Uncertain Voronoi Diagram

Mohammadreza Jooyandeh [a,*], Ali Mohades [a],
Maryam Mirzakhah [a]

[a] *Faculty of Mathematics and Computer Science,*
Amirkabir University of Technology, PO Box 15875-4413, Tehran, Iran

Abstract

In this paper, we introduce the *fuzzy Voronoi diagram* as an extension of the Voronoi diagram. We assume Voronoi sites to be fuzzy points and then define the Voronoi diagram for this kind of sites, then we provide an algorithm for computing this diagram based on Fortune's algorithm which costs $O(n \log n)$ time. Also we introduce the fuzzy Voronoi diagram for a set of fuzzy circles, rather than fuzzy points, of the same radius. We prove that the boundary of this diagram is formed by the intersection of some hyperbolae, and finally we provide an $O(n^3 \log n)$-time algorithm to compute the boundary.

Key words: Fuzzy Voronoi Diagram, Voronoi Diagram, Fuzzy Geometry, Computational Geometry, Fuzzy Set.

1 Introduction

In this paper we work on fuzzy Voronoi diagrams. We extend the definition of the Voronoi diagram for fuzzy sites. First we define the Voronoi diagram for a fuzzy subset of $\mathbb{R}^2$ and then we provide an $O(n \log n)$-time algorithm for computing it. This algorithm adds a post-process to Fortune's algorithm [1] which computes the standard Voronoi diagram. Furthermore, we define the fuzzy Voronoi diagram for a set of fuzzy circles. And finally we provide a geometrical algorithm to compute the boundary of the diagram. In this algorithm we use Mathematica [2] software to solve a system of equations.

* Corresponding author
Email addresses: mohammadreza@jooyandeh.info (Mohammadreza Jooyandeh), mohades@aut.ac.ir (Ali Mohades), mirzakhah@aut.ac.ir (Maryam Mirzakhah).
URL: http://www.jooyandeh.info (Mohammadreza Jooyandeh).

2 Preliminaries

Before discussing the main result it is better to have a review of some fundamental definitions of the Voronoi diagram and the fuzzy set theory.

Let P be a discrete subset of a metric space X. For each point p in P, the set of all points x in X whose distance from p is less than (or equal to) their distance from other points in P is called its *Voronoi cell* (or *Diricle domain*) and is shown by $V(p)$.

$$V(p) = \{x \in X \,|\forall q \in P\,[d(p,x) \le d(q,x)]\}\,. \tag{1}$$

Let P be a discrete subset of a metric space X. The *Voronoi diagram* of P is the set of all Voronoi cells of its points, which is shown by $V(P)$. The members of P are also called the *Voronoi sites* of the Voronoi diagram of P.

$$V(P) = \{V(p)\,|p \in P\}\,. \tag{2}$$

Because of the topological properties of X, we can equivalently define the Voronoi diagram of P as the set of cell boundaries. In this paper, we work with this second definition. Also, this boundary could be defined as the points which possess the property of having maximum distance from the nearest sites. If we assume $\mathbb{R}^2$ as the space and use Euclidean metric, the boundary of a Voronoi cell would be made up of some line segments.

Let U be a set. $\tilde{F}$ is called a *fuzzy set* over U if

- $\tilde{F} \subseteq U \times (0,1]$
- A partial function $f : U \longrightarrow (0,1]$ exists such that

$$\tilde{F} = \{(x, f(x))\,|x \in U\}\,. \tag{3}$$

Let $\tilde{P}$ be a fuzzy set. The *non-fuzzy image* of $\tilde{P}$ is shown by P, and is defined as follows:

$$P = \left\{x \,\middle|\, \exists\lambda \in (0,1]\left[(x,\lambda) \in \tilde{P}\right]\right\}. \tag{4}$$

Let $\tilde{x} = (x, f(x))$ be a member of a fuzzy set. The *non-fuzzy image* of $\tilde{x}$ is its first coordinate (i.e. its value without its membership degree).

Let $\tilde{F}$ be a fuzzy set over a set U. The *characteristic function* of $\tilde{F}$ is shown by $\chi_{\tilde{F}}$ and is defined as follows:

$$\chi_{\tilde{F}} : U \longrightarrow [0,1]$$

$$\chi_{\tilde{F}}(x) = \begin{cases} f(x), & (x, f(x)) \in \tilde{F} \\ 0, & otherwise \end{cases} \tag{5}$$

A *t-norm* is a function $\top : [0,1] \times [0,1] \longrightarrow [0,1]$ which satisfies the following properties:

- Commutativity: $\top(a,b) = \top(b,a)$
- Monotonicity: $(a \leq c \wedge b \leq d) \Longrightarrow \top(a,b) \leq \top(c,d)$
- Associativity: $\top(a, \top(b,c)) = \top(\top(a,b), c)$
- Identity element: $\top(a,1) = a$

A *t-conorm* is a function $\perp : [0,1] \times [0,1] \longrightarrow [0,1]$ which satisfies the following properties:

- Commutativity: $\perp(a,b) = \perp(b,a)$
- Monotonicity: $(a \leq c \wedge b \leq d) \Longrightarrow \perp(a,b) \leq \perp(c,d)$
- Associativity: $\perp(a, \perp(b,c)) = \perp(\perp(a,b), c)$
- Identity element: $\perp(a,0) = a$

One can see that the t-norm and the t-conorm are duals of each other with regard to the generalized De Morgan's law:

$$\perp(a,b) = 1 - \top(1-a, 1-b). \tag{6}$$

In fuzzy logic and the fuzzy set theory, the t-norm (t-conorm) is the extension of $\wedge$ ($\vee$) in the set theory.

Let k be a positive integer. We denote the set of all positive integers smaller than $k+1$ by $\mathbb{N}_k$.

3 Fuzzy Voronoi diagram for fuzzy points

In this section, we define the Voronoi diagram for a fuzzy discrete set of sites. For this purpose, we first extend the definition of the Voronoi cell to the fuzzy Voronoi cell. Based on this definition we can define the fuzzy Voronoi diagram.

Definition 1 *Let $\tilde{P}$ be a discrete fuzzy subset of $\mathbb{R}^2$ and let $\tilde{p} \in \tilde{P}$. The fuzzy*

Voronoi cell of $\tilde{p}$ is shown by $\tilde{V}(\tilde{p})$ and is defined as follows:

$$\tilde{V}(\tilde{p}) = \left\{(x, \chi_{\tilde{P}}(p)) \middle| \forall \tilde{q} \in \tilde{P}\, [d(x,p) \leq d(x,q)] \right\}. \tag{7}$$

Definition 2 *Let $\tilde{P}$ be a discrete subset of $\mathbb{R}^2$, $\tilde{p} \in \tilde{P}$ and (x, α) be a boundary member of $\tilde{V}(\tilde{p})$. (x, φ) will be a member of the fuzzy Voronoi diagram of P, shown by $\tilde{V}(\tilde{P})$, in which*

$$\varphi = \top_{(x,\alpha)\in\tilde{V}(\tilde{p})\wedge\tilde{p}\in\tilde{P}} (\alpha). \tag{8}$$

$\top$ could be any t-norm and is selected based on the application.

Theorem 1 *The non-fuzzy image of the fuzzy Voronoi diagram of a fuzzy set of points is the Voronoi diagram of the non-fuzzy image of the set of points.*

Proof: Let $\tilde{x} = (x, \varphi)$ be a member of the fuzzy Voronoi diagram of $\tilde{P}$ $(\varphi > 0)$. So $\tilde{x}$ is a member of the boundary of at least two Voronoi cells. Let $\tilde{p}_1, \cdots, \tilde{p}_k$ be the sequence of all such sites. According to Definition 1:

$$\begin{aligned}
\forall i \in \mathbb{N}_k \;\; & \forall \tilde{q} \in \tilde{P}\, [d(x,p_i) \leq d(x,q)] \\
& \Longrightarrow \forall i \in \mathbb{N}_k \, \forall q \in P \, [d(x,p_i) \leq d(x,q)] \\
& \Longrightarrow \forall i \in \mathbb{N}_k \, [x \in V(p_i)] \\
& \Longrightarrow x \in V(P).
\end{aligned}$$

To prove the reverse, suppose x is a member of $V(P)$. So there must exist sites $p_1, \cdots, p_k$ $(k \geq 2)$ such that x belongs to the boundaries of their Voronoi cells. By the definition of the Voronoi cell and Definition 1, one can see that, $\forall i \in \mathbb{N}_k \left[(x, \chi_{\tilde{P}}(p_i)) \in \tilde{V}(\tilde{p}_i)\right]$ and by Definition 2, $(x, \top_{(x,\alpha)\in\tilde{V}(\tilde{p})\wedge\tilde{p}\in\tilde{P}}(\alpha)) \in \tilde{V}(\tilde{P})$. Also we know

$$\begin{aligned}
\forall i \in \mathbb{N}_k \, & [\chi_{\tilde{P}}(p_i) > 0] \\
& \Longrightarrow \top_{(x,\alpha)\in\tilde{V}(\tilde{p})\wedge\tilde{p}\in\tilde{P}}(\alpha) > 0 \\
& \Longrightarrow \chi_{\tilde{P}}(x) > 0.
\end{aligned}$$

So $\forall x \in \mathbb{R}^2 \left[x \in V(P) \Longleftrightarrow \left(\exists \alpha > 0 \; (x, \alpha) \in \tilde{V}(\tilde{P})\right)\right]$. □

4 Computing the fuzzy Voronoi diagram of a fuzzy set of points

Regarding Theorem 1 we can compute the fuzzy Voronoi diagram of a fuzzy set of points easily. We can compute the classic Voronoi diagram for the non-fuzzy image of the set of sites by Fortune's algorithm, and then we can compute the degree of membership for each line segment in $O(1)$. Because each line segment is a part of the boundary of two Voronoi cells, computing the t-norm will cost $O(1)$ time. Also for each Voronoi vertex we should compute the t-norm for three sites (the points are in general position). So the total time cost of the algorithm would be equal to the time cost of Fortune's algorithm, which is $O(n \log n)$.

5 Fuzzy Voronoi diagram for fuzzy circles

In Section 4, we introduced an algorithm for computing the fuzzy Voronoi diagram of a fuzzy set of points. In this section, we extend the definition of the Voronoi diagram to cover a larger set of objects, not only a set of points. Suppose the members of the set of sites to be sets. This means that each site is a set of points which can be any geometric objects, such as circles, rectangles, etc.

This extension of the Voronoi diagram to non-fuzzy objects is studied in many researches, e.g. [3] and [4]. But all these works provide the non-fuzzy (standard) Voronoi diagram. The Voronoi diagram of a set of fuzzy objects is not much similar to its non-fuzzy version. To clarify the difference, consider the following example. Assume that we have only two sites both of which are circles with equal radii. Also, both have continuous characteristic functions which converge to zero at the boundary of the circles. Let us check what the non-fuzzy Voronoi diagram for the non-fuzzy image of these sites would be like. It is clear that the diagram would be the bisector of the line segment joining their centers. But if we want to convert it to the fuzzy version, like previous sections, we will see that the degree of membership of this line's points in the diagram would be zero because their membership degree is computed from points whose membership degree is lower than any $\epsilon > 0$ (according to the continuity assumption of the membership functions).Therefore, the line is not part of the diagram!

The problem occurs because the Voronoi diagram of fuzzy objects has been created only using the boundaries, while it must be created using all points of sites and according to their degree of membership. If we consider all points and their membership degrees, as we will show later the fuzzy Voronoi diagram of two such fuzzy circles is similar to Figure 1.

Definition 3 *Let P be a finite family of fuzzy subsets of $\mathbb{R}^2$ and let $x \in \mathbb{R}^2$. We define $N_P(x)$ as the set of the nearest neighbors of x in P.*

$$N_P(x) = \{\tilde{p} \in P \,|\forall \tilde{q} \in P \; \exists \tilde{y} \in \tilde{q} \; \exists \tilde{y}' \in \tilde{p} \, [d(x, y) \geq d(x, y')] \} \,. \tag{9}$$

Definition 4 *Let P be a finite family of fuzzy subsets of $\mathbb{R}^2$ and let $x \in \mathbb{R}^2$. x is a member of the fuzzy Voronoi diagram of P, shown by $\tilde{V}(P)$, if $|N_P(x)| > 1$. In other words, x is in the diagram if and only if it has at least two nearest neighbors in P. The degree of membership of x in $\tilde{V}(P)$ is defined as follows:*

$$\chi_{\tilde{V}(P)}(x) = \begin{cases} \underset{\substack{\tilde{p},\tilde{q}\in N_P(x) \\ \tilde{p}\neq\tilde{q}}}{\perp} \left(\underset{\substack{\tilde{y}\in\tilde{p},\tilde{y}'\in\tilde{q} \\ d(x,y)=d(x,y')}}{\sup} \top\left(\chi_{\tilde{p}}(y), \chi_{\tilde{q}}(y')\right) \right) & ,|N_P(x)| > 1 \\ 0 & ,|N_P(x)| = 1 \end{cases} \tag{10}$$

Definition 5 *A fuzzy circle is a fuzzy subset of $\mathbb{R}^2$, say $\tilde{C}$, if there exists a disk $D \subset \mathbb{R}^2$ which satisfies the following properties:*

- *D contains all points of C ($C \subseteq D$).*
- *In any open neighborhood of a boundary point of D, there exists at least one point of C.*

$$\forall x \in \partial D \; \forall \epsilon > 0 \; \exists p \in C \left[d(x,p) < \epsilon \right]. \tag{11}$$

Also we define the center of this fuzzy circle as the center of D (D is the closure of C and so it is unique).

Note that if we remove the second condition from Definition 5, any bounded fuzzy subset of $\mathbb{R}^2$ becomes a fuzzy circle. Because any bounded subset of $\mathbb{R}^2$ can be covered by a disk.

Let us see which points of $\mathbb{R}^2$ can form the boundary of this kind of Voronoi diagram. In this paper we assume that all circles have the same radius (denoted by r) and the distance between any two circles is greater than their diameter. In the first step we check only two sites, then we extend it to a finite family of sites. Consider $P = \left\{\tilde{A}, \tilde{B}\right\}$ in which $\tilde{A}$ and $\tilde{B}$ are two fuzzy circles with centers $a = (a_x, a_y)$ and $b = (b_x, b_y)$, respectively. Let p be a point in $\mathbb{R}^2$. p is in $V(P)$, with some membership degree, if and only if there exist points $a' \in A$ and $b' \in B$ such that p is on the bisector of the line going through them. In

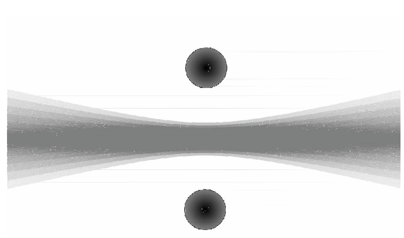

Fig. 1. Fuzzy Voronoi diagram of two fuzzy circles.

other words, p is in $V(P)$, with some membership degree, if and only if there exists a circle C with center p such that it intersects both A and B. So the boundary of $V(P)$ is formed by the centers of circles that intersect both A and B only at one point such that one lies inside and the other one lies outside the circle. So

$$\begin{aligned} &p \in \partial V(P) \\ &\qquad \Longleftrightarrow [d(p,a) + r = d(p,b) - r] \vee [d(p,a) - r = d(p,b) + r] \\ &\qquad \Longleftrightarrow |d(p,a) - d(p,b)| = 2r \end{aligned}$$

Therefore, the boundary of this diagram would be a hyperbola. If we use symbols α, β and c for $(a_x + b_x)/2$, $(a_y + b_y)/2$ and $d(a,b)/2$, respectively, and assume a and b to be on a horizontal line, with some simple computation one can see that the formula of the hyperbola is:

$$\frac{(x-\alpha)^2}{r^2} - \frac{(y-\beta)^2}{c^2 - r^2} = 1. \tag{12}$$

Moreover, we can conclude that the Voronoi cell of $\tilde{A}$ is the interior of the branch of hyperbola whose focus is a (denoted by $V_B(A)$). Generally, if $P = \{\tilde{C}_1, \ldots, \tilde{C}_n\}$, for each site $\tilde{C}_i \in P$ we have:

$$V(\tilde{C}_i) = \bigcap_{j \in \mathbb{N}_n \setminus \{i\}} V_{C_j}(C_i) \tag{13}$$

Let $\tilde{C}_i \in P$ be a fuzzy circle. We denote by $H_{i,j}$ that branch of the hyperbola formed by C_i and C_j whose focus is the center of C_i. Also we denote $\mathbf{H}_{C_i} = \{H_{i,j}\}_{j \in N_n \setminus \{i\}}$. With these symbols we can see that $V(\tilde{C}_i)$ is equal to the intersection of the interiors of the members of $\mathbf{H}_{C_i}$. An example is shown in Figure 2. Note that if $|P| = 2$, obviously the boundary of the diagram is $H_{1,2} \cup H_{2,1}$. So, from now on, we address the cases of more than two sites.

Each hyperbola whose one branch is $H_{i,j}$ can be parameterized with a single variable t (using cosh and sinh functions) in a way that when t increases, its

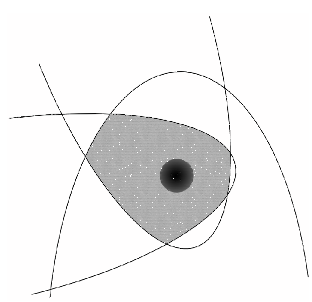

Fig. 2. A site and its Voronoi cell

corresponding point on $H_{i,j}$ moves round C_i clockwise. For each $H_{i,j} \in \mathbf{H}_{C_i}$ we put its intersections with other members of $\mathbf{H}_{C_i}$ into a list $L_{i,j}$, which is sorted in ascending order with respect to parameter t. Also, by $L^*_{i,j}$ denote $L_{i,j}$ with $-\infty$ and $+\infty$ added to the beginning and the end, respectively. We refer to the members of $L^*_{i,j}$ as event points.

6 Algorithm for computing the boundary of the fuzzy Voronoi diagram of fuzzy circles

For each site $\tilde{C}_i \in P$, we compute its boundary, which is denoted by B_i. First, we compute $L^*_{i,j}$ for all $i \neq j \in \mathbb{N}_n$. We have used the following Mathematica command to find the intersection points:

```
simplify[Solve[{x^2/a^2-y^2/b^2 == 1,
        c*x^2+d*x*y+e*y^2+f*x+g*y+h == 0}, {x, y}]]
```

We start with the nearest point p_0 in $\bigcup_{j\in\mathbb{N}_n} L_{i,j}$ and set $B_i =< p_0 >$. Then we start a loop. In each iteration we choose the last point p in the sequence of B_i. Consider $T = \{H_{i,j} | p \in H_{i,j}\}$. Then we select the $H_{i,j} \in T$ that is the nearest one to the center of C_i in the interval between p and its next event point in $L^*_{i,j}$ (p is in $L^*_{i,j}$). Let $H_{i,j}$ be the selected hyperbola branch. We add $H_{i,j}$ and the next event point for p in $L^*_{i,j}$ into B_i. We repeat this procedure until we reach p_0 or $+\infty$. If we reach p_0, B_i will be the boundary of C_i. If we reach $+\infty$, it is sufficient to follow the same procedure starting at p_0 but in counterclockwise direction until we reach $-\infty$ and join the two sequences.

7 Time complexity of the algorithm

The total running time of the algorithm is $O(n^3 \log n)$ as described below. For a set of n sites, the preprocess step computes $L^*_{i,j}$ for all $i \neq j \in \mathbb{N}_n$. It computes $n(n-1)$ lists, each one in $O(n)$ time and then sorts them (which totally costs $O(n^3 \log n)$). Then for each site:

- Find the nearest intersection, p_0, to C_i (costs $O(n^2)$).
- Set $B_i =< p_0 >$
- Until p_0 or $+\infty$ is reached ($O(n)$ times)
  - · Find the next hyperbola branch and the next point (costs $O(n)$).
  - · Add both to the end of B_i.
- If B_i ends with $+\infty$, until $-\infty$ is reached ($O(n)$ times)
  - · Find the previous hyperbola branch and the previous point (costs $O(n)$).
  - · Insert both into the beginning of B_i.

8 Conclusion

In this paper, we introduced the fuzzy Voronoi diagram and studied it for point and circle sites and showed that these diagrams can be computed in $O(n\log(n))$ and $O(n^3\log(n))$ time, respectively. Some other works can still be done on this diagram and related topics, some of which are:

- Computing the degree of membership of interior points for the fuzzy Voronoi diagram of circles
- Studying the dual of this diagram which may lead to the extension of Delony triangulation
- Studying the fuzzy Voronoi diagram for other shapes
- Studying the fuzzy Voronoi diagram with other metrics or in higher dimensions

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