The snake lemma

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> "the serpent was more subtil than any beast of the field which the LORD God had made."

1 Proof of the snake lemma

Our proof will use the following lemmata, which are simple consequences of the snake lemma. They are also easy to prove without recourse to the snake lemma, and we will see in a moment that they (and their duals) imply it.

Lemma 1 *If*

$$A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is exact and $X \to B$ is any morphism then

$$A \longrightarrow B/X \longrightarrow C/X \longrightarrow 0$$

is exact.

This is just the right exactness of the cokernel.

Lemma 2 *Suppose $A \to B \to C$ is a sequence of morphisms in an abelian category. Then*

$$\ker(B \to C)/A \to \ker(B/A \to C)$$

is an isomorphism.

Nothing will be changed if we replace A by the image of $A \to B$ and replace C by the image of $B \to C$. This is Axiom AB2 of an abelian category [Gro57].

Lemma 3 *If*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C$$

is exact and $X \to A$ is any morphism then

$$0 \longrightarrow A/X \longrightarrow B/X \longrightarrow C$$

is exact.

The exactness in the middle is obvious, since $(B/X)/(A/X) = B/A$ is a sub-object of C. For the injectivity of $A/X \to B/X$, note that the kernel of $B/X \to C$ is the same as

$$\ker(B \to C)/X = A/X.$$

We'll use these to prove the snake lemma. Now suppose that the following diagram is exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & K' \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & K'' \arrow[d] & \\
 & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B'' \arrow[d] & \\
 & L' \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & L'' \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Theorem 1 (Snake lemma) *There is a natural map $K'' \to L'$ making the sequence*

$$K' \to K \to K'' \to L' \to L \to L''$$

exact.

We shall mimic the element-chasing proof by taking kernels and cokernels in such a way as not to lose the exactness of the rows and columns, until we arrive at a diagram where the snake lemma is obvious.

Every object in this diagram has a map to L'' and a map from K'. We replace each object X of the above diagram by $\ker(X/K' \to L'')$.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & K_1 \arrow[r] \arrow[d] & K'' \arrow[d] & \\
 & A_1' \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B' \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & B_1'' \arrow[d] & \\
 & L' \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

Note that this diagram still has exact rows and exact columns. Everything in the second two columns has a map from K_1, so we replace every object X in one of these columns by X/K_1.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & K_2'' \arrow[d] & \\
 & A_1' \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A_2'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B' \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & B_1'' \arrow[d] & \\
 & L' \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

Note that the rows and columns are still exact. Taking the kernel of the map into L_1 in the first two columns gives another diagram with exact rows and columns.

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & K''_2 \arrow[d] & \\
 & A'_1 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & A''_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B'_2 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & B''_1 \arrow[d] & \\
 & L'_2 \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}
$$

Now take kernel into B'' in the first three rows and divide by A' in the last three rows.

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & 0 \arrow[d] & K''_2 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & A''_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B'_2 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & 0 & \\
 & L'_2 \arrow[d] & 0 & & \\
 & 0 & & &
\end{tikzcd}
$$

But now all the non-zero maps are isomorphisms, so we get an isomorphism $K''_2 \to L'_2$. If we then work out what we've done, we discover that $K''_2 = K''/K$ and $L'_2 = \ker(L' \to L)$ so this proves the snake lemma.

2 Proof of the salamander lemma

The same ideas can be used to prove the salamander lemma. We follow the notation of [Ber]. Suppose that we have a bicomplex containing the diagram Fig. 7 (a) in degrees $[0,1]\times[0,2]$. Truncating the complex to these degrees does not change $C_\Box$, $A^=$, $A_\Box$, ${}^\Box B$, $B^=$, or ${}^\Box D$. It's therefore sufficient to prove the salamander lemma for a bicomplex of the shape

$$\begin{tikzcd}
C \arrow[r, "\alpha"] \arrow[d] & E \arrow[d, "\beta"] \\
A \arrow[r] \arrow[d, "\gamma"'] & B \arrow[d] \\
F \arrow[r, "\delta"] & D
\end{tikzcd}$$

that is zero in all terms that aren't displayed above. Replace C by $C_1 = \ker(\alpha)$, B by $B_1 = \operatorname{coker}(\beta)$, A by $A_1 = \ker(\gamma)$, D by $D_1 = \operatorname{coker}(\delta)$ so that we have

$$\begin{tikzcd}
C_1 \arrow[d, "u"'] \arrow[dr, "0"] & \\
A_1 \arrow[r, "v"] \arrow[dr, "0"'] & B_1 \arrow[d, "w"] \\
& D_1.
\end{tikzcd}$$

Since $C_\Box$, $A^=$, $A_\Box$, ${}^\Box B$, $B^=$, and ${}^\Box D$ are unchanged by this, it is sufficient to prove the salamander lemma in this case, i.e., that the sequence

$$C_1 \to \ker(v) \to A_1/C_1 \to \ker(w) \to B_1/A_1 \to D_1$$

is exact. Exactness at $\ker(v)$ and at A_1/C_1 follows from Lemma 3 applied to the exact sequence

$$0 \to \ker(v) \to A_1 \to \ker(w)$$

and the map $C_1 \to \ker(v)$. Duality implies exactness at $\ker(w)$ and B_1/A_1.

References

[Ber] George M. Bergman, *On diagram-chasing in double complexes*, `http://sbseminar.files.wordpress.com/2007/11/diagramchasingbergman.pdf`.

[Gro57] Alexander Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–221. MR MR0102537 (21 #1328)