Equivalence of Norms in Finite Dimension

Theorem 0.1. *If H is a normed linear space of finite dimension, then all norms on H are equivalent.*

Proof. Let $\Phi = \{\phi_1, \phi_2, \cdots, \phi_n\}$ be a basis of H so for any $x \in H$ we have

$$x = \sum_{i=1}^{n} a_i \phi_i$$

for some set of $(a_1, a_2, \cdots, a_n)$. Define a function $\rho : H \to \mathbb{R}$ by

$$\rho(x) = \sqrt{\sum_{i=1}^{n} |a_i|^2}.$$

We can prove that ρ is a norm. Since Φ is a basis of H, then $\{\phi_i\}$ are linearly independent and

$$x = \sum_{i}^{n} a_i \phi_i = 0 \Leftrightarrow a_i = 0 \ \forall\ i \Leftrightarrow \rho(x) = 0.$$

It can also be verified that $\rho(x)$ satisfies linearity property and the triangular inequality. Hence $\rho(x)$ is a norm. Now let $\|\cdot\|$ be an arbitrary norm on H. By definition

$$\|x\| = \|\sum_{i}^{n} a_i \phi_i\| \leq \sum_{i}^{n} \|a_i\| \cdot \|\phi_i\|.$$

By Cauchy inequality,

$$\sum_{i}^{n} \|a_i\| \|\phi_i\| \leq \sqrt{\sum_{i}^{n} \|a_i\|^2} \cdot \sqrt{\sum_{i}^{n} \|\phi_i\|^2}.$$

Therefore

$$\|x\| \leq M \rho(x)$$

where

$$M = \sqrt{\sum_{i}^{n} \|\phi_i\|^2}.$$

Now define $f : S \to \mathbb{R}$ on

$$S = \left\{ (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n : \sqrt{\sum_{i}^{n} |b_i|^2} = 1 \right\}$$

by

$$f(a) = f(a_1, a_2, \cdots, a_n) = \left\| \sum_{i}^{n} a_i \phi_i \right\|.$$

Since S is compact and f is continuous, f attains a minimum on S for some

$$\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$$

with

$$\sqrt{\sum_{i}^{n} |\alpha_i|^2} = 1.$$

Suppose that $f(\alpha) = m$. Then for any $x \in H$ we have

$$\begin{aligned}
\|x\| &= \left\| \sum_i^n a_i \phi_i \right\| \\
&= \left(\frac{\sqrt{\sum_i^n \|a_i\|^2}}{\sqrt{\sum_i^n \|a_i\|^2}} \right) \cdot \left\| \sum_i^n a_i \phi_i \right\| \\
&= \sqrt{\sum_i^n \|a_i\|^2} \cdot \left\| \sum_i^n \frac{a_i}{\sqrt{\sum_i^n \|a_i\|^2}} \phi_i \right\| \\
&\geq \sqrt{\sum_i^n \|a_i\|^2} \cdot m = m\rho(x).
\end{aligned}$$

In summary there are m, M such that

$$m\rho(x) \leq \|x\| \leq M\rho(x).$$

$\square$