intersection pairing $A^r(X) \otimes A^s(X) \to A^{r+s}(X)$, $[V_1] \cdot [V_2] = [V_1 \cap V_2]$ and we call $A^*(X) := \bigoplus_k A^k(X)$ the *Chow ring.*

Fact 4.3.1 (cf. (4.2.8))**.** *Chow rings are functorial with respect to flat pullbacks $f^*([V]) = [f^{-1}(V)]$ and proper pushforwards*

$$f_*([V]) = \begin{cases} \deg(V/f(V))[f(V)] & \text{if } \dim(f(V)) = \dim(V); \\ 0 & \text{if } \dim(f(V)) < \dim(V) \end{cases}$$

where $\deg(V/f(V)) = [K(V) : K(f(V))]$ is the (finite) degree of the induced extension of function fields. [**Ful98**, Theorems 1.4 and 1.7]

Examples 4.3.2. Here are some examples of Chow rings:

(1) $A^*(\mathbb{P}^n_{\mathbb{C}}) = \mathbb{Z}[y]/(y^{n+1})$ where y corresponds to the subscheme $\mathbb{P}^{n-1}_{\mathbb{C}} \subset \mathbb{P}^n_{\mathbb{C}}$ (a hyperplane section). So y^i corresponds to the intersection of i generic linear hyperplanes which is just the class of $\mathbb{P}^{n-i}_{\mathbb{C}} \subseteq \mathbb{P}^n_{\mathbb{C}}$.

(2) For a general smooth scheme X, $A^0(X)$ is the free abelian group on $[X]$, so $A^0(X) \cong H^0(X, \mathbb{Z}) \cong \mathbb{Z}$. Also, $A^1(X)$ is the group of Weil divisors modulo linear equivalence which is known to be isomorphic to $\mathrm{Pic}(X)$, the group of algebraic line bundles on X, which in turn conincides with $H^1(X, \mathcal{O}_X^\times)$. Thus $A^1(X) \cong H^1(X, \mathcal{O}_X^\times) \cong \mathrm{Pic}(X)$. It is also true that the higher Chow rings can be described as cohomology rings (cf. "Bloch's formula").

We now proceed to define the Chern classes and the Chern character of algebraic vector bundles using a Leray-Hirsch type theorem. There are Leray-Hirsch type theorems for singular cohomology and topological K-theory (2.2.3). What these theorems have in common is that they express the cohomology theory of the projectivization of a bundle in terms of the cohomology theory of the base space and a polynomial relation whose coefficients can be used to define characteristic classes. This is one way to define Chern classes of topological vector bundles and this was also used when we defined the Adams operations.

To state the Leray-Hirsch theorem for Chow rings, we need to make sense of the projectivization of an algebraic line bundle. We begin by constructing the "total space" of an algebraic vector bundle. If M is a free R-module of rank n then the symmetric algebra $\mathrm{Sym}_R(M)$ is isomorphic to $R[x_1, \dots, x_n]$, where $x_1, \dots, x_n$ is a basis of M. This yields a natural projection map $\mathrm{Spec}(\mathrm{Sym}_R(M)) \to \mathrm{Spec}(R)$ since $\mathrm{Spec}(\mathrm{Sym}_R(M)) \cong \mathrm{Spec}(R[x_1, \dots, x_n]) \cong$

$\mathbb{A}^n_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(R)$. We readily globalize this construction: if $\mathcal{E}$ is an algebraic vector bundle over X, then $p_{\mathcal{E}} : \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_X} \mathcal{E}^\vee) \to X$ is locally the product projection of the previous construction. Setting $V(\mathcal{E}) := \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_X} \mathcal{E}^\vee)$, we may thus think of an algebraic vector bundle $\mathcal{E}$ on X as a map of varieties $p_{\mathcal{E}} : V(\mathcal{E}) \to X$ satisfying properties which are the algebro-geometric anologs of the properties of a topological bundle projection. We consider the dual sheaf $\mathcal{E}^\vee := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$ so that the association $\mathcal{E} \mapsto V(\mathcal{E}^\vee)$ is covariantly functorial.

Similarly, we define the projectivization of $\mathcal{E}$ as $\pi_{\mathcal{E}} : P(\mathcal{E}) \to X$ where $P(\mathcal{E}) := \mathrm{Proj}(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E}))$. Then $P(\mathcal{E})$ comes equipped with a canonical line bundle $\mathcal{O}_{P(\mathcal{E})}(1)$ which we can identify with a divisor class in $A^1(P(\mathcal{E}))$ as mentioned in example (2) above.

Theorem 4.3.3 (cf. 2.2.3)**.** *Let $\mathcal{E}$ be a rank n vector bundle on a smooth scheme X and let $\zeta \in A^1(P(\mathcal{E}))$ be the divisor class associated to the canonical line bundle $\mathcal{O}_{P(\mathcal{E})}(1)$. Then $A^*(P(\mathcal{E}))$ is the free $A^*(X)$-module with basis $\{1, \zeta, \ldots, \zeta^{n-1}\}$ and module structure induced by pullback $\pi^*_{\mathcal{E}}$.* [**Ful98**, Theorem 3.3]

We can thus express ζ^n as a linear combination of $1, \zeta, \ldots, \zeta^{n-1}$ with coefficients in $A^*(X)$. These coefficients are by definition the *Chern classes* $c_i(\mathcal{E}) \in A^i(X)$ of $\mathcal{E}$:

$$A^*(P(\mathcal{E})) = A^*(X)[\zeta]/\sum_{i=0}^{n}(-1)^i c_i(\mathcal{E})\zeta^{n-i}.$$

In this equation by $c_i(\mathcal{E})$ we really mean $\pi^*_{\mathcal{E}}(c_i(\mathcal{E}))$. Chern classes are natural with respect to pullbacks since pullback and projectivization commute. Moreover, using the algebro-geometric analog of the splitting principle (2.2.2), one can show that the higher Chern classes are uniquely determined by the assignment of the first Chern class to line bundles.

Theorem 4.3.4 (Splitting Principle)**.** *Let $\mathcal{E}$ be an algebraic vector bundle of rank n on a scheme X. Then there exists a splitting scheme $F(\mathcal{E})$ and a flat morphism $p : F(\mathcal{E}) \to X$ such that the induced map $p^* : A^*(X) \to A^*(F(\mathcal{E}))$ is injective and $p^*\mathcal{E}$ splits, i.e. it has a filtration by subbundles $p^*\mathcal{E} = E_n \supseteq E_{n-1} \supseteq \cdots \supseteq E_1 \supseteq E_0 = 0$ whose successive quotients are line bundles $\mathcal{L}_i \cong E_i/E_{i-1}$. Thus, $[p^*(\mathcal{E})] = \sum_i [\mathcal{L}_i]$ in $K_0(F(\mathcal{E}))$.* [**Ful98**, Theorem 3.2]

Using this theorem, we can also define the *Chern character* $ch : K_0(X) \to A^*(X) \otimes \mathbb{Q}$ for a scheme X in exact analogy to how we defined it for topological spaces, namely by imposing additivity $ch(\mathcal{E} \oplus \mathcal{E}') = ch(\mathcal{E}) + ch(\mathcal{E}')$

and declaring

$$ch(\mathcal{L}) = e^{c_1(\mathcal{L})} = 1 + c_1(\mathcal{L}) + c_1(\mathcal{L})^2/2! + \cdots \in A^*(X) \otimes \mathbb{Q}$$

for a line bundle $\mathcal{L}$. Note that this definition is natural with respect to pullbacks. For a general vector bundle $\mathcal{E}$ of rank n we are thus led to set $ch(\mathcal{E}) = \sum_i e^{\alpha_i}$ where $\alpha_i = c_1(\mathcal{L}_i)$ are the Chern roots of $\mathcal{E}$ obtained from the splitting principle. Using the Newton polynomials s_k this can be expanded as

$$ch(\mathcal{E}) = \operatorname{rank} \mathcal{E} + \sum_{k>0} s_k(c_1(\mathcal{E}), \ldots, c_k(\mathcal{E}))/k!. \tag{3}$$

By the splitting principle, $ch : \mathbf{Vect}(X) \to A^*(X) \otimes \mathbb{Q}$ is also multiplicative since the first Chern class is additive on tensor products [**Ful98**, Proposition 2.5(e)]. Moreover:

Proposition 4.3.5. *The Chern character $ch : \mathbf{Vect}(X) \to A^*(X) \otimes \mathbb{Q}$ is additive on short exact sequences.*

Proof. Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be an exact sequence of vector bundles on X, and let $p : F(X) \to X$ be a simultaneous splitting morphism for $\mathcal{E}'$ and $\mathcal{E}''$. Recall that $0 \to p^*\mathcal{E}' \to p^*\mathcal{E} \to p^*\mathcal{E}'' \to 0$ is exact by (4.2.8). We have filtrations $p^*\mathcal{E}' = E'_n \supseteq E'_{n-1} \supseteq \cdots \supseteq E'_1 \supseteq E'_0 = 0$ and $p^*\mathcal{E}'' = E''_m \supseteq E''_{m-1} \supseteq \cdots \supseteq E''_1 \supseteq E''_0 = 0$. By exactness $p^*\mathcal{E}/(p^*\mathcal{E}' \oplus E''_{m-1}) \cong p^*\mathcal{E}''/E''_{m-1} \cong \mathcal{L}''_m$, and so these induce a filration

$$0 = E'_0 \subseteq \cdots \subseteq E'_n = p^*\mathcal{E}' \subseteq p^*\mathcal{E}' \oplus E''_1 \subseteq \cdots \subseteq p^*\mathcal{E}' \oplus E''_{m-1} \subseteq p^*\mathcal{E}.$$

Hence $ch(\mathcal{E}) = \sum_k e^{c_1(\mathcal{L}_k)} = \sum_i e^{c_1(\mathcal{L}'_i)} + \sum_j e^{c_1(\mathcal{L}''_j)} = ch(\mathcal{E}') + ch(\mathcal{E}'')$. □

Thus, ch factors through $K_0(X)$ by the universal property of K_0 and becomes a ring homomorphism.

We now have almost all the necessary constructions in place to state Grothendieck's Riemann-Roch theorem. This theorem describes how the Chern character commutes with the pushforward (4.2.8) induced by a proper morphism $f : X \to Y$ of smooth varieties. The defect to commute is measured by a characteristic class called the *Todd class*. Just like the Chern character, by the splitting principle the Todd class $td : K_0(X) \to A^*(X) \otimes \mathbb{Q}$ is characterized by the following properties:

(1) $td \circ f^* = f^* \circ td$ for all proper morphisms $f : X \to Y$ of smooth schemes (naturality),
(2) $td(\mathcal{E} \oplus \mathcal{E}') = td(\mathcal{E}) \cdot td(\mathcal{E}')$,
(3) $td(\mathcal{L}) = c_1(\mathcal{L})/(1 - e^{-c_1(\mathcal{L})}) = \sum_{i=0}^{\infty} (-1)^i B_i c_1(\mathcal{L})^i/i! = 1 + c_1(\mathcal{L})/2 + c_1(\mathcal{L})^2/12 - c_1(\mathcal{L})^4/720 + \cdots$ where B_i is the i^{th} Bernoulli number.

This leads to $td(\mathcal{E}) = \prod_i \alpha_i/(1 - e^{-\alpha_i})$ where again α_i are the Chern roots of $\mathcal{E}$. Expanding this we get

$$td(\mathcal{E}) = 1+\frac{1}{2}c_1+\frac{1}{12}(c_1^2+c_2)+\frac{1}{24}c_1c_2+\frac{1}{720}(-c_1^4+4c_1^2c_2+3c_2^2+c_1c_3-c_4)+\cdots,$$

where $c_i := c_i(\mathcal{E})$. Moreover, if $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ is exact, then one can show similarly to how was done for the Chern character that $td(\mathcal{E}) = td(\mathcal{E}')\cdot td(\mathcal{E}'')$. So $td : \mathbf{Vect}(X) \to A^*(X)\otimes\mathbb{Q}$ descends to $K_0(X)$ as a group homomorphism from the additive to the multiplicative structure.

Recall that the tangent sheaf of a smooth k-scheme X is defined as the locally free $\mathcal{O}_X$-module $\mathcal{T}_X := \Omega_{X/k}^\vee$ of rank $n = \dim X$, the dual sheaf of the cotangent sheaf $\Omega_{X/k}$.

Theorem 4.3.6 (Grothendieck's Riemann-Roch Theorem)**.** *Let $f : X \to Y$ be a proper morphism of smooth varieties over an algebraically closed field k. Then for any $\mathcal{E} \in K_0(X)$,*

$$ch(f_!(\mathcal{E}))\cdot td(\mathcal{T}_Y) = f_*(ch(\mathcal{E})\cdot td(\mathcal{T}_X)).$$

A modern proof can be found in [**Ful98**, Theorem 15.2]. See [**Vak04**, Classes 18 and 19] for the standard proof.

As a first application we show that Grothendieck's Riemann-Roch theorem generalizes

Theorem 4.3.7 (Hirzebruch's Riemann-Roch Theorem)**.** *Let $\mathcal{E} \in K_0(X)$ be a vector bundle on a smooth variety X of dimension n over an algebraically closed field k. Then*

$$\chi(X,\mathcal{E}) = \int_X ch(\mathcal{E})\cdot td(\mathcal{T}_X).$$

A few comments are in order. The left-hand side is the *Euler-Poincaré characteristic* of $\mathcal{E}$ defined as $\chi(X,\mathcal{E}) = \sum_i(-1)^i \dim H^i(X,\mathcal{E})$. That this is well-defined (i.e. that $\chi(X,-)$ is an additive function) follows from the long exact sequence in sheaf cohomology.

The integral on the right-hand side is called the *top graded degree* and defined as follows: given a cycle $Z \in A^*(X)\otimes\mathbb{Q}$, take its top graded piece $\sum q_p p$ in $A^n(X)\otimes\mathbb{Q}$ which is a finite formal sum of points with rational coefficients and consider $\int_X Z := \sum q_p[K(p):k] = \sum q_p$ where again $[K(p):k]$ is the degree of the induced field extension which is 1 in our case because k is algebraically closed.

Proof of (4.3.7). Apply Grothendieck's Riemann-Roch theorem to the projection to a point $f : X \to * = \operatorname{Spec}(k)$ and recall the pushforward of cycles defined on a single subvariety $V \subseteq X$ by:

$$f_*([V]) = \begin{cases} \deg(V/f(V))[f(V)] & \text{if } \dim(f(V)) = \dim(V); \\ 0 & \text{if } \dim(f(V)) < \dim(V). \end{cases}$$

Since $f(V) = \operatorname{Spec}(k)$, $K(f(V)) = k$ and $\dim(f(V)) = 0$. So this becomes

$$f_*([V]) = \begin{cases} [K(V) : k] = 1 & \text{if } 0 = \dim(V); \\ 0 & \text{if } 0 < \dim(V) \end{cases}$$

where we used that k is algebraically closed. It follows that for a general cycle $Z \in A^*(X)$ $f_*(Z) = \int_X Z$.

On the left-hand side, it is clear that $td(\mathcal{T}_*) = 1$. Moreoever, it turns out that for each $i \geq 0$ and for each sheaf $\mathcal{F}$, $R^i f_*(\mathcal{F})$ is the sheaf associated to the presheaf $V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$ [**Har06**, III.8.1]. Thus, in our case we have $R^i f_*(\mathcal{E})$ is the sheaf $\operatorname{Spec}(k) \mapsto H^i(X, \mathcal{E})$ and so $f_!(\mathcal{E}) = \sum(-1)^i R^i f_*(\mathcal{E}) = \sum(-1)^i H^i(X, \mathcal{E})$. Finally $ch(f_!(\mathcal{E})) = \operatorname{rank}(f_!(\mathcal{E}))$ by (3) since $A^i(*) = 0$ for $i > 0$ and so all Chern classes are zero. This finishes the proof. $\square$

This theorem in turn generalizes

Theorem 4.3.8 (Riemann-Roch Theorem). *For any divisor D on a compact Riemann surface X of genus $g := \dim H^1(X, \mathcal{O}_X)$, one has*

$$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = \deg D + 1 - g.$$

Proof. Apply Hirzebruch's Riemann-Roch theorem to the sheaf $\mathcal{O}_D$ associated to the divisor D. Since $c_1(\mathcal{O}_D) = D$, we have $ch(\mathcal{O}_D) = 1 + D$. The tangent sheaf $\mathcal{T}_X$ is the dual of Ω_X. Thus $\mathcal{T}_X \cong \mathcal{O}_{-K}$ where K is the canonical divisor (any two canonical divisors are linearly equivalent). But then $td(\mathcal{T}_X) = 1 + \frac{1}{2}c_1(\mathcal{O}_{-K}) = 1 - \frac{1}{2}K$. By Hirzebruch's Riemann-Roch theorem we obtain

$$\begin{aligned} \chi(X, \mathcal{O}_D) &= \dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) \\ &= \int_X (1 + D)\left(1 - \frac{1}{2}K\right) = \deg\left(D - \frac{1}{2}K\right). \end{aligned}$$

Setting $D = 0$ in the above lines so that $\mathcal{O}_D = \mathcal{O}_X$ we obtain $1 - g = -\frac{1}{2}\deg(K)$. Thus, we can then rewrite the preceeding equation as

$$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = \deg D + 1 - g.$$

$\square$

Another important corollary of Grothendieck's Riemann-Roch theorem is

Theorem 4.3.9 (cf. (3.3.5)). *Let X be a smooth variety. Then $ch : K_0(X) \otimes \mathbb{Q} \to A^*(X) \otimes \mathbb{Q}$ is a ring isomorphism.* [**Ful98**, Example 15.2.6]

We apply this to establish another connection between algebraic and topological vector bundles. Let X be a scheme of finite type over $\mathbb{C}$, such as a subvariety of $\mathbb{P}^n_{\mathbb{C}}$ or $\mathbb{A}^n_{\mathbb{C}}$. Consider the closed $\mathbb{C}$-valued points $X(\mathbb{C})$ of X. $X(\mathbb{C})$ is covered by open sets $U(\mathbb{C})$ homeomorphic to analytic subsets (zero loci of holomorphic functions) of $\mathbb{A}^n_{\mathbb{C}}(\mathbb{C})$, and $\mathbb{A}^n_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}^n$. Thus, $X(\mathbb{C})$ has the structure of an *analytic space* and we write $X^{an} := (X(\mathbb{C}), \mathcal{O}_{an})$ for the ringed space $X(\mathbb{C})$ with the analytic topology and the sheaf of holomorphic functions. We can of course also consider $X(\mathbb{C})$ as a topological space only and will write $X^{top} := (X(\mathbb{C}), \mathcal{O}_{top})$ for $X(\mathbb{C})$ as a ringed space with the sheaf of continuous functions. If we require X to be projective, then $X(\mathbb{C})$ is compact because it is a closed subspace of the compact space $\mathbb{P}^n_{\mathbb{C}}(\mathbb{C}) \cong \mathbb{C}P^n$.

The inclusion map $X(\mathbb{C}) \to X$ induces morphisms of ringed spaces $X^{top} \to X^{an} \to X$ which in turn yields functors $\mathbf{Vect}(X) \to \mathbf{Vect}(X^{an}) \to \mathbf{Vect}(X^{top})$. By (4.2.2) $\mathbf{Vect}(X^{top}) \cong \mathbf{Vect}(X(\mathbb{C}))$ where the latter is the category of complex topological vector bundles over the topological space $X(\mathbb{C})$. We thus obtain a natural map

$$K_0(X) \to K^0(X(\mathbb{C})).$$

It is an open problem to understand the kernel and image of this map, especially after tensoring with $\mathbb{Q}$:

$$A^*(X) \otimes \mathbb{Q} \cong K_0(X) \otimes \mathbb{Q} \to K^0(X(\mathbb{C})) \otimes \mathbb{Q} \cong H^{\text{even}}(X(\mathbb{C}); \mathbb{Q}), \tag{4}$$

where we have used the Chern character isomorphisms (4.3.9) and (3.3.5).

The kernel of (4) is the subspace of $A^*(X) \otimes \mathbb{Q}$ consisting of rational equivalence classes of algebraic cycles on X which are homologically equivalent to 0. The image of (4) can be identified with those classes in $H^{\text{even}}(X(\mathbb{C}); \mathbb{Q})$ represented by algebraic cycles - the subject of the Hodge conjecture!

4.4. K_1 of Rings and Simple-Homotopy Theory

The following definition of K_1 of a ring may seem ad hoc at first but we will see in the next section that this is not so. It turns out to be the correct definition for turning algebraic K-theory into a cohomology theory with associated spectrum just like we did for topological K-theory in the first chapter. K_1 is of particular importance for simple-homotopy theory as it houses the Whitehead torsion invariant which we discuss below.

Recall that the *commutator subgroup* $[G, G]$ of a group G is the subgroup generated by its *commutators* $[g, h] = ghg^{-1}h^{-1}$. It is a normal subgroup and has the universal property that every homomorphism from G to an abelian group factors through $G/[G, G]$. Define $K_1(R)$ of a ring (associative and with unit) as the abelian group

$$K_1(R) := \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$$

where $\mathrm{GL}(R)$ is the infinite general linear group. The group operation in $K_1(R)$ will usually be written additively $[A] + [B] = [AB]$ with unit $[1] = 0$.

A ring homomorphism $R \to S$ gives a natural homomorphism $\mathrm{GL}(R) \to \mathrm{GL}(S)$ and thus a map $K_1(R) \to K_1(S)$ by the universal property. K_1 is thus a functor from rings to abelian groups.

If R is commutative, then the determinant of a matrix provides a group homomorphism $\mathrm{GL}(R) \to R^\times$ onto the group $R^\times$ of units of R. By the universal property this induces a surjection $\det : K_1(R) \to R^\times$ and we write $SK_1(R) := \ker(\det)$. The natural inclusion of $R^\times = \mathrm{GL}_1(R)$ into $\mathrm{GL}(R)$ splits the exact sequence $SK_1(R) \hookrightarrow K_1(R) \twoheadrightarrow R^\times$ so that $K_1(R) \cong R^\times \oplus SK_1(R)$.

A matrix with coefficients in a ring R is called *elementary* if it coincides with the identity matrix except for one off-diagonal element $r \in R$. We will use the notation $e_{ij}(r)$ for the elementary matrix with element r in the (i, j)-position. Let $E_n(R) \leq \mathrm{GL}_n(R)$ denote the subgroup generated by these elementary matrices and let $E(R) \leq \mathrm{GL}(R)$ be their colimit.

Interpreting matrices as linear operators on column vectors, $e_{ij}(r)$ is the elementary row operation of adding r times row j to row i and $E_n(R)$ is the subgroup of all matrices that can be reduced to the identity matrix via these row operations only. $\mathrm{GL}(R)/E(R)$ measures the obstruction to such a reduction. In this light, it is easy to see that $e_{ij}(r)e_{ij}(s) = e_{ij}(r + s)$ and that $e_{ij}(-r)$ is the inverse of $e_{ij}(r)$.

Recall that a group is called *perfect* if $G = [G, G]$. Clearly, every perfect subgroup of a group is contained in the commutator of the group.

Lemma 4.4.1. *If $n \geq 3$ then $E_n(R)$ is perfect.*

PROOF. If i, j, k are distinct then $e_{ij}(r) = [e_{ik}(r), e_{kj}(1)]$. □

Proposition 4.4.2 (Whitehead's Lemma). *For any ring R, $E(R)$ is the commutator subgroup of $\mathrm{GL}(R)$. Hence $K_1(R) = \mathrm{GL}(R)/E(R)$.*

PROOF. By the previous lemma we know $E(R) \subseteq [\mathrm{GL}(R), \mathrm{GL}(R)]$. Conversely, every commutator in $\mathrm{GL}_n(R)$ can be expressed as a product in

$\mathrm{GL}_{2n}(R)$:

$$[g,h] = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} (hg)^{-1} & 0 \\ 0 & hg \end{pmatrix}.$$

But each matrix of this form can be expressed as a product in $E_{2n}(R)$

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -g^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

since $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)$ and by Gaussian elimination every matrix with 1's on the diagonal belongs to $E(R)$ (inductively kill off all super-diagonals). □

There is a convenient way for adding elements in $K_1(R)$: given $A, B \in \mathrm{GL}(R)$ form their *block sum* $A \oplus B = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) = \left(\begin{smallmatrix} AB & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} B^{-1} & 0 \\ 0 & B \end{smallmatrix}\right)$. Since $\left(\begin{smallmatrix} B^{-1} & 0 \\ 0 & B \end{smallmatrix}\right) \in E(R)$ by the argument from the above proposition, we have $[A \oplus B] = [AB \oplus 1] = [AB]$.

Example 4.4.3. If F is a field, we show that $K_1(F) = F^\times$. Indeed, by Gaussian elimination every invertible matrix can be turned into the identity matrix by a sequence of row operations which correspond to elementary matrices of three types:

(1) adding one row to another row ($e_{ij}(r)$ for $i \neq j$),
(2) row swaps (a block $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on the diagonal of the identity matrix),
(3) and taking multiplies of one row with itself ($e_{ii}(r)$).

If a matrix has determinant 1, then there must be an even number of row swaps and for every row that gets multiplied by r, another row must get multiplied by $1/r$. It is not hard to show that two consecutive type (2) and two consecutive type (3) operations (r and $1/r$) are products of type (1) operations so that in fact $E(F) = \mathrm{SL}(F)$. But then $K_1(F) = \mathrm{GL}(F)/E(F) \cong \mathrm{GL}(F)/\mathrm{SL}(F) \cong F^\times$ where the last isomorphism is given by taking determinant. Similarly, if R is a Euclidean domain then Gaussian elimination still works using the Euclidean algorithm to find least common multiplies needed for type (1) and (3) operations. So then again $E(R) = \mathrm{SL}(R)$, and $K_1(R) = R^\times$. Thus for instance $K_1(\mathbb{Z}) = \mathbb{Z}^\times = \mathbb{Z}/2$ and $K_1(F[x]) = F^\times$.

The perhaps most important application of K_1 to topology comes from *simple-homotopy theory*. As in the case of Wall's finiteness obstruction (4.1.8), the rings of interest are integral group rings $\mathbb{Z}[G]$ where the group G is often the fundamental group of some topological space. We define the *Whitehead group* $Wh(G)$ of a group G as the abelian group which is the quotient of $K_1(\mathbb{Z}[G])$ by the image of the *trivial units* $\pm G = \{\pm g : g \in G\} \subseteq \mathbb{Z}[G]^\times = \mathrm{GL}_1(\mathbb{Z}[G])$.

Examples 4.4.4. Here are some examples of Whitehead groups:

(1) If $G = \{1\}$, then $Wh(G) = K_1(\mathbb{Z})/[\pm 1] = \{[1]\}$ is trivial by the above example.
(2) If G is finite abelian, $\mathbb{Z}[G]$ is a commutative ring so that we can define a determinant as in (4.4.3). $\mathbb{Z}[G]$ is also a Euclidean domain by taking the absolute value of the *augmentation map* $\epsilon(\sum n_g g) = \sum n_g$ as the Euclidean function and using the Euclidean algorithm of $\mathbb{Z}$. Hence $K_1(\mathbb{Z}[G]) \cong \mathbb{Z}[G]^\times$ and $Wh(G)$ is the group of units of $\mathbb{Z}[G]$ modulo trivial units.
(3) Let $G = \pi_1(X, x_0)$ be the fundamental group of some path-connected space X computed with respect to some basepoint x_0. Then there is an inner automorphism $f : \pi_1(X, x_0) \to \pi_1(X, x_1)$, $\sigma \mapsto \phi^{-1}\sigma\phi$ for any other basepoint x_1 and so $Wh(\pi_1(X, x_0)) \cong Wh(\pi_1(X, x_1))$ and we may write $\pi_1(X)$ without reference to the basepoint. Moreover, $f_* : Wh(\pi_1(X)) \to Wh(\pi_1(X))$ is the identity since the corresponding automorphism of $\mathrm{GL}_n(\mathbb{Z}[\pi_1(X)])$ is given by

$$(a_{ij}) \mapsto \begin{pmatrix} \phi & & \\ & \ddots & \\ & & \phi \end{pmatrix}^{-1} (a_{ij}) \begin{pmatrix} \phi & & \\ & \ddots & \\ & & \phi \end{pmatrix}$$

and $Wh(\pi_1(X))$ is commutative. Hence, $X \to Wh(\pi_1(X))$ gives a well-defined functor from the category of path-connected spaces to the category of abelian groups.

A priori it may not seem clear that the Whitehead group is ever nontrivial so consider the following

Example 4.4.5. Let C_5 be the cyclic group of order 5 with generator t. We exhibit an element of infinite order in $Wh(C_5) = \mathbb{Z}[C_5]^\times / \pm C_5$. Let $a = 1 - t - t^{-1}$, then one can check that $(1 - t - t^{-1})(1 - t^2 - t^3) = 1$ and so $a \in \mathbb{Z}[C_5]^\times$. Define a homomorphism $\alpha : \mathbb{Z}[C_5] \to \mathbb{C}$ by $t \to e^{2\pi i/5}$, then $\pm C_5$ maps into the roots of unity and so α induces a homomorphism $\beta : Wh(C_5) \to \mathbb{R}_+^\times$, $b \mapsto |\alpha(b)|$. Now $\beta(a) = |1 - e^{2\pi i/5} - e^{-2\pi i/5}| = |1 - 2\cos(2\pi/5)| \approx 0.4$ and so a cannot be of finite order.

The motivation for defining Whitehead groups is that they house an algebraic obstruction to a homotopy equivalence between two manifolds to be "simple". Simple-homotopy type is a finer invariant than homotopy type and can be used to distinguish homotopy equivalent spaces which are not homeomorphic.

A famous application of simple-homotopy theory is the s-cobordism theorem, "s" for simple. First recall that an *h-cobordism* is a cobordism $(W; M, M')$ such that the inclusions $M \hookrightarrow W$ and $M' \hookrightarrow W$ are homotopy equivalences. An h-cobordism is an *s-cobordism* if the homotopy equivalences are simple. The main result of simple-homotopy theory to be discussed below is that a homotopy equivalence such as $M \hookrightarrow W$ is simple if and only if an associated invariant $\tau(W, M) \in Wh(\pi)$ named Whitehead torsion vanishes ($\pi := \pi_1(M) = \pi_1(W)$).

Theorem 4.4.6 (s-Cobordism Theorem of Barden-Mazur-Stallings)**.** *Let* **CAT** *be the category of topological manifolds, smooth manifolds or PL manifolds and let M be a compact* **CAT** *manifold of dimension $n \geq 5$. Then Whitehead torsion defines a one-to-one correspondence*

$$\begin{aligned} \{h\text{-}cobordisms\ on\ M\}/(isomorphisms \operatorname{rel} M) &\to Wh(\pi) \\ [(W; M, M')] &\mapsto \tau(W, M) \end{aligned}$$

between isomorphism classes of h-cobordisms on M and elements of the Whitehead group. In particular, an h-cobordism is trivial, i.e. W is **CAT** *isomorphic to $M \times [0, 1]$ $(\operatorname{rel} M)$ if and only if its Whitehead torsion vanishes.*

It follows that any simply-connected h-cobordism is trivial. Moreover, just like in the case of Wall's finiteness obstruction it is conjectured that $Wh(\pi) = 0$ for any torsion-free group π. Thus, any h-cobordism with torsion-free fundamental group would be trivial.

Corollary 4.4.7 (Poincaré Conjecture for $n \geq 5$)**.** *For $n \geq 5$, a closed n-manifold Σ which has the homotopy type of S^n is homeomorphic to S^n.*

Proof. First assume that $n \geq 6$. Cut out two open disks D_1^n, D_2^n from Σ, viewed as "polar caps" of the homotopy sphere. What remains is a manifold W with the homotopy type of a cylinder and with two boundary components each homeomorphic to S^{n-1} with $n - 1 \geq 5$. Since $\pi := \pi_1(W) = \pi_1(S^{n-1}) = 1$, $Wh(\pi) = 1$ as we saw in (4.4.4). But then the Whitehead torsion which is an element of this group must vanish and we are in a position to apply the s-cobordism theorem to conclude that there is a homeomorphism $h : \Sigma \setminus (D_1^n \cup D_2^n) \to S^{n-1} \times [0, 1]$ with $h|_{\partial D_1^n} = \mathrm{id}$. Extend h to $\Sigma \setminus D_2^n$ by taking the identity map on D_1^n. On the other end, $h|_{\partial D_2^n}$ is a homeomorphism $\widetilde{f} : S^{n-1} \to S^{n-1}$. So the problem is whether we can extend this homeomorphism to a homeomorphism $f : D^n \to D^n$. Indeed, this can be done by radial extension, i.e. $f(re^{i\theta}) = r\widetilde{f}(e^{i\theta})$. We have arrived at a homeomorphism $\Sigma \to D_1^n \cup_f D_2^n$. Such a manifold is called a *twisted sphere*. The proof is completed by showing that any twisted sphere $D_1^n \cup_f D_2^n$ is homeomorphic to S^n. To see this, define a map $g : D_1^n \cup_f D_2^n \to S^n$ as follows. Let $e_{n+1} = (0, \dots, 0, 1) \in \mathbb{R}^{n+1}$, let $i : D_1^n \hookrightarrow S^n$ be the embedding of D_1^n onto the southern hemisphere ($x_{n+1} \leq 0$) of S^n and write every point of D_2^n as tv, $0 \leq t \leq 1$ with $v \in \partial D_2^n$. Then

$$g(u) = \begin{cases} i(u) & \text{if } u \in D_1^n; \\ i(f^{-1}(v))\sin(\frac{\pi t}{2}) + e_{n+1}\cos(\frac{\pi t}{2}) & \text{if } u = tv \in D_2^n \end{cases}$$

is a one-to-one, continuous map onto S^n and hence is a homeomorphism.

For $n = 5$, we use the fact that all 5-manifolds with the homotopy type of S^5 bound a 6-manifold [**KM63**], i.e. that in that case there exists some manifold V such that $\partial V = \Sigma$. Since $\Sigma \simeq S^5$, it follows that V is contractible. That is, the homotopy equivalence $\partial V \simeq \partial D^6$ extends to a homotopy equivalence $V \simeq D^6$. Cutting out a D^6 from the interior of V, we obtain an h-cobordism between Σ and $\partial D^6 = S^5$. Since everything is simply-connected, we apply the s-cobordism theorem in the category of smooth manifolds to find a diffeomorphism between Σ and S^5. Note that this means that there are no exotic 5-spheres! $\square$

The main idea of simple-homotopy theory is to build up homotopy equivalences as composites of simple moves. This works particularly well in the category of finite, connected relative CW complexes (X, A), i.e. A is a Hausdorff space and X is obtained from A by attaching finitely many cells. We will say the inclusion $A \hookrightarrow X$ is an *elementary collapse*, denoted $X \searrow_e A$, if X is obtained from A by attaching two cancelling cells in adjacent dimensions. By this we mean that $X = (A \cup_f e^{k-1}) \cup_g e^k$ for some k, such that $f : S^{k-2} \to A$ is the attaching map for the $(k-1)$-cell and $g : S^{k-1} \to (A \cup_f B^{k-1})$ is the attaching map for the k-cell, and that g maps one hemisphere of S^{k-1} identically onto the $(k-1)$-cell and the other hemisphere into A. Thus, X can be viewed as the mapping cylinder of a map $D_-^{k-1} \to A$ and A is a deformation retract of X.

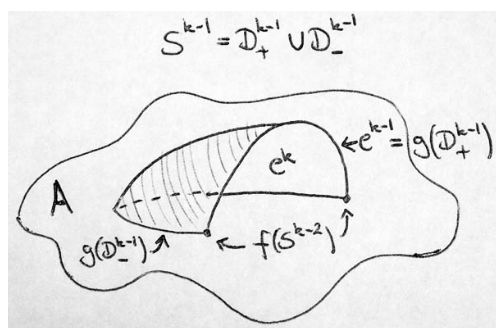
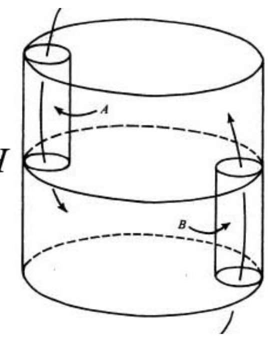


Figure 1. An elementary collapse

More generally, we say X *collapses to* A or A *expands to* X and write $X \searrow A$ or $A \nearrow X$ if $X = X_0 \searrow_e X_1 \searrow_e X_2 \searrow_e \cdots \searrow_e X_n = A$. Finally, the inclusion $A \hookrightarrow X$ is called a *simple-homotopy equivalence* if it is in the equivalence relation generated by $\searrow$, i.e. $X = X_0 \nearrow X_1 \searrow X_2 \nearrow \cdots \searrow X_n = A$.

Example 4.4.8. The "house with two rooms" H shown on the right is contractible and $* \hookrightarrow H$ is a simple-homotopy equivalence. However, H is not *collapsible*, i.e. some expansions are needed. To see this, pour cement through cylinder A until the lower room and A are filled up. This corresponds to an elementary expansion with a 3-cell. Do the same with the other cylinder labeled B. Then H expands to D^3. Now D^3 clearly collapses to a point $*$. Hence $H \nearrow D^3 \searrow *$.

H



Every simple-homotopy equivalence is a homotopy equivalence. The converse is not true in general. Moreover, by a theorem of Chapman [**Cha74**] homeomorphic finite CW complexes are simple-homotopy equivalent. It is in this sense that simple-homotopy type is a finer invariant than just homotopy type in trying to decide whether homotopy equivalent spaces are homeomorphic.

The Whitehead torsion of $A \hookrightarrow X$ is defined from the relative cellular chain complex $C_*(\widetilde{X}, \widetilde{A})$ of universal covers. We begin by defining torsion of a general chain complex.

Let $C : C_n \to C_{n-1} \to \cdots \to C_0$ be a chain complex of modules over a ring R such that each C_i is free with a preferred basis c_i, and each homology group $H_i(C)$ vanishes. Such a chain complex is called *acyclic* and *based*. We wish to define the torsion of C in $\widetilde{K}_1(R) := K_1(R)/[(-1)]$ where $(-1) \in \mathrm{GL}_1(R)$. The reason that we use $\widetilde{K}_1$ rather than K_1 is that it is both messy and unnecessary for us to deal with ordered bases.

Two pathologies can occur in dealing with free modules over a ring R. The first is that $R^m \cong R^n$ may not imply $m = n$. However, we will only consider group rings $R = \mathbb{Z}[G]$ which admit an augmentation map $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ so that $\mathbb{Z}[G]^m \cong \mathbb{Z}[G]^n$ implies $\mathbb{Z}^m = \mathbb{Z} \otimes_{\mathbb{Z}[G]} (\mathbb{Z}[G])^m \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} (\mathbb{Z}[G])^n = \mathbb{Z}^n$.

The second pathology does occur for group rings, so we cannot assume it away: it is not necessarily true that a submodule of a free module is free. In particular, let B_i denote the image of the boundary homomorphism

$\partial : C_{i+1} \to C_i$ and let $Z_{i+1} = B_{i+1}$ denote its kernel, then we cannot assume that B_i is free. However, B_i is stably free. To see this, we use the following

Lemma 4.4.9. *Consider a short exact sequence $0 \to X \to Y \to Z \to 0$ of R-modules. If Y and Z are stably free, then X is also stably free.*

PROOF. Since Z is projective the exact sequence splits so that $Y \cong X \oplus Z$. Thus if $Z \oplus F \cong F'$ and $Y \oplus F \cong F''$ where F, F', F'' are free, it follows that $X \oplus F' \cong X \oplus Z \oplus F \cong Y \oplus F \cong F''$. □

Returning to the acyclic, based chain complex $C_n \to \cdots \to C_0$ it then follows by induction using the exact sequences $0 \to Z_{i+1} \to C_{i+1} \to B_i \to 0$ that B_i is stably free.

Given a free R-module F, with two different bases $b = (b_1, \ldots, b_k)$ and $c = (c_1, \ldots, c_k)$ we can assign an element in $\widetilde{K}_1(R)$ by considering the change of basis matrix. That is, let $c_i = \sum r_{ij} b_j$ to obtain a nonsingular matrix (r_{ij}) with entries in R. Write $[c/b] := [(r_{ij})] \in \widetilde{K}_1(R)$ for the corresponding element in $\widetilde{K}_1$. The identities $[d/c] + [c/b] = [d/b]$ and $[b/b] = 0$ show that this is an equivalence relation.

We would like to do something similar with stably free modules by considering bases of the free module of which they are a summand of. So let F_i denote the standard free module of rank i, with standard basis $f_1, \ldots, f_i$. An *s-basis* b for a stably free module M is a basis $(b_1, \ldots, b_{r+t})$ for some free module $F' \cong M \oplus F_t$ where t can be any nonnegative integer. Given two s-bases $b = (b_1, \ldots, b_{r+t})$ and $c = (c_1, \ldots, c_{r+u})$ for M, choose an integer $v \geq \max(t, u)$. Extend b to a basis for $M \oplus F_v$ by setting $b_{r+i} = 0 \oplus f_i$ for $i \geq t + 1$. Similarly extend c to a basis for $M \oplus F_v$. Let (r_{ij}) be the change of basis matrix in $\mathrm{GL}_{r+v}(R)$ of these two extended bases and let $[c/b]$ be the corresponding element in $\widetilde{K}_1(R)$. This construction does not depend on the choice of v since we are working in the infinite general linear group.

We can now proceed with our acyclic, based chain complex $C : C_n \to \cdots \to C_0$. Choose an s-basis b_i for each B_i. Since $C_i/Z_i \cong C_i/B_i \cong B_{i-1}$, we see that the bases b_i and b_{i-1} combine to yield a new basis b_i, b_{i-1} for C_i. Define the *torsion* of C as

$$\tau(C) = \sum (-1)^i [b_i, b_{i-1}/c_i].$$

This does not depend on the choice of the b_i since, choosing different bases $\widetilde{b}_i$, we have

$$\sum (-1)^i [\widetilde{b}_i, \widetilde{b}_{i-1}/c_i] = \sum (-1)^i ([b_i, b_{i-1}/c_i] + [\widetilde{b}_i/b_i] + [\widetilde{b}_{i-1}/b_{i-1}])$$

where the last two terms sum up to zero. Of course, $\tau(C)$ does depend on the basis of C. The motivation for defining the Whitehead group the way we did is to eliminate this dependence as we shall see now.

So consider the situation of a finite, connected relative CW complex (X, A) where the inclusion $f : A \hookrightarrow X$ is a homotopy equivalence so that $\pi := \pi_1(A) = \pi_1(X)$. We consider the associated relative cellular chain complex $C_*(\widetilde{X}, \widetilde{A})$ of the universal covers $\widetilde{X}$ and $\widetilde{A}$. As before, $C_*(\widetilde{X}, \widetilde{A})$ is a complex of free $\mathbb{Z}[\pi]$-modules, and is of finite type since X is finite. It is also acyclic because the homology groups $H_i(\widetilde{X}, \widetilde{A}; \mathbb{Z}) = H_i(X, A; \mathbb{Z}[\pi])$ of this complex are zero since A is a deformation retract of X.

If we were given a preferred basis c_p for each module $C_p(\widetilde{X}, \widetilde{A})$ then the torsion $\tau(C_*(\widetilde{X}, \widetilde{A})) \in \widetilde{K}_1(\mathbb{Z}[\pi]) \in \widetilde{K}_1(\mathbb{Z}[\pi])$ would be defined. Conveniently, the geometry of the situation determines a preferred basis as follows: let $e_1, \ldots, e_\alpha$ denote the k-cells of $X \setminus A$. For each e_i choose a representative cell $\widetilde{e}_i$ of $\widetilde{X}$ lying over e_i. Furthermore, choose an orientation ± 1 so that $\widetilde{e}_i$ determines a basis element of $C_k(\widetilde{X}, \widetilde{A})$ which we may also denote by $\widetilde{e}_i$. Then $c_p = (\widetilde{e}_1, \ldots, \widetilde{e}_\alpha)$ is the required basis for $C_p(\widetilde{X}, \widetilde{A})$.

Example 4.4.10. Let $X = \mathbb{R}P^n$ with $n > 1$. Then $X = e^0 \cup e^1 \cup \cdots \cup e^n$ and $\widetilde{X} = S^n$. The lifted CW structure is $S^n = e^0_+ \cup e^0_- \cup e^1_+ \cup e^1_- \cup \cdots \cup e^n_+ \cup e^n_-$ with $e^i_\pm$ being the upper and lower hemispheres of S^i. As a $\mathbb{Z}[\pi]$-module, $C_i(S^n)$ is then free and of rank 1 with basis either e^i_+ or e^i_-.

As mentioned before, the only arbitrariness in defining torsion comes from how we choose the basis c_p. Any other lift of e_i differs from $\widetilde{e}_i$ by an element of the fundamental group and any other orientation by -1 so that we see now that this indeterminacy is precisely removed by quotienting out by the subgroup $\{\pm g : g \in \pi\}$. Denote the image of $\tau(C_*(\widetilde{X}, \widetilde{A}))$ in $Wh(\pi)$ by $\tau(X, A)$ and call it the *Whitehead torsion.*

Theorem 4.4.11 (Fundamental Theorem of Simple-Homotopy Theory)**.** *Let (X, A) be a finite, connected relative CW complex where the inclusion $f : A \hookrightarrow X$ is a homotopy equivalence so that $\pi := \pi(A) = \pi(X)$. Then f is simple if and only if $\tau(X, A) = 0$.*

Proof Sketch. If $X \searrow_e A$, then $\tau(X, A) = 0$. This follows since the cellular chain complex of an elementary collapse is given by $\partial_k : C_k(X, A) \to C_{k-1}(X, A)$ so that $B_k = B_{k-2} = 0$ and $B_{k-1} = \ker(\partial_{k-1}) = C_{k-1}(X, A)$. Hence $[b_{k-1}, b_{k-2}/c_{k-1}] = [b_{k-1}/c_{k-1}] = [c_{k-1}/c_{k-1}] = [1]$. The only other nontrivial term is $[b_k, b_{k-1}/c_k] = [b_{k-1}/c_k] = [c_{k-1}/c_k]$ which is given by the boundary map $\partial_k : C_k(X, A) \to C_{k-1}(X, A)$. But by definition ∂_k maps the noncollapsed hemisphere of $\partial e^k = S^{k-1}$ identically onto e^{k-1} as illustrated in figure (1). Hence $[b_k, b_{k-1}/c_k] = [1]$ also and $\tau(X, A) = 0$.

Next observe that if $A = X_n \hookrightarrow X_{n-1} \hookrightarrow \cdots \hookrightarrow X_0 = X$ and all the inclusions are homotopy equivalences of finite CW complexes, then $\tau(X, A) = \tau(X, X_1) + \cdots + \tau(X_{n-1}, A)$. This follows from the fact that $C_i(X, A; R) = C_i(X, X_1; R) \oplus \cdots \oplus C_i(X_{n-1}, A; R)$. In particular, if $X \searrow A$ so that $X = X_0 \searrow_e X_1 \searrow_e \cdots \searrow_e X_n = A$ then $\tau(X, A) = 0$ since each piece vanishes.

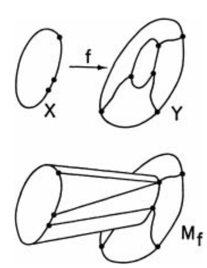
More generally, if $f : A \hookrightarrow X$ is simple, then we might have to deal with a mix of expansions and collapses, e.g. $X \nearrow Y \searrow A$. But then $0 = \tau(Y, A) = \tau(Y, X) + \tau(X, A) = \tau(X, A)$. Hence the Whitehead torsion vanishes in this case also.

Conversely, suppose that $\tau(X, A) = 0$. The first step in showing that $f : A \hookrightarrow X$ is a simple-homotopy equivalence is called *cell-trading* [**Coh73**, 7.3]. If e is a cell of $X \setminus A$ of minimal dimension i, one constructs a simple-homotopy equivalence $X \to X'$ rel A so that X' has one less i-cell and one more $(i+1)$-cell than X and all other cells remain unchanged. It follows that one may assume that all the cells added to A to form X are in two consecutive dimension k and $k-1$. Thus, the chain complex $C_*(X, A)$ is described by an invertible matrix $\partial_k : C_k(X, A) \to C_{k-1}(X, A)$. Since the torsion is zero, we may assume that $[(\partial_k)] = [c_{k-1}/c_k] \in Wh(\pi)$ is a product of elementary matrices $e_{ij}(r)$, $r \in \mathbb{Z}[\pi]$ for $i \neq j$, and matrices $e_{ii}(\pm g)$, $g \in \pi$. Let M be a matrix of one of these types and write $\partial_k = \partial'_k \circ M$, then there is a technique called *cell-sliding* [**Coh73**, 8.3] by which one can produce a simple-homotopy equivalence $X \to X'$ rel A so that $C_*(X', A)$ has boundary map ∂'_k. We have thus reduced the situation to the case where $A \hookrightarrow X$ has the chain complex

$$C_*(X, A) : \cdots \to 0 \to \mathbb{Z}[\pi]^m \xrightarrow{1} \mathbb{Z}[\pi]^m \to 0 \to \cdots .$$

There is one last technique, *cell-cancellation* [**Coh73**, 8.2], which then says that $A \hookrightarrow X$ is a simple-homotopy equivalence. □

The concept of Whitehead torsion can be carried over from inclusions to general homotopy equivalences $f : X \to Y$ between finite, connected CW complexes. As often in such situations, we begin by forming the mapping cylinder M_f. The cell structure on M_f is chosen in the obvious way so that $X (= X \times \{0\})$ and Y are disjoint subcomplexes of M_f (see the figure on the right). Clearly Y is a deformation retract of M_f and we could consider $\tau(M_f, Y)$. However:



Lemma 4.4.12. *The torsion $\tau(M_f, Y)$ is zero.*

PROOF. Let $f(p) : X^p \to Y$ denote the restriction of f to the p-skeleton of X, so that

$$Y = M_{f(-1)} \subseteq M_{f(0)} \subseteq M_{f(1)} \subseteq \cdots \subseteq M_{f(n)} = M_f.$$

Then $\tau(M_f, Y) = \sum_p \tau(M_{f(p)}, M_{f(p-1)})$ and each term is zero since $M_{f(p)}$ collapses to $M_{f(p-1)}$ and we have seen in the last theorem that this means the torsion $\tau(M_{f(p)}, M_{f(p-1)})$ is zero. □

Instead, note that f being a homotopy equivalence implies that Y deformation retracts to $f(X)$ and so X is also a deformation retract of M_f. Define the Whitehead torsion of a cellular homotopy equivalence $f : X \to Y$ as $\tau(f) := \tau(M_f, X) \in Wh(\pi)$ where $\pi := \pi_1(M_f) \cong \pi_1(Y) \cong \pi_1(X)$.

This definition agrees with our old definition when f is an inclusion since then $X \hookrightarrow Y \hookrightarrow M_f$ so that $\tau(f) = \tau(M_f, X) = \tau(M_f, Y) + \tau(Y, X) = \tau(Y, X)$.

One can go even further and define Whitehead torsion for a homotopy equivalence $f : X \to Y$ which is not cellular. By the cellular approximation theorem [**AGP02**, Theorem 5.1.44] f is homotopic to a cellular map f_0 so define $\tau(f) := \tau(f_0)$. This is well-defined since Whitehead torsion is homotopy invariant, i.e. if $f_0 \simeq f_1$ then $\tau(f_0) = \tau(f_1)$. To see this, note that C_{f_0} and C_{f_1} differ by a homotopy F of the attaching maps. It is enough to consider the case $C_{f_0} = X \cup_{f_0} D^k$ and $C_{f_1} = X \cup_{f_1} D^k$ where $F : S^{k-1} \times I \to X$ is the homotopy. Define $W = X \cup_F (D^k \times I)$ where we glue $D^k \times I$ to X along $S^{k-1} \times I$. Then (W, A) is a finite relative CW complex and $C_{f_0} \nearrow_e W \searrow_e C_{f_1}$ so that $\tau(C_{f_0}, A) = \tau(C_{f_1}, A)$ since the torsion of any elementary collapse is zero and $\tau(W, A) = \tau(W, C_{f_0}) + \tau(C_{f_0}, A) = \tau(W, C_{f_1}) + \tau(C_{f_1}, A)$.

Finally, how does simple-homotopy theory apply to manifolds? One can give a smooth manifold the structure of a simplicial complex and hence that of a CW complex by constructing a triangulation. Triangulations are unique up to subdivision and one can show that the torsion of a homotopy equivalence $X \to Y$ of smooth manifolds is invariant under subdivision of the pair (X, Y) [**Mil66**, Theorem 7.1]. Compact smooth manifolds thus have a well-defined simple-homotopy type. By the theory of Kirby and Siebenmann [**KS69**] the same also holds for topological manifolds.

4.5. Higher K-Theory and its Geometric Motivation

In this section we will see the main geometric motivation for defining higher K-theory. We will do this in a somewhat roundabout fashion by going from the most modern (Waldhausen K-theory) to the more classical ideas (Quillen K-theory) and see how they are interrelated. **CAT** will denote any of the categories of topological, smooth or PL manifolds.

Recall that the s-cobordism theorem (4.4.6) settled the existence question of product structures on an h-cobordism. An h-cobordism $(W; M, M')$ is **CAT** isomorphic to $M \times [0,1]$ if and only if it is an s-cobordism. One may ask about uniqueness: given two product structures $f, g : W \overset{\approx}{\to} M \times [0,1](\mathrm{rel}\, M)$, when is f isotopic to g? To answer this question we look at the topological group $P(M) := \mathbf{CAT}(M \times I, M \times 0)$ of **CAT** automorphisms of $M \times I$ restricting to the identity on $M = M \times 0$. Note that $f \circ g^{-1}$ belongs to $P(M)$ and the uniqueness problem becomes a question about the path-connected components of $P(M)$, i.e. what is $\pi_0(P(M))$?

$P(M)$ is called the space of *pseudo-isotopies*. The reason for the name is that if a pseudo-isotopy $F \in P(M)$ commutes with the projection map $M \times I \to I$ (i.e. it preserves the level sets $M \times t$ for all $t \in [0,1]$), then it induces an isotopy between id_M and $F|_{M \times 1}$.

If M is simply-connected, it is a theorem of Cerf [**Cer70**] that $P(M)$ is path-connected and thus every pseudo-isotopy is an isotopy. In general the obstruction for this to happen is the the following:

Theorem 4.5.1 (Pseudo-Isotopy Theorem of Hatcher-Wagoner)**.** *If M is a smooth compact connected manifold of dimension $n \geq 5$ with fundamental group π, then there is a surjection of $\pi_0(P(M))$ onto $Wh_2(\pi)$.* [**Hat73**]

In analogy to the aforementioned $Wh(\pi)$ which we will from now on write as $Wh_1(\pi)$, $Wh_2(\pi)$ is defined as a quotient of the second higher K-group $K_2(\mathbb{Z}[\pi])$ which we won't say anything particular about. Instead, we attack the more general question of higher K-theory head on by extracting its main geometric motivation from a recent book by Waldhausen, Jahren, and Rognes [**JRW12**].

Here is the general idea. One begins by seeing the s-cobordism theorem as a computation of the set of path components of the space $H^{\mathbf{CAT}}(M)$ of h-cobordisms built on M such that $\pi_0 H^{\mathbf{CAT}}(M) \cong Wh_1(M)$ whenever $\dim M \geq 5$. The goal of the *parametrized* h-cobordism theorem is to compute the homotopy type of $H^{\mathbf{CAT}}(M)$ in general. Unfortunately, one needs to settle for a *stable* parametrized h-cobordism theorem.

Let us explain these terms a little. Let M be a compact **CAT** manifold. Define the **CAT** h-cobordism space $H(M) = H^{\mathbf{CAT}}(M)$ of M as a simplicial set. Its 0-simplices are the compact **CAT** h-cobordisms on M. For each $q \geq 0$, a q-simplex of $H(M)$ is a **CAT** bundle of h-cobordisms over Δ_q, the standard topological q-simplex. We get a topological space by geometric realization.

We remove concerns about the validity of certain statements only in particular dimensions by *stabilizing* the problem. Consider the map

$$\sigma : H(M) \to H(M \times I)$$

where $I = [0, 1]$, sending an h-cobordism W on M to the h-cobordism $W \times I$ on $M \times I$. The stable h-cobordism space of M is the colimit

$$\mathcal{H}^{\mathbf{CAT}}(M) = \operatorname{colim}_k H^{\mathbf{CAT}}(M \times I^k).$$

The model for the homotopy type of $\mathcal{H}^{\mathbf{CAT}}(M)$ is

Theorem 4.5.2 (Stable Parametrized h-Cobordism Theorem)**.** *There is a natural homotopy equivalence*

$$\mathcal{H}^{\mathbf{CAT}}(M) \simeq \Omega Wh^{\mathbf{CAT}}(M)$$

for each compact **CAT** *manifold M.*

Here $Wh^{\mathbf{CAT}}(M)$ is the **CAT** *Whitehead space* defined in terms of Waldhausen's $A(M)$ known as the *algebraic K-theory of spaces* [**Wal85**]. To define $A(M)$, let M be a CW complex and let $\mathcal{R}(M)$ be the category of CW complexes Y obtained from M by attaching cells, and having M as a retract. We require some sort of finiteness condition on these CW complexes to avoid an *Eilenberg swindle* which would make our K-theory trivial. One way to do this is to impose that only finitely many cells be attached to M to obtain Y. This category is denoted by $\mathcal{R}_f(M)$. Another option is to require that all such Y are finitely dominated. We then write $\mathcal{R}_{fd}(M)$.

In any case, all variants of $\mathcal{R}(M)$ are *Waldhausen categories* [**Wei12**, §II.9], that is categories with cofibrations and weak equivalences, which in our case are cellular inclusions fixing M and (weak) homotopy equivalences respectively. We continue with $\mathcal{R}_f(M)$. Waldhausen's *$S_\bullet$-construction* of $\mathcal{R}_f(M)$ is then defined as a simplicial Waldhausen category $S_\bullet\mathcal{R}_f(M)$, and the *algebraic K-theory space of M* is defined to be the loop space of the geometric realization of the simplicial subcategory of weak equivalences h in $S_\bullet\mathcal{R}_f(M)$

$$A(M) = \Omega|hS_\bullet\mathcal{R}_f(M)|.$$

The K-groups of a Waldhausen category as above are the homotopy groups of the K-theory space, e.g.

$$K_i(\mathcal{R}_f(M)) = \pi_i(A(M)),$$

and one computes $K_0(\mathcal{R}_f(M)) = \mathbb{Z}$. Similary, one finds $K_0(\mathcal{R}_{fd}(M)) = \mathbb{Z}[\pi_1(M)]$.

Recall the $S_\bullet$-construction on a Waldhausen category $\mathcal{C}$. Its output is a simplicial Waldhausen category $S_\bullet\mathcal{C}$ defined as follows.

- $S_0\mathcal{C}$ is the zero category.
- $S_1\mathcal{C}$ is the category $\mathcal{C}$, but whose objects A are thought of as cofibrations $0 \rightarrowtail A$.
- $S_2\mathcal{C}$ is the *extension category* of $\mathcal{C}$. Its objects are cofibration sequences $E : A_1 \rightarrowtail A_2 \twoheadrightarrow A_{12}$ in $\mathcal{C}$ (the axioms of a Waldhausen category imply that every cofibration has a cokernel $A_{12} := A_2/A_1$). A morphism $E \to E'$ is a commutative diagram

$$\begin{tikzcd}
 & & & & & A_{12} \arrow[dl, "u_{12}"'] \\
 & & & & A'_{12} & \\
 & E: \arrow[dl] & & A_1 \arrow[rr, tail] \arrow[dl, "u_1"'] & & A_2 \arrow[uu, two heads] \arrow[dl, "u_2"] \\
E': & & A'_1 \arrow[rr, tail] & & A'_2 \arrow[uu, two heads] &
\end{tikzcd}$$

  A morphism is a cofibration if u_1, u_2, and the pushout map $A'_1 \cup_{A_1} A_2 \to A'_2$ are cofibrations in $\mathcal{C}$. A morphism is a weak equivalence if u_1, u_2 (and hence u_{12}) are weak equivalences in $\mathcal{C}$.
- $S_n\mathcal{C}$ is the category whose objects $A_\bullet$ are sequences of n cofibrations in $\mathcal{C}$:
$$A_\bullet : 0 = A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n$$
  together with a choice of every subquotient $A_{ij} = A_j/A_i$ $(0 < i \leq j \leq n)$. These choices are to be compatible in the sense that there is a commutative diagram

$$\begin{tikzcd}
 & & & & A_{(n-1)n} \\
 & & A_{23} \arrow[r, tail] & \cdots \arrow[r, tail] & A_{2n} \arrow[u, dashed, no head] \\
 & A_{12} \arrow[r, tail] & A_{13} \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A_{1n} \arrow[u, two heads] \\
A_1 \arrow[r, tail] & A_2 \arrow[r, tail] \arrow[u, two heads] & A_3 \arrow[r, tail] \arrow[u, two heads] & \cdots \arrow[r, tail] & A_n \arrow[u, two heads]
\end{tikzcd} \tag{5}$$

  and a morphism $A_\bullet \to B_\bullet$ is a natural transformation of sequences (and hence of the above commutative diagrams). A morphism is a cofibration when for every $0 \leq i < j < k \leq n$ the map of cofibration sequences $(A_{ij} \rightarrowtail A_{ik} \twoheadrightarrow A_{jk}) \to (B_{ij} \rightarrowtail B_{ik} \twoheadrightarrow B_{jk})$

is a cofibration in $S_2\mathcal{C}$. A morphism is a weak equivalence if each $A_i \to B_i$ (and hence each $A_{ij} \to B_{ij}$) is a weak equivalence in $\mathcal{C}$.

It remains to specify the face and degeneracy maps to produce a simplicial category. For each $n \geq 0$ and for each $0 \leq i \leq n$, define an exact functor $\partial_i : S_n\mathcal{C} \to S_{n-1}\mathcal{C}$ by omitting the row A_{i*} and the column containing A_i in (5), and reindexing the A_{jk} as needed. Similarly, define exact functors $s_i : S_n\mathcal{C} \to S_{n+1}\mathcal{C}$ by duplicating A_i in $A_\bullet$, and reindexing with the normalization $A_{i(i+1)} = 0$.

Then the $S_n\mathcal{C}$ fit together to form a simplicial Waldhausen category $S_\bullet\mathcal{C}$, and the subcategories $wS_n\mathcal{C}$ of weak equivalences fit together to form a simplicial category $wS_\bullet\mathcal{C}$.

Having produced a simplicial Waldhausen category $S_\bullet\mathcal{R}_f(M)$, we can reiterate the $S_\bullet$-construction to obtain a sequence of spaces

$$\Omega|hS_\bullet\mathcal{R}_f(M)|, |hS_\bullet\mathcal{R}_f(M)|, |hS_\bullet S_\bullet\mathcal{R}_f(M)|, \ldots, |hS_\bullet S_\bullet \cdots S_\bullet\mathcal{R}_f(M)|, \ldots$$

with appropriate structure maps defining an Ω-spectrum $\mathbf{A}(M)$, which has $A(M)$ as its underlying infinite loop space. $Wh^{\mathbf{CAT}}(M)$ is then defined as the homotopy cofiber of a spectrum map to $\mathbf{A}(M)$. See [**JRW12**, Definition 1.3.2] for details.

$Wh^{\mathbf{CAT}}(M)$ is so defined that

$$\pi_0(\mathcal{H}^{\mathbf{Diff}}(M)) = \pi_1(Wh^{\mathbf{Diff}}(M)) = Wh_1(\pi_1(M)),$$

and

$$\pi_1(\mathcal{H}^{\mathbf{Diff}}(M)) = \pi_2(Wh^{\mathbf{Diff}}(M)) = \pi_0(\mathcal{P}^{\mathrm{Diff}}(M))$$

where $\mathcal{P}^{\mathrm{Diff}} = \operatorname{colim}_k P^{\mathrm{Diff}}(M \times I^k)$ is the stable pseudo-isotopy space. We thus have agreement with the earlier discussion. Hopefully this doesn't come as a complete surprise considering that

$$K_0(\mathcal{R}_{fd}(M)) = \pi_0(A(M)) = \pi_1(|hS_\bullet\mathcal{R}_{fd}(M)|) = \mathbb{Z}[\pi_1(M)].$$

Higher homotopy groups extract other geometric information.

The above definition of K-theory of a Waldhausen category generalizes that of an exact category given by Quillen. Quillen's $\mathcal{Q}$-*construction* takes an exact category $\mathcal{C}$ and produces an auxiliary category $\mathcal{Q}(\mathcal{C})$. This category has the same objects as $\mathcal{C}$ but a morphism from A to B in $\mathcal{Q}(\mathcal{C})$ is an equivalence class of zig-zag diagrams

$$A \overset{j}{\twoheadleftarrow} Q \overset{i}{\rightarrowtail} B$$

where j is an admissible epimorphism and i is an admissible monomorphism in $\mathcal{C}$. Recall that a monomorphism is *admissible* if it can be completed to a short exact sequence and similarly for an admissible epimorphism. Two zig-zags are equivalent if there is an isomorphism between them which is the identity on A and B. To compose $A \twoheadleftarrow Q_1 \rightarrowtail B$ with $B \twoheadleftarrow Q_2 \rightarrowtail C$ we form the pullback and compose:

$$\begin{tikzcd}
& Q_1 \times_B Q_2 \arrow[r, tail] \arrow[d, two heads] & Q_2 \arrow[r, tail] \arrow[d, two heads] & C \\
A & Q_1 \arrow[l, two heads] \arrow[r, tail] & B &
\end{tikzcd}$$

The K-theory space of an exact category is then the loop space of the classifying space of the category $\mathcal{Q}(\mathcal{C})$

$$K(\mathcal{C}) = \Omega \, \mathrm{B} \, \mathcal{Q}(\mathcal{C}).$$

The K-groups of $\mathcal{C}$ are the homotopy groups of the K-theory space

$$K_i(\mathcal{C}) = \pi_i(K\mathcal{C}).$$

Every exact category defines a Waldhausen category with cofibrations being admissible monomorphisms and weak equivalences being isomorphisms i.

Theorem 4.5.3. *For any exact category $\mathcal{C}$, there is a natural homotopy equivalence $|iS_\bullet\mathcal{C}| \xrightarrow{\sim} \mathrm{B}\,\mathcal{Q}(\mathcal{C})$.* [**Wal85**, §1.9]

Quillen's $\mathcal{Q}$-construction in turn generalizes Quillen's $+$-construction first used to define higher K-theory of rings.

Let X be a pointed connected CW complex and P a perfect normal subgroup of $\pi_1(X)$. A map $X \to X^+$ is said to be a *$+$-construction relative to P* when all the following hold:

(i) X^+ is a connected CW complex (based at the image of the base point of X).
(ii) The map $\pi_1(X) \to \pi_1(X^+)$ is surjective with kernel P.
(iii) The map $X \to X^+$ induces an isomorphism on homology for any local coefficient system on X^+.

The last requirement is equivalent to the homotopy fiber $F(X \to X^+)$ being homologically acyclic, i.e. $\widetilde{H}_*(F(X \to X^+); \mathbb{Z}) = 0$. X^+ is called the *$+$-construction* and its main feature is that the perfect normal subgroup P has been killed from its fundamental group.

Theorem 4.5.4 (Quillen)**.** *The $+$-construction exists and can be obtained by attaching only 2-cells and 3-cells to X. Moreover, X^+ is unique up to homotopy equivalence* rel X.

Proof Sketch. One begins by forming a complex Y by attaching one 2-cell e_p for each element $p \in P$ along a chosen 1-cell representing p. Then $\pi_1(Y) = \pi_1(X)/P$. Next one shows that $H_2(Y)$ is isomorphic to the direct sum of $H_2(X)$ and the free abelian group generated by the classes $[e_p]$, and that each $[e_p]$ lies in the image of the Hurewicz homomorphism $\pi_2(Y) \to H_2(Y)$. This enables us to choose representing maps $S^2 \to Y$, along which we can attach 3-cells to form a complex Z which is a +-construction relative to P. See [**Ros94**, Theorem 5.2.2] for details. $\square$

Recall from (4.4.1) that $E(R) = [\mathrm{GL}(R), \mathrm{GL}(R)]$ is a perfect normal subgroup of $\mathrm{GL}(R)$. We define the K-theory space of a ring R to be

$$K(R) = \mathrm{B}\,\mathrm{GL}(R)^+ \times K_0(R),$$

where the +-construction on $\mathrm{B}\,\mathrm{GL}(R)$ is taken relative to $E(R)$. Having seen the idea already two times now, it is not surprising anymore that the K-groups of R are defined to be the homotopy groups of the K-theory space, i.e.

$$K_i(R) = \pi_i(K(R)).$$

Clearly, $\pi_0(K(R)) = K_0(R)$ by construction. Furthermore, $\pi_1(K(R)) = \pi_1(\mathrm{B}\,\mathrm{GL}(R)^+) = \pi_1(\mathrm{B}\,\mathrm{GL}(R))/E(R) = \mathrm{GL}(R)/E(R) = K_1(R)$, so this definition of K-groups is consistent with our previous definitions. Recall that $\mathbf{P}(R)$ is the exact category of finitely generated projective modules over R. We then have

Theorem 4.5.5 (Quillen)**.** *For any ring R, there is a natural homotopy equivalence $K(\mathbf{P}(R)) = \Omega\,\mathrm{B}\,\mathcal{Q}(\mathbf{P}(R)) \xrightarrow{\sim} K(R)$.* [**Sri91**, Theorem 7.7]

We have thus established agreement between the three versions of higher K-theory introduced.

CHAPTER 5

The Equivariant Story

5.1. Equivariant Homotopy Theory

The aim of this chapter is to explain the basics of equivariant algebraic topology, in particular equivariant K-theory.

We begin with some facts from equivariant homotopy theory. Let G be a fixed topological group. We work in the category $\mathbf{Top}_G$ of G-spaces and G-maps. The usual constructions on spaces apply. In particular, G acts diagonally on Cartesian products of G-spaces and acts by conjugation on the space $\mathrm{Map}(X,Y)$ of (nonequivariant) maps between G-spaces X and Y, i.e. $(g \cdot f)(x) := gf(g^{-1}x)$. As usual, we take all spaces to be Hausdorff and compactly generated (which means that a subspace is closed if its intersection with each compact subspace is closed). We then have the familiar adjunction

$$\mathrm{Map}(X \times Y, Z) \approx_G \mathrm{Map}(X, \mathrm{Map}(Y, Z))$$

which is a G-homeomorphism.

Subgroups of G are assumed to be closed. For $H \subset G$, we have the fixed point functor $(-)^H : \mathbf{Top}_G \to \mathbf{Top}$ where $X^H = \{x : hx = x \text{ for } h \in H\}$. For $x \in X$, $G_x = \{g \in G : gx = x\}$ is called the *isotropy group* of x. We will soon see that a lot of equivariant homotopy theory reduces to ordinary homotopy theory of fixed point spaces. The *Weyl group* associated to H is $W_GH := N_GH/H$, where $N_GH = \{g \in G : gH = Hg\}$ is the *normalizer* of H in G, will appear frequently. Note that X^H and the orbit space X/H are W_GH-spaces.

Given a subgroup $H \subset G$, we have some important adjunctions. First, the forgetful functor $U : \mathbf{Top}_G \to \mathbf{Top}_H$ is right adjoint to the *induced G-space* functor $G \times_H - : \mathbf{Top}_H \to \mathbf{Top}_G$ where $G \times_H X$ is the quotient of $G \times X$ where we identify (gh, x) with (g, hx). Then

$$\mathrm{Map}_G(G \times_H X, Y) \cong \mathrm{Map}_H(X, UY), \tag{6}$$

where these are now sets of equivariant maps. Of course, these sets can be given the structure of an equivariant space. In particular, U is also a left adjoint to the *coinduced G-space* functor $\mathrm{Map}_H(G, -) : \mathbf{Top}_H \to \mathbf{Top}_G$ with left G-action given by $(g \cdot f)(g') = f(g'g)$:

$$\mathrm{Map}_H(UX, Y) \cong \mathrm{Map}_G(X, \mathrm{Map}_H(G, Y)).$$

Another important adjunction follows by observing that

$$\mathrm{Map}_G(K, X) \cong \mathrm{Map}(K, X^G) \text{ and } G \times_H K \approx_G G/H \times K$$

when K is a space regarded as a G-space with the trivial action. By using (6) we then obtain that $(-)^H$ is right adjoint to the functor $G/H \times - : \mathbf{Top} \to \mathbf{Top}_G$.

$$\mathrm{Map}_G(G/H \times X, Y) \cong \mathrm{Map}(X, Y^H) \tag{7}$$

A *G-homotopy* between G-maps $X \to Y$ is a homotopy $h : X \times I \to Y$ that is a G-map, where G acts trivially on I. A G-map $f : X \to Y$ is said to be a *weak G-equivalence* if $f^H : X^H \to Y^H$ is a weak equivalence for all $H \subset G$.

One can also develop the theory of pointed G-spaces by replacing products with smash products and all of the above results go through unchanged. In either case, a cofibration is defined by the homotopy extension property and a fibration by the homotopy lifting property analogous to the nonequivariant case, except that all maps in sight are G-maps.

In equivariant algebraic topology, orbits G/H play the role of points, and the set of G-maps $G/H \to G/H$ can be identified with $W_G H$. Staying true to this slogan, the analog of a nonequivariant CW complex is a *G-CW complex* which is a G-space X with a decomposition $X = \operatorname{colim} X^k$ such that

$$X^0 = \coprod_{\alpha \in A_0} G/H_\alpha, \qquad X^{n+1} = X^n \cup_{\phi_n} \left(\coprod_{\alpha \in A_{n+1}} D^{n+1} \times G/H_\alpha \right)$$

where $D^{n+1} \times G/H_\alpha$ are G-cells and ϕ_n is made up of attaching G-maps $\phi_{n,\alpha} : S^n \times G/H_\alpha \to X^n$. By (7) these attaching maps are determined by nonequivariant maps $S^n \to (X^n)^{H_\alpha}$ which allows the inductive analysis of G-CW complexes by reduction to nonequivariant homotopy theory.

Many of our favorite nonequivariant CW complex theorems go through with similar proofs. Let $[X, Y]_G$ denote set of G-homotopy classes of G-maps $X \to Y$.

Theorem 5.1.1 (Whitehead Theorem)**.** *If X is a G-CW complex and $f : Y \to Z$ is a weak G-equivalence, then*

$$f_* : [X, Y]_G \to [X, Z]_G$$

is a bijection. [**May96**, Corollary 3.3]

It follows that a weak G-equivalence $Y \to Z$ between G-CW complexes is a G-homotopy equivalence by taking $X = Z$ and then $X = Y$ in the previous theorem.

Theorem 5.1.2 (Cellular Approximation)**.** *Any G-map $f : X \to Y$ between G-CW complexes is G-homotopic to a cellular map.* [**May96**, Chapter 1, Corollary 3.5]

Theorem 5.1.3 (CW Approximation)**.** *For any G-space X, there is a G-CW complex Y and a weak G-equivalence $Y \to X$.* [**May96**, Chapter 1, Theorem 3.6]

A G-space is said to be *G-connected* if G^H is connected for all $H \subset G$. In contrast to the nonequivariant world, it is often insufficient to consider only G-connected spaces. Another important theorem is

Theorem 5.1.4. *Let G be a compact Lie group. Then any compact smooth G-manifold has a finite G-CW complex structure.* [**Mat71**, Proposition 4.4]

Unfortunately, Kirby-Siebenmann theory does not hold in this context and while topological G-manifolds have the homotopy types of G-CW complexes they may not be finite. On that note, Milnor's results on spaces of the homotopy type of CW complexes [**Mil59**] discussed in the context of Wall's finiteness obstruction in the previous chapter generalize to G-spaces [**Wan80**]. In particular, $\mathrm{Map}(X, Y)$ has the homotopy type of a G-CW complex if X is a compact G-space and Y has the homotopy type of a G-CW complex.

5.2. Equivariant K-Theory

We continue the discussion for vector bundles with group actions. A *G-vector bundle* over a G-space X is a G-space E with a G-map $p : E \to X$ such that

(i) $p : E \to X$ is an ordinary (nonequivariant) complex vector bundle;
(ii) for each $g \in G$ and $x \in X$ the map $g : E_x \to E_{gx}$ is a vector space homomorphism.

This is not to be confused with the notion of a *principal G-bundle* which is a fiber bundle $p : E \to X$ with E a G-space such that G preserves the fibers of p and acts freely and transitively on them.

A *section* of a G-vector bundle $p : E \to X$ is a (nonequivariant) map $s : X \to E$ such that $p \circ s = \mathrm{id}_X$. We denote the space of sections by ΓE and the subspace of equivariant sections $\Gamma^G E$. As with ordinary bundles we can form new bundles from old ones by operations from linear algebra such as direct sum, tensor product, and Hom. A morphism between G-vector bundles $p : E \to X$ and $q : F \to X$ in the category $\mathbf{Vect}_G(X)$ of G-vector bundles is a G-map $\phi : E \to F$ such that

(i) $q \circ \phi = p$;
(ii) the restriction $\phi_x : E_x \to F_x$ is a vector space homomorphism.

If V is any complex representation of G then we can form the G-vector bundle $X \times V \to X$. Any such bundle is called *trivial* and denoted by $\mathbf{V}$. Given any G-vector bundle $p : E \to X$ and any G-map $f : Y \to X$ we can also consider the pullback $f^*(E)$ in the category of G-spaces.

For the rest of this chapter we will assume that G is compact and X is a fixed compact G-space unless otherwise stated. $\mathbf{Vect}_G(X)$ is symmetric monoidal with respect to $\oplus$ and we can consider its group completion to obtain an abelian group $K_G(X)$ called the *equivariant K-theory of X*. Tensor product makes $K_G(X)$ into a commutative ring.

Examples 5.2.1. (1) Let $*$ be a point then $\mathbf{Vect}_G(*) = \mathbf{Rep}_{\mathbb{C}}(G)$ and $K_G(*) \cong R(G)$ (cf. 4.2.4).

(2) More generally, consider G-vector bundles over the homogeneous space G/H. Given an H-module V form the bundle $G \times_H V \to G \times_H * = G/H$. Conversely, given a bundle $p : E \to G/H$ form the H-module $p^{-1}(H)$ which is the fiber over the trivial coset. Then these maps are inverses of each other and so $K_G(G/H) = R(H)$.

Since pullback preserves direct sums and tensor product, $K_G(-)$ becomes a contravariant functor from compact G-spaces to commutative rings. That $K_G(-)$ is also a functor on the homotopy category of compact G-spaces follows from the the same three propositions as in the nonequivariant case after some adjustments.

Lemma 5.2.2. *Let Y be a closed G-subspace of a compact G-space X and let $E \to X$ be a G-vector bundle over X. Then any equivariant section of the restriction E_Y extends to an equivariant section of E.*

Proof. Proceed as in (1.1.4) to obtain a section s. Average s by using the Haar measure on G to obtain an equivariant section

$$s^G = \int_G s \circ g dg.$$

Here we need the compactness of G. $\square$

Lemma 5.2.3. *Let Y be a closed G-subspace of a compact G-space X and let $E \to X$ and $F \to X$ be two G-vector bundles over X. Then any isomorphism*

$s : E_Y \to F_Y$ extends to an isomorphism $E_U \to F_U$ for some G-neighborhood U containing Y.

Proof. Apply the previous lemma to the G-vector bundle $\mathrm{Hom}(E, F)$ and proceed as in (1.1.5). □

This in turn implies just as in (1.1.6):

Proposition 5.2.4. *Let Y be a compact G-space, $f : Y \times I \to X$ be a G-homotopy, and E a G-vector bundle over X. Then $f_0^*E \cong f_1^*E$.*

Similarly to the nonequivariant case, equivariant K-theory is in fact a multiplicative *G-cohomology theory*. By this we mean a sequence of contravariant functors h^n_G $(-\infty < n < \infty)$ on the homotopy category of pairs of G-CW complexes into the category of commutative rings together with natural transformations $\delta^n : h^n_G(A) \to h^{n+1}_G(X, A)$ satisfying equivariant exactness and excision axioms just like in the nonequivariant case.

Examples 5.2.5. Other examples of G-cohomology theories:

(1) Cohomology of orbit spaces $h^n_G(X) = H^n(X/G; \mathbb{Z})$.
(2) The Borel cohomology $h^n_G(X) = h^n(EG \times_G X)$, where EG is the universal principal G-vector bundle and h^n is a cohomology theory of spaces.

There is a *reduced* version of equivariant K-theory, $\widetilde{K}_G(-)$, defined on the homotopy category of compact pointed G-spaces. It is defined just as in the ordinary case as the group of stable equivalence classes of G-vector bundles over X. For this one needs the following generalization of (1.1.2):

Fact 5.2.6. *For each G-vector bundle $E \to X$ there exists a G-vector bundle $E' \to X$ and a G-module V such that $E \oplus E' \cong \mathbf{V}$, i.e. $E \oplus E'$ is trivial.* [**Seg68**, Proposition 2.4]

The proof uses the Peter-Weyl theorem to define E' as the orthogonal complement to E. $\widetilde{K}_G(X)$ can be naturally identified with the kernel of the map $K_G(X) \to R(G)$ induced by inclusion of a basepoint.

As in the nonequivariant case, there is an exact sequence of a pair [**Seg68**, Proposition 2.6] used to define $\widetilde{K}^{-n}_G(X) := \widetilde{K}_G(S^n X)$ for $n \in \mathbb{N}$ via suspension. One-point compactification extends the definition of equivariant K-theory to locally compact spaces without basepoints: $K^{-q}_G(X) := \widetilde{K}^{-q}_G(X_+)$. When X is compact the new $K^0_G(X)$ and the original $K_G(X)$ coincide as before. By

Fact 5.2.7 (Equivariant Bott Periodicity)**.** *$\widetilde{K}^{-q}_G(X)$ is naturally isomorphic to $\widetilde{K}^{-q-2}_G(X)$.* [**Seg68**, Proposition 3.5] we can extend the theories to positive integers. Finally, the map collapsing X to a point induces a map $R(G) \to K_G^*(X)$ given by $V \to \mathbf{V}$. In summary, $K_G^*(X)$ is thus a $\mathbb{Z}/2$-graded $R(G)$-algebra. This allows us to localize and complete at ideals of $R(G)$. An interesting candidate is $I = \ker(\epsilon : R(G) \to \mathbb{Z})$ known as the *augmentation ideal*. Here are some useful properties of equivariant K-theory.

(1) **Free Action:** If G acts freely on X then there is a canonical ring isomorphism $K(X/G) \cong K_G(X)$. [**Seg68**, Proposition 2.1]

(2) **Trivial Action:** When G acts trivially on X we have a homomorphism $K(X) \to K_G(X)$ giving a vector bundle the trivial G-action. This map induces a ring homomorphism $\mu : R(G) \otimes K(X) \to K_G(X)$ which is an algebra isomorphism. [**Seg68**, Proposition 2.2]

Example 5.2.8. This enables us to determine the remaining equivariant K-theory of a point:

$$\begin{aligned} K_G^1(*) = \widetilde{K}_G^1(*_+) &= \widetilde{K}_G(S(*_+)) \\ &= \widetilde{K}_G(S^1) \\ &= \ker(K_G(S^1) \to R(G)) \\ &= \ker(R(G) \otimes K(S^1) \to R(G)) \text{ by above} \\ &= \ker(\mathrm{id} : R(G) \to R(G)) \text{ by (1.1.9)} \\ &= 0. \end{aligned}$$

(3) If H is a closed subgroup of G and X is an H-space, then we have an inclusion $i : X \approx H \times_H X \hookrightarrow G \times_H X$ which induces an isomorphism $i^* : K_G^*(G \times_H X) \xrightarrow{\sim} K_H^*(X)$. $K_G^*(G/H) \cong K_H^*(*)$ is a special case of this.

(4) **Thom Isomorphism Theorem:** The Thom homomorphism $\phi_* : K_G^*(X) \to K_G^*(E)$ is an algebra isomorphism for any G-vector bundle E on a locally compact G-space X. [**Seg68**, Proposition 3.2]

(5) **Atiyah-Hirzebruch Spectral Sequence:** Let X be a finite G-CW complex. Then associated to the skeletal filration

$$X^0 \subset X^1 \subset \cdots \subset X^n \subset \cdots \subset X$$

there exists a multiplicative spectral sequence with

$$E_2^{p,q} = H_G^p(X, \mathcal{K}_G^q) \Rightarrow K_G^{p+q}(X)$$

where $H_G^p(X, \mathcal{K}_G^q)$ is Bredon cohomology with coefficient system $\mathcal{K}_G^q$ defined by $G/H \mapsto K_G^q(G/H)$. By our previous remarks $K_G^q(G/H)$ is $R(H)$ for q even and vanishes for q odd. [**Mat73**, Theorem 8.1]

(6) **Hodgkin Spectral Sequence:** This is the analog of the Künneth Theorem. Let X and Y be locally compact G-spaces and let G be a compact connected Lie group such that $\pi_1(G)$ is torsion free. Then there exists a spectral sequence with

$$E_2^{p,q} = \operatorname{Tor}^{p,q}_{R(G)}(K^*_G(X), K^*_G(Y)) \Rightarrow K^{p+q}_G(X \times Y).$$

[**BZ00**, Theorem 2.3] Sadly, not much is currently known for G with torsion or not connected, e.g. when G is a finite group. Rosenberg worked out the case $G = \mathbb{Z}/2$ in [**Ros12**].

Example 5.2.9. Let G be a compact connected Lie group with torsion free fundamental group acting on itself by conjugation. Using the Hodgkin Spectral Sequence it can be shown that $K^*_G(G) \cong \Omega^*_{R(G)/\mathbb{Z}}$ as $R(G)$-modules, where $\Omega^*_{R(G)/\mathbb{Z}}$ is the algebra of Grothendieck differentials. In fact, there is an algebra isomorphism. [**BZ00**]

(7) **Localization:** If X is a locally compact G-space, and $\mathfrak{p}$ is a prime of $R(G)$ with support H, a closed subgroup of G, then the restriction

$$K^*_G(X)_{\mathfrak{p}} \to K^*_G(G.X^H)_{\mathfrak{p}}$$

is an isomorphism. Here the *support* of a prime of $R(G)$ is the smallest subgroup of G such that $\mathfrak{p}$ is the inverse image of a prime in $R(H)$ under $i^* : R(G) \to R(H)$. [**Seg68**, Proposition 4.1]

Example 5.2.10. Recall that $R(G)$ can also be interpreted as the *character ring*, i.e. the ring generated by characters $\chi_V : G \to \mathbb{C}$ of complex representations V of G. If $\mathfrak{p}$ is the ideal of all characters vanishing at some $g \in G$, then $S = \langle g \rangle$. Indeed, $(i^*)^{-1}(0) = \mathfrak{p}$.

(8) **Atiyah-Segal Completion Theorem:** Let X be a finite G-CW complex. Then $K^*(X \times_G EG) \cong K^*_G(X)^{\widehat{}}_I$ where I is the aforementioned augmentation ideal. In particular, if $X = *$ is a point then $K^0(BG) = R(G)^{\widehat{}}_I$ and $K^1(BG) = 0$.

Example 5.2.11. We can use this theorem to compute the non-equivariant K-theory of $\mathbb{C}P^\infty$ (cf. 3.3.3):

$$K^*(\mathbb{C}P^\infty) = K^*(BS^1) = \begin{cases} R(S^1)^{\widehat{}}_I & \text{if } * = 0; \\ 0 & \text{if } * = 1. \end{cases}$$

Now the irreducible complex representations of S^1 are given by the characters $z \mapsto z^m$, $m \in \mathbb{Z}$, generated by $x : z \mapsto z$. So $R(S^1) = \mathbb{Z}[\mathbb{Z}] \cong \mathbb{Z}[x, x^{-1}]$. The augmentation ideal I is $(x - 1)$

since the sum of coefficients of a Laurent polynomial $f(x)$ is zero, i.e. $f(1) = 0$, if and only if $x - 1$ divides $f(x)$. Thus,

$$R(S^1)\widehat{_I} = \mathbb{Z}[x, x^{-1}]\widehat{_{(x-1)}} = \mathbb{Z}[t+1, (t+1)^{-1}]\widehat{_{(t)}} = \mathbb{Z}[[t]]$$

since $1 + t$ is invertible in the formal power series ring.

(9) **Leray-Hirsch Theorem:** Let $E \to X$ be a rank n G-vector bundle and let H be the canonical line bundle over the projectivization $P(E) \to X$. Then $K_G^*(P(E))$ is generated as a $K_G^*(X)$-algebra by H, modulo the relation $\sum_{i=0}^n (-1)^i \Lambda^i(E) H^i = 0$. [**Seg68**, Proposition 3.9]

Example 5.2.12. We use this to compute the equivariant K-theory of the action of S^1 on S^2 by rotation about the z-axis. Let $x : \mathbb{C} \to *$ be the bundle with S^1 acting on $\mathbb{C}$ by complex multiplication and let $1 : \mathbb{C} \to *$ denote the bundle with the trivial action on $\mathbb{C}$. Then $P(x \oplus 1)$ is S^2 with the action we are considering. Thus

$$K_{S^1}^*(S^2) = K_{S^1}^*(*)[H] \Bigg/ \left(\sum_{i=0}^{2} (-1)^i \Lambda^i (x \oplus 1) H^i \right)$$

which means $K_{S^1}^1(S^2) = 0$. Moreover, recall that

$$\sum_{i=0}^{2} (-1)^i \Lambda^i (x \oplus 1) H^i = \sum_{i=0}^{2} (-1)^i \sigma_i(x, 1) H^i = (H - x)(H - 1)$$

since x and 1 are both line bundles (cf. 3.3.3). Thus

$$K_{S^1}^0(S^2) = \mathbb{Z}[x, x^{-1}][H] \Big/ (H^2 - H(x+1) + x) \cdot$$

Bibliography

[AB64] Atiyah, M. F., Bott, R. *On the periodicity theorem for complex vector bundles.* Acta Math. 112 (1964), 229-247.

[Ada62] Adams, J. F. *Vector Fields on Spheres.* Ann. of Math. 75 (1962), 603-632.

[AGP02] Aguilar, M., Gitler, S., and Prieto, C. *Algebraic Topology from a Homotopical Viewpoint.* New York: Springer-Verlag, 2002.

[BM58] Bott, R., Milnor, J. *On the parallelizability of the spheres.* Bull. Amer. Math. Soc. 64 (1958), 87-89.

[BT82] Bott, R., Tu, L. W. *Differential Forms in Algebraic Topology.* New York: Springer-Verlag, 1982.

[Bt85] Bröcker, T., tom Dieck, T. *Representations of Compact Lie Groups.* New York: Springer-Verlag, 1985.

[BZ00] Brylinski, J., Zhang, B. *Equivariant K-Theory of Compact Connected Lie Groups.* K-Theory 20 (2000), no. 1, 23-36.

[Cer70] Cerf, J. *La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie.* Publ. Math. Inst. Hautes Études Sci. 39 (1970), 5-173.

[Cha74] Chapman, T. A. *Topological invariance of Whitehead torsion.* Amer. J. Math. 96 (1974), 488-497.

[Coh73] Cohen, M. *A Course in Simple-Homotopy Theory.* New York: Springer-Verlag, 1973.

[DK01] Davis, J. F., Kirk, P. *Lecture Notes in Algebraic Topology.* Graduate Studies in Math. v. 35, AMS, Providence, RI, 2001.

[Fri07] Friedlander, E. M. *An Introduction to K-Theory.* 2007. Available at http://users.ictp.it/~pub_off/lectures/lns023/Friedlander/Friedlander.pdf.

[Ful98] Fulton, W. *Intersection Theory.* New York: Springer-Verlag, 1998.

[Gor70] Gordon, R. *Vector Fields on Spheres.* Lect. Notes in Math. Berlin/Heidelberg: Springer, 1970.

[Har06] Hartshorne, R. *Algebraic Geometry.* New York: Springer-Verlag, 2006.

[Hat73] Hatcher, A. *Pseudo-isotopy and K_2.* Lecture Notes in Mathematics, Volume 342 (1973), 328-336.

[Hat02] Hatcher, A. *Algebraic Topology.* 2002. Available at http://www.math.cornell.edu/~hatcher/AT/AT.pdf.

[Hat09] Hatcher, A. *Vector Bundles and K-Theory.* 2009. Available at http://www.math.cornell.edu/~hatcher/VBKT/VB.pdf.

[Hus93] Husemöller, D. *Fibre Bundles.* New York: Springer-Verlag, 1993.

[Jam76] James, I. M. *The Topology of Stiefel Manifolds.* Cambridge: Cambridge University Press, 1976.

[JRW12] Jahren, B., Rognes, J., Waldhausen, F. *Spaces of pl manifolds and categories of simple maps*. 2012. Available at http://folk.uio.no/rognes/papers/plmf.pdf.

[Kar78] Karoubi, M. *K-Theory: An Introduction*. New-York: Springer-Verlag, 1978.

[Ker58] Kervaire, M. A. *Non-parallelizability of the n-sphere for $n > 7$*. Proc. Nat. Acad. Sci. U.S.A. 44 (1958), 280-283.

[KM63] Kervaire, M. A., Milnor J. *Groups of Homotopy Spheres*. Annals of Math. 77 (1963), 504-537.

[KS69] Kirby, R. C., Siebenmann, L. C. *On the triangulation of manifolds and the Hauptvermutung*. Bull. Amer. Math. Soc. 75 (1969), no. 4, 742-749.

[Mat71] Matumoto, T. *Equivariant K-theory and Fredholm operators*. J. Fac. Sci. Univ. Tokyo Sect. I. 18 (1971), 109-125.

[Mat73] Matumoto, T. *Equivariant Cohomology Theories on G-CW complexes*. Osaka J. Math. 10 (1973), 51-68.

[Mau81] Mauro, D. J. *Cross-sections of the Sphere and J-Theory*. 1981. Available at https://circle.ubc.ca/bitstream/handle/2429/22572/UBC_1981_A6_7%20M39.pdf.

[May96] May, J. P. *Equivariant Homotopy and Cohomology Theory*. American Mathematical Society, 1996.

[May99] May, J. P. *A Concise Course in Algebraic Topology*. Chicago and London: The University of Chicago Press, 1999.

[Mil59] Milnor, J. *On spaces having the homotopy type of a CW complex*. Trans. Amer. Math. Soc. 90 (1959), 272-280.

[Mil65] Milnor, J. *Lectures on the h-Cobordism Theorem*. Princeton University Press, 1965.

[Mil66] Milnor, J. *Whitehead Torsion*. Bull. Amer. Math. Soc. 72 (1966), 358-426.

[Mil10] Miller, H. *Vector Fields on Spheres, etc.*. 2010. Available at http://math.berkeley.edu/∼ericp/latex/haynes-notes/haynes-notes.pdf.

[Nor01] Nørgaard, E. *Vector Fields on the Sphere*. 2001. Available at http://www.locolomo.org/pub/vfs/vfs_en.pdf.

[Ran02] Ranicki, A. *Algebraic and Geometric Surgery*. Oxford University Press, 2002.

[Rog12] Rognes, J. *The Adams Spectral Sequence*. 2012. Available at http://www.uio.no/studier/emner/matnat/math/MAT9580/v12/undervisnings-materiale/notes.240512.pdf.

[Ros12] Rosenberg, J. *The Künneth Theorem in equivariant K-theory for actions of a cyclic group of order 2*. arXiv:1208.6355, 31 Aug 2012.

[Ros94] Rosenberg, J. *Algebraic K-Theory and Its Applications*. New York: Springer-Verlag, 1994.

[Seg68] Segal, G. *Equivariant K-Theory*. Publ. Math. Inst. Hautes Études Sci. 34 (1968), 129-151.

[Ser77] Serre, J.-P. *Linear Representations of Finite Groups*. New York: Springer-Verlag, 1977.

[Sha11] Shah, J. *Vector Fields on Spheres*. 2011. Available at http://www.math.uchicago.edu/∼may/VIGRE/VIGRE2011/REUPapers/Shah.pdf.

[Sri91] Srinivas, V. *Algebraic K-Theory*. Progress in Mathematics, 90. Birkhäuser Boston, 1991.

[Vak04] Vakil, R. *Math 245A: Topics in Algebraic Geometry*. 2004. Available at http://math.stanford.edu/∼vakil/245/.

[Vak12] Vakil, R. *Foundations of Algebraic Geometry.* 2012. Available at http://math.stanford.edu/∼vakil/216blog/FOAGmay1612public.pdf.

[Var89] Varadarajan, K. *The Finiteness Obstruction of C.T.C. Wall.* Wiley, New York, 1989.

[Vic94] Vick, J. W. *Homology Theory.* New York: Springer-Verlag, 1994.

[Wal85] Waldhausen, F. *Algebraic K-theory of spaces, Algebraic and geometric topology.* Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 318-419.

[Wal65] Wall, C. T. C. *Finiteness conditions for CW complexes.* Ann. of Math. 81 (1965), 55-69.

[Wal66] Wall, C. T. C. *Finiteness conditions for CW complexes. II.* Proc. Royal Soc. Ser. A 295 (1966), 129-139.

[Wan80] Waner, S. *Equivariant homotopy theory and Milnor's theorem.* Trans. Amer. Math. Soc. 258 (1980), 351-368.

[Wei12] Weibel, C. A. *Algebraic K-Theory.* 2012. Available at http://www.math.rutgers.edu/ weibel/Kbook.html.