

Example of Homoclinic Bifurcation

```
sysid  
Mathematica 6.0.3, DynPac 11.02, 3/25/2009
```

This example is taken from **Nonlinear Dynamics and Chaos**, Steven Strogatz, Addison Wesley 1994, p. 262-263. We define the system for DynPac.

```
setstate[{x, y}];  
setparm[{\mu}];  
slopevec = {y, \mu y + x - x^2 + x y};
```

We start our analysis by finding the equilibrium states -- easily done by inspection or with DynPac.

```
equils = findpolyeq  
{ {0, 0}, {1, 0} }  
  
eq1 = equils[[1]]  
{0, 0}  
  
eq2 = equils[[2]]  
{1, 0}
```

We calculate the derivative matrix at each of the equilibria.

```
d1 = dermatval[eq1]  
{ {0, 1}, {1, \mu} }  
  
d2 = dermatval[eq2]  
{ {0, 1}, {-1, 1 + \mu} }  
  
eigval[eq1]  
{ \frac{1}{2} \left( \mu - \sqrt{4 + \mu^2} \right), \frac{1}{2} \left( \mu + \sqrt{4 + \mu^2} \right) }
```

```
eigval[eq2]
```

$$\left\{ \frac{1}{2} \left(1 + \mu - \sqrt{-3 + 2 \mu + \mu^2} \right), \frac{1}{2} \left(1 + \mu + \sqrt{-3 + 2 \mu + \mu^2} \right) \right\}$$

It is easy to see that for eq1, the eigenvalues are always real, and that one is always positive, one always negative. Hence the origin remains a saddle throughout the bifurcations. The equilibrium at {1,0} has a more diverse history as μ changes. Analysis of the eigenvalues yields the following results: for $\mu < -3$ a stable node; for $-3 < \mu < -1$ a stable spiral; for $-1 < \mu < 1$ an unstable spiral; $1 < \mu$ an unstable node. We see that there is a Hopf bifurcation at $\mu = -1$. We do a spot check on these with classify2D.

```
parmval = {-4}; classify2D[eq2]
strictly stable - node

parmval = {-2}; classify2D[eq2]
strictly stable - spiral

parmval = {0}; classify2D[eq2]
unstable - spiral

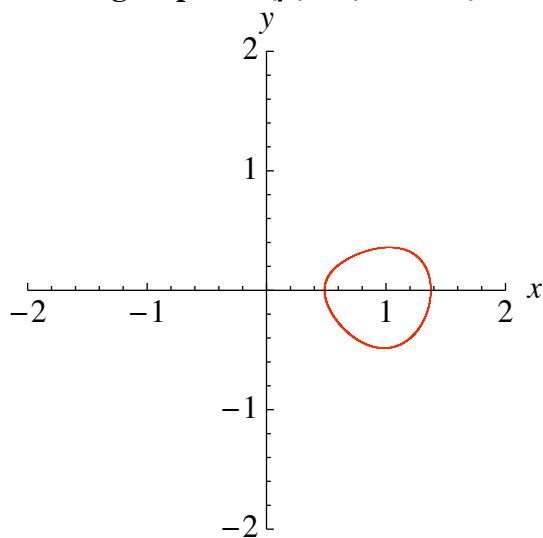
parmval = {2}; classify2D[eq2]
unstable - node
```

We begin the study of the Hopf bifurcation by looking for a stable limit cycle for μ slightly larger than -1. We use an initial value close to the equilibrium.

```
sysname = "Strogatz p. 263";
plrange = {{-2, 2}, {-2, 2}};
setcolor[{Red}];
decdig = 4;
imsize = 200;
```

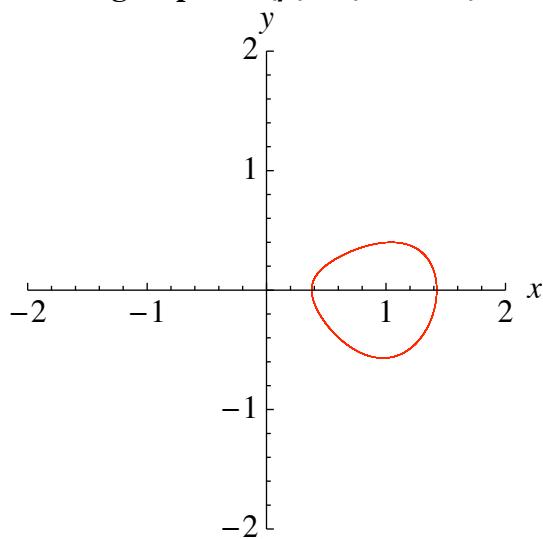
```
parmval = {-0.95}; lim1 = phaser[limcyc[{1.5, 0}, 0, 0.0005, 100000]]
```

Strogatz p. 263 $\{\mu\} = \{-0.9500\}$



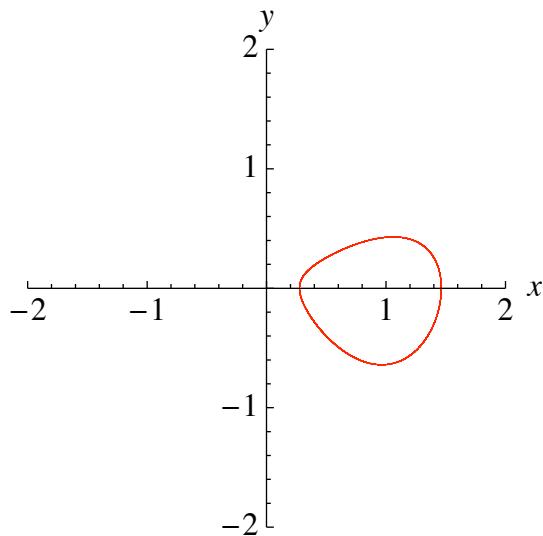
```
parmval = {-0.93}; lim2 = phaser[limcyc[{1.5, 0}, 0, 0.0005, 100000]]
```

Strogatz p. 263 $\{\mu\} = \{-0.9300\}$



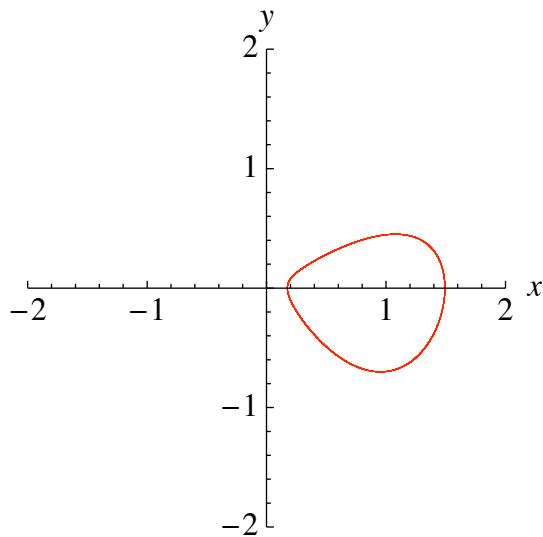
```
parmval = {-0.91}; lim3 = phaser[limcyc[{1.5, 0}, 0, 0.0005, 100000]]
```

Strogatz p. 263 $\{\mu\} = \{-0.9100\}$



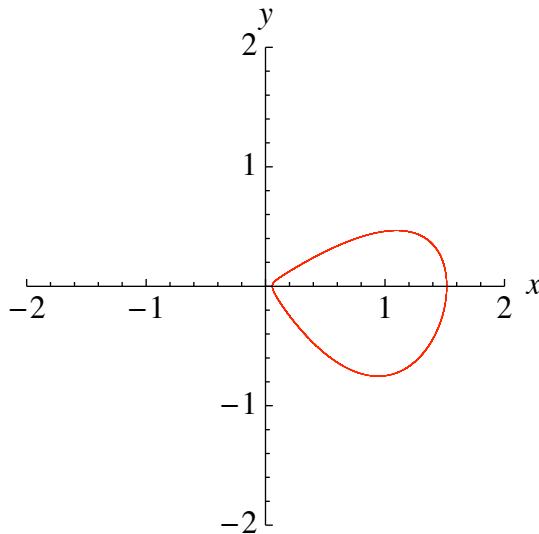
```
parmval = {-0.89}; lim4 = phaser[limcyc[{1.5, 0}, 0, 0.0005, 100000]]
```

Strogatz p. 263 $\{\mu\} = \{-0.8900\}$



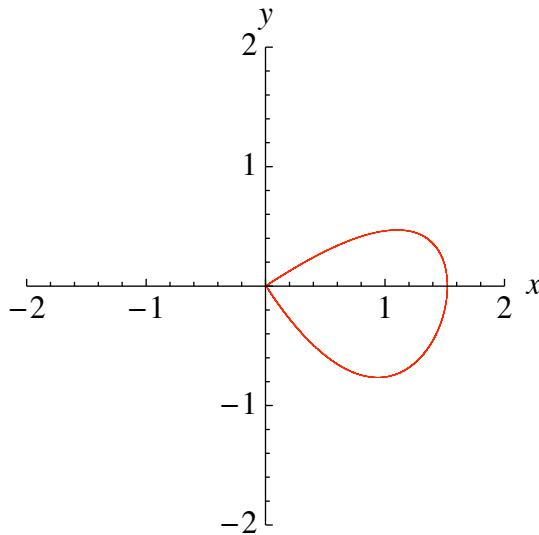
```
parmval = {-0.87}; lim5 = phaser[limcyc[{1.5, 0}, 0, 0.0005, 100000]]
```

Strogatz p. 263 $\{\mu\} = \{-0.8700\}$



```
parmval = {-0.865}; lim6 = phaser[limcyc[{1.5, 0}, 0, 0.0005, 100000]]
```

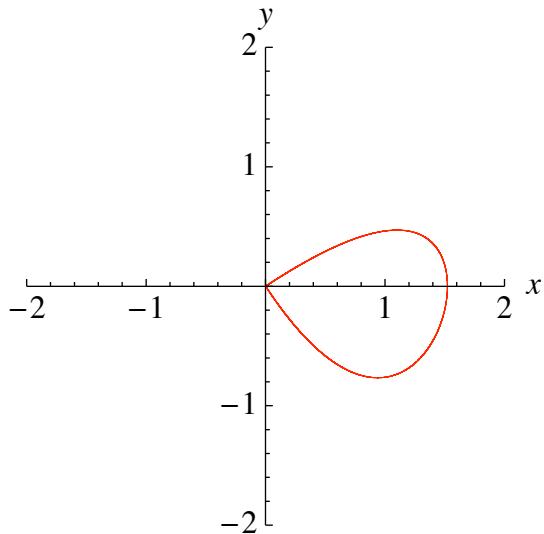
Strogatz p. 263 $\{\mu\} = \{-0.8650\}$



Something strange is happening to our limit cycle. We try one more integration, using a value very slightly less than the bifurcation value given by Strogatz, namely -0.8645.

```
parmval = {-0.8646}; lim7 = phaser[limcyc[{1.5, 0}, 0, 0.0005, 100000]]
```

Strogatz p. 263 $\{\mu\} = \{-0.8646\}$

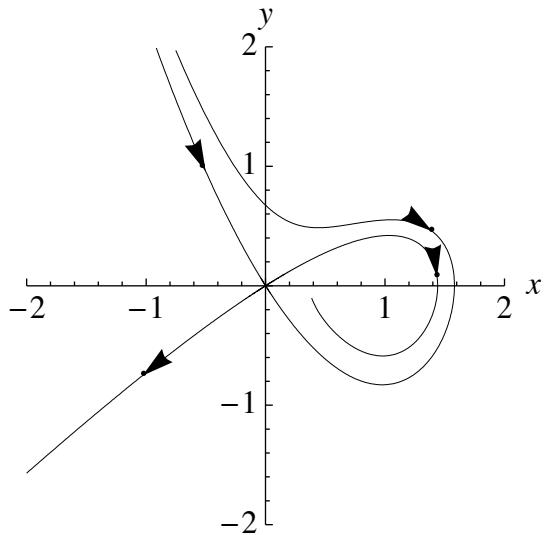


Our limit cycle has become a homoclinic loop. Beyond this value of μ , the limit cycle no longer exists, nor does the homoclinic loop. We now add to our pictures the configuration of the saddle point.

```
setcolor[{Black}];
```

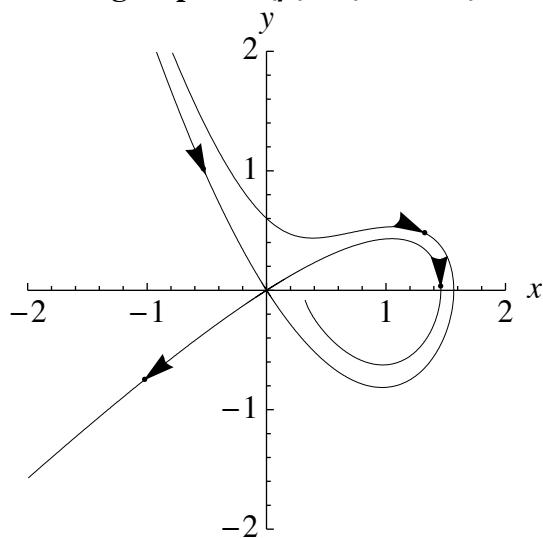
```
parmval = {-0.95}; sad1 = saddleportraitman[eq1, plrange]
```

Strogatz p. 263 $\{\mu\} = \{-0.9500\}$



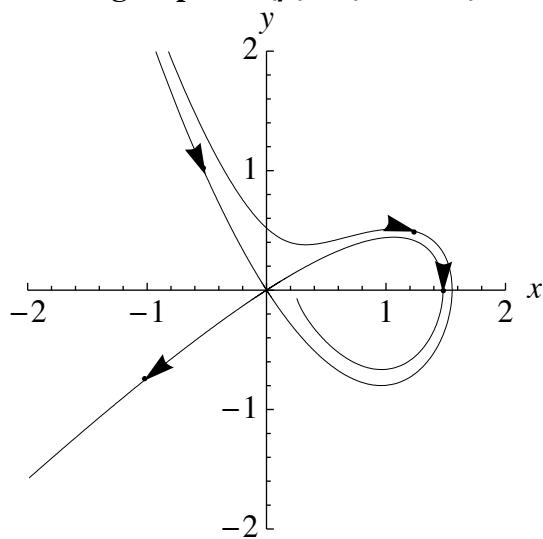
```
parmval = {-0.93}; sad2 = saddleportraitman[eq1, plrange]
```

Strogatz p. 263 $\{\mu\} = \{-0.9300\}$



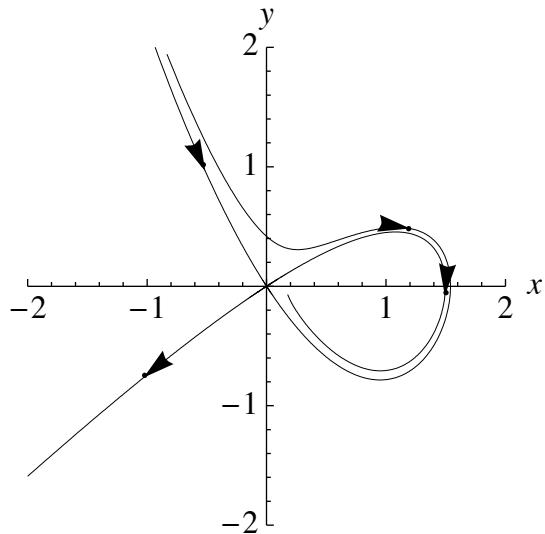
```
parmval = {-0.91}; sad3 = saddleportraitman[eq1, plrange]
```

Strogatz p. 263 $\{\mu\} = \{-0.9100\}$



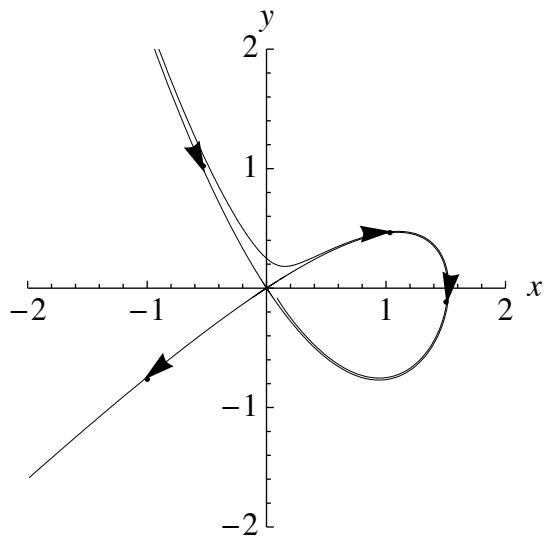
```
parmval = {-0.89}; sad4 = saddleportraitman[eq1, plrange]
```

Strogatz p. 263 $\{\mu\} = \{-0.8900\}$



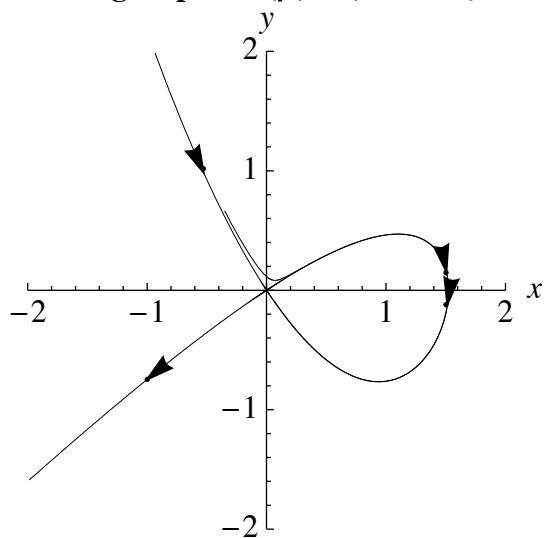
```
parmval = {-0.87}; sad5 = saddleportraitman[eq1, plrange]
```

Strogatz p. 263 $\{\mu\} = \{-0.8700\}$



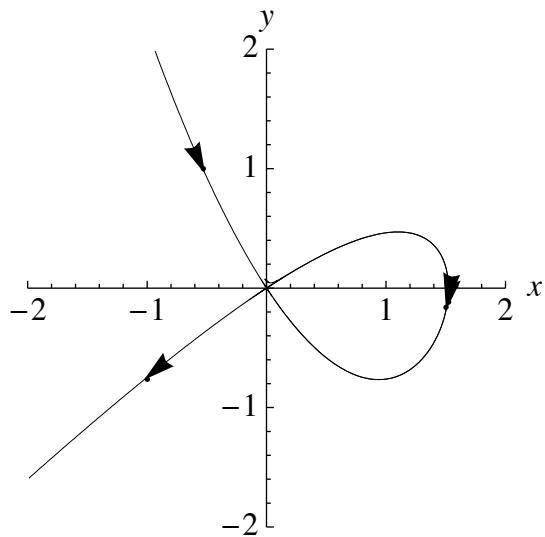
```
parmval = {-0.865}; sad6 = saddleportraitman[eq1, plrange]
```

Strogatz p. 263 $\{\mu\} = \{-0.8650\}$



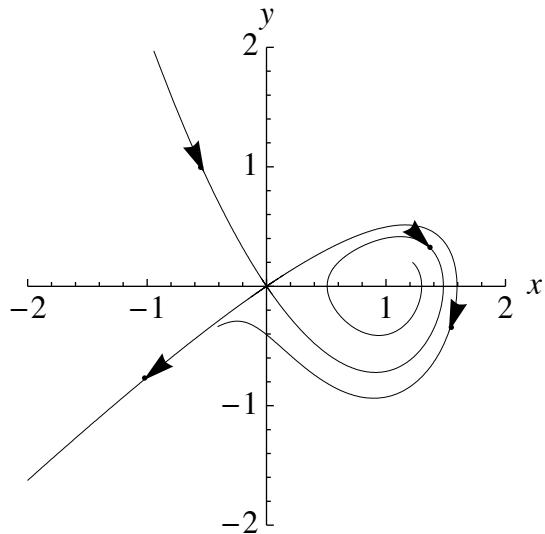
```
parmval = {-0.8646}; sad7 = saddleportraitman[eq1, plrange]
```

Strogatz p. 263 $\{\mu\} = \{-0.8646\}$



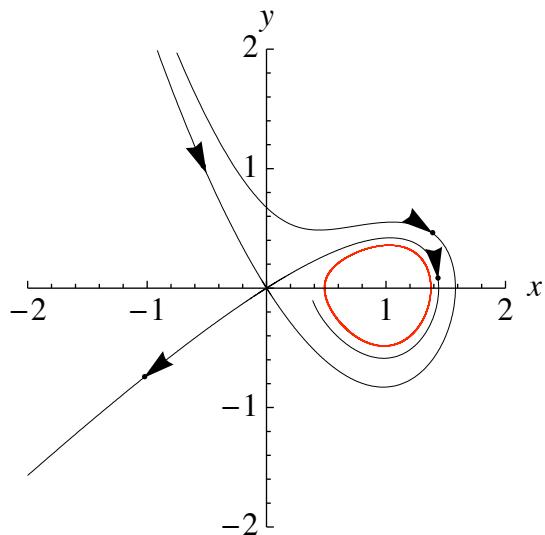
```
parmval = {-0.8}; sad8 = saddleportraitman[eq1, plrange]
```

Strogatz p. 263 $\{\mu\} = \{-0.8000\}$



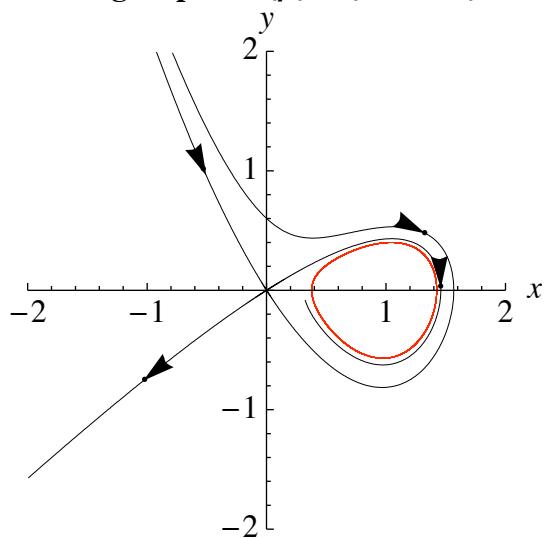
```
parmval = {-0.95}; show[sad1, lim1]
```

Strogatz p. 263 $\{\mu\} = \{-0.9500\}$



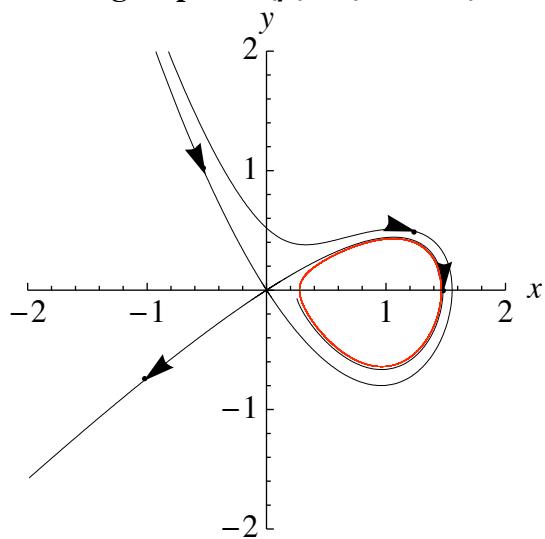
```
parmval = {-0.93}; show[sad2, lim2]
```

Strogatz p. 263 $\{\mu\} = \{-0.9300\}$



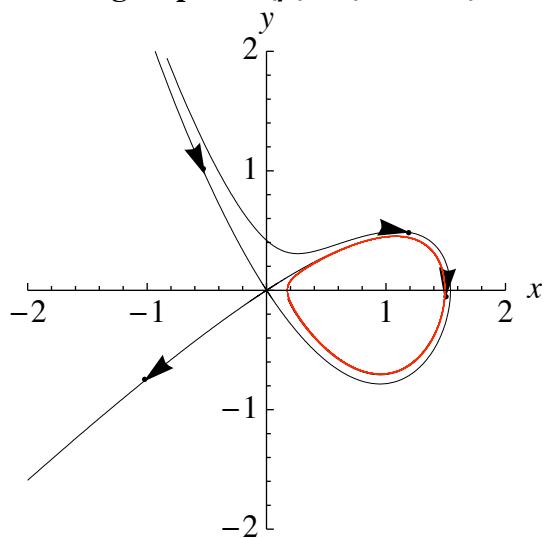
```
parmval = {-0.91}; show[sad3, lim3]
```

Strogatz p. 263 $\{\mu\} = \{-0.9100\}$



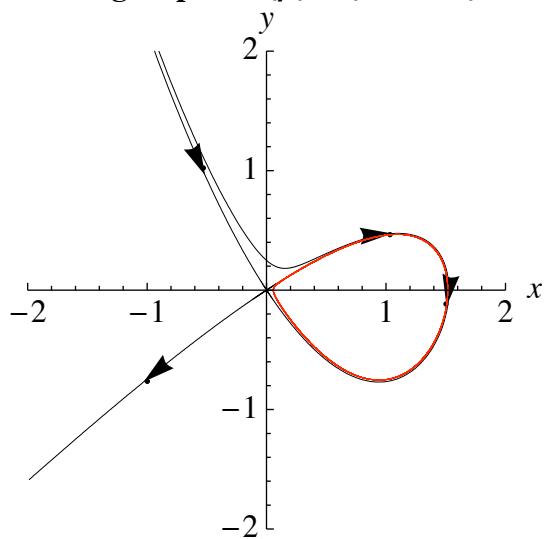
```
parmval = {-0.89}; show[sad4, lim4]
```

Strogatz p. 263 $\{\mu\} = \{-0.8900\}$



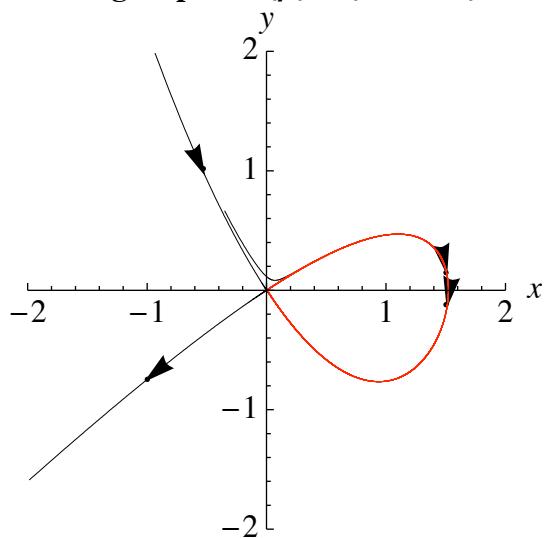
```
parmval = {-0.87}; show[sad5, lim5]
```

Strogatz p. 263 $\{\mu\} = \{-0.8700\}$



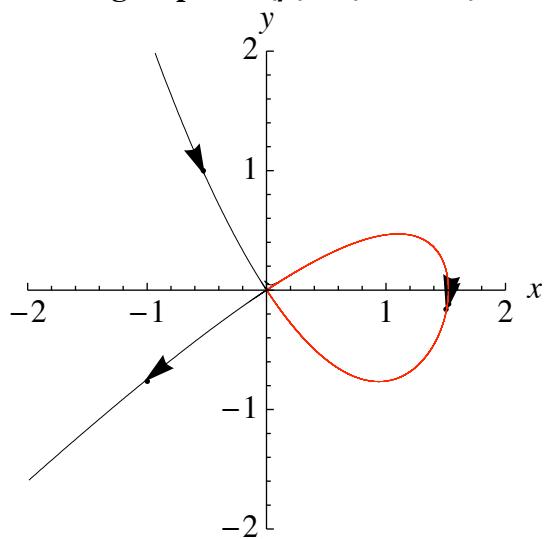
```
parmval = {-0.865}; show[sad6, lim6]
```

Strogatz p. 263 $\{\mu\} = \{-0.8650\}$



```
parmval = {-0.8646}; show[sad7, lim7]
```

Strogatz p. 263 $\{\mu\} = \{-0.8646\}$



```
parmval = {-0.8}; show[sad8]
```

Strogatz p. 263 $\{\mu\} = \{-0.8000\}$

