Erdös' Minimum Overlap Problem

Steven Finch

July 2, 2004

Let A, B be disjoint, complementary subsets of the set $\{1, 2, 3, \ldots, 2n\}$ with cardinality $|A| = |B| = n$. Let M_k denote the number of solutions of the equation $a_i - b_j = k$, where k is an integer between $-2n$ and $2n$. Define

$$M(n) = \min_{A,B} \max_k M_k.$$

We wish to estimate $M(n)$ as n grows large [1, 2, 3]. The work of Erdös, Scherk and others [4, 5, 6] implies that

$$\mu_L = \liminf_{n\to\infty} \frac{M(n)}{n} \geq \sqrt{4 - \sqrt{15}} > 0.35639$$

and specific examples provide that [7]

$$\mu_R = \limsup_{n\to\infty} \frac{M(n)}{n} \leq \frac{2}{5} = 0.4.$$

Haugland [6] recently demonstrated that $\mu_L = \mu_R$ (meaning that the limit exists) and, using a theorem of Swinnerton-Dyer, obtained the improvement

$$0.35639 < \mu = \lim_{n\to\infty} \frac{M(n)}{n} < 0.38201.$$

No one has conjectured an exact value for this limiting ratio.

Observe that M_{-k} is the cardinality of the set $A_k \cap B$, where A_k is the translated set $\{a + k : a \in A\}$. Mycielski and Świerczkowski [4] considered a continuous analog of Erdös' problem. Let X, Y be disjoint, complementary measurable subsets of the interval $[0, 1]$ with Lebesgue measure $|X| = |Y| = 1/2$. It is not surprising that

$$\inf_{X,Y} \sup_t |X_t \cap Y| = \frac{\mu}{2}$$

where X_t is the translated set $\{x + t : x \in X\}$. Hence the bounds $0.17819 < \mu/2 < 0.19101$ carry over from before.

Moser and Murdeshwar [8, 9, 10] studied the following generalization. Let f, g be Lebesgue integrable functions on $\mathbb{R}$ satisfying

$$0 \le f(x) \le 1 \quad \text{for } 0 \le x \le 1, \quad f(x) = 0 \quad \text{otherwise;}$$

$$0 \le g(x) \le 1 \quad \text{for } 0 \le x \le 1, \quad g(x) = 0 \quad \text{otherwise;}$$

$$\int_0^1 f(x)\,dx = \frac{1}{2} = \int_0^1 g(x)\,dx.$$

(This scenario reduces to the preceding case by taking f to be the characteristic function of X and g to be the characteristic function of Y; clearly $f(x) + g(x) = 1$ for all $0 \le x \le 1$.) Define

$$\lambda = \inf_{f,g} \sup_t \int_0^1 f(x+t)\,g(x)\,dx.$$

It is known [10] that $0.136 \le \lambda \le 0.166$, but it is not presently known whether Swinnerton-Dyer's theorem [6] can be applied here (in some extended form) to improve these bounds.

Here is a related problem due to Czipszer [3, 11]. Let $\tilde{a}_1 < \tilde{a}_2 < \tilde{a}_3 < \cdots < \tilde{a}_n$ be arbitrary integers and define $\tilde{A}_k = \{\tilde{a}_j + k : 1 \le j \le n\}$ for each integer k. Let $\tilde{M}_k$ denote the cardinality $|\tilde{A}_k - \tilde{A}_0|$, that is, the number of elements of $\tilde{A}_k$ not in $\tilde{A}_0$. Define

$$\tilde{M}(n) = \min_{\tilde{A}} \max_{-n \le k \le n} \tilde{M}_k$$

and $\tilde{\mu}_L$, $\tilde{\mu}_R$ as earlier. It is known that $1/2 \le \tilde{M}(n)/n \le 2/3$ and, further, that $\tilde{M}(n)/n \ge 3/5$ for all $n \ge 26$ [12]. It is conjectured that $\tilde{\mu}_L = \tilde{\mu}_R = 2/3$. We give the corresponding functional version. Let $\tilde{f}$ be a Lebesgue integrable function on $\mathbb{R}$ satisfying

$$0 \le \tilde{f}(x) \le 1, \qquad \int_{-\infty}^{\infty} \tilde{f}(x)\,dx = 1.$$

Define

$$\tilde{\lambda} = \inf_{\tilde{f}} \left\{ 1 - \inf_{-1 \le t \le 1} \int_{-\infty}^{\infty} \tilde{f}(x+t)\,\tilde{f}(x)\,dx \right\}.$$

It is known that $0.5892 \le \tilde{\lambda} \le 2/3$ [11]. As a corollary, if $\tilde{X}$ is a measurable subset of $\mathbb{R}$ with Lebesgue measure $|\tilde{X}| = 1$, then

$$0.5892 \le \inf_{\tilde{X}} \sup_{-1 \le t \le 1} \left| \tilde{X}_t - \tilde{X} \right| \le \frac{2}{3}.$$

The discrete and continuous analogs do not appear to be as closely linked as before. Again, we wonder whether recent techniques [6] can be invoked to sharpen these bounds.

REFERENCES

[1] P. Erdös, Some remarks on number theory (in Hebrew), *Riveon Lematematika* 9 (1955) 45–48; MR0073619 (17,460d).

[2] P. Erdös, Problems and results in additive number theory, *Colloque sur la Théorie des Nombres*, Proc. 1955 Bruxelles conf., Liège, G. Thone, 1956, pp. 127–137; MR0079027 (18,18a).

[3] R. K. Guy, *Unsolved Problems in Number Theory*, 2^{nd} ed., Springer-Verlag, 1994, sect. C17; MR 96e:11002.

[4] S. Świerczkowski, On the intersection of a linear set with the translation of its complement, *Colloq. Math.* 5 (1958) 185–197; MR0103172 (21 #1955).

[5] L. Moser, On the minimal overlap problem of Erdös, *Acta Arith.* 5 (1959) 117–119; MR0106864 (21 #5594).

[6] J. K. Haugland, Advances in the minimum overlap problem, *J. Number Theory* 58 (1996) 71–78; MR1387725 (97c:11031).

[7] T. S. Motzkin, K. E. Ralston and J. L. Selfridge, Minimal overlapping under translation, *Bull. Amer. Math. Soc.* 62 (1956) 558.

[8] W. O. J. Moser, A generalization of some results in additive number theory, *Math. Z.* 83 (1964) 304–313; MR0167475 (29 #4748).

[9] L. Moser and M. G. Murdeshwar, On the overlap of a function with the translation of its complement, *Colloq. Math.* 15 (1966) 93–97; MR0196023 (33 #4218).

[10] M. G. Murdeshwar, On the maximum correlation between functions, *Prace Mat.* 11 (1968) 193–197; MR0225955 (37 #1545).

[11] L. Moser and M. G. Murdeshwar, On the overlap of a function with its translates, *Nieuw Arch. Wisk.* 14 (1966) 15–18; MR0193201 (33 #1422).

[12] M. Katz and F. Schnitzer, On a problem of J. Czipszer, *Rend. Sem. Mat. Univ. Padova* 44 (1970) 85–90; MR0299492 (45 #8540).