Finsler and sub-Finsler geometries

New Directions in Exterior Differential Systems Conference

In honor of Robert Bryant's 60th birthday

Chris Moseley

Calvin College

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Finsler geometry is just Riemannian geometry without the quadratic restriction.

– *S.-S. Chern*

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- defined the general notion of a metric (line element) on a manifold;
- solved the equivalence problem for Riemannian metrics:
  - showed the existence of $\frac{n(n-1)}{2}$ scalar invariants;
  - if these invariants all vanish, metric is flat;
  - in the 2-dimensional case, one scalar invariant (Gauss curvature K)

Riemann's conception of a metric

- Line element on a curve $\gamma : [a, b] \to M$ parametrized by t:

$$ds = F(x, \dot{x})\, dt$$

where F is chosen so that $L(\gamma) = \int_b^a F(x, \dot{x})\, dt$ is a plausible measure of length.

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  - $F(x, \dot{x}) = F(x, -\dot{x})$.
  - Riemann's simplest case:

$$F(x, \dot{x}) = \sqrt{g_{ij}(x) x^i x^j}$$

    where the g_{ij} are continuous and non-negative

The next case in simplicity includes those manifolds in which the line-element may be expressed as the fourth root of a quartic differential expression. The investigation of this more general kind would require no really different principles, but would take considerable time and throw little new light on the theory of space....

- Riemann (1854)

Paul Finsler

- *On Curves and Surfaces in Generalized Spaces*, PhD thesis (1919)
- Returned to Riemann's general notion of a metric with a modified homogeneity condition:

  $$F(x, \lambda\dot{x}) = \lambda F(x, \dot{x})$$

  for $\lambda > 0$ only.
- Illustration (Finsler): when walking different paths on a hillside, speed depends on direction of path

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- *Strong convexity: The $n \times n$ Hessian matrix*

$$(g_{ij}) = \left[\frac{\partial^2 (\frac{1}{2} F^2)}{\partial y^i \, \partial y^j} \right]$$

is positive definite at every point of $TM \setminus 0$.

A Finsler metric F determines its *indicatrix bundle* $\Sigma \subset TM$ defined on each fiber T_xM by

$$\Sigma_x = \{y \in T_xM | F(x,y) = 1\}.$$

The strong convexity condition implies that Σ_x is a smooth, strongly convex hypersurface enclosing $0_x \in T_xM$.

In the Riemannian case the indicatrix must be an ellipsoid centered at 0_x, but in the Finsler case it may be much more general. In particular, it need not even be symmetric about the origin.

Conversely, suppose $\Sigma \subset TM$ is such that each Σ_x is a smooth, strongly convex hypersurface enclosing $0_x \in T_x M$. Then Σ uniquely determines a Finsler metric F on M, and Σ is called a *Finsler structure on M.*

It is often convenient to define a Finsler metric in this manner.

Examples of Finsler metrics on surfaces

Perturbed quartic metric:

$$F(x^1, x^2, p, q) = \sqrt{\sqrt{p^4 + q^4} + \lambda(p^2 + q^2)}, \quad \lambda > 0$$

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Note that

$$\det(g_{ij}) = \lambda^2 + \lambda \frac{(p^2 + q^2)^3}{(p^4 + q^4)^{3/2}} + \frac{3p^2q^2}{p^4 + q^4}$$

$$\mathrm{tr}(g_{ij}) = 2\lambda + \frac{(p^2 + q^2)^3}{(p^4 + q^4)^{3/2}}$$

Examples of Finsler metrics on surfaces, cont'd

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Examples of Finsler metrics on surfaces, cont'd

- The **limaçon metric** with indicatrix given in polar coordinates by
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- **Randers metrics:** family of metrics defined by choosing indicatrices of the form
$$R = \frac{1}{1 + B\cos\theta}, \quad 0 < B < 1$$
(off center ellipses).

Local geometry of Riemannian surfaces

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We may also think of Σ as the *oriented orthonormal frame bundle* of M, i.e., the bundle over M whose fiber over each point $x \in M$ consists of the oriented, orthonormal frames $\{\mathbf{e}_1, \mathbf{e}_2\}$ for the tangent space T_xM, via the identification

$$\{\mathbf{e}_1, \mathbf{e}_2\} \leftrightarrow \mathbf{e}_1.$$

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The forms ω^1, ω^2 are called the *dual forms* on Σ.

Furthermore, there exists a unique 1-form α on Σ which is linearly independent from ω^1, ω^2, and a function K on Σ so that

$$\begin{aligned} d\omega^1 &= -\alpha \wedge \omega^2 && (1) \\ d\omega^2 &= \alpha \wedge \omega^1, \\ d\alpha &= K\, \omega^1 \wedge \omega^2. && (2) \end{aligned}$$

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α is the Levi-Civita connection form; the function K is semi-basic and is the Gauss curvature of the Riemannian metric on M.

Together, equations (1) and (2) form the *structure equations* for the canonical coframing $(\omega^1, \omega^2, \alpha)$ on Σ.

The Gauss curvature K, together with its covariant derivatives K_1, K_2, defined by the condition

$$dK = K_1\,\omega^1 + K_2\,\omega^2,$$

form a complete set of local invariants for the Riemannian metric on M.

Geodesics for Riemannian surfaces

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Any curve $\gamma : [a, b] \to M$ that is parametrized by arc length has a canonical lift

$$\gamma' : [a, b] \to \Sigma.$$

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In terms of the canonical coframing, the geodesic vector field is tangent to the line field given by $\{\omega^2, \alpha\}^\perp$. The geodesic equations may be written in the form

$$\omega^1 = ds, \qquad \omega^2 = \alpha = 0.$$

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Defining a canonical coframing on Σ requires a bit more work than in the Riemannian case, because there is no notion of orthogonality for a Finsler metric.

Following (Bryant 1995), we say that a vector $\mathbf{v} \in T_{\mathbf{u}}\Sigma$ is *monic* if

$$\pi'(\mathbf{u})(\mathbf{v}) = \mathbf{u}$$

(where $\pi'(\mathbf{u}) : T_{\mathbf{u}}\Sigma \to T_{\pi(\mathbf{u})}M$). The set of monic vectors forms an affine line in $T_{\mathbf{u}}\Sigma$.

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Any two null 1-forms are linearly dependent, and the difference of any two monic 1-forms is a null 1-form.

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- *$\phi \wedge d\eta^1 = \phi \wedge d\eta^2 = 0$.*

This coframing satisfies the structure equations

$$\begin{aligned} d\eta^1 &= -\phi \wedge \eta^2 \\ d\eta^2 &= \phi \wedge \eta^1 - I\,\phi \wedge \eta^2 \\ d\phi &= K\,\eta^1 \wedge \eta^2 + J\,\phi \wedge \eta^2. \end{aligned} \tag{3}$$

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The functions I, J, K with their covariant derivatives form a complete set of local invariants for the Finsler structure.

- The Finsler structure is Riemannian if and only if $I \equiv 0$ (and $J \equiv 0$).
- K is the *flag curvature*. It is a function on M if and only if Σ is Riemannian, in which case it is the Gauss curvature.

As in the Riemannian case, the geodesic equations on Σ take the form

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In contrast to Riemannian geodesics, Finsler geodesics are not necessarily *reversible*.

Definition . *A Finsler metric is geodesically reversible if each oriented geodesic can be reparametrized as a geodesic with the reverse orientation.*

Finsler surfaces with constant non-positive flag curvature

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Theorem (Akbar-Zadeh 1988). *Let M^2 be compact. Then*

1. *a Finsler structure on M with $K \equiv -1$ is necessarily Riemannian;*
2. *if $K \equiv 0$, M is either the torus or the Klein bottle.*

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This raised the question: what if $K \equiv 1$?

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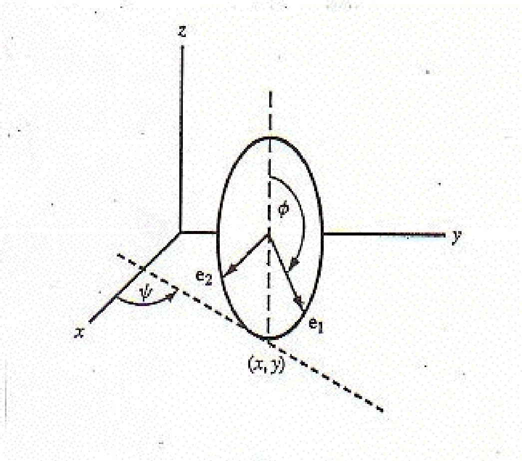
you are free to ask Robert about these results!

Sub-Finsler geometry

Control theory example: wheel rolling without slipping on a plane



State of system: (x, y, ψ, ϕ) (coordinates on $Euc(2) \times S^1$).

Velocity vectors for the wheel on the plane

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \dot{\phi}\cos\psi \\ \dot{\phi}\sin\psi \\ \dot{\psi} \\ \dot{\phi} \end{pmatrix} = \dot{\phi}X_1 + \dot{\psi}X_2$$

where

$$X_1 = \cos\psi\frac{\partial}{\partial x} + \sin\psi\frac{\partial}{\partial y} + \frac{\partial}{\partial \phi},$$
$$X_2 = \frac{\partial}{\partial \psi}$$

The 2-plane field D locally spanned by X_1 and X_2 is *bracket-generating*: set

$$X_3 = -[X_1, X_2] = \cos\psi \frac{\partial}{\partial y} - \sin\psi \frac{\partial}{\partial x},$$
$$X_4 = -[X_2, X_3] = \cos\psi \frac{\partial}{\partial x} + \sin\psi \frac{\partial}{\partial y}$$

then $\{X_1, X_2, X_3\}$ has rank 3 everywhere and $\{X_1, \ldots, X_4\}$ spans TM.

Definition . *A k-plane field $D \subset TM$ spanned locally by $X_1, \ldots, X_k$ is* ***bracket-generating*** *if these vector fields and their iterated Lie brackets locally span TM.*

Definition . *A curve* $\gamma : [a,b] \to M$ *is a horizontal curve if whenever* $\dot{\gamma}(t)$ *exists,* $\dot{\gamma}(t) \in D_{\gamma(t)}$.

Theorem (Chow, 1937) Let M be a connected manifold and let $D \subset TM$ be a smooth k-plane field on M. If D is bracket-generating, then any two points $p, q \in M$ can be joined by a piecewise-smooth horizontal curve.

Thus any two points in a manifold M with a bracket-generating k-plane field D are connected by a horizontal curve. It makes sense, therefore, to define a length functional for these curves.

Definition . *Let M be a smooth manifold, and let $D \subset TM$ be a bracket-generating k-plane field.*

1) *A **sub-Finsler metric** on M is a smoothly varying Finsler metric L on each subspace $D_x \subset T_xM$. The triple (M, D, L) is called a **sub-Finsler manifold**.*
2) *The **sub-Finsler length** of a smooth horizontal curve $\gamma : [a, b] \to M$ is*

$$\mathcal{L}(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t))\, dt.$$

   (If γ is piecewise-smooth, extend this definition in the obvious manner.)
3) *The **sub-Finsler distance** between two points p and q in an sub-Finsler manifold is*

$$\mathcal{D}(p, q) = \inf\{\mathcal{L}(\gamma) : \gamma \textit{ is a horizontal curve connecting } p \textit{ and } q\}$$

Special case: sub-Riemannian metrics

If L is a Riemannian metric on D, it is called a **sub-Riemannian metric** on M and M is a **sub-Riemannian manifold.**

We will explore sub-Finsler structures in two cases:

- A rank 2 contact distribution on a 3-manifold
- A rank 2 Engel distribution on a 4-manifold

Local geometry of sub-Finsler contact 3-manifolds

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The following result is due to Pfaff.

Let (M, D) be a contact 3-manifold. For every point $p \in M$, there exists a neighborhood U of p and a local coordinate system (x, y, z) on U, based at p, such that

$$D = \{dz + \tfrac{1}{2}(x\,dy - y\,dx)\}^{\perp}.$$

Theorem (Hughen, 1995) Let $(M, D, \langle,\rangle)$ be a sub-Riemannian contact 3-manifold. There are canonical 1-forms $\omega^1, \omega^2, \omega^3, \alpha$ on the unit circle bundle Σ over D satisfying the structure equations

$$\begin{aligned} d\omega^1 &= -\alpha \wedge \omega^2 + A_1\, \omega^2 \wedge \omega^3 + A_2\, \omega^3 \wedge \omega^1 \\ d\omega^2 &= \alpha \wedge \omega^1 + A_2\, \omega^2 \wedge \omega^3 - A_1\, \omega^3 \wedge \omega^1 \\ d\omega^3 &= \omega^1 \wedge \omega^2 \\ d\alpha &= S_1\, \omega^2 \wedge \omega^3 + S_2\, \omega^3 \wedge \omega^1 + K\, \omega^1 \wedge \omega^2. \end{aligned} \tag{4}$$

The functions A_1, A_2, K, together with their covariant derivatives, form a complete set of local invariants for the sub-Riemannian structure $(M, D, \langle,\rangle)$.

Theorem (Clelland, M) Let (M, D, L) be a contact 3-manifold with a sub-Finsler metric L, and let Σ be the indicatrix bundle over M. There are canonical 1-forms $(\eta^1, \eta^2, \eta^3, \phi)$ on Σ that satisfy the following structure equations:

$$\begin{aligned} d\eta^1 &= -\phi \wedge \eta^2 + A_1\, \eta^2 \wedge \eta^3 + (A_2 + \tfrac{1}{2} IK)\, \eta^3 \wedge \eta^1 + J_1\, \phi \wedge \eta^3 \\ d\eta^2 &= \phi \wedge \eta^1 + (A_2 - \tfrac{1}{2} IK)\, \eta^2 \wedge \eta^3 - A_1\, \eta^3 \wedge \eta^1 + J_2\, \phi \wedge \eta^3 \\ &\qquad - I\, \phi \wedge \eta^2 \\ d\eta^3 &= \eta^1 \wedge \eta^2 - I\, \phi \wedge \eta^3 \\ d\phi &= S_0\, \eta^3 \wedge \phi + S_1\, \eta^2 \wedge \eta^3 + S_2\, \eta^3 \wedge \eta^1 \\ &\qquad - J_1\, \phi \wedge \eta^1 - 2J_2\, \phi \wedge \eta^2 + K\, \eta^1 \wedge \eta^2. \end{aligned} \tag{5}$$

Moreover, the indicatrix bundle Σ is the unit circle bundle for a sub-Riemannian metric if and only if $I \equiv 0$.

Geodesics of sub-Finsler contact 3-manifolds

The problem of finding length-minimizing curves is complicated by the fact that we do not allow arbitrary variations of γ, but rather only variations of γ through *horizontal* curves of D.

By the *Griffiths formalism*, we may convert this constrained variational problem on Σ to an unconstrained problem on an affine sub-bundle

$$Z \subset T^*\Sigma.$$

Extremals of the unconstrained length functional on Z then project to extremals of the original, constrained variational problem on Σ.

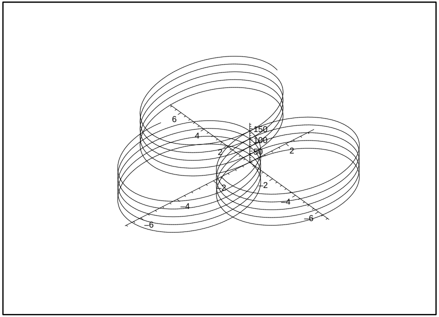
Example: the Heisenberg group

Let $\mathcal{H}$ be the Heisenberg group, defined by

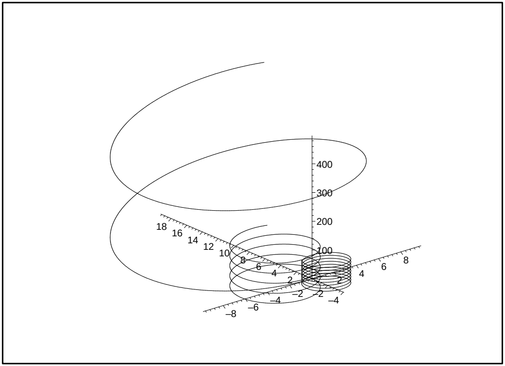
$$\mathcal{H} = \left\{ \begin{bmatrix} 1 & y & z + \frac{1}{2}xy \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\} \cong \mathbb{R}^3,$$

and let the contact structure on $\mathcal{H}$ be the rank 2 distribution

$$D = \{dz + \tfrac{1}{2}(x\,dy - y\,dx)\}^{\perp}.$$



Geodesics of flat sub-Riemannian metric

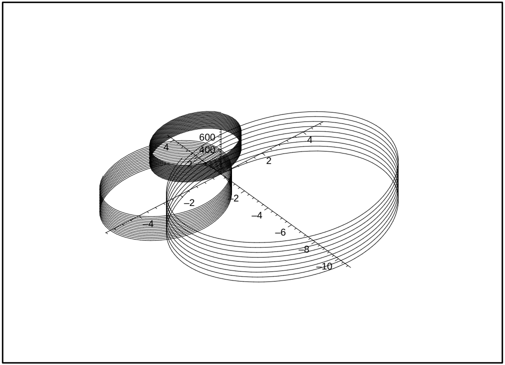


Geodesics of Randers-type metric

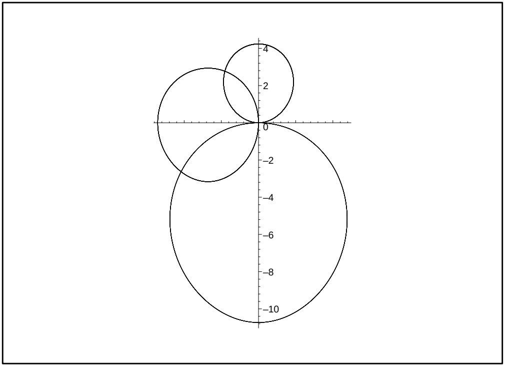
Example: A "limaçon metric" on $\mathcal{H}$.

For an example in which the geodesics are not liftings of conic sections, consider the sub-Finsler metric whose indicatrix is the convex limaçon with polar equation

$$R = 3 + \cos\theta.$$



Geodesics of limaçon metric



Projections onto the xy-plane

Theorem (Clelland, M 2006)

For any homogeneous sub-Finsler metric F on the Heisenberg group $\mathcal{H}$, the sub-Finsler geodesics are straight lines parallel to the xy-plane or liftings of simple closed curves in the xy-plane. In the latter case, the simple closed curves are the curves of minimal Finsler length enclosing a given area in the plane.

Local geometry of sub-Finsler Engel manifolds

Definition . *An* Engel manifold *is a 4-manifold* $\mathcal{X}$ *equipped with a rank 2 distribution* D *satisfying the conditions that*

- $rank([D, D]) = 3$, *and*
- $[D, [D, D]] = T\mathcal{X}$

at each point of $\mathcal{X}$.

Example: $Euc(2) \times S^1$ with the 2-plane field D defined by the wheel-on-the-plane control system.

Theorem (Engel)

Let $(\mathcal{X}, D)$ be an Engel manifold. For every point $p \in \mathcal{X}$, there exists a neighborhood U of p and a local coordinate system (x, y, z, w) on U, based at p, such that

$$D = \{dy - z\,dx,\ dz - w\,dx\}^{\perp}.$$

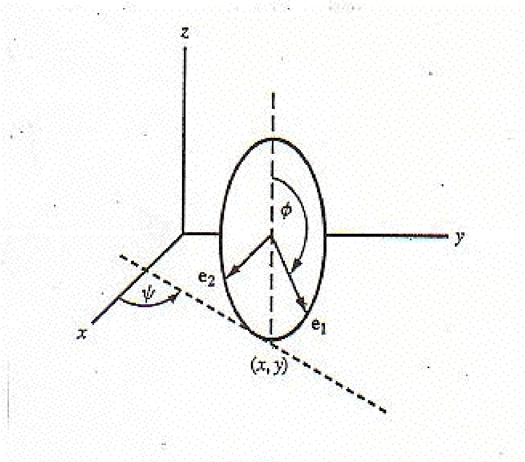
Geodesics of Sub-Finsler Engel Manifolds

Note: the variations of a horizontal curve must themselves must be horizontal curves, that is, tangent to the 2-plane Engel system at each point. Such variations are not guaranteed to exist.

Definition . *A horizontal curve* $\gamma : [a, b] \to M$ *is rigid if* γ *has no* C^1 *horizontal variations other than reparametrizations. A horizontal curve* γ *is locally rigid if every point of* $[a, b]$ *lies in a subinterval* $J \subset [a, b]$ *so that* γ *restricted to* J *is rigid.*

Theorem (R. Bryant, L. Hsu 1994). Every Engel manifold has a canonical foliation by curves that are locally rigid.

The wheel on the plane, again



Rigid curves for this system project to lines in the plane.

A natural Engel structure from surface geometry

Proposition (M, 2001) Let N be a surface with a Riemannian metric $\langle\, ,\, \rangle$, and let $\mathcal{F}(N)$ be the orthonormal frame bundle over N. There is a canonical Engel system on $\mathcal{F}(N) \times \mathbb{R}$ so that

1) regular curves on N lift to horizontal curves of the Engel system on $\mathcal{F}(N) \times \mathbb{R}$;
2) the locally rigid curves on $\mathcal{F}(N) \times \mathbb{R}$ project to Riemannian geodesics on N;
3) the regular sub-Riemannian geodesics on $\mathcal{F}(N) \times \mathbb{R}$ project to local extremals of the functional

$$F(\alpha) = \int_\alpha 1 + \kappa^2\, ds$$

where κ is the geodesic curvature of $\alpha : [c, d] \to N$.

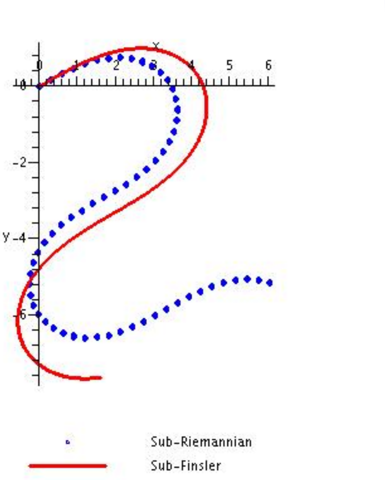
Theorem (Clelland, M, Wilkens 2007) For a “tame” sub-Finsler structure, locally there are canonical 1-forms $\eta^1, \eta^2, \eta^3, \eta^4, \phi$ on Σ that satisfy the structure equations:

Theorem (Clelland, M, Wilkens 2007) For a "tame" sub-Finsler structure, locally there are canonical 1-forms $\eta^1, \eta^2, \eta^3, \eta^4, \phi$ on Σ that satisfy the structure equations:

$$\begin{aligned}
d\eta^1 &= -\phi \wedge \eta^2 + (T^1_{03}\,\phi + T^1_{13}\,\eta^1 + T^1_{23}\,\eta^2) \wedge \eta^3 \\
&\qquad + (T^1_{04}\,\phi + T^1_{14}\,\eta^1 + T^1_{24}\,\eta^2 + T^1_{34}\,\eta^3) \wedge \eta^4 \\
d\eta^2 &= \phi \wedge \eta^1 - I\,\phi \wedge \eta^2 + (T^2_{03}\,\phi + T^2_{13}\,\eta^1 + T^2_{23}\,\eta^2) \wedge \eta^3 \\
&\qquad + (T^2_{04}\,\phi + T^2_{14}\,\eta^1 + T^2_{24}\,\eta^2 + T^2_{34}\,\eta^3) \wedge \eta^4 \\
d\eta^3 &= \eta^1 \wedge \eta^2 + (-I\,\phi + T^3_{13}\,\eta^1 + T^3_{23}\,\eta^2) \wedge \eta^3 \\
&\qquad + (T^3_{04}\,\phi + T^3_{14}\,\eta^1 + T^3_{24}\,\eta^2 + T^3_{34}\,\eta^3) \wedge \eta^4 \\
d\eta^4 &= \big((\sin\Theta)\,\eta^1 + (\cos\Theta)\,\eta^2\big) \wedge \eta^3 \\
&\qquad + \big(-I(\cos^2\Theta + 1)\,\phi + T^4_{14}\,\eta^1 + T^4_{24}\,\eta^2 + T^4_{34}\,\eta^3\big) \wedge \eta^4 \\
d\phi &= \phi \wedge (T^1_{03}\,\eta^1 + T^0_{02}\,\eta^2 + T^0_{03}\,\eta^3 + T^0_{04}\,\eta^4).
\end{aligned} \tag{6}$$

Back to the wheel on the plane, again

We can define a sub-Finsler metric for the wheel-on-the-plane system that modifies the sub-Riemannian metric in a way that makes curvature more costly.



Note that the sub-Finsler geodesic doesn't curve as sharply as the sub-Riemannian one.

Control systems

A control system may be described in local coordinates by an underdetermined system of ordinary differential equations

$$\dot{x} = F(x, u), \tag{7}$$

where $x \in \mathbb{R}^n$ represents the *state* of the system and $u \in \mathbb{R}^s$ represents the *controls*. More generally, x and u may take values in an n-dimensional manifold $\mathcal{X}$ and and s-dimensional manifold $\mathcal{U}$, respectively.

Control linear and control affine systems

- **Control linear**: can be locally represented as

$$\dot{x} = f(x)u, \tag{8}$$

  where $f(x)$ is an $n \times s$ matrix whose entries are smooth functions of x.

- **Control affine (control system with drift)**:

$$\dot{x} = v(x) + f(x)u, \tag{9}$$

  where $f(x)$ is as above and $v(x)$ is a so-called *drift* vector field that is not subject to control.

Control linear systems are well understood, but control affine systems are ubiquitous and yet not as well understood.

Affine distributions

Definition . *A rank s affine distribution F on an n-dimensional manifolds M is a smoothly varying family of s-dimensional affine subspaces $F_p \subset T_pM$.*

- *F is strictly affine if none of the affine subspaces F_p are linear subspaces. (Studied in depth by V. Elkin.)*
- *A point-affine distribution is an strictly affine distribution F with a fixed vector field $a_0 \in F$.*
- *Associated to F is the linear distribution*

$$L_F = \{\xi_1 - \xi_2 : \xi_1, \xi_2 \in F\},$$

called the direction distribution of F.

Almost bracket generating affine distributions

Set

$$F^1 = F$$
$$F^{i+1} = F^i + [F, F^i],\ i \geq 1.$$

Definition . *An affine distribution F on an n-dimensional manifold M is almost bracket generating if rank $(F^\infty) = n - 1$ and for each $p \in M$ and any vector $\xi(p) \in F_p$, span $(\xi(p), (L_{F^\infty})_p) = T_pM$.*

Point-affine equivalence

- When are two seemingly different control systems actually the same kind of system (locally)?
- In other words, can we describe different equivalence classes of point-affine systems?
- If we can, we should be able to obtain **normal forms** corresponding to the equivalence classes for these systems.

Rank 1 point-affine distributions on 3-manifolds

Theorem . *(Clelland, M, Wilkens 2009) Let F be a rank 1 point-affine distribution on a 3-manifold M.*

1. *If F is almost bracket generating, then there are local coordinates (x^1, x^2, x^3) so that*

$$F = \left(\frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + J \frac{\partial}{\partial x^3}\right) + span\left(\frac{\partial}{\partial x^3}\right).$$

2. *If F is bracket generating and L_{F^2} is integrable, then*

$$F = \left(x^2 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} + J \frac{\partial}{\partial x^3}\right) + span\left(\frac{\partial}{\partial x^3}\right).$$

Rank 1 point-affine distributions on 3-manifolds

Theorem . *(continued)*

3. *If F is bracket generating and L_{F^2} is not integrable, then*

$$F = \left(\frac{\partial}{\partial x^1} + J\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\right)\right) + span\left(x^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + H\frac{\partial}{\partial x^3}\right)$$

where H, J are arbitrary functions on M satisfying $\frac{\partial H}{\partial x^1} \neq 0$.

Example: single spin 1/2 quantum system

The time evolution of a quantum spin system is governed by Schrödinger's equation

$$\dot{U}(t) = -i\,H(t)U(t) \tag{10}$$

where $U(t) \in SU(n)$ for each t. The *Hamiltonian* $H(t)$ is the $n \times n$ matrix

$$H(t) = H_d(t) + \sum_{j=1}^{m} u_j(t)H_j.$$

where H_d is the drift vector field of the system.

Example: single spin 1/2 quantum system

In the case of NMR control of a single spin 1/2 system, the state space is $SU(2)$. There are local coordinates so that the Hamiltonian is

$$H = I_z + u(t) I_x$$

where

$$I_z = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad I_x = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus Schrödinger's equation (10) becomes

$$\dot{U} = -i\, I_z U + (-i\, I_x U) u$$

Example: single spin 1/2 quantum system

The matrices $-i\, I_x, -i\, I_z$ are in the Lie algebra $su(2)$ with commutator

$$[-i\, I_z, -i\, I_x] = -i\, I_y \tag{11}$$

where

$$I_y = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

The point affine distribution is $F = a_0 + \mathrm{span}(a_1)$ where a_0, a_1 are the right-invariant vector fields

$$a_0 = -i\, I_z U,\ a_1 = -i\, I_x U,$$

and by (11) F is bracket generating.

Example: single spin 1/2 quantum system

This example falls into Case 3 of the normal form theorem for rank 1 point-affine distributions on 3-manifolds. There are local coordinates x^1, x^2, x^3 so that the invariants of the system are

$$\begin{aligned} H &= -e^{2f(x^2,x^3)} \tan\left(x^1 + g(x^2,x^3)\right) + h(x^2,x^3), \\ J &= 0 \end{aligned}$$

where f, g, and h satisfy the PDEs

$$\begin{aligned} e^{2f} f_{x^3} + g_{x^2} + h g_{x^3} + x^3 &= 0, \\ e^{2f} g_{x^3} + f_{x^2} + h f_{x^3} - h_{x^3} &= 0. \end{aligned}$$

Optimal control of control-affine systems

Theorem (Clelland, M, Wilkens 2013). *1. Let $\mathcal{F}$ be a rank 1 strictly affine point-affine distribution of constant type on a surface M^2 and let Q be a positive definite quadratic cost functional on $\mathcal{F}$. If the structure $(\mathcal{F}, Q)$ is homogeneous, then $(\mathcal{F}, Q)$ is locally point-affine equivalent to one of two normal forms, with one local invariant.*

2. Let $\mathcal{F}$ be a rank 1 strictly affine, bracket generating point-affine distribution of constant type on a manifold M^3 and let Q be a positive definite quadratic cost functional on $\mathcal{F}$. If the structure $(\mathcal{F}, Q)$ is homogeneous, then $(\mathcal{F}, Q)$ is locally point-affine equivalent to one of six normal forms, with three local invariants.