



UNIVERSITY OF CAPE TOWN

Methods of Pricing Convertible Bonds

Author:

Ariel Zadikov
ZDKARI001

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The aim of this dissertation was to build a basic understanding of hybrid securities with a focus on convertible bonds. We look at various methods to price these complex instruments and learn of the many subtleties they exhibit when traded in the market.

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Chapter 1

Introduction

“He has the deed half done who has made a beginning” Horace

To raise capital in financial markets, companies may choose among three major asset classes: equity, bonds, and hybrid instruments, such as convertible bonds. While issues arising from valuing equity and bonds are extensively studied by researchers, convertible bonds research still trails a short way behind. This is surprising as convertible bonds cannot simply be considered as a combination of equity and bonds but present their own specific pricing challenges. Even though they have delivered admirable results, convertibles still receive less media coverage and are sometimes overlooked by investors.

The strongest growth point of the convertibles market came in the 1990’s with the growth of hedge funds and the development of the ‘convertible bond arbitrage’, where managers bought the debt while shorting the corresponding stock. Demand then dipped in 2007 as stock markets fell and many long-only managers started selling when returns turned negative for the year. However, when performance turned around in early 2009, interest in the security picked up once more. But issuance rates for convertibles remain low relative to conventional debt. The market capitalisation for convertibles globally is approximately \$571 billion spread across 2,535 issuances worldwide with South Africans beginning to take to this quirky investment class (Redgrave [2010] and Reynolds [2010]).

Convertible Bonds are innovative financial instruments which despite their name, have greater similarities with derivatives than with conventional bonds. One of the most attractive features of convertible bonds is that they can be customised to be

either more debt like or more equity like in various ways in order to meet the needs of investors and issuers. The hybrid feature of convertible bonds provides investors with the downside protection of bonds and the upside return of equities. For this reason, convertible bonds are an efficient way to implement some asset allocation strategies that take advantage of both fixed income and equity markets. For equity investment, convertible bonds offer protection against losses, enabling portfolio managers to invest in riskier firms without exceeding their risk-exposure limits. For fixed income investors, the equity kicker offers upside returns that offset the firm's credit risk. For these reasons, convertible bonds provide managers with investment choices not often available in equity or fixed income markets. There are a variety of reasons that issuers would prefer to issue convertibles over straight debt. The main reason is the opportunity to issue a bond at a lower coupon rate than would be required for a straight bond.

Essentially, a convertible bond is a bond that can be converted into shares, a feature which allows the collective interests of the three parties involved - the issuing company, the equity investor and the fixed-income investor - to be struck more efficiently than was the case when equity and fixed income were treated as separate investment categories, involving different, if not incompatible, standards. It would be wrong to conclude that convertibles provide investors with absolute advantages over bonds or shares. Indeed equities will typically outperform convertibles in a rising share price environment, while a company's straight debt is likely to fair less badly than its convertible debt in a falling share price environment. It is the uncertain nature of investment returns that lies at the core of the rationale for the convertible asset class.

As hybrid instruments, convertible bonds are difficult to value because they depend on variables related to the underlying equity model, the vanilla corporate bond part and the risks associated to it such as interest rate risk and credit risk, and the interaction between these components. Embedded options, such as conversion, call and put provisions often are restricted to certain periods, may vary over time, and are subject to additional path dependent features of the state variables. Sometimes, individual convertible bonds contain innovative, pricing-relevant specifications that require flexible valuation models. All of this adds up and can make pricing convertible bonds a fairly difficult exercise. In addition, investors may desire valuation models that would allow them to hedge the convertible bond with a combination

of bonds and stock, and possibly place a value on each of the special features of a convertible bond.

Theoretical research on convertible bond pricing can be divided into three branches. The first pricing approach implies finding a closed-form solution to the valuation equation. The second pricing approach values convertible bonds numerically, using numerical partial differential equation approaches. This is the most common approach used in practice and by commercial pricing models used by the likes of Bloomberg. In this paper, most of our valuation models belong to this category. The third class of convertible bond pricing methods uses monte carlo simulation - we touch on this approach in the chapter dealing with convertible bond pricing models.

It is expected that the equity level will be the driving factor in the valuation of convertible bonds. This is the reason why the quantitative analysis of convertible bonds lends itself naturally to the Black Scholes analysis where the share price is the state variable, and dynamic hedging strategies are the basis for the valuation of the embedded option. Not only will the convertible bond value depend on the volatility of the share, but we shall expect the share price itself to set the dividing line between equity behavior and bond behavior. It turns out indeed that the Black-Scholes analysis provides the right framework to formulate the convertible bond pricing problem. However once the share becomes the driving factor, we must consider the impact it may have on the issuer's credit quality. For that reason, questions of how to deal with the coupled nature of the convertible bond and how to include credit risk into its value are at the forefront of innovation in convertible bond pricing.

The paper is organised as follows: Chapter 2 gives an introduction to convertible bonds. It explains basic terms and definitions related to convertible bonds and describes the fundamental behaviour of convertible bonds in the market. Chapter 3 touches on a range of convertible bond pricing models using different approaches. In Chapter 4 we analyse the effect that changes in input parameters have on the value of a convertible bond. We also evaluate the greeks of convertibles using two practical examples from the American and European convertibles markets. We conclude Chapter 4 by comparing the results of certain mathematical models to a simplified model which we have built; a short explanation and code for this model may be

found in the appendix. Chapter 5 is a small analysis of the convertible bond market in South Africa today. Finally, conclusions are drawn in Chapter 6.

Chapter 2

Convertibles for Beginners

“There are no classes in life for beginners; right away you are always asked to deal with what is most difficult.” Rainer Maria Rilke

2.1 The Basics

Convertible Bonds are creative hybrid securities that are relatively straightforward in concept: A convertible bond is simply a corporate bond that gives the holder the additional right to give up the coupons received from the bond in exchange for a fixed number of shares of common stock which is defined by the issuing corporation at issuance. All possible conversion dates and restrictions are decided at the outset of the convertible bond issue and are stipulated within the convertible bond prospectus. At maturity, convertibles are worth the greater of their cash redemption value or the market value of the shares into which they are convertible.

Convertible bonds have attractive features to both investors and issuers of the convertibles and can be customised to accommodate both parties' needs. From the investor's point of view, convertibles have several appealing features. They offer a higher yield than obtainable on the shares into which the bonds convert. They provide downside asset protection since the value of the convertible bond will only ever fall to the value of the bond floor. At the same time, convertible bonds can provide the possibility of high equity-like returns with a greater stability of income than regular shares. On the other hand, if the company stock rises, the investor can participate in this increase by converting the bonds to stock. Finally, because they are often theoretically underpriced, convertibles provide a cheap source of common

stock volatility.

Issuers on the other hand, have several reasons to favour convertible financing. The issuing company is able to offer the bond at a lower coupon rate - less than it would have to pay on a straight bond. Convertible bond issue can be regarded as a contingent issue of equity. If a company's investment opportunity expands, its stock price is likely to increase, leading to conversion. Thus the company can defer equity financing to a time when growth has been achieved. Small and growth firms are typically less known and have more expansion opportunities. Therefore, it is not surprising to see they are the main issuers of convertible bonds as this is a way to enter the debt market. In addition, the relatively low coupon rate on convertible bonds may also be attractive to small, growth firms facing heavy cash constraints. One of the cons of issuing convertible bonds is that conversion is often met by the issuance of new stock. This inevitably causes dilution to the existing stock holder and impacts the return on equity. Also, coupon payments do not disappear when they are least desired: when the stock price falls.

Convertible bonds combine the features of bonds and stocks in one instrument and its price will be affected by interest rates, share prices, stock volatilities, dividend yields and issuer's credit spread. Although there are many exceptions to the plain-vanilla convertible structure through different call and put provisions, it is important to be familiar with the basic model. Throughout this section we will use the real example of the industrial company Steinhoff International's convertible bond issue: **Steinhoff 5.7% 2013 ZAR**. The example is used to describe a range of terms useful in explaining and valuing convertible bonds in general. In order to understand how this convertible works, we will first analyze the fixed terms of the convertible bond followed by its market valuation. A summary of the bond offering can be found in Figure 2.1.

2.1.1 The Fixed Terms

Name Convention

Convertible bonds are usually identified by the issuer name (Steinhoff), coupon (5.7%), maturity date (2013) and currency (ZAR). They may also be identified by their International Securities Identification Number (ISIN) which uniquely identifies

Steinhoff 5.7% 2013 ZAR - Convertible Data at Issue	
Coupon	5.7%
Coupon Frequency	2 (semi-annual)
Issue Amount	ZAR 1,500,000,000
Denominations	ZAR 1,000,000 and multiples of ZAR 10,000 in excess thereof
Issue Date	30 June 2006
Issue Price	100%
First Coupon Date	31 January 2007
Maturity	31 July 2013
Nominal Value	ZAR 100
Redemption Value	100%
Conversion Ratio	3.64964
Conversion Price	ZAR 27.40 per Ordinary Share
Call Features	Hard Call 3 years Soft Call 130% trigger
Call Price	Principal amount plus accrued interest
Put Features	-
ISIN	XS0257978337
Listing	SGX-ST (Stock Exchange in Singapore)

Figure 2.1: Summary of the principal features of the bond offering

a security. In our case, the ISIN is XS0257978337.

Coupon

Coupons are the interest payments on the bond made by the issuer. These coupon payments can be made annually, semi-annually (as in our example) or even quarterly with the amount fixed for the life of the bond. The coupon rate payable on the bonds will be 5.70% per annum on an assumed nominal amount of ZAR 100¹, payable semiannually in arrear in equal instalments of ZAR 2.85, on 31 January and 31 July in each year (each an ‘Interest Payment Date’).

¹The actual denominations of the convertible are a minimum of ZAR 1,000,000 as stated in the prospectus

Traders often think in terms of ‘clean price’ quotation convention. The clean price is the price without the next coupon having an impact. We thus need to take into account the accrued interest on a coupon amount when the settlement date falls between two coupon dates. As with normal corporate bonds, interest accrues according to relevant market convention. In South Africa, the convention is to use an **Actual/365 days** day count convention.

Assume a settlement date of 4 June 2009. This would imply that the last coupon date is 31 January 2009 and hence 124 days of accrued interest. The accrued interest calculation is as follows:

$$\begin{aligned} \text{Accrued Interest} &= (\text{Accrued Interest days}/365) \times \text{coupon} \\ &= 124/365 \times \text{ZAR } 5.7 \\ &= \mathbf{1.93644} \end{aligned}$$

Thus, an investor buying Steinhoff 5.7% 2013 ZAR for settlement 4 June 2009 for 100% would actually pay 101.93644% after accrued interest i.e. ZAR 101.93644 (ignoring withholding tax).

Maturity

The date (31 July 2013) on which the issuer must offer to redeem the convertibles for their redemption amount (100% of the nominal value). If the value of the shares underlying the bond on this date exceeds the redemption amount, the holder will convert the bonds into shares. However, there may be cases for conversion prior to maturity as will be discussed in the coming text.

Issue Price

In our example, the issue price is 100% and hence the bonds are said to be ‘issued at par’. Issue prices below nominal value are bonds ‘issued at discount’ and prices above nominal value are ‘issued at a premium’. Market forces will cause fluctuations in the price of the convertible bond over the duration of its life.

Conversion Ratio

This is the number of ordinary shares into which each nominal value (ZAR 100) bond is convertible. The conversion ratio of 3.64964 means that each bond can be converted into 3.64964 ordinary Steinhoff (SHF) shares. The conversion ratio established at issue remains fixed throughout the life of the instrument although it

is adjusted for stock splits, special dividends and other dilutive events and any reset clauses.

Conversion Price

The price at which shares are ‘bought’ upon conversion, assuming the bond principal is used to pay for the shares. The relationship between the conversion ratio and the conversion price is given by the following:

$$\begin{aligned}\text{Conversion Price} &= (\text{Nominal Value}/\text{Conversion Ratio}) \\ &= 100/3.64964 \\ &= 27.4\end{aligned}$$

In a single currency convertible, the optimal strategy would be to convert rather than allow the bonds to mature if the share price is greater than the conversion price. In that sense, the conversion price is seen as a sort of ‘strike price’:

- A convertible is ‘in the money’ if Share Price > Conversion Price
- A convertible is ‘out of the money’ if Share Price < Conversion Price

Call Provisions

Call features of convertible bonds give the issuer the right to redeem a convertible before maturity at a predetermined call price which is usually at par plus any accrued interest. In addition, the convertibles will almost certainly contain a call protection scheme. Call protection (Hard Call) is important for investors as it guarantees the optionality of the convertible and whatever yield advantage it has over the underlying shares for a fixed period of time. The longer the call protection period, the greater the benefit for investors.

The 3 year ‘Hard’ call protection means that the bonds cannot be called under any circumstances in the first 3 years. Thereafter there is a ‘soft’ (or ‘provisional’) call period which means that the bond cannot be called unless the stock trades above the specified level of 130% of the conversion price for a certain period of time. To prevent abnormal trading patterns prompting a provisional call, the prospectus states that the stock must trade above this trigger price for at least 20 dealing days during any period of 30 consecutive dealing days ending not earlier than 7 days prior to the giving of the notice of redemption. Added to this, the call is only set in motion after a certain period of time following the issuer’s call declaration, called

the call-notice-period². This gives the investor time to consider whether to convert his bonds or not. When a bond is called, investors almost always have the right to choose whether to accept redemption price plus accrued interest or to convert into the underlying shares. Rational investors would take the value that is greater, naturally.

In reality, soft call provisions are a way for the issuer to implicitly force conversion. At high prices, the convertible behaves almost identically to the underlying share except that one does not receive dividends. At these prices the investor is almost certain to convert the bond but his decision is based on whether he chooses to surrender the coupon payments of the bond in favour of receiving dividends instead; should the dividend payments be greater. Calling the bond (and thus forcing conversion which is a much more profitable outcome for the investor) protects the issuer from continuously paying the coupon despite the bond destined to be converted in the future.

Sometimes it may be rational for an issuer to call a bond even if the conversion value is well below the call price. This may be the case if either interest rates have fallen or if the company's credit has improved placing the company in a position to refinance on more attractive terms. Calls for cash redemption are less common than calls forcing conversion.

Put Provisions

Although our example does not contain any put provision, such a provision provides additional downside protection for the holder of the convertible and hence increases the value of the bond. A put feature gives the investor the right to require the issuer to redeem the convertible. The bond is usually redeemed for cash although some convertibles give the issuer the option of delivering shares or a combination of the two. Put provisions usually occur on an exact date or a number of dates and not over a continuous period of time. This gives some certainty to the issuing company and defends their cash flow positions. It would be unreasonable for the company to continuously maintain the cash on hand in order to redeem outstanding bonds.

²For simplicity, when pricing convertible bonds in this paper, we will be doing so without any triggers, notice periods and restrictions to the exercise time unless otherwise stated.

2.1.2 The Market Valuation

This section examines the characteristics of the bond on a specific day (trade date 5 June 2009). A summary of the relevant market variables are given in Figure 2.2³.

Secondary Market Valuation	
Convertible Price	63.1288
Parity	49.27
Conversion Premium	28.128%
Absolute Premium	13.8588
Current/Running Yield	9.029%
Yield to maturity	19.292%
Ordinary Share Data	
Stock Price	13.50
Stock Volatility	36.3484%
Dividend Yield	4.37%
Breakeven	4.71 years
Fixed Income Analysis	
Credit Spread	9.5%
Bond Floor Value	50.89
Risk Premium	24.05%

Figure 2.2: Summary of the Underlying Stock - 5 June 2009

We now take a closer look at some of the terminology.

Convertible Price

In most markets around the world, convertibles are quoted using the fixed income percentage of par approach. Therefore, 63.1288 means 63.1288% of the nominal value of the bond i.e ZAR 63.1288 per bond.

Prices are also quoted as ‘clean’ prices in that they do not include accrued interest. An investor purchasing this particular bond would actually pay slightly more than the quoted ZAR 63.1288 per bond because of this effect of the next coupon. The price of the convertible bond is displayed in Figure 2.3.⁴

³Source: Bloomberg

⁴Steinhoff 5.7% 2013 ZAR Convertible Price information retrieved from Bloomberg



Figure 2.3: Steinhoff 5.7% 2013 ZAR Convertible Bond price from 8 December 2008 to 5 June 2009.

Parity

Parity is a very important term for understanding how convertibles perform. Parity is the market value of the shares into which the bond can be converted at the time. As with convertible prices, parity is also expressed on a percentage of par basis premise. 113.78 is actually a market abbreviation for 113.78% of ZAR 100.

$$\begin{aligned} \text{Parity} &= \text{Conversion Ratio} \times \text{Current Share Price} \\ &= 3.64964 \times \text{ZAR } 13.5 \\ &= \text{ZAR } 49.27 \end{aligned}$$

Parity is usually given as a percentage of par amount:

$$\begin{aligned} \text{Parity} &= \text{ZAR } 49.27 / \text{ZAR } 100 \\ &= 49.27\% \end{aligned}$$

Parity is vital because it will impact not only the price of the bond but also whether

the bond will be converted at maturity or not. The Table in Figure 2.4 below demonstrates how this works by showing the optimal actions of the bondholders at maturity for different possible scenarios of the share price at maturity⁵.

Stock Price at Maturity	Parity	Redemption Amount	Investor's Decision
ZAR 20	72.99	100	Redeem for Cash
ZAR 27.4	100	100	Indifferent between cash redemption and share conversion
ZAR 30	109.49	100	Convert into shares

Figure 2.4: What happens to Steinhoff 5.7% 2013 ZAR at Maturity ?

It is important to note that in a single currency convertible, parity moves with the underlying share price and that parity is normally less than the convertible price otherwise the bond would be converted into ordinary shares.

Premium

Premium is the difference between parity and the bond price, expressed as a percentage of parity. Premium expresses how much more an investor has to pay to control the same number of shares via a convertible rather than buying the shares outright.

$$\begin{aligned}
 \text{Premium} &= (\text{Convertible Bond Price} - \text{Parity}) / \text{Parity} \\
 &= (63.1288 - 49.27) / 49.27 \\
 &= 13.8588 / 49.27 \\
 &= 28.128\%
 \end{aligned}$$

An investor buying Steinhoff 5.7% 2013 ZAR for ZAR 63.1288 controls 3.64964 shares per bond. However, an investor buying 3.64964 shares in the market would only pay ZAR 49.27 (3.64964×13.5). The additional ZAR 13.8588 represents the premium of the convertible bond.

Premium also gives a guide as to how the convertible will perform in relation to

⁵The analysis in table 2.4 is simplified to exclude the final coupon payment. In reality, the final coupon payment is lost on conversion so an investor would rather take the cash in Scenario 2 in the table.

the underlying shares. Convertibles with low premiums should be more sensitive to movements in the underlying share price than convertibles when premium is high.

Absolute premium is the difference between the convertible price and parity. It is used in some calculations such as Breakeven.

$$\begin{aligned}\text{Absolute Premium} &= (\text{Convertible Bond Price} - \text{Parity}) \\ &= (63.1288 - 49.27) \\ &= 13.8588\end{aligned}$$

Current Yield

Investors should think of current yield (sometimes also referred to as running yield) in the same way as dividend yield on equity.

$$\begin{aligned}\text{Current Yield} &= \text{Coupon/Current Price of Convertible} \\ &= 5.7/63.1288 \\ &= 9.029\%\end{aligned}$$

The convertible's current yield will change over its life as the convertible price fluctuates over time.

Yield Advantage

The advantage gained by purchasing convertibles instead of common stock, which equals the difference between the yield of the convertible bond and ordinary shares.

$$\begin{aligned}\text{Convertible Yield Advantage} &= \text{Current Yield} - \text{Dividend Yield} \\ &= 9.029\% - 4.37\% \\ &= 4.659\%\end{aligned}$$

Yield to Maturity

The yield to maturity (YTM) is the rate of return that an investor will receive if the bond is held to maturity. The YTM of the convertible is inversely correlated to its price.

If a convertible has a put before maturity, investors will also look at yield to put (YTP).

YTM is calculated through a process of iteration via the formula below. This simplified version of the convertible bond pricing formula represents the price of a convertible bond with annual coupon payments, no broken periods and a nominal

annual compounded annually YTM:

$$\text{Convertible Price} = \sum_{i=1}^m \frac{\text{Coupon}}{(1 + YTM)^i} + \frac{\text{Redemption Value}}{(1 + YTM)^m}$$

Bond Floor

A convertible's bond floor or investment value is calculated by considering the fixed income attributes of the convertible in isolation. To calculate a bond floor it is necessary to discount to present value the coupons and redemption value of the convertible bond at a discount rate which we expect to find on comparable straight Steinhoff bonds. The calculation is the same as would be applied to determine the value of a normal fixed income security. The discount rate that is commonly used in the bond floor calculation is the sum of the risk-free rate (in our case the 10 year government bond rate) and a credit spread which exhibits the issuer's credit quality. The bond floor of a convertible should provide a price floor if the interest rates and credit perceptions of the issuer remained constant and is calculated via the formula below. The formula assumes that coupon payments are made annually, there are no broken periods and the discount rate applied is a nominal annual rate compounded annually. These assumptions can be expanded to include more complex versions of the convertible bond pricing formula.

$$\text{Bond Floor} = \sum_{i=1}^m \frac{\text{Coupon}}{(1 + d)^i} + \frac{\text{Redemption Value}}{(1 + d)^m}$$

Risk Premium

Just as premium expresses the premium investors pay to own a fixed number of shares via a convertible, risk premium refers to the premium of the convertible price over its bond floor. The difference between the convertible price and its bond floor can be viewed as the value that the market places on the option to convert to the underlying shares.

$$\begin{aligned} \text{Risk Premium} &= (\text{Convertible Price} - \text{Bond Floor}) / \text{Bond Floor} \\ &= (63.1288 - 50.89) / 50.89 \\ &= 24.05\% \end{aligned}$$

2.2 Convertibles in the Market

In Figure 2.5 below, we have a comprehensive representation of convertible bond prices in the secondary market. It is a very important diagram as it illustrates how convertibles behave in almost all possible scenarios.

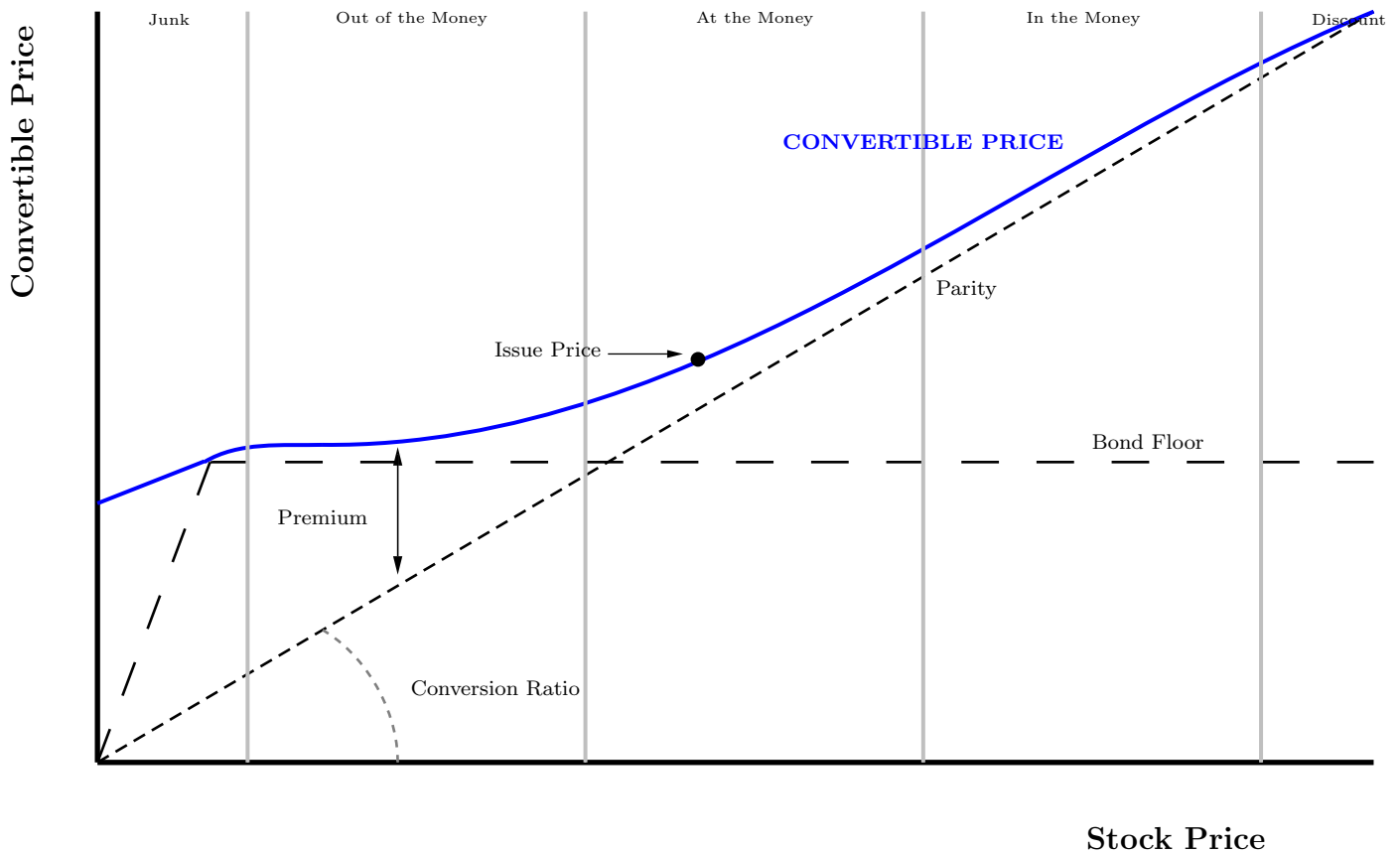


Figure 2.5: How Convertible Bonds Perform in the Secondary Market

As can be seen, we have divided the diagram into 5 main sections:

- (i.) Junk Convertible Bonds (Distressed Debt)
- (ii.) Out of the Money Convertible Bonds
- (iii.) At the Money Convertible Bonds
- (iv.) In the Money Convertible Bonds
- (v.) Discount Convertible Bonds

It is important to understand the difference between the terms ‘out of’ the money, ‘in’ the money and ‘at’ the money with regard to stock price and conversion price:

Out of the Money \Rightarrow stock price $<$ conversion price

At the Money \Rightarrow stock price = conversion price

In the Money \Rightarrow stock price $>$ conversion price

However, ‘at the money’ is usually considered to be slightly broader and includes instruments whose stock prices are fairly close to the conversion price.

We will now discuss some of the features of convertible bonds in the secondary market for each possible state that the convertible may be in. In the following few sections we also speak of Delta. Delta is a measure of the sensitivity of a convertible’s price to changes in parity - It will be explained more accurately further in the paper.

(i) Junk Convertibles

Parity = 0-40

Premium $>$ 100%

Delta N/A

Risk Premium $<$ 5%

When the underlying share price falls considerably or the issuer’s ability to meet its debt obligation is called into question, the convertible bond enters the ‘Junk’ class. The major concern for convertible investors and bondholders alike becomes the creditworthiness of the company. Junk convertible bonds usually trade with a large premium to parity, at the fixed income value for the equivalent high yield instrument. There will exist a large credit spread over the risk-free rate indicating the high probability of default. In this regard, distressed bonds are often more sensitive to the credit perceptions of the underwriter than to interest rates.

As with other non-investment grade debt, junk convertibles should be approached with much caution unless extensive and reliable research is done on this highly speculative investment. In recent years there has been an increase in the number of high yield funds that are preferring to switch in and out of convertibles as high yield alternatives. This has brought greater efficiency into convertible bond pricing. The value of high yield bonds is affected to a higher degree than investment grade bonds by the possibility of default. We can take the state of the world economy today as an example. In recessionary times, interest rates tend to tumble and this fall results in a rise in the value of investment grade bonds. However, a recession also tends to increase the possibility of default in speculative-grade bonds. Other considerations for

determining junk convertibles include accounting issues and bankruptcy laws which vary greatly across the world and must be considered before making any investment decisions. In addition, the recovery amount that will be returned to the investor in the event of default must be considered when pricing the convertible bond. This is the value of the bond when the stock price is equal to 0 and can be in the form of cash, other securities, or even physical assets.

(ii) Out of the Money Convertibles

Parity	= 40-70
Premium	> 35%
Delta	= 10% - 40%
Risk Premium	= 5% - 20%

Unlike junk convertibles, out of the money convertibles generally have issuers with some creditworthiness. The reason for them being classified as out of the money is attributed to the underperforming share price since the issue of the convertible and the share price still being below the conversion price. This results in parity being well below 100. From the definition of Parity from earlier we may also be represent it as:

$$\text{Parity} = \frac{\text{Stock Price} \times \text{Conversion Ratio}}{\text{Nominal Value}}$$

and therefore:

$$\text{Parity} = \frac{\text{Stock Price}}{\text{Conversion Price}}$$

Using this final representation of parity, we can better understand why parity is low for out the money convertibles.

We recall that the bond floor of the convertible is ZAR 50.89 using the appropriate credit spread. The convertible should never trade below this value and in fact, the convertible trades at a slight premium to its straight bond value to reflect the time value of the equity call option embedded within the convertible bond. The exact price of this convertible will be discussed in detail in the chapters dealing with the pricing of Convertible Bonds.

Contrary to what may be perceived by their name, out of the money convertibles will provide profitable returns under four different scenarios:

- (1.) If the underlying share price rises somewhat
- (2.) If interest rates fall. This will cause bond prices to rally, thus raising the bond floor and hence increasing the price of the convertible bond.
- (3.) If the market's perception of the credit quality of the company improves. This reduces the credit spread of the bond and raising the bond floor. As before, this increases the price of the convertible.
- (4.) If the convertible's premium to its bond floor increases as a result of a general richening of convertible valuations.

Out of the money convertible bonds have become progressively more appealing to fixed income funds who switch their straight debt positions into these hybrid securities in order to exploit any sizeable rally in the underlying share price.

(iii) At the Money Convertibles

Parity	= 70 - 130
Premium	= 10% - 35%
Delta	= 40% - 80%
Risk Premium	= 20% - 40%

At the money convertible bonds have a higher equity sensitivity (delta) when the underlying share price rises than when the share price falls. From Figure 2.5 we see that as parity increases, the convertible price line is steeper, whereas when parity decreases, the convertible price line tends to be flatter. This means that the convertible will participate in more equity upside than downside.

At the money convertible bonds are thus often regarded as 'balanced' convertibles, as they tend to have asymmetric risk/return profiles (upside participation with downside protection versus stock price movements). We can demonstrate this risk-adjusted profile by examining how Steinhoff 5.7% 2013 ZAR performs under different stock price scenarios when parity starts at 100. The convertible values calculated in figure 2.6 below are derived from a binomial tree valuation model which is discussed further in the paper. Naturally, as the stock price moves, the bond price will move.

When considering at the money profile convertible bonds with a parity of 100, Steinhoff 5.7% 2013 ZAR participates in 66.28% of the 20% increase in the stock price yet only suffers a 58.86% decrease in the equivalent 20% decrease. This is the

Stock Price	Parity	Theoretical Convertible Value	Participation with Stock move
ZAR 21.92	80	105.22	-58.86%
ZAR 27.4	100	119.26	0%
ZAR 32.88	120	135.07	66.28%

Figure 2.6: Asymmetric Risk/Return Profile of ‘At the Money’ Steinhoff 5.7% 2013 ZAR

kind of asymmetric risk/return profile that knowledgeable convertible investors and hedge funds look for. Convertible bonds are usually issued with an at the money profile and parity in the region between 70 and 90.

(iv) In the Money Convertibles

Parity	> 130
Premium	= 0% - 10%
Delta	> 80%
Risk Premium	> 40%

The convertible is classed as ‘In the Money’ as the stock price begins to rise above the conversion price of ZAR 27.40 for Steinhoff 5.7% 2013 ZAR. Deep in the money convertibles will almost surely be converted at maturity (unless they are called prior to this). As these convertibles move further away from the bond floor, the premium that investors will pay over parity begins to decrease. There would only be 2 cases when in the money convertibles trade well above parity. Firstly, if the bond has a large income advantage over the underlying shares. Alternatively, if there is still a considerable amount of time until the first call date or maturity (Time Value of the option).

The value of the convertible’s inherent put is the cash redemption amount at maturity. The value of this put naturally declines as parity rises and increases in significance with a longer time until maturity. For example, a put at 100 is worth much more when parity is 100 than when parity is 150. When a convertible is deep in the money the put is effectively worthless and thus an investor should pay as a premium only the present value of the expected income advantage over the underlying shares.

Most convertibles have a call feature as part of the convertible which allow the issuer of the bond to redeem the bond at some point prior to the maturity date. Indeed investors will convert rather than accept the cash redemption for convertibles which are In the Money. Figure 2.7 (a) below demonstrates how premium (the difference between the convertible price and parity) is retained if long call protection exists within a convertible whereas Figure 2.7 (b) shows how it collapses when considering a convertible bond with no call protection.

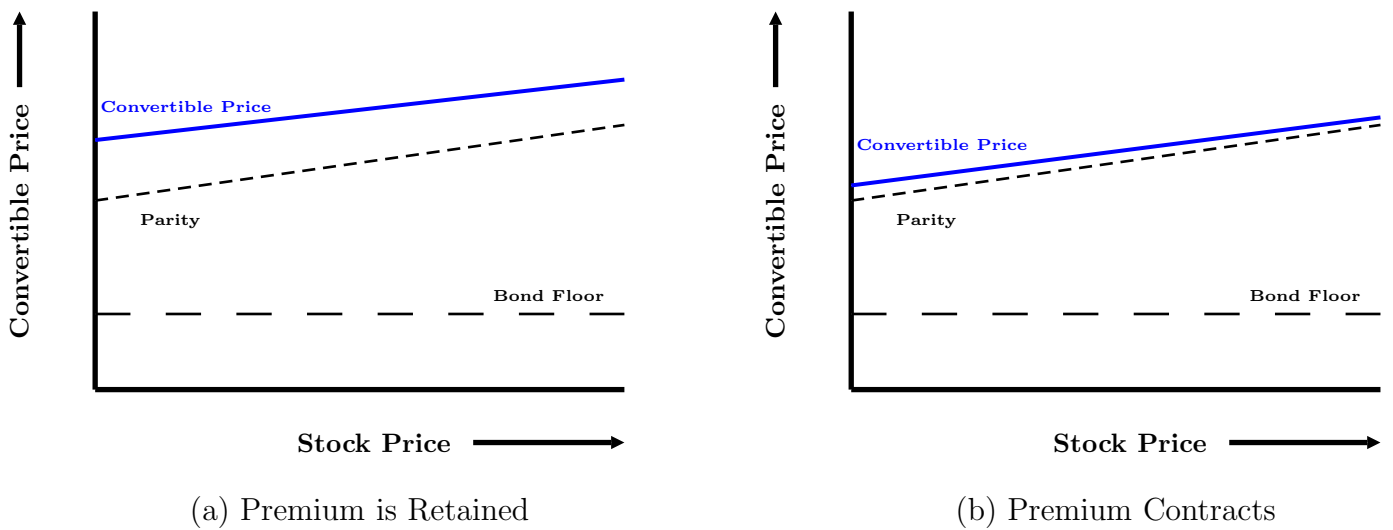


Figure 2.7: The effect on premium with and without call protection

Investors will pay more for a bond with longer hard call protection because the optionality of the bond lasts longer and the income advantage of the convertible over the underlying shares lasts longer with this extended call protection. Figure 2.8 on page 22 exhibits the significant effect hard call protection has on the theoretical value of a convertible bond.

Convertibles with reasonable call protection remaining and a significant yield advantage over the underlying shares are extremely attractive to equity investors as they offer upside participation with downside protection and yield enhancement.

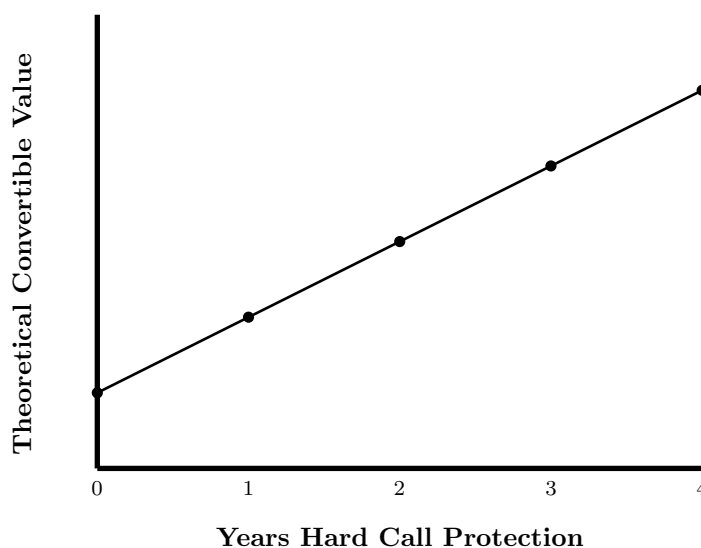


Figure 2.8: Theoretical convertible bond value rises as call protection period is extended

(v) Discount Convertibles

Parity	High
Premium	$\leq 0\%$
Delta	100% (can even be $> 100\%$)
Risk Premium	$> 40\%$

A discount convertible is one in which parity is greater than the convertible price. For example, if parity rose to 150 for Steinhoff 5.7% 2013 ZAR and the convertible had a market value of 148, the bond would be trading at a 1.35% discount to parity. It may seem strange for a convertible to trade at a discount to parity as this gives rise to arbitrage profits. Such a risk free profit could be made by buying the convertible in the market, converting the shares immediately and selling those shares to make the profit. In theory this would work however in the real world there are certain obstacles that are likely to prevent acquiring such effortless profits.

For example, in certain Asian convertible markets (such as Singapore where Steinhoff 5.7% 2013 ZAR is listed) conversion into the underlying shares is not so straightforward. In some cases investors must first convert into ‘Entitlement Certificates’ which are then convertible into the underlying shares only on set dates throughout the year. In other markets there is a delay between lodging bonds for conversion and receiving shares. Even this barrier would not cause bonds to trade at a discount

if sufficient stock to borrow were available. Investors in Steinhoff 5.7% 2013 ZAR in the above scenario would simply sell short 100% of the conversion ratio, buy the bond for 148, then lodge the bonds for conversion in order to cover the short position, resulting in a risk free profit. Even though there is a time delay in receiving shares the investor is protected through his short position.

In many convertible markets however, short selling of stock is either inadequate or slowly becoming non-existent, and investors facing lengthy conversion procedures are exposed to adverse movements in the share price with no means of hedging themselves. This may result in the bond trading at a slight discount. In other situations, lack of liquidity in the underlying shares can give the appearance of a discount when none really exists. The time and size of price quotations can assist in differentiating between this and a true discount bond.

In reality, discount bonds are extremely rare in developed markets except where a callable convertible trades at 'net parity' (parity less accrued interest) to reflect the loss of accrued interest upon conversion.

Chapter 3

Convertible Bond Pricing Models

“The purpose of science is not to analyze or describe but to make useful models of the world. A model is useful if it allows us to get use out of it.” Edward de Bono

Given the many challenges associated with the valuation of convertibles and the importance of the convertible debt market size, the valuation of convertible bonds has had extensive research time awarded it.

Pricing models range from the simple to the extremely complicated and differ even more so when market realities such as stochastic interest rates, credit spreads, issuer calls, investor puts and conversion rate resets are taken into account. Tree structure models such as binomial and trinomial trees have become the most popular pricing approaches throughout the world as they are relatively easy to understand and can deal with the most common features found in convertibles. However, even this approach may be problematic as trees may not cope competently with discrete events such as stock dividends or reset clauses.

Most of the early research on convertible bond pricing was based on the firm value approach which emanated from Black and Scholes [1973] and Merton [1974]. Securities were thought of as derivatives on the firm value and thus the value of the firm’s assets were the main risk factor. The pricing partial differential equation (PDE) was solved by Ingersoll [1977] and it was proven that a non-callable convertible bond with no dividends is akin to a portfolio of a straight bond and a European call option.

Firm value models, whilst theoretically sound, are difficult to use in practice because of the need to estimate the uncertain firm value as well as its volatility. Hence, different pricing approaches were needed to more accurately price these hybrid instruments. One needed to consider the four stochastic factors that affect a convertible's price before coming up with a more advanced approach. These factors are: stock price, the probability of default of the bond, volatility of the stock price and stochasticity of interest rates. Surprisingly, it is known that the stochastic nature of interest rates is of less importance than the other factors and hence the assumption of a flat yield curve will be made (unless otherwise stated) for the sake of simplicity. For this reason, McConnell and Schwartz [1986] introduced a single factor equity-based model which most of the successive literature on convertible pricing is based.

A comprehensive group of in-use models follow the structure where the convertible is split into a bond component (which is subject to credit risk) and an equity portion (which is deemed risk-free). Many of these are in the form of different binomial tree models such as Goldman Sachs [1994] which uses a Cox, Ross and Rubenstein [1979] stock price binomial tree, Hung and Wang [2002] and Hull [2005] to name a few. In the same trend, Tsiveriotis and Fernandes [1998] proposed splitting the convertible into equity and bond components, each discounted at a different rate.

More recent advances in credit risk literature and the introduction of the 'reduced form' approach provides for more consistent handling of default risk in equity based models by allowing the stock price to jump downwards at the moment of default. This approach allowed for the introduction of two-factor models to be explored. Takahashi et al. [2001] were able to take advantage of this approach to price a convertible bond with default risk.

Ayache et al. [2002] introduce a single-factor model that divides the convertible into a bond and equity component much in the same way to Tsiveriotis and Fernandes [1998], but allows for the stock price to jump partially upon default, and the hazard rate to vary as a function of the stock price. They consider both, the market value and the face value recovery assumptions. Other proponents of the proportional recovery (in this case, of bond face value) include Anderson and Buffum [2002].

The aforementioned models represent a few of the many available pricing models

in use. In this chapter, we discuss several of these and other models in far greater detail. We will be considering convertibles that offer uniform coupon payments at regular intervals that if not put by the investor or called by the issuer, are convertible into a predetermined number of shares at the discretion of the investor. Naturally, the price which the convertible is puttable by the investor must be less than the price that the issuer may call the bond at for feasible pricing analysis. Along with our assumption of flat yield curves we make one final assumption: that both parties to the convertible bond always act in a rational, optimal manner.

3.1 Elementary Models

3.1.1 Component Model

The component model, or synthetic model as it is sometimes referred to, is a popular method for pricing convertibles in the market due mainly to its simplicity. The convertible is divided into a straight bond component, denoted by B_t , and a call option component, denoted by K_t , on the conversion price $\gamma_t S_t$ with a strike price X_t that is $X_t = B_t$. V_t can be defined as

$$V_t = \max(B_t, S_t) = B_t + \max(S_t - B_t, 0) \quad (3.1)$$

We use standard formulae to calculate the fair value of the straight bond component and call option component thus making the model straightforward to implement and solve for V_t .

The fair value of the straight bond with face value N , continuously compounded risk-free interest rate from time t to time t_i , r_{t,t_i} , coupon rate c and a credit spread ξ_t is given by:

$$B_t = Nc \sum_{i=1}^n e^{-(r_{t,t_i} + \xi_{t_i})(t_i - t)} + N e^{-(r_{t,T} + \xi_t)(T - t)} \quad (3.2)$$

where t_i represents the coupon payment dates. The credit spread is a very important measure and is defined by Landskroner and Raviv [2003] as the yield differential between non-treasury securities and treasury securities, that are identical in all aspects other than the risk of default. This differential is the credit risk premium. We also assume that the underlying stock price process follows Geometric Brownian motion under the risk-neutral measure

$$dS_t = (r_{t,T} - q_S)S_t dt + \sigma_S S_t dW_t^S \quad (3.3)$$

where q_S represents the continuously compounded dividend yield and W_t^S is the Wiener process. Using the Black-Scholes pricing formula we attain the value of the option:

$$K_t = e^{-q_S(T-t)} S_t \mathcal{N}(d_1) - X_t e^{-r_{t,T}(T-t)} \mathcal{N}(d_2) \quad (3.4)$$

where

$$d_{1,2} = \frac{\ln \frac{S_t}{X_t} + (r_{t,T} - q_S \pm \frac{\sigma_{S,t}^2}{2})(T-t)}{\sigma_{S,t} \sqrt{T-t}}$$

By adding the straight bond component and the call option on the bond we are left with the fair value of the convertible, V_t :

$$V_t = B_t + K_t$$

The component model is simple to implement however it does contain certain drawbacks. Firstly, splitting the convertible into components as we have done relies on certain restrictive assumptions such as the absence of embedded options. Callability by the issuer and investor putability are examples of convertible bond features that cannot be considered under the above separation. In addition, the Black-Scholes closed-form solution for the option part of the convertible will only work for plain-vanilla European style bonds whilst convertible bonds are typically of American type. Lastly, unlike call options where strike prices are known in advance, convertibles have stochastic strike prices as the straight value of the bond (which is the future strike price) depend on the future development of interest rates and the future credit spread.

3.2 Firm Value Models

Ingersoll [1977] was amongst the first to value convertible bonds based on Black-Scholes literature. In his paper, Ingersoll develops arbitrage arguments to derive several results concerning the optimal conversion strategy for the holder of the bond as well as the best call strategy for the issuer. He also produces analytical solutions

for convertibles in a variety of special cases. Ingersoll is able to solve analytically for the price of the convertible because of his assumption of no dividends and no coupons. Brennan and Schwartz [1977] consider the valuation of convertible bonds within the same framework as Ingersoll. Although several of the results are the same in both papers, the major difference is that Ingersoll concentrates on deriving closed form solutions for the bond value in a collection of special cases whereas Brennan and Schwartz offer a general algorithm for determining the value of the convertible. It was actually Brennan and Schwartz that found the additional factor representing stochastic interest rates had little significant impact on the price of the convertible bond.

3.2.1 Brennan and Schwartz (1977)

In the paper by Brennan and Schwartz [1977], the authors use finite difference methods to solve the partial differential equation for the price of a convertible bond with call provisions, coupons and dividends. The model also permits the possibility that the firm will default on the bond by bankruptcy either prior to or at maturity. In the paper, it is also assumed that the firm's outstanding securities consist solely of common stock and convertible securities (although even this assumption could be relaxed by modification of the boundary conditions) and that a constant risk-free interest rate is used.

According to Brennan and Schwartz, a convertible bond can be valued only if the call strategy of the bond issuer and the conversion strategy of the investor can be determined. Naturally, it is assumed that both the issuing firm and the investor pursue an optimal strategy. This gives rise to two important lemmas. The first gives a stronger condition on the value of the bond while the second represents the firm's optimal call strategy which minimizes the value of the convertible bond.

- (1.) It will never be optimal to convert an uncalled convertible bond except immediately prior either to a dividend date or to an adverse change in the conversion terms, or at maturity.
- (2.) The firm's optimal call strategy is to call the bond as soon as its value if it is not called is equal to the call price.

Together, these two lemmas give additional boundary conditions on the value of the bond and help in the pricing process. Finally, since there exists no known analytical

solution to the differential equation that we would like to solve, numerical methods are then used to solve the equation.

The papers of Ingersoll and Brennan and Schwartz assume the value of the firm is composed of equity and convertible bonds only and they model the value of the firm as geometric Brownian motion. Although firm value models are relatively easy to implement, especially in times of financial distress, they have one major flaw - the value of the firm is not observable unlike the value of the firm's equity which is traded on the market. Subsequent literature on the topic focuses on the convertible bond being a security contingent on the underlying equity (and certain more complicated models based on interest rates) rather than firm value.

3.3 Equity Value Models

3.3.1 Goldman Sachs (1994)

In their Quantitative Strategies Research Notes, Goldman Sachs [1994] consider the issue of which discount rate to use when valuing a convertible bond and make use of the theory of options to value and hedge convertibles. It is assumed that the only source of uncertainty is the price of the underlying share which Goldman Sachs believes captures the major source of the options value. Another significant assumption of the model is that all future interest rates (the riskless rate, stock loan rate and the issuer's credit spread) along with stock volatility are known with certainty. With these assumptions, we can hedge a convertible by shorting the underlying stock to create an instantaneously riskless hedge. You can value the bond by calculating its expected value over all future stock price scenarios, provided they are consistent with the known forward prices of the stock and its volatility.

A Cox, Ross and Rubenstein univariate binomial tree is built in formulating the Black-Scholes equation. The up and down jump size that the stock price can take at each discrete time step is given by

$$u = e^{\sigma s \sqrt{\Delta t}} \quad \text{and} \quad d = \frac{1}{u}$$

In order to calculate the risk-neutral probability of an up movement of the stock price, we define a new variable, which is the value of a risk free security at the end

of the time interval

$$a = e^{r_{t,t+\Delta t} - qS}$$

The risk-neutral probability of the stock price increasing at the next time step, i.e. an upward movement is given by p_u while the probability of a downward movement is p_d :

$$p_u = \frac{a - d}{u - d} \quad \text{and} \quad p_d = 1 - p_u = \frac{u - a}{u - d}$$

Figure 3.1 below shows one period of the stock tree we have considered. The stock starts out at price S . After a short period Δt , the stock can move to either S_u with a probability p_u or S_d with a probability p_d . The difference between S_u and S_d is determined by the stock volatility. The expected value of S_u and S_d is the stock's forward price after each time level.

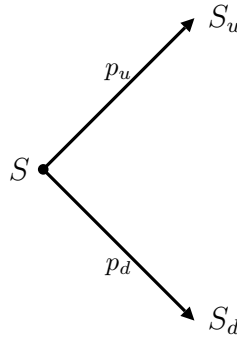


Figure 3.1: One-period stock tree

We start by calculating the value of the convertible bond at maturity. It is the greater of its fixed redemption value and its conversion value represented by:

$$V_T = \max(\kappa N, \gamma_T S_T) \tag{3.5}$$

This terminal condition needs to be considered at all end nodes of the tree.

From the endnodes, we begin to move backwards in time down the tree, one level at a time to calculate the convertible's value at each interior node. Figure 3.2 shows one period in the convertible tree where at each point, there are 5 possibilities:

- (1.) The convertible is neither called nor converted and continues as a convertible bond.

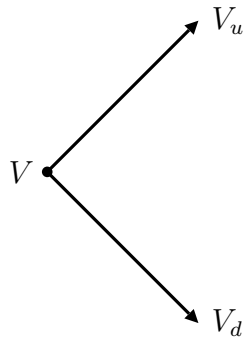


Figure 3.2: One-period convertible tree

- (2.) The convertible is converted by the holder of the bond.
- (3.) The convertible bond is called by the issuer.
- (4.) The convertible bond is called by the issuer and immediately converted by the holder. This is referred to as a forced conversion.
- (5.) The convertible is put by the holder of the bond.

In order to determine what course of action will be taken, one needs to calculate the holding value of the convertible bond. The holding value of the convertible bond at the start of the period is the expected present value of V_u and V_d plus the present value of any convertible coupons paid during this period, discounted at the credit-adjusted discount rate. The holding value at time t , H_t , represents the expected value the investor can realise by waiting for one further time period without converting, assuming no provisions are applicable during that time. The following shows how to calculate the value of the convertible at the current node for all combinations of provisions that may be in effect¹:

- **No call or put provisions**

The investor may either hold the convertible bond for one more period or convert it into stock. Therefore:

$$V_t = \max(H_t, \gamma_t S_t)$$

¹Note that put and call provisions allow the investor to receive accrued interest. An investor who converts will forfeit the accrued interest

- **Convertible is puttable at price P_t**

The investor can hold the bond, convert it according to the bond agreement or put the bond for amount P_t . Therefore:

$$V_t = \max(H_t, \gamma_t S_t, P_t)$$

- **Convertible is puttable at P_t and callable at C_t**

The issuer of the bond will only call the bond if the call price is less than the holding value. If the bond is called, the investor still has the option of whether to convert it into stock, put the bond to the issuer or accept the issuer's call - naturally whichever is the greatest. Therefore in this scenario:

$$V_t = \max(\gamma_t S_t, P_t, \min(H_t, C_t))$$

It is important to note that if the value of the convertible at any node results from the bond being put, then the conversion probability is set to zero at that node. Similarly, if the value at the node results from conversion, then the conversion probability is set to one.

All that is left to discuss is the credit-adjusted discount rate that is used to calculate the present values throughout the convertible tree. For an ordinary option that exercises into stock, the appropriate discount rate is the risk free rate r . However, convertible bonds pay coupons and return principal which are both subject to default so the risk free rate is not entirely appropriate for discounting these payoffs. At the extreme cases of very high or very low stock prices the problem is not too difficult to handle. When the stock is far above the conversion price, the investor is certain to obtain stock with no default risk hence the appropriate discount rate is the riskless rate, r . Alternatively, if the next stock price at the next node is deep out of the money, where eventual conversion is overwhelmingly unlikely, the appropriate discount rate is obtained by adding the issuer's credit spread ξ to the riskless rate r .

At intermediate stock price levels, we let p be the probability at a given node that the convertible will convert to stock in the future, hence $(1 - p)$ is the probability that it will remain a coupon-bearing bond. Our goal of a credit-adjusted discount rate y with a weighting factor p is given by:

$$y = p \times r + (1 - p) \times (r + \xi) \tag{3.6}$$

So, starting at maturity, where the value of the convertible bond is known with certainty and working recursively through the tree through each node, the value for the starting node is then calculated. This gives the current value for the convertible bond.

There are certain subtleties of convertibles too. One, which we briefly mention here but which this paper ignores, is that a convertible bond is really a compound derivative claim on the company's assets. This may cause problems with the distribution of future stock prices as they may not be lognormally distributed. Another assumption we have made is that credit spreads remain constant throughout the convertible's lifetime. It is feasible to modify the model to account for these feature but the resultant model would take longer to build and run on a computer. The model of Goldman Sachs [1994] disregards these complexities in the favour of having a relatively straightforward model that can be used with easily available security prices, interest rates and credit ratings.

The one major drawback of the Goldman Sachs model is that the methodology used seems somewhat incoherent i.e. the investor is assumed to receive stock through conversion even in the event of default but the stock is not explicitly modeled as having zero value in the case of default. Moreover, the intensity rate is not introduced into the drift of the stock as one would expect. Finally the model makes no mention of any recovery in the event of default on the debt.

3.3.2 Tsiveriotis and Fernandes (1998)

The approach used by Goldman Sachs [1994] is formalised by Tsiveriotis and Fernandes [1998]. The Tsiveriotis and Fernandes framework is a very popular choice of model amongst practitioners for pricing convertible bonds due to its relative simplicity and its ability to incorporate the fundamental traits of convertible bonds that have limited market data. In this section we give a detailed overview of the Tsiveriotis and Fernandes approach which is able to provide accurate and practical valuation of convertibles consistent with the market values.

The value of a convertible bond, u is governed by the Black-Scholes equation as

$$\frac{\partial u}{\partial t} + r_g S \frac{\partial u}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} - (r + r_c)u + f(t) = 0 \quad (3.7)$$

where S , r , r_g and r_c represent the underlying stock, risk-free rate, growth rate of the stock and the credit spread respectively. $f(t)$ represents the coupon payments c_i made at times t_i .

Equation 3.7 is sufficient to value any convertible bond with a set of conversion, call and put conditions. However, it does not account for the different credit quality of the sources of value for the convertible and therefore a more general method is needed to include the issuer's credit spread into the valuation model.

We define the 'cash only portion of the convertible bond' (COCB) as only the cash flows that an optimally behaving holder of the corresponding convertible would receive. By definition, this value denoted v , is determined by the behaviour of u , S and t . Therefore, like u , the COCB price v should also follow the Black-Scholes equation and should involve the issuer's credit spread in some way as we are dealing with 'risky' cash payments only. Naturally, $(u - v)$ represents the equity portion of the convertible and may be discounted using the risk-free rate. This leads to a new formulation of the convertible bond valuation as a system of two coupled Black-Scholes equations:

$$\frac{\partial u}{\partial t} + r_g S \frac{\partial u}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} - r(u - v) - (r + r_c)v + f(t) = 0 \quad (3.8)$$

$$\frac{\partial v}{\partial t} + r_g S \frac{\partial v}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} - (r + r_c)v + f(t) = 0 \quad (3.9)$$

As expected, equations 3.8 and 3.9 differ only in their discounting terms $r(u - v) + (r + r_c)v$ and $(r + r_c)v$ which reflect the different risk treatment of cash payments and equity upside.²

Equations 3.8 and 3.9 are parabolic partial differential equations in inverse time $\tau = T - t$, where T is the tenor of the convertible bond. This means that given any set of final conditions such as payoff expirations and any set of boundary conditions represented by certain early exercise features, the system of PDE's (3.8 and 3.9) is guaranteed to have a solution.

The coupling of the 3.8 and 3.9 is due to the fact that the convertible bond valuation problem is an American-style derivative where early call, put and conversion

²In this case, r_c represents the observable credit spread implied by the straight bonds of the same issuer for similar maturities with the convertible.

is possible. The stock prices at which these early termination events occur are the free boundaries for equations 3.8 and 3.9 and the equations are not valid beyond those boundaries.

Together with u and v , the free boundaries form the unknowns to the problem for equation 3.8. Therefore, early exercise conditions involving the u , S and t define the location of the free boundaries at each time which subsequently define the boundary conditions for equation 3.9. Therefore, it is through their common free boundaries that equations 3.8 and 3.9 are coupled and hence solved simultaneously.

The above explanation is shown below on a standard convertible bond maturing at time T . We define the call price of the bond to be B_c at time T_c and the bond is puttable for B_p after T_p . Final conditions at expiration are given by:

$$u(S, T) = \begin{cases} \gamma S & \text{if } S \geq B/\gamma \\ 0 & \text{elsewhere} \end{cases} \quad (3.10)$$

$$v(S, T) = \begin{cases} 0 & \text{if } S \geq B/\gamma \\ B & \text{elsewhere} \end{cases} \quad (3.11)$$

where γ is the conversion ratio. Upside constraints due to conversion are:

$$\begin{aligned} u &\geq \gamma S && \text{for } t \in [0, T] \\ v = 0 & \text{ if } u \leq \gamma S && \text{for } t \in [0, T] \end{aligned} \quad (3.12)$$

Equations 3.13 below represent the upside constraints due to callability and equations 3.14 represent the downside constraints due to putability by the holder of the convertible:

$$\begin{aligned} u &\leq \max(B_c, \gamma S) && \text{for } t \in [T_c, T] \\ v = 0 & \text{ if } u \geq B_c && \text{for } t \in [T_c, T] \end{aligned} \quad (3.13)$$

$$\begin{aligned} u &\geq B_p && \text{for } t \in [T_p, T] \\ v = B_p & \text{ if } u \leq B_p && \text{for } t \in [T_p, T] \end{aligned} \quad (3.14)$$

We see that the COCB is non-zero in these equations only when a cash payment takes place such as the cases of cash redemption at maturity in equation 3.11 and when the bond is put in 3.14.

Contractual features such as callability and putability make it impossible to price

modern convertibles by some closed formula or reasonable number of partial differential equations and hence Tsiveriotis and Fernandes [1998] approximate the solution by using the explicit finite difference method. Hull [2005] gives a simple description of this model in the binomial context by using a Cox, Ross and Rubenstein binomial tree. In their paper ‘Pricing Convertible Bonds with Default Risk: A Duffie-Singleton Approach’, Takahashi et al. [2001] also test this model empirically by using Japanese convertible bonds prices. A one-dimensional tree is built containing both the ‘cash only’ and the ‘share only’ components discounted at the credit spread adjusted rate and the risk free rate respectively. The sum of the two components at each node is equal to the value of the convertible bond at that node. This single factor model based on share prices ignores other random factors affecting the convertibles price such as interest rates and volatility however this simplified model is extremely useful as it allows one to focus on the effect of the credit risk on the bond price. We provide the methods for the numerical solution of the coupled PDEs.

First, we let $x = \ln S$ and $\tau = T - t$. These transformations simplify equations 3.8 and 3.9 to the diffusion equations:

$$\frac{\partial u}{\partial \tau} = \left(r_g - \frac{\sigma^2}{2}\right) \frac{\partial u}{\partial S} + \frac{1}{2} \sigma^2 \frac{\partial^2 u}{\partial x^2} - r(u - v) - (r + r_c)v + f(t) \quad (3.15)$$

$$\frac{\partial v}{\partial \tau} = \left(r_g - \frac{\sigma^2}{2}\right) \frac{\partial v}{\partial S} + \frac{1}{2} \sigma^2 \frac{\partial^2 v}{\partial x^2} - (r + r_c)v + f(t) \quad (3.16)$$

The solution $u(x, \tau)$ and $v(x, \tau)$ is then discretised on a set of equally spaced grid points $(x_i, i = 1, \dots, N)$, spaced a distance of h from each other. The unknowns become two N -dimensional vectors $\mathbf{u}(\tau)$ and $\mathbf{v}(\tau)$. Only finite time steps are used and therefore $\mathbf{u}^k = \mathbf{u}(k\Delta\tau)$ and $\mathbf{v}^k = \mathbf{v}(k\Delta\tau)$.

Using explicit time stepping, the partial differential equations are transformed to difference equations as:

$$\begin{aligned} \frac{u_i^{k+1} - u_i^k}{\Delta\tau} &= \left(r_g - \frac{\sigma^2}{2}\right) \frac{u_{i+1}^k - u_{i-1}^k}{2h} + \frac{1}{2} \sigma^2 \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{h^2} \\ &\quad - r(u_i^k - v_i^k) - (r + r_c)v_i^k + f(k\Delta\tau) \end{aligned} \quad (3.17)$$

$$\frac{v_i^{k+1} - v_i^k}{\Delta\tau} = \left(r_g - \frac{\sigma^2}{2}\right) \frac{v_{i+1}^k - v_{i-1}^k}{2h} + \frac{1}{2} \sigma^2 \frac{v_{i+1}^k - 2v_i^k + v_{i-1}^k}{h^2}$$

$$-(r + r_c)v_i^k + f(k\Delta\tau) \quad (3.18)$$

The solution proceeds as follows. At time step $k + 1$ ($\tau = (k + 1)\Delta\tau$), start with $(\mathbf{u}^k, \mathbf{v}^k)$. Using 3.17, we can solve for \mathbf{u}^{k+1} and apply the relevant conditions of 3.12, 3.13 and 3.14. Similarly, compute \mathbf{v}^{k+1} from equation 3.9 and again the relevant conditions of 3.12, 3.13 and 3.14 are applied to \mathbf{v}^{k+1} . This completes the method for each time step. The final conditions from 3.10 and 3.11 are used to calculate the starting values $(\mathbf{u}^0, \mathbf{v}^0)$.

Throughout, discrete coupon payments are added to the solution. In particular, each time that we are within $\Delta\tau$ from a coupon payment date τ_c (i.e. $k\Delta\tau < \tau_c \leq (k + 1)\Delta\tau$), the time step is temporarily adjusted to $\Delta\tau' = \tau_c - k\Delta\tau$ so that the algorithm has a step exactly on the coupon payment date.

The main aim of this single factor model is to incorporate issuer's credit spread into the pricing of convertible bonds which needs to be estimated in some way as it cannot be measured directly from market data. It is a better model than the one proposed by Goldman Sachs in a sense that the correct weighting rather than a probability weighting is used to discount the different components of the convertible. However, the model is still handicapped by some of the same theoretical conflicts that affect the Goldman Sachs model. Some of the bigger drawbacks are that the intensity rate does not enter the drift on the equity process, the share price is not explicitly modeled as jumping to zero in the event of default and any fraction of a bond recovery is omitted.

3.3.3 Ho and Pfeffer (1996)

In this article, Ho and Pfeffer [1996] present a valuation model that prices the convertible with all the important bond features, calculates the effective combination of stocks and bonds needed to hedge the convertible and calculates the value of the features such as callability and putability. The model is based on a two-factor, arbitrage-free framework that uses lattice methodologies to solve for the pricing equation numerically where the two factors are the stock price process as well as the interest rate risk. Unlike some other two factor models such as that of Brennan and Schwartz [1980] (which does not take the initial market yield curve as given

and therefore may not ensure that the bond valuation is consistent with the market time value of money), the Ho and Pfeffer model can be calibrated to the initial term structure. The interest rate factor is modeled using the Ho and Lee [1986] model which is the first arbitrage-free interest rate model widely employed in derivatives valuation.

Ho and Pfeffer use a two dimensional binomial tree as their pricing algorithm. Because this structure incorporates both stock and interest rate risks, the model can take into account the embedded stock and interest rate options among the convertible's special features. And as mentioned above, the model is arbitrage free in that it prices the convertible bond relative to the corporate bond market as well as to the stock price. Thus, the fair convertible bond price is defined as the cost of replicating the convertible bond by having a combination of shares and straight bonds from the market and following the basic principle of the Black-Scholes model of relative valuation. Relative valuation does not necessarily believe that the market prices these shares and bonds perfectly. The significance of this arbitrage-free, relative valuation model is its ability to isolate the analysis of the value of the underlying securities from the contingent claim, which in this case will enable one to value a convertible.

In the paper, all cash flows are discounted at the rate including the credit spread (a constant credit spread is assumed) which implies the share price drops to zero in the event that the issuer defaults on the bond. Hence the intensity rate enters into the drift on the equity. However, this is not stated in the paper but rather a result of studying the paper. Additionally, any recovery on the bond in the event of default is absent from the model. The assumption of a fixed credit spread is a feature that is also evident in the papers of Goldman Sachs [1994] and Tsiveriotis and Fernandes [1998] regardless of the level of the share price. However, through empirical observation, it is known that financial practitioners allow the credit spread to vary inversely with the level of equity.

3.4 A Pricing Model Having Other Models as Special Cases

3.4.1 The ‘Reduced Form’ Model

Jarrow and Turnbull [1995] spearheaded a new methodology for pricing and hedging derivative securities involving credit risk. Their model uses the firm credit spread and the term structure of interest rates as inputs.

The default event is modeled as a point process with one jump to default in the period $u \in [0, \tau]$ where the default event occurs at the stopping time τ .

$$N(u) = \mathbf{1}_{\{\tau \leq u\}} \quad (3.19)$$

A compensating intensity process $\lambda(u)$ (also called the arrival rate or hazard rate process) drives $N(u)$ such that

$$N(u) - \int_0^u \lambda(s) ds \quad (3.20)$$

is a martingale. Let $N(u) = \sum_{n \geq 1} \mathbf{1}_{\{\tau \leq u\}}$ and let the compensated process be $N(u) - \lambda(u)$ with the arrival rate λ constant, then $N(u)$ is a standard Poisson process. Therefore, the probability of i jumps occurring between time t and time u for any $u, t \in [0, \tau]$ such that $u > t$ is:

$$\mathbb{P}[N(u) - N(t) = i] = \frac{(\int_t^u \lambda(s) ds)^i}{i!} e^{-\int_t^u \lambda(s) ds}, \quad \forall i \in \mathbb{N}^+ \quad (3.21)$$

Only the first jump in the time interval $[t, u]$ is relevant as the jump is into bankruptcy and therefore we are interested only in the case when $i = 0$. The survival probability at time u (i.e. no bankruptcy up to time u) conditional on survival up to time t is given by:

$$\mathbb{P}[N(u) - N(t) = 0] = e^{-\int_t^u \lambda(s) ds} \quad (3.22)$$

Over a small time horizon the probability of default during the interval $[t, t + \Delta t]$ is, to a first order approximation, proportional to the intensity rate $\lambda(t)$,

$$\mathbb{P}[N(u) - N(t) = 1] = \lambda(t) \Delta t \quad (3.23)$$

3.4.2 Equity, Interest Rate and Intensity Rate Processes

A stochastic process is specified under the risk-neutral measure \mathbb{Q} for the equity price and the interest rate and possibly even the intensity rate. In this paper however, we will not deal with stochastic intensity rates. However, the exact form of the interest rate is defined in a way so as to allow different models to be included as special cases.

Equity Process

Under the risk neutral measure \mathbb{Q} the equity process is represented by the following stochastic differential equation for our nested convertible bond model

$$dS(t) = (r(t) + \lambda(t) - q(t))S(t)dt + \sigma_1 S(t)dW_1(t) - S(t-)dN(t) \quad (3.24)$$

Equation 3.24 is almost identical to equation 3.3 except for its behaviour when default occurs. For this reason, the stock price jumps to zero by subtracting the stock price immediately prior to default $S(t-)$ as represented by the last term in equation 3.24. Conditional on default not having occurred the stock has the usual solution except the return is increased by the intensity rate $\lambda(t)$ to compensate for the risk of default thereby giving

$$S(t) = S(0)e^{\int_0^t (r(s) + \lambda(s) - q(s))ds - \frac{1}{2}\sigma_1^2 t + \sigma_1 W_1(t)} \quad (3.25)$$

The ‘reduced form’ model as well as the other models we use to compare to this model all follow this stochastic equity process.

Interest Rate Process

Generally, our models will follow a deterministic short rate process. However, it is possible for the spot rate $r(t)$ to follow the following stochastic differential equation.

$$dr(t) = c(t, r)dt + d(t, r)dW_2(t) \quad (3.26)$$

where $c(t, r)$ is the drift of the spot rate (possibly mean reverting), $d(t, r)$ is the volatility of the spot rate and $W_2(t)$ is the Wiener process relating to the stochasticity of interest rates. The price at time t of a bond maturing at time T is given by

$$B^T(t) = \mathbb{E}^{\mathbb{Q}}[e^{-\int_t^T r(s)ds}].$$

Of the models we have discussed, only the Ho and Pfeffer (1996) model uses a stochastic interest rate model based on the Ho Lee interest rate model with $c(t, r) = \theta(t)$ and $d(t, r) = \sigma_2$. The others all use a deterministic interest rate process.

Intensity Rate Process

In order to model the volatility of credit spreads the intensity rate and recovery rate processes must be specified. Many assumptions can be made about the intensity rate process. For example, Jarrow and Turnbull [1995] allow the intensity rate process to be an arbitrary random process. Other approaches include letting the intensity rate be a function of credit ratings or even firm values. There are other approaches but for the purpose of comparing convertible bond models in this paper, if there exists an intensity rate, then our models will assume a deterministic intensity rate $\lambda(t)$. The recovery rate δ is assumed to be a predetermined fraction of the notional amount of the convertible bond. Hence, in the event of default the price of the convertible bond drops to the recovery value of δK (K being the notional amount) which is assumed to be invested at the risk free rate.

3.4.3 Convertible Bond Boundary Conditions

The following boundary conditions are not only relevant for the ‘reduced form’ model. Some have already been mentioned but are repeated here as they are applicable across all convertible bond pricing models.

The parity relationship of the convertible bond gives a minimum boundary. If the convertible falls to a price below parity then it is possible make arbitrage profits by buying the convertible bond in the market, short selling the underlying stock and replacing the borrowed shares by immediately converting the bond. However, in reality both transaction costs and any accrued interest lost on conversion have to be taken into account. In addition, shorting the underlying stock may not be possible but the theory still stands. Therefore,

$$V(t) \geq \gamma(t)S(t). \quad (3.27)$$

At maturity, the bond is either converted or worth the principal amount P plus any final coupon $CF(T)$:

$$V(T) = \begin{cases} \gamma(t)S(t) & \text{if } \gamma(t)S(t) \geq P + CF(T) \\ P + CF(T) & \text{if } \gamma(t)S(t) < P + CF(T) \end{cases} \quad (3.28)$$

If the bond is neither callable nor puttable then:

$$V(t) \sim \gamma(t)S(t) \quad \text{as } S \rightarrow \infty \quad (3.29)$$

$$V(t) \sim \mathbb{E}^{\mathbb{Q}}\left[P + \sum_{i=1}^n CF(t_i)\right] \quad \text{as } S \rightarrow 0 \quad (3.30)$$

Equation 3.30 represents the bond floor and is the second minimum arbitrage boundary. This is the value of the convertible bond if it were just a straight coupon bond without any conversion features. If the interest rate changes, the level of the bond floor will also change. As interest rates increase, the bond floor will decrease and vice versa. The increase or decrease will depend on the interest rate process i.e. whether it is mean-reverting or not.

Using these universal boundary conditions and considering restrictions brought on by callability and putability, then ultimately optimal conversion is given by

$$V(t) = \max(\gamma(t)S(t), P(t), \min(V(t), C(t))) \quad (3.31)$$

3.4.4 Treatment of different Cash-flows

Different pricing models make different assumptions about the intensity rate $\lambda(s)$ and the recovery rate δ . We have also seen that within each model different assumptions are made about the valuation of cash flows depending on whether they are related to equity or debt. This can be straightforward at certain times in the life of the convertible bond where the nature of the cash flow is clear cut (the best example of this is at maturity), but can become quite complex as some of the alternative models attempt to capture what happens prior to maturity. Jarrow and Turnbull [1995] provide the following expression which values the convertible at any time u with $i = 1, \dots, n$ cash flows

$$V(u) = \mathbb{E}^{\mathbb{Q}} \left[\sum_{i=1}^n \left[e^{-\int_u^{t_i} r(s) ds} \left[V(t_i) e^{-\int_u^{t_i} \lambda(s) ds} + V(t_i) \delta (1 - e^{-\int_u^{t_i} \lambda(s) ds}) \right] \right] \right] \quad (3.32)$$

The parameter values $\lambda(s)$ and δ are a function of the model and nature of the particular cash flow. The models that we have dealt with, all assume a recovery rate of $\delta = 0$ in all cases. The risk-free “naive” model assumes that all cash flows are valued with $\lambda(s) = 0$. Under the Goldman Sachs [1994] model, all cash flows

that are weighted by the conversion probability p are valued with $\lambda(s) = 0$ and all cash flows weighted by $(1 - p)$ are valued with $\lambda(s) \neq 0$. Equity related cash flows are valued with $\lambda(s) = 0$ and debt related cash flows are considered with $\lambda(s) \neq 0$ under the Tsiveriotis and Fernandes [1998] model. Finally, the Ho and Pfeffer [1996] model assumes that all cash flows are to be valued with $\lambda(s) \neq 0$.

The above framework for thinking about the different models in terms of equity and debt cash flows is in the spirit of Goldman Sachs [1994] and Tsiveriotis and Fernandes [1998] papers. However, a more illuminating framework for comparing the different models is presented in the next section.

3.4.5 Analysis using the Margrabe Model

Margrabe [1978] generalises the Black Scholes option pricing formula to price options which give the holder the right to exchange one asset for another. A convertible can be viewed as a portfolio of a risky straight bond (worth B at $t = 0$ which pays principal K at time T_2) and the option to exchange the straight bond for a certain amount of stock. Once again, since both the price of the bond component and the exchange option component can be evaluated, the fair value of the convertible is just the sum of the two. Margrabe shows that the fair value of the exchange option to exchange asset S_2 for asset S_1 at expiration, T_1 is given by

$$E(t) = Q_1 S_1 e^{(b_1 - r)(T_1 - t)} N(d_1) - Q_2 S_2 e^{(b_2 - r)(T_1 - t)} N(d_2) \quad (3.33)$$

where

$$d_{1,2} = \frac{\ln \frac{Q_1 S_1}{Q_2 S_2} + (b_1 - b_2 \pm \frac{\hat{\sigma}^2}{2})(T_1 - t)}{\hat{\sigma} \sqrt{T_1}} \quad (3.34)$$

$$\hat{\sigma} = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \quad (3.35)$$

where Q_1 and Q_2 are the quantities of the assets S_1 and S_2 respectively and $N(x)$ is the cumulative probability distribution function for a standardised normal distribution. The Margrabe Model assumes that both assets follow Geometric Brownian Motion with correlation ρ . Although intuition would contradict the idea of Geometric Brownian Motion applying to bonds, it does so in this situation and leads to

sensible results.

The models of Goldman Sachs, Tsiveriotis and Fernandes and Ho and Pfeffer can all be interpreted with reference to a simplified convertible bond contract using the philosophy of Margrabe as a tool. Some simplifying assumptions are needed such as assuming that the exchange option has a European style, that no callability is allowed by the issuer and that the bond pays no coupons throughout its life. Under these assumptions of the Margrabe model, S_1 can be interpreted as the price of our stock S , Q_1 is the conversion ratio $\gamma(t)$, S_2 is the bond B and since the quantity of the bond is unity we have $Q_2 = 1$. The price of the bond B at $t = 0$ is given by $B = Ke^{-yT_2}$, with a continuously compounded yield of y . The option replicating portfolio can be seen from equation 3.33 to consist of an amount of $\gamma(t)e^{((b_1-r)(T_1-t))}N(d_1)$ of equity and $e^{((b_2-r)(T_1-t))}N(d_2)$ of borrowed money where the values of b_1 and b_2 depend on the model being used.

When $S \rightarrow \infty$ then $N(d_1); N(d_2) \rightarrow 1$. This leaves the replicating portfolio to be long in equity worth $S\gamma e^{(b_1-r)(T_1-t)}$ and short a cash amount worth $Ke^{-yT_2}e^{((b_2-r)(T_1-t))}$ which is exactly offset by the long risky bond position. Therefore $V(t) \rightarrow S\gamma e^{(b_1-r)(T_1-t)}$ as $S \rightarrow \infty$. If the option to exchange is American in nature (as is usually the case) then $V(t) \rightarrow \max(S\gamma e^{(b_1-r)(T_1-t)}, S\gamma)$ as $S \rightarrow \infty$. Thus in our simple model, if there exists a continuous dividend rate q then $b_1 = r - q$ and $V(t) \rightarrow S\gamma$ for the American option to exchange.

In the other extreme case when $S \rightarrow 0$ then $N(d_1); N(d_2) \rightarrow 0$. This makes $E(t) = 0$ i.e. our exchange option is worthless. The corresponding replicating portfolio is equal to zero as a result and hence $V(t) \rightarrow Ke^{-y(T_2-t)}e^{((b_2-r)(T_1-t))}$. Since we assume a flat yield curve then $y = b_2$ and $V(t) \rightarrow Ke^{-b_2(T_2-t)}e^{((b_2-r)(T_1-t))}$.

The “naive” riskfree model assumes the forward bond price, and therefore also the cash hedge, grows at a conditional expectation adjusted rate which is the riskfree rate. This is represented by $b_2 = r$. The forward bond price is also discounted at the same risk-free rate. The forward equity price grows at a conditional expectation adjusted rate which is $b_1 = r - q$ and is discounted at the riskfree rate, r . This is the simplest “naive” model and is thought of as such as it is clearly not realistic for the forward price of the risky bond to grow at the riskfree rate.

Figure 3.3 is a summary of the parameter values b_1 and b_2 used under each model. Goldman Sachs and Tsiveriotis and Fernandes both assume $b_1 = r - q$ (equity growth

Model	b_1	b_2
“Naive” risk-free	$r - q$	r
Goldman Sachs	$r - q$	$r + \lambda$
Tsiveriotis and Fernandes	$r - q$	$r + \lambda$
Ho and Pfeffer	$r + \lambda - q$	$r + \lambda$

Figure 3.3: Margrabe interpretation of convertible bond models

rate) and $b_2 = r$ (growth rate of the forward bond price). Although, these models are realistic in evolving the forward price of the risky bond at $b_2 = r + \lambda$ they do not consider any recovery on the risky bond. In addition, the forward equity price is conditional on no default occurring and does not make an allowance for an intensity rate λ . If a conditional expectation adjusted rate including the possibility of default is used for the risky bond of a company then to be consistent it must be used for the equity component. Ho and Pfeffer also assume the forward bond price grows at a conditional expectation adjusted rate of $b_2 = r + \lambda$. However, they seem to discount all cash flows at a risky rate $r + \lambda$. This suggests that they must have $b_1 = r + \lambda - q$ in order for their model to stand.

Model	$S \rightarrow 0$	$S \rightarrow \infty$ (European)	$S \rightarrow \infty$ (American)
“Naive” risk-free	Ke^{-rT_2}	$S\gamma e^{(-q)(T_1-t)}$	$\max(S\gamma, S\gamma e^{(-q)(T_1-t)})$
Goldman Sachs	$Ke^{-r(T_2-t)}e^{(-\lambda)(T_2-T_1)}$	$S\gamma e^{(-q)(T_1-t)}$	$\max(S\gamma, S\gamma e^{(-q)(T_1-t)})$
Tsiveriotis and Fernandes	$Ke^{-r(T_2-t)}e^{(-\lambda)(T_2-T_1)}$	$S\gamma e^{(-q)(T_1-t)}$	$\max(S\gamma, S\gamma e^{(-q)(T_1-t)})$
Ho and Pfeffer	$Ke^{-r(T_2-t)}e^{(-\lambda)(T_2-T_1)}$	$S\gamma e^{(\lambda-q)(T_1-t)}$	$\max(S\gamma, S\gamma e^{(\lambda-q)(T_1-t)})$

Figure 3.4: Asymptotic behaviour of convertible bond models under Margrabe assumptions

As noted above, $V(t) \rightarrow Ke^{-b_2(T_2-t)}e^{(b_2-r)(T_1-t)}$ as $S \rightarrow 0$. Hence the asymptotic behaviour is model dependent and depends on the value of b_2 . From the table in figure 3.4 above, we observe that $V(t)$ is maximised under the naive model and minimised for each other choice of model. As $S \rightarrow \infty$ then for a European style exchange option, the convertible’s price (which is dependent on b_1) is maximised

under the assumptions of Ho and Pfeffer and minimised elsewhere. For the same extreme of $S \rightarrow \infty$ for American style exchange options, for $q \neq 0$ then the “naive” risk-free, Goldman Sachs and Tsiveriotis and Fernandes models all give convertible values $V(t) \rightarrow S\gamma$. For the Ho and Pfeffer model, the situation is somewhat more involved and depends on the relative sizes of the intensity rate λ and q , the dividend rate. If $\lambda \geq q$ then the option will not be exercised until maturity whereas when $\lambda < q$ then early exercise would be optimal.

3.5 Other Notable Models

3.5.1 Ayache, Forsyth, Vetzal (2002; 2003)

In the publications of Ayache et al. [2002] and Ayache et al. [2003], the approach of Tsiveriotis and Fernandes [1998] is brought fundamentally into question, in that their approach is claimed to be internally inconsistent. One major issue with the Tsiveriotis and Fernandes model is that the stock price does not jump down in the case of default and in the case that default does occur, there is no recovery on the bond. It is not only Tsiveriotis and Fernandes’s approach that is criticised but all of the papers mentioned above that arbitrarily (according to Ayache et al. [2002] and Ayache et al. [2003]) split the convertible into separate debt and equity portions and then only apply the credit spread when considering the bond component. Fundamentally, this is self-referential and hence is not really a linear problem. The approach of Ayache et al. [2002] is to treat the entire convertible bond as a contingent claim and to derive a Black Scholes type PDE which among other things models the residual value of the convertible bond in the event of default. In particular, default occurs with an intensity rate which is inversely related to the current stock price.

In creating their model, Ayache et al. [2002] want to provide a way to allow the recovery fraction F to be freely determined by the value of the convertible bond itself. It must be noted that the holder of a risky bond with an embedded option will want to argue that he was owed more prior to default than just the present value of the fixed income part of the bond. Having agreed to exclude the option to convert from the treatment of recovery, this means that the contingent cash-flows (puts and calls) must in some way be integrated in the holder’s claim to recovery.

The problem is that their precise value will depend on whatever optimal exercise policy the holder was supposed to follow prior to default.

Ayache et al. [2002] propose the following in dealing with default and how to apply it when valuing convertible bonds ($V(S, t)$). Firstly, we assume that:

$$S^+ = S^-(1 - \eta) \quad (3.36)$$

where S^- is the stock price immediately prior to default and S^+ is the stock price of the underlying immediately after default. η can take on any value between 0 and 1 with $\eta = 1$ implying a total default with the stock price dropping to zero. Alternatively, $\eta = 0$ means that the issuing firm defaults on the bond but stock price remains unaffected. As usual, a hedging portfolio is constructed in 3.37 and we call p the instantaneous probability of default.

$$\Pi = V - \beta S \quad (3.37)$$

If there is no default risk (i.e. $p = 0$) we follow the following process:

$$d\Pi = dV - \beta dS \quad (3.38)$$

3.38 represents the portfolio and again the share price satisfies

$$dS = \mu S dt + \sigma S dW \quad (3.39)$$

By Ito's Formula,

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} dS^2 \\ &= \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dW \end{aligned} \quad (3.40)$$

Hence

$$d\Pi = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} - \beta \mu S \right) dt + \sigma S \left(\frac{\partial V}{\partial S} - \beta \right) dW \quad (3.41)$$

Choosing $\beta = \frac{\partial V}{\partial S}$, so the portfolio is short $\frac{\partial V}{\partial S}$ shares, then

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt \quad (3.42)$$

For the more general case when the hazard rate p is non-zero we make some assumptions:

- In the case of default, the stock price jumps according to equation 3.36
- In the case of default, the convertible bondholder may choose between:
 - The recovery fraction of the bond, F or:
 - Shares worth $\gamma S^-(1 - \eta)$

Under these assumptions, the change in value of the hedging portfolio during the time period $[t; t + dt]$ is:

$$\begin{aligned}
 d\Pi &= (1 - p \, dt) \left(\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt - p \, dt (V - \beta S \eta) + p \, dt \max[\gamma S(1 - \eta), F] \\
 &= \left(\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt - p \, dt \left(V - \frac{\partial V}{\partial S} S \eta \right) + p \, dt \max[\gamma S(1 - \eta), F]
 \end{aligned}
 \tag{3.43}$$

For a brief instant, the portfolio is risk free. By the no-arbitrage argument, it must earn the same return as the risk free bank account, i.e

$$d\Pi = r\Pi dt = r \left(V - \frac{\partial V}{\partial S} S \right) dt \tag{3.44}$$

By equating 3.43 and 3.44 we obtain:

$$r \left(V - \frac{\partial V}{\partial S} S \right) dt = \left(\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} \right) dt - p \, dt \left(V - \frac{\partial V}{\partial S} S \eta \right) + p \, dt \max[\gamma S(1 - \eta), F]
 \tag{3.45}$$

This implies that

$$\frac{\partial V}{\partial t} + (r + p\eta) S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r + p)V + p \max[\gamma(1 - \eta)S, F] = 0
 \tag{3.46}$$

What is still of concern is how to model the recovery fraction of the bond, F . It may be argued that F should be a fraction of the face value of the bond. There are also those, such as Takahashi et al. [2001], who believe that the holder of the bond recovers a fraction of the convertible bond value prior to default. What the holder should recover in the case of early termination due to default is the recovery fraction of the expected value of cash flows he would have received had default not occurred plus the value of other convertible bond features, such as putability. Ayache et al. [2002] believe that the optimal model is one where the recovery fraction F is freely

determined by the value of the convertible bond itself and their approach is summed up in what follows.

The following constraints need to be considered in their approach:

- Split the convertible into a debt and equity component such that $V = C + E$.
- C is the value of debt that the holder will argue he was owed just prior to default, and consequently will claim he must recover a fraction of according to some recovery rate of the bond R . Therefore $F = R \times C$.
- C will be worth at least the present value of the cash flows of the underlying straight bond. The increased value is a result of embedded options such as a put.
- C should not include the option to convert. Rather, the option to convert acts externally to the process of recovery, for the holder will retain the right to convert at the residual value of the share once default and recovery have taken place.
- E would then have to incorporate this conversion option and would consequently finish as the holders last option to convert into the residual value of the share when default takes place.

Therefore, the general PDE for the convertible bond would be:

$$\frac{\partial V}{\partial t} + (r + p\eta)S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r + p)V + p\max[\gamma(1 - \eta)S, RC] = 0 \quad (3.47)$$

subject to the early call and put constraints

$$\begin{aligned} V(t) &\geq \max(P(t), \gamma S(t)) \\ V(t) &\leq \max(C(t), \gamma S(t)). \end{aligned}$$

Therefore, to correctly value the convertible bond under default risk, the following coupled PDE's need to be solved:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r + p\eta)S \frac{\partial C}{\partial S} - (r + p)C + pRC = 0 \quad (3.48)$$

$$\frac{\partial E}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 E}{\partial S^2} + (r + p\eta)S \frac{\partial E}{\partial S} - (r + p)E + \max[\gamma(1 - \eta)S - RC, 0] = 0 \quad (3.49)$$

with the initial conditions:

$$C(S, T) = N \quad \text{and} \quad B(S, t) = \max[\gamma S - N, 0]$$

(where N is the face value of the convertible) and subject to the following algorithm which results in non-linearity:

- if $P(t) > \gamma S$ and continuation value $C + E < P(t)$ then $C := P(t) - E$
- if $P(t) \leq \gamma S$ and continuation value $C + E < \gamma S$ then $E := \gamma S - C$
- if $C(t) < \gamma S$ then $E := \gamma S - C$
- if $C(t) \geq \gamma S$ and $C + E > C(t)$ then $E := C(t) - C$
- $C := C + \text{Coupon}$ on coupon dates.

It is important to notice that the term that multiplies the hazard rate (p) in each PDE's expresses the recovery value of each one of the two components after default. For the debt component C this is the usual term however for the equity component E , this is the intrinsic value of the convertible holder's last opportunity to convert the bond into the residual value of the share.

The model of Tsiveriotis and Fernandes [1998] is a special case of the model proposed by Ayache et al. [2002] and Ayache et al. [2003] which assumes that the value of the stock price is unchanged in the event of default and that there is no entitled recovery on the bond should the bond default i.e. when $R = 0$ and $\eta = 0$, the Ayache et al. [2002] model becomes the Tsiveriotis and Fernandes [1998] model. Ayache et al. [2002] and Ayache et al. [2003] subsequently show that the model of Tsiveriotis and Fernandes [1998] predicts sub-optimal behavior by the bondholder.

3.5.2 PDE method, Tavella and Randall (2000)

Many convertible bond results most notably the dirty price, but also the probabilities of a conversion, call, put, redemption or default in the bond lifetime, and the expected lifetime of the bond (until an exercise or a default) can all be expressed as the solution to some differential equation as shown above. These equations are similar to the Black Scholes equation but not identical because of credit risk and

other bond features such as recovery payments. Most of the equations can be solved by finite difference techniques and here we present an overview of the method found in ‘Pricing Financial Instruments: The PDE Method’ by Tavella and Randall [2000]. We start with the simplest case of a two-dimensional PDE in stock price and time. The PDE is solved on a two-dimensional grid in stock price and time. Grid points in the two directions are constructed separately, by specifying the number of time steps and stock steps in each. The time axis extends from valuation date to the convertible maturity date. These two endpoints, plus key event dates (such as coupon and dividend payment dates, put dates, the start and end of call) are taken to be grid points. Then extra points are distributed evenly between event times. Finally extra points are added around valuation date and maturity to improve convergence. The stock price axis is initially centred on the instrument conversion price and extends to suitably extreme minimum and maximum prices. These end points are taken to be some reasonable number of stock price standard deviations (four, for example) from the conversion price. If this grid not does enclose the spot price, the appropriate end point is moved by whole numbers of stock standard deviations until it does enclose the spot price. Conversion price is then no longer the central point but that is not a problem. Remaining stock step points are distributed flatly between these.

The PDE is solved backwards from maturity date to valuation date. At maturity, the convertible value is known as it is simply the payoff. So the vector holding prices for each gridded stock price at the current computational time step is initialised to this payoff. Similarly, the probabilities of conversion or redemption are known (0 or 1 depending on the stock price at each grid point); all other probabilities such as callability and putability are zero. So given the values (stock prices or probabilities) at some grid time, the values at the preceding time are found by solving the simultaneous linear equations which are the finite difference approximations to the PDE (i.e. rolling back). We can do this by using the Crank-Nicolson finite difference scheme, solving the resulting equations by the method of Gauss with the appropriate constraints (conversion, putability or callability) imposed immediately after each rollback.

Starting at the instrument maturity and having carried out the rollback, application of constraints and processing of coupons and dividends for every grid time step, we arrive at time zero with vectors of values (specifically for the convertible price and the various probabilities) for all gridded stock prices. Each of these vectors is splined and this is used to calculate output values for the requested stock prices. Primarily

this gives the convertible fair value at the spot stock price. In addition, the delta, gamma and theta can also be calculated in a similar simple fashion.

3.5.3 Other Tree Methods

John C. Hull (1988)

Tree methods pre-date finite difference methods in finance and are well described in ‘Options, Futures and Other Derivatives’ by Hull [1988]. As can be seen from above, they are very common place amongst many different convertible bond valuation models and are based on a more overtly probabilistic, and intuitive, approach to pricing, but it can be shown that they are equivalent to PDE methods in terms of the underlying equation being solved. In fact, when described at an algorithmic level the likelinesses become very clear. Both employ a grid (rectangular for the PDE, triangular for the Tree), both employ rollback techniques (which differ in detail and performance, though not in concept) and both impose constraints and process cash flows in the same fashion. Perhaps the most obvious difference is that the Tree produces results for a single stock price only (by the very nature of the triangular tree grid) while the PDE produces results for all gridded stock prices as a result of its rectangular grid. Although we have kept our convertible’s relatively simple, tree methods are also capable of dealing with the more complicated cases of convertible valuations which may contain path dependent features and stochastic interest rates. Hull uses a binomial tree in his valuation. The tree nodes are constructed in the usual fashion, from the spot stock price at valuation date to bond maturity. Each node connects to two others at the next time step, in a recombining fashion. The stock prices at these nodes, and the transition probabilities (assuming no default) to them, are determined by the stock dynamics (i.e. its mean and variance). To allow for default, each node has an additional associated ‘default node’. The transition probability for this node is simply the probability of a default, and the bond value on this node is the recovery value for default at that point. We assume that there is a probability of $\lambda\Delta t$ that there will be a default in each period of time Δt .³ As with the PDE valuation above, bond values are initialised to the payoff at each node at maturity. In addition, the bond values on each default node can be initialised at this stage to the recovery value at that time.

Given values at time t_i , the tree algorithm computes the values at the preceding

³ λ is the risk neutral default intensity

time t_{i-1} . Quite simply the value at a node $f(S, t_{i-1})$ is calculated as the discounted expectation of the values on the three connecting nodes at the next time step. We discount the up movement of size u with a probability of p_u ($f(S_u, t_i)$), the down movement of size d with a probability of p_d ($f(S_d, t_i)$) and also the default node with a probability of $\lambda\Delta t$ as discussed. The parameter values chosen to match the first two moments of the stock price distribution are:

$$p_u = \frac{a - de^{-\lambda\Delta t}}{u - d}, \quad p_d = \frac{ue^{-\lambda\Delta t} - a}{u - d}, \quad u = e^{\sqrt{(\sigma^2 - \lambda)\Delta t}}, \quad d = \frac{1}{u}$$

where $a = e^{(r-q)\Delta t}$ and r and q are the risk free rate and dividend yield on the stock respectively. So the expectation is easily found. This is done for each node at the current time step while still considering put, call and conversion constraints along the way. At valuation date, where the tree consists of a single node at the spot stock price, we have the convertible fair value. A sample of what the tree would look like is given in figure 3.5

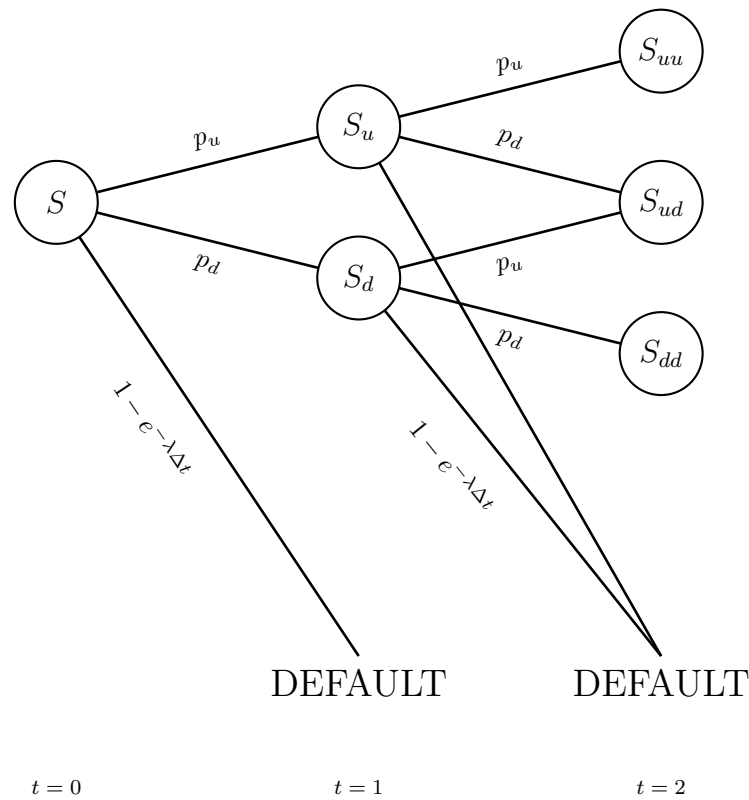


Figure 3.5: Three step tree for valuing convertible bonds according to Hull.

Hung and Wang (2002)

Yet another slight adjustment on the standard tree method is produced by Hung and Wang [2002]. In their paper, they focus on finding a new method of pricing convertibles subject to default risk. They use the reduced form model to model the credit risk which specifies the default process and the recovery rate. The authors make careful distinction between the risky discount rate from the risk-free interest rate and extend the Jarrow and Turnbull [1995] methodology to price convertible bonds that may be defaultable. The literature recognises two main approaches to pricing convertible bonds. The first approach distinguishes the risky discount rate from the risk-free interest rate and takes into account the stochastic characteristics of both the stock price and interest rates. The second approach is the traditional tree model which has been widely adopted. This popular method differentiates the risk-free rate from the risky discount yield but ignores the stochastic characteristics of the two processes.

Hung and Wang's model builds on the traditional tree model for pricing convertibles by adopting Jarrow and Turnbull's model and combines the stochastic risk-free and risky discount rates into one tree. A corporate bond has a positive probability $\lambda_{\mu,t}$ of default with a recovery rate of δ .

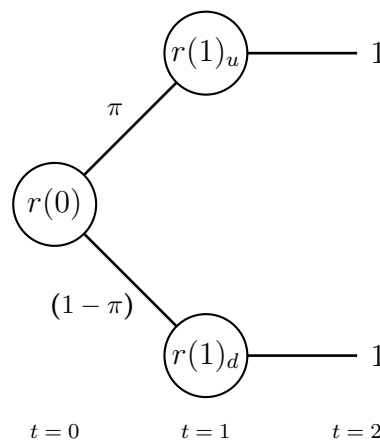


Figure 3.6: Two-period risk-free interest rate tree

The original risk-free interest rate tree is plotted in Figure 3.6 and Jarrow and Turnbull's model in Figure 3.7 where π is the pseudo-probability for the risk-free interest rate process. For simplicity, Hung and Wang [2002] assume $\pi = 0.5$ and

the recovery rate δ is given exogenously if default occurs. Using the values for π and δ and including real data such as the prices of default-free zero coupon bonds ($p(0, T)$) and the prices of the company's defaultable zero coupon bonds ($v(0, T)$), the default probability at time t , $\lambda_{\mu,t}$, can be derived. An example of this is the two period tree where $\lambda_{\mu,0}$ and $\lambda_{\mu,1}$ are solved can be solved. The price relationship between period 0 and 1 can be described as:

$$\begin{aligned} v(0, 1) &= e^{-r(0)}[\lambda_{\mu,0} \times \delta + (1 - \lambda_{\mu,0})1] \\ &= p(0, 1)[\lambda_{\mu,0} \times \delta + (1 - \lambda_{\mu,0})1] \end{aligned} \tag{3.50}$$

Therefore, we can solve for $\lambda_{\mu,0}$, the default probability in the first period by using the prices $p(0, 1)$, $v(0, 1)$ and the recovery rate δ .

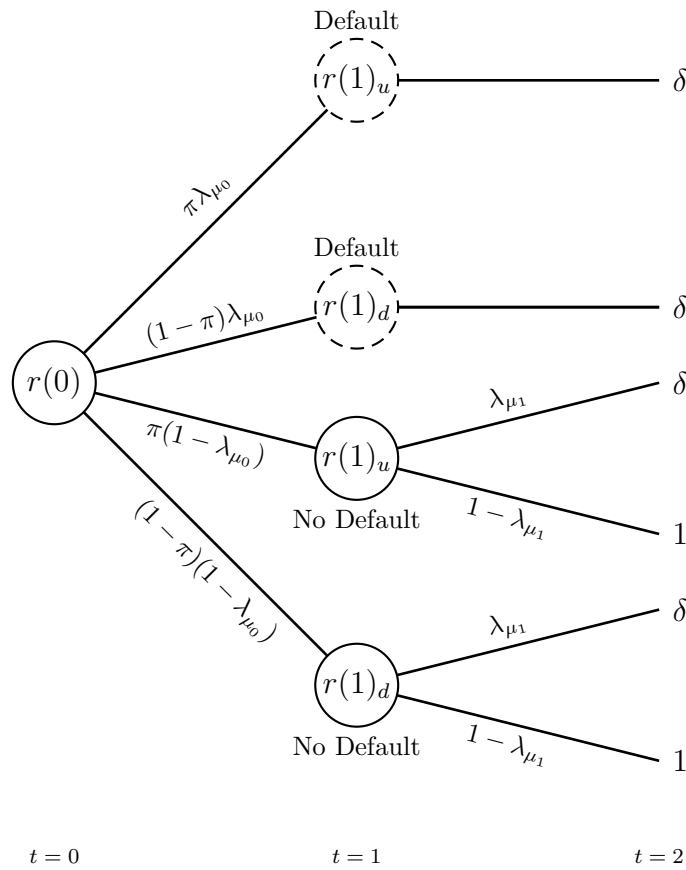


Figure 3.7: Two-period risky interest rate tree

Similarly, we consider the time interval between 0 and 2. This time, the pricing relationship is as follows:

$$v(0, 2) = e^{-r(0)} \left[e^{-r(1)_u} \pi \times \lambda_{\mu,0} \times \delta + e^{-r(1)_d} (1 - \pi) \times \lambda_{\mu,0} \times \delta + \right.$$

$$\begin{aligned}
& e^{-r(1)u} \pi \times (1 - \lambda_{\mu,0}) [\lambda_{\mu,1} \delta + (1 - \lambda_{\mu,1}) 1] + \\
& e^{-r(1)d} (1 - \pi) (1 - \lambda_{\mu,0}) [\lambda_{\mu,1} \delta + (1 - \lambda_{\mu,1}) 1] \Big] \\
= & p(0, 2) \left[\lambda_{\mu,0} \times \delta + (1 - \lambda_{\mu,0}) [\lambda_{\mu,1} \times \delta + (1 - \lambda_{\mu,1}) 1] \right] \tag{3.51}
\end{aligned}$$

where

$$p(0, 2) = e^{-r(0)} [e^{-r(1)u} \times \pi \times 1 + e^{-r(1)d} \times (1 - \pi) \times 1] \tag{3.52}$$

And because $\lambda_{\mu,0}$ has been derived and $p(0, 2)$ and $v(0, 2)$ are given, then $\lambda_{\mu,1}$ can easily be found.

Using the above procedure, $\lambda_{\mu,1}$ is calculated recursively at each time period. After that, the two models are combined to derive a new tree model used for pricing convertible bonds. We are therefore left with three kinds of probabilities to consider within the convertible bond pricing tree. Firstly, p_{r_t} is the probability of a share price increase when the risk-free rate at period t is r_t ; π is the probability that the risky yield and the risk-free yield increase; and finally the default probability of the corporate bond for the period $(t, t + 1)$ is $\lambda_{\mu,t}$. Once again, each $\lambda_{\mu,t}$ is calculated recursively from within the simplified pricing tree.

The simplified convertible bond pricing tree is shown in figure 3.8 on page 57. There are three possibilities at each node as one moves through the tree. The first case occurs if no default has occurred at the node and there are six possible branches from this node, each representing a unique situation at the next node:

- Default occurs; r goes up; S drops to 0.
- Default occurs; r goes down; S drops to 0.
- No Default occurs; r goes up; S goes up.
- No Default occurs; r goes up; S goes down.
- No Default occurs; r goes down; S goes up.
- No Default occurs; r goes down; S goes down.

The assumption made is that in the event of bankruptcy, the stockholder receives nothing and the stock price jumps to zero.

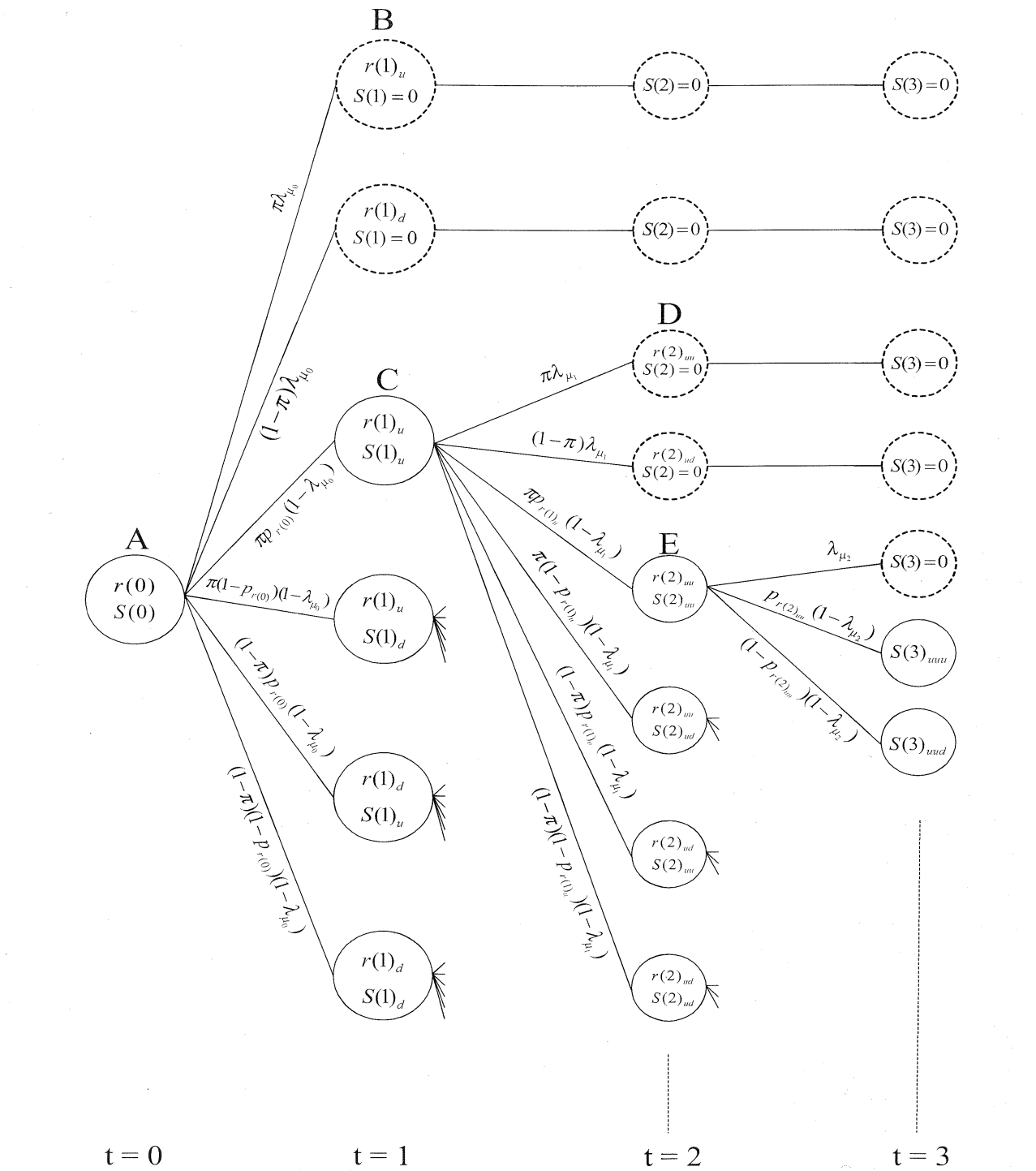


Figure 3.8: Constructing tree for Hung and Wang Model

The second possibility materialises if default occurs. From this node onwards the bond and the discount yield will not fluctuate again. Instead the bond price will equal the product of the recovery rate and the bond's face value.

The third scenario is a special case and only occurs at penultimate nodes. If no default has occurred then these nodes will only contain three branches, unlike the six branches seen in the first case. The reason being is that the discount yield $r(t)$ already represents the rate for the period $(t, t + 1)$ and hence only the stochasticity of the stock price and the probability of default need to be considered. Therefore, the possibilities for the three branches are:

- Default occurs; S drops to 0.
- No Default occurs; S goes up.
- No Default occurs; S goes down.

The tree is constructed just as in a traditional tree model with the payoffs of the terminal nodes decided first. This is followed by rolling back through the tree and once again each node will contain a stock price, equity component, debt component and total value for the convertible. However, in this case, only the risk-free rate is used as the discount yield without the need to adjust for credit spreads. This is because the risky rate has already been represented in each period's default probability $\lambda_{\mu t}$ and recovery rate δ . By using $\lambda_{\mu t}$ and δ , they are able to combine the stock price process, risk-free process and risky discount rate process to form one tree. The value at the origin of the tree gives the price of the convertible.

3.5.4 Monte Carlo Model

The Monte Carlo algorithm for pricing convertibles is based on the least-squares approach developed by Longstaff and Schwartz [2001]. The objective of the algorithm is to provide a pathwise approximation to the exercise rules for all options embedded in the convertible bond. Since convertibles are American in nature, a technique to compute the optimal stopping time needs to be included to the usual Monte Carlo method. For example, in the case of a vanilla convertible, at every conversion time before default the investor compares the payoff from immediate conversion to the expected present value of future payoffs from the bond and naturally converts if the payoff is greater. Thus the optimal conversion rule is essentially determined by the conditional expectation of discounted future payoffs from the bond. The key insight

of Longstaff and Schwartz is that the optimal stopping time needed for the Monte Carlo algorithm can be estimated based on their least squares regression approach used in both papers by Ammann et al. [2001] and Lvov et al. [2004].

We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Ω is the set of all possible paths ω , \mathcal{F} is the sigma field of disjoint events and \mathbb{P} is the probability measure corresponding to \mathcal{F} . n is the number of days until maturity and hence we have a discrete number of stopping times, $0 = t_0 < t_1 < \dots < t_n = T$. This makes sense as the conversion ratio is evaluated once a day at the close of the day.

Table 3.9 below represents the optimal option exercise behavior of both the issuer and investor and the corresponding payoffs at any exercise date t_k . $F(\omega, t_i)$ is the conditional expected value of continuation, i.e. the value of holding the convertible bond for one more time period instead of exercising it immediately and represents the optimal stopping time in each stock price path. Ω_{conv} represents the conversion period for the convertible. Similarly define for Ω_{put} and Ω_{call} .

Payoff at t_k	Condition	Exercise Restrictions	Action
$\gamma_{t_k} S_{t_k}$	if $\gamma_{t_k} S_{t_k} > F(\omega, t_k)$ and $P_{t_k} \leq \gamma_{t_k} S_{t_k}$	$t_k \in \Omega_{conv}$ $t_k \in \Omega_{put} \cap \Omega_{conv}$	Voluntary Conversion
P_{t_k}	if $P_{t_k} > F(\omega, t_k)$ and $P_{t_k} \geq \gamma_{t_k} S_{t_k}$	$t_k \in \Omega_{put}$ $t_k \in \Omega_{put} \cap \Omega_{conv}$	Put
C_{t_k}	if $C_{t_k} < F(\omega, t_k)$ and $C_{t_k} \geq \gamma_{t_k} S_{t_k}$	$t_k \in \Omega_{call}$ $t_k \in \Omega_{call} \cap \Omega_{conv}$	Call
$\gamma_{t_k} S_{t_k}$	if $C_{t_k} < F(\omega, t_k)$ and $C_{t_k} < \gamma_{t_k} S_{t_k}$	$t_k \in \Omega_{call}$ $t_k \in \Omega_{call} \cap \Omega_{conv}$	Forced Conversion
κN	if $\kappa N > \gamma_{t_k} S_{t_k}$	$t_k = T \in \Omega_{conv}$	Redemption
0	otherwise		Continuation

Figure 3.9: Optimal Option Exercise Behaviour

Once the optimal stopping time is found, then all subsequent values in the stock price path are set to zero. A backward recursion algorithm is then used to determine the continuation value $F(\omega, t_i)$ and those cash flows are then used to calculate the cash flows at one prior time step $F(\omega, t_{i-1})$. Naturally, each path will have a different stopping time and we denote this stopping time by τ_i^* . $CF_{total}(\tau_i^*)$, which is the total

cash flows received from a convertible bond at the stopping time τ_i^* is the sum of the payoffs in table 3.9 ($\text{Payoff}(\tau_i^*)$) and the present value of all coupons received and accrued interest gained in the period $[t_0, \tau_i^*]$, $c(\tau_i^*)$.

Using this to calculate the optimal stopping times and corresponding cash flows of all possible paths, the price of the convertible is then given as the average over all the simulated paths i.e.:

$$V_t = \frac{1}{n} \sum_{i=1}^n e^{-r_{t, \tau_i^*}(\tau_i^* - t)} CF_{total}(\tau_i^*) \quad (3.53)$$

This however depends on $CF_{total}(\tau_i^*)$ which in turn relies on $\text{Payoff}(\tau_i^*)$ and the continuation value $F(\omega, \tau_i^*)$ which needs to be estimated. By no arbitrage arguments, $F(\omega, \tau_i^*)$ is equal to the expected value of all future cash flows under the risk neutral measure \mathbb{Q} , on condition that exercise is only possible after t_i :

$$F(\omega, t_i) = E^{\mathbb{Q}} \left[\sum_{j=i+1}^n e^{-r_{t_i, t_j}(t_j - t_i)} CF_{total}(t_j) | \mathcal{F}_{t_i} \right] \quad (3.54)$$

Clearly, the accuracy of the entire model depends on the quality of the above approximation.

As was seen from the other models presented, credit risk plays a major role in convertibles analysis. This is an extremely complex business within the Monte Carlo realm so the paper makes a simple (not so accurate) assumption in order to build a pricing model. It assumes that those paths that ultimately lead to redemption get discounted at the credit adjusted rate while the paths that led to conversion are adjusted at the risk free rate. The biggest drawback of this assumption is that coupons payments which occur at all paths are not taken into consideration. The Monte Carlo method may be the faster than many models to calculate convertible prices but the increase in speed is often coupled with poor pricing results as a result of credit risk assumptions.

Chapter 4

Sensitivity Analysis

“People that are really very weird can get into sensitive positions and have a tremendous impact on history” George W. Bush

In this chapter, we perform several sensitivity analyses on the parameters of convertible bonds and assess the affect these changes have on the overall value of the convertible bond. Sensitivity analysis is used to determine how susceptible a model is to changes in the value of the parameters of the model and to changes in the structure of the model. In this paper, we focus on parameter sensitivity. Parameter sensitivity is performed as a series of tests in which we set different parameter values to see how a change in the parameter causes a change in the dynamic behaviour of the convertible bond. By showing how the model behaviour responds to changes in parameter values, sensitivity analysis is a useful tool in model building as well as in model evaluation. Sensitivity analysis helps to build confidence in the model by studying the uncertainties that are often associated with parameters in models.

We will begin by representing asymptotic analyses on certain convertible features and then move on to more specific sensitivity analysis and in particular look at the Greeks. Throughout, we will use different real life convertible bonds as basis for our comparisons as we attempt to accurately report on the sensitivities of convertibles. In our analysis, we will be dealing with three main convertible bond issues.

The first issue is the french computer software company Business Objects (a division of SAP) and their convertible bond ‘Business Objects 2.25% 01 January 2027 EUR’. The main characteristic of ‘Business Objects 2.25% 01 January 2027 EUR’ is that it contains both call and put features. Our second convertible is ‘Rhodia 0.5% 1 January 2014 EUR’. Rhodia is an international chemical company based in France

and focuses on the development and production of speciality chemicals. ‘Rhodia 0.5% 1 January 2014 EUR’ contains call features but has no putability. Finally, we consider ‘ViroPharma Inc 2% 15 March 2017 USD’, an american based pharmaceutical and biotechnology company. The issue has no call or put features in its basic set up. Unless otherwise stated, the valuations are done on the 21 September 2009. Any further information needed for the analysis will be given within the chapter. Throughout, we will be making use of MONIS Software Limited when doing the analysis on the different convertible bonds. Unless otherwise stated, the method explained by Hull [2005] is the model of choice for the analysis.

4.1 Parameter Sensitivity Analysis

Business Objects 2.25% 01 January 2027 EUR

We start off by displaying how the value of the convertible might vary with the share price some time before maturity. Figure 4.1 below demonstrates this while also plotting the bond floor and parity (as defined in Chapter 2). This is a real representation of what was laid out in Figure 2.5.

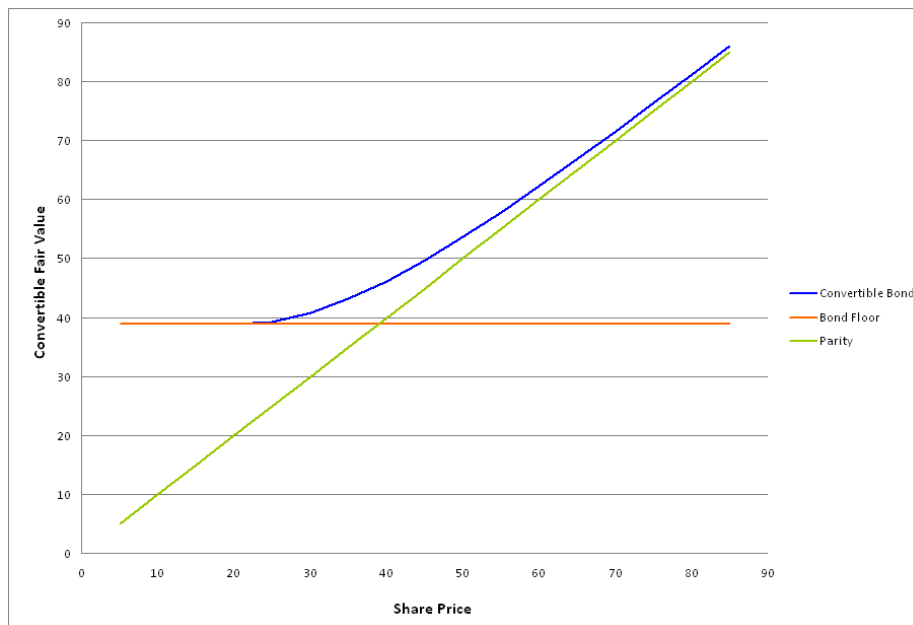


Figure 4.1: Convertible bond fair value with changes in the share price.

We notice that the share price has no effect on the value of the bond floor except for

at very low share prices where the low price is perceived to affect the credit quality of the issue and thus decreases the value of the bond floor. Parity is a linear function and is proportional to the share price where the slope of the parity line represents the conversion ratio. Figure 4.1 also proves that investors who decide to purchase shares via a convertible forfeit a degree of equity appreciation as the convertible price has less than 100% share sensitivity through scope of share prices.

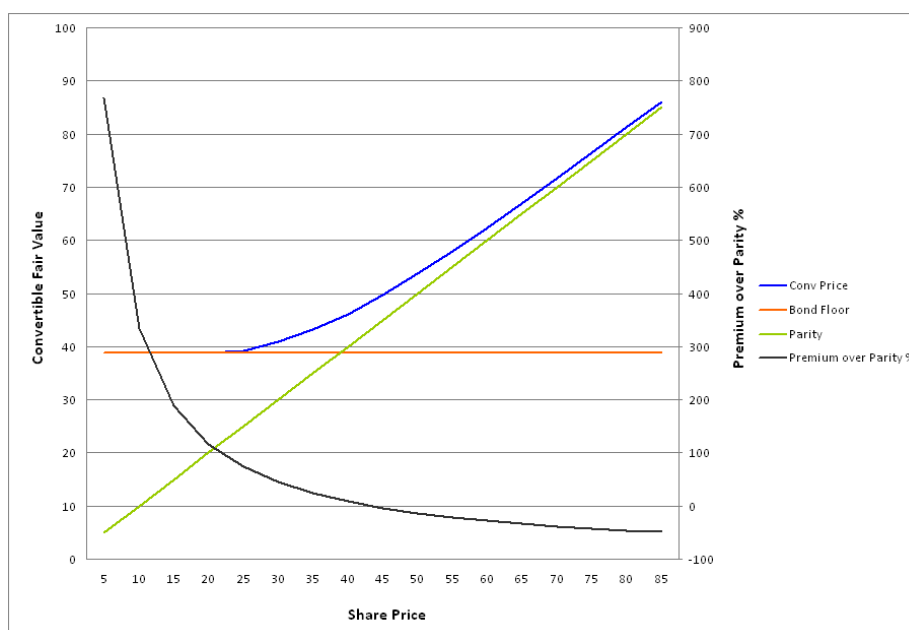


Figure 4.2: Convertible bond fair value and premium.

In figure 4.2, we include the premium over parity on the same diagram as the convertible to illustrate that except for very low share prices, then as the share price increases, the premium contracts to a point where the convertible bond value is derived solely from the equity component. As the share price rises, it becomes more likely that the investor will convert to ordinary shares and thus the convertible behaves more like equity than like the bond. Similarly as the share prices declines, the premium expands and the price of the convertible trades more in line with its bond floor.

The original features of ‘Business Objects 2.25% 01 January 2027 EUR’ include a hard no call period up until 11 May 2012. Thereafter the bond is callable for 125% of the conversion price. In addition, the convertible bond is puttable at 100% on the 11 May 2012, 2017 and 2022. These calls and puts have a very large impact

on the convertible fair value and we will be assessing this impact in the next few diagrams.

First, we will look at the impact of callability on a convertible's price and so in the following diagram we deactivate the put features.

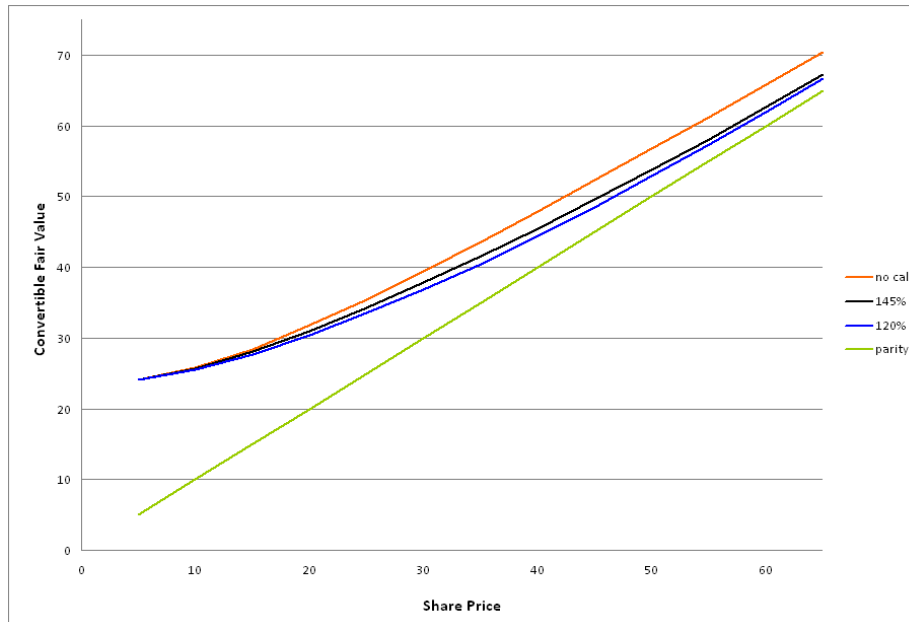


Figure 4.3: Variation with share price of the convertible fair value in the presence of call provisions.

Figure 4.3 illustrates how call features reduce the fair value of the convertible. Lower call levels give the issuer a greater chance of being able to force early conversion thereby diminishing more of the time value of the conversion option. This decreases the value of the convertible. The reduction in value due to calls is more evident at high share prices.

Figure 4.4 is an analysis of put prices and hence callability is deactivated. The diagram shows the added value the convertible obtains from a put provision. A bond with a higher put level has a greater value because of the additional protection puts provide in declining markets. The contribution of the puts to convertible fair value is thus greater at low equity levels.

Figures 4.5 and 4.6 endorse the fact that put provisions increase the fair value of the convertible while call provisions decrease the value. If both a call and a put provision are present then the value is slightly greater than when only a call provision is present but it is lower than when there is only a put provision. When

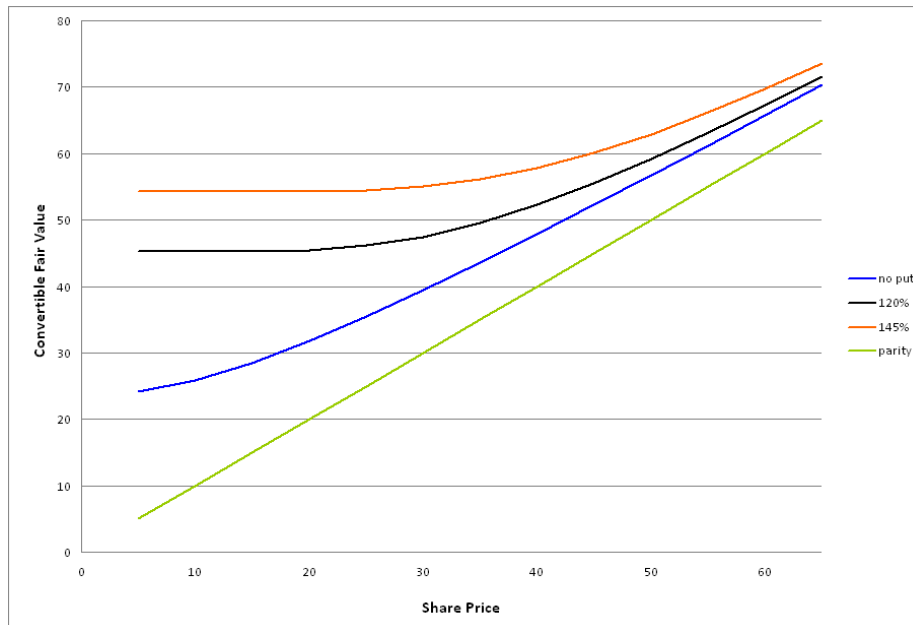


Figure 4.4: Variation with share price of the convertible fair value in the presence of put provisions.

we compare the convertible value which has call and put provisions deactivated to the original convertible value which has both call and put provisions, the results we obtain are somewhat engaging, in particular the behaviour of the convertible bond value relative to share price (Figure 4.5). At low share prices the value of the convertible bond with both put and call provisions is greater than that of the case when there are no provisions. However, as the share price increases they cross and the one without any provisions attached has a greater value. In other words, at low share prices the put feature dominates, while at relatively high prices, the call feature dominates. This will not always be the case, as it depends on the call price and the put price. This same crossing over would also be present if we were to assess the convertible fair value compared to the time until maturity of the convertible.

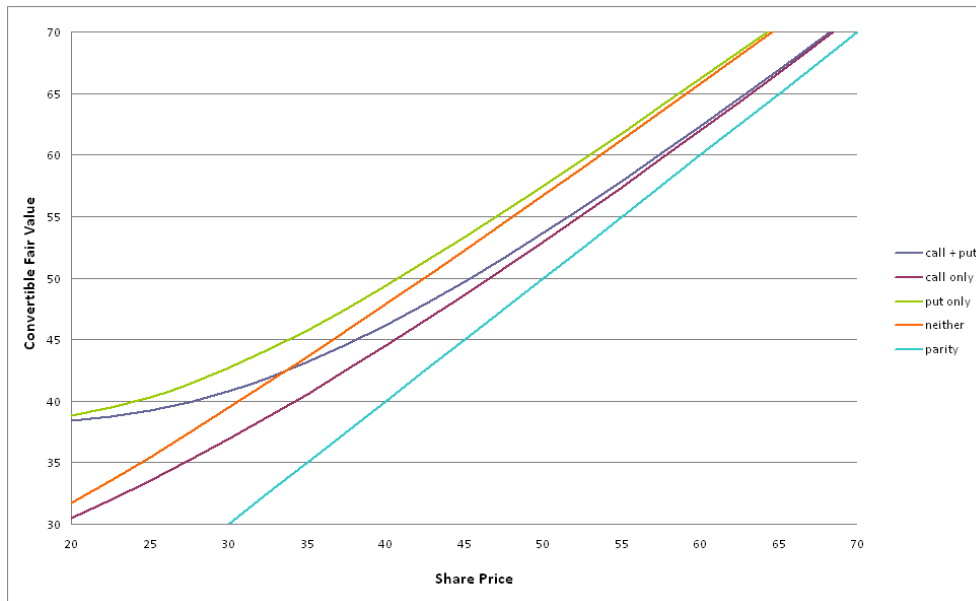


Figure 4.5: Effects of put and call provisions on the convertible fair value with a change in share price.

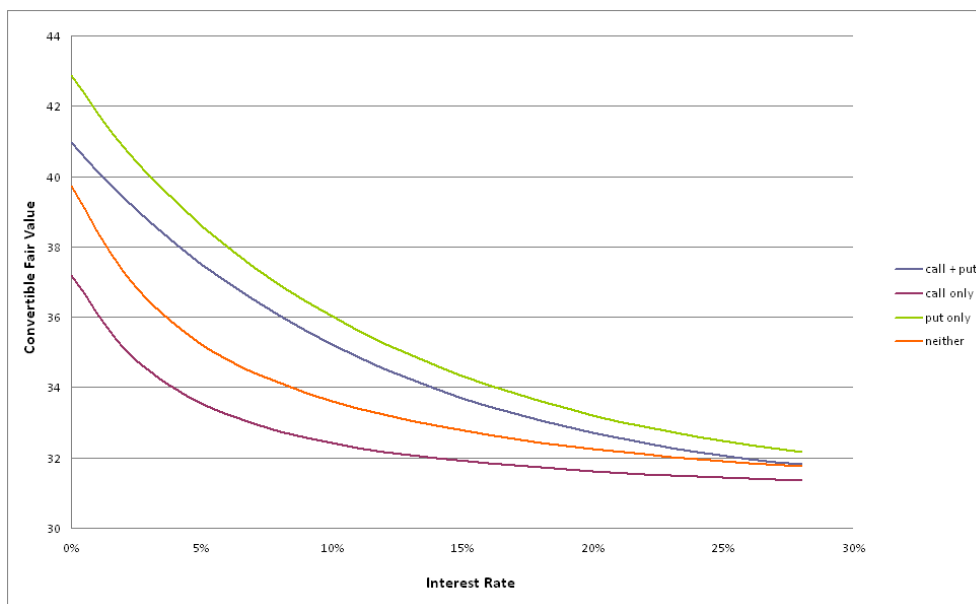


Figure 4.6: Effects of put and call provisions on the convertible fair value with a change in interest rates.

Rhodia 0.5% 1 January 2014 EUR

As mentioned above, ‘Rhodia 0.5% 1 January 2014 EUR’ is callable but not putable. The call terms are as follows: The bond is not callable until 27 April 2009. It is then

callable at 170% for two years and from that date is callable at 135% until maturity. Figures 4.7 until 4.12 are all an analysis of ‘Rhodia 0.5% 1 January 2014 EUR’.

Figure 4.7 shows the relationship between the share price and the convertible bond value when the call trigger is active or not on the valuation date. When the trigger is active, the curve passes through the point at which the conversion value is equal to the call price. Since the bond is called as soon as it reaches the call price, this is the maximum it will reach unless the share price is large enough so that it becomes more profitable to convert the bond into shares. The value of the bond then moves with parity as seen in the graph.

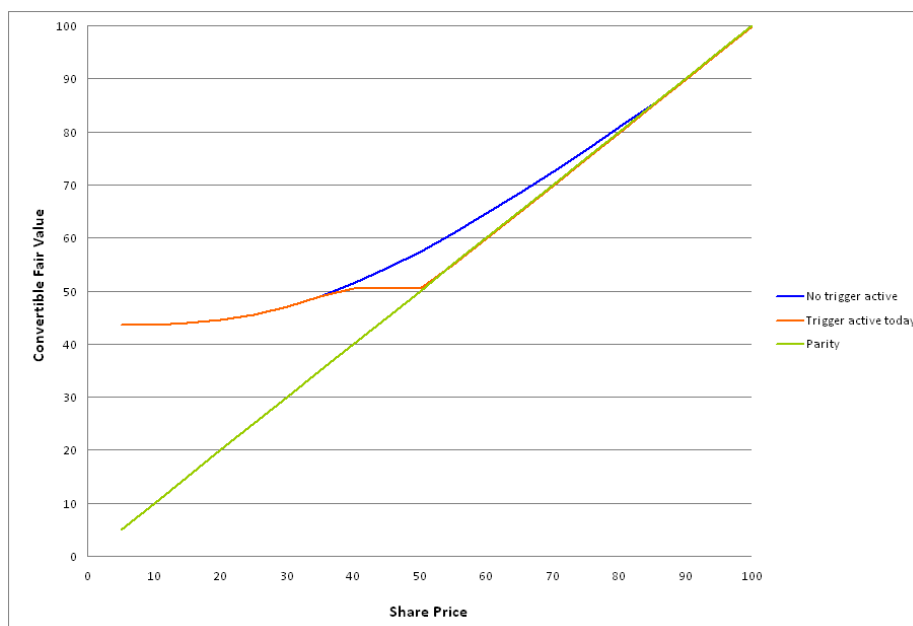


Figure 4.7: Convertible fair value when calls triggers are active or not.

Since convertible bonds are not protected against the dividends paid by the firm, an increase in the dividend yield would therefore decrease the value of the convertible. This effect is present irrespective of the level of the share price as illustrated in Figure 4.8. The main reason for the inverse relationship between dividend yield and convertible bond value is that a greater dividend yield affects the straight debt value of the bond by increasing the probability of default and by reducing the assets available for the bondholders in the event of default.

Figures 4.9 and 4.10 show the sensitivity of convertible values to changes in the share volatility. We define volatility as the relative rate at which the price of the

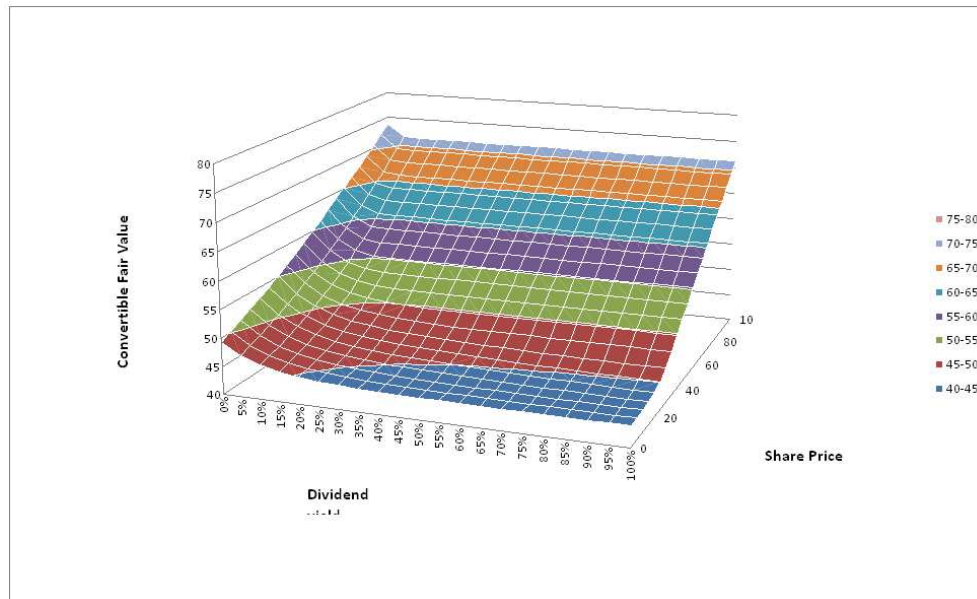


Figure 4.8: Effect of dividend payments on the convertible fair value.

security moves up and down and is calculated by computing the annualised standard deviation of the daily change in price¹. The figures clearly indicate that an increase in the share volatility increases the value of the bond. An increase in the volatility both raises the expected loss through default of the bond portion and increases the expected gain for conversion. The latter effect predominates the former however resulting in the convertible value to increase as volatility rate increases.

¹Although we are looking at the effect of different volatility rates, throughout this paper we have been dealing with flat volatility rate models which are used for instruments that are expected to have a volatility which will remain at a constant level throughout the lifetime of the instrument.

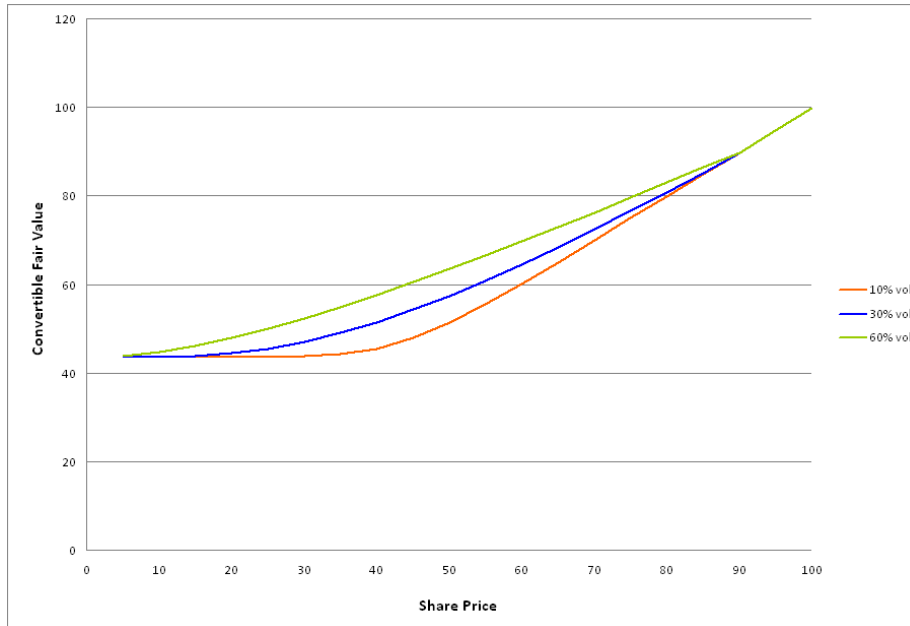


Figure 4.9: Convertible bond fair value with different volatility rates.

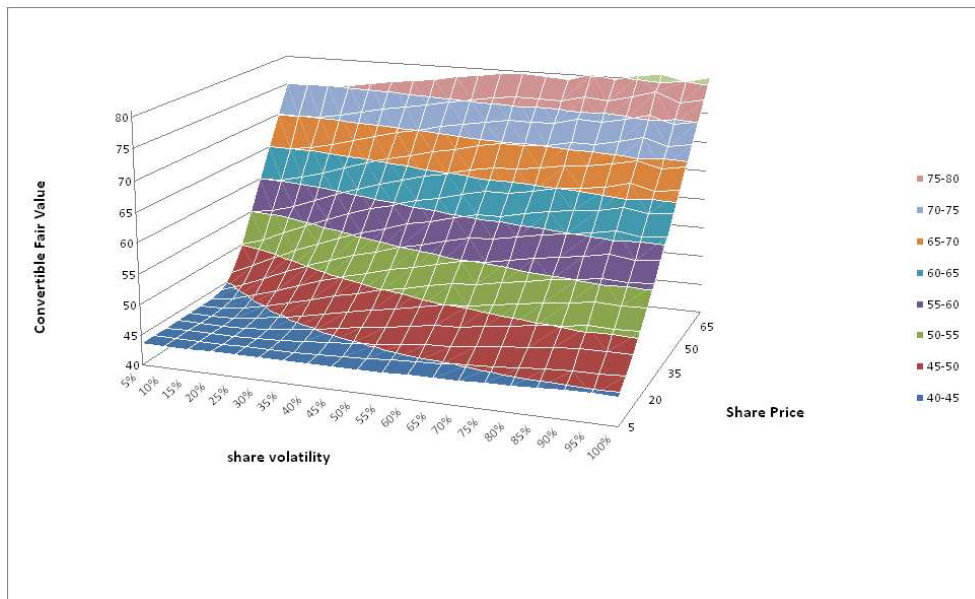


Figure 4.10: Effect of volatility rates on the convertible fair value.

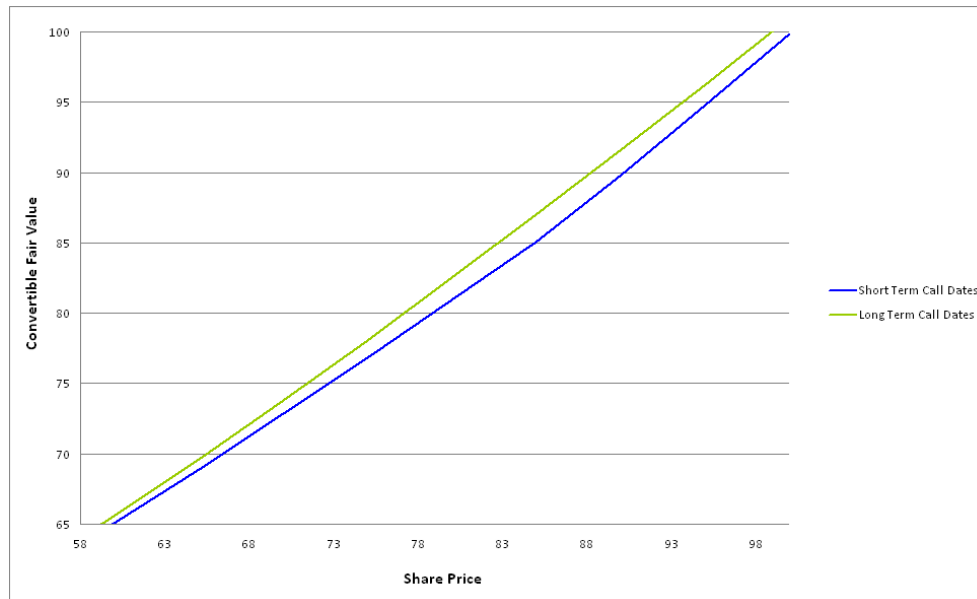


Figure 4.11: Effect of call deferral periods on the convertible fair value.

Figure 4.11 illustrates the effect of varying the date of the first call on the value of the bond. This has no effect on the value of the bond for low share prices where the prospect of conversion is remote and thus we only display the convertible's fair value for greater share prices. In the diagram, 'Short term call dates' contains the original call structure of 'Rhodia 0.5% 1 January 2014 EUR' as described whereas for 'long term call dates' we adjust the first call date to only start on 1 Jan 2011. From this we can clearly see that for a greater deferral of a call dates, the convertible fair value increases.

In Figure 4.12 the fraction of the firm's shares into which the bond is convertible is varied. At extremely low conversion rates, for example when the conversion ratio is equal to 0.1, the convertible behaves as if it were a straight bond. The vertical difference between this lowest curve and any of the others corresponds to the value of the conversion privilege. The original conversion ratio for 'Rhodia 0.5% 1 January 2014 EUR' is equal to 1.

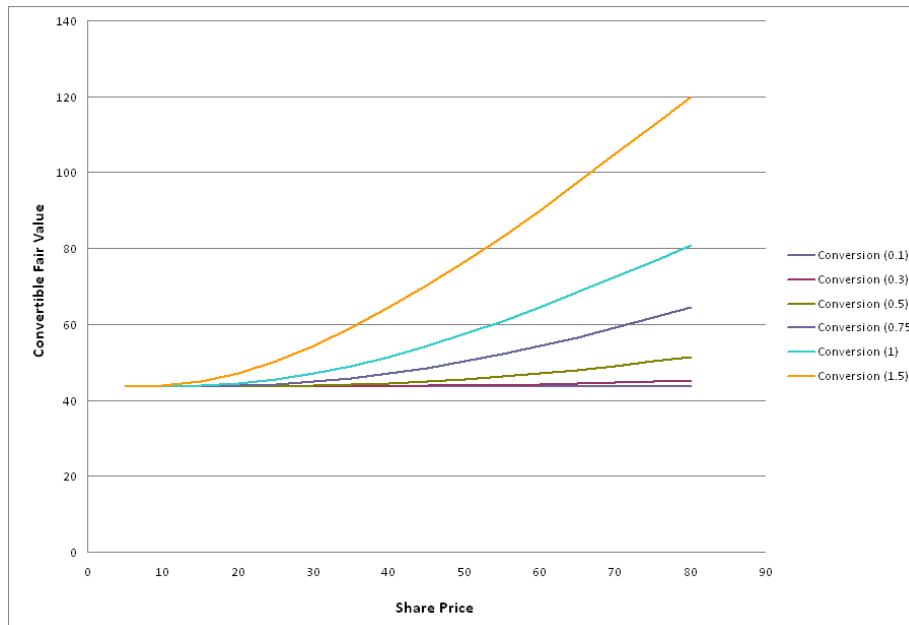


Figure 4.12: Effect of conversion ratios on the convertible fair value.

ViroPharma Inc 2% 15 March 2017 USD

From this point on until the end of the section we will be dealing primarily with ‘ViroPharma Inc 2% 15 March 2017 USD’ unless otherwise stated. Although it has no call or put provisions, ‘ViroPharma Inc 2% 15 March 2017 USD’ still allows us to accurately analyse many other features of convertible bonds. In fact, in many occasions it allows us to meticulously assess changes in the convertible’s value without the distraction of callability and putability. It is noted that at the valuation date used in the analysis (21 September 2009), ‘ViroPharma Inc 2% 15 March 2017 USD’ was trading ‘In the money’ and the underlying share was trading at 14.76 USD, above the conversion price.

Figure 4.13 is easily understood and shows the value of ‘ViroPharma Inc 2% 15 March 2017 USD’ with changing credit spreads. ‘ViroPharma Inc 2% 15 March 2017 USD’ is initially calculated with a credit spread of 0.69% and we adjust this value as seen in the graph. An increased credit spread results in a lower convertible fair value and this impact is more apparent at lower share values. As the share price increases, this difference becomes smaller and smaller as the bond price tends to the conversion value.

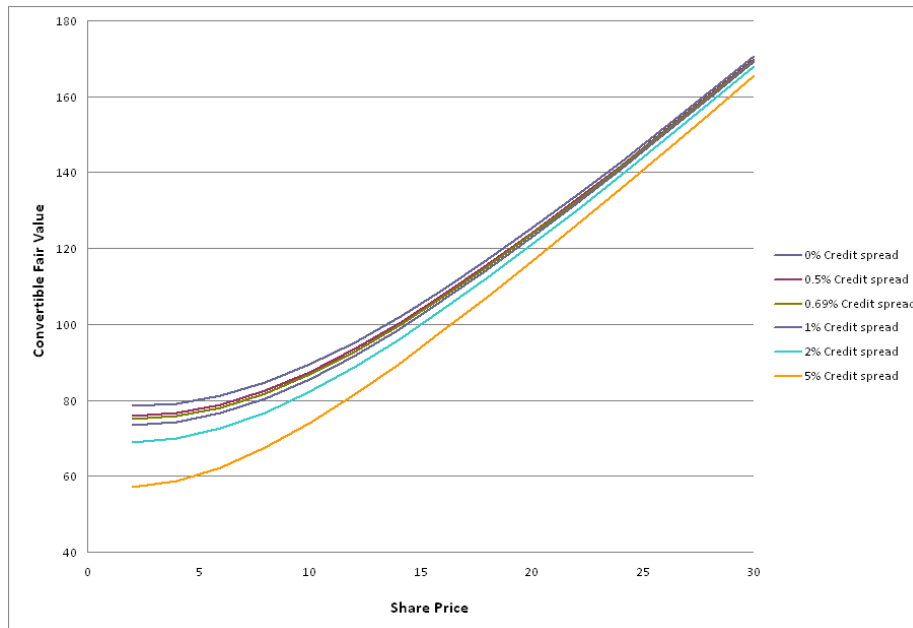


Figure 4.13: Effects of a credit spread on the value of a convertible bond with no call or put features.

Figure 4.14 below shows the sensitivity of the convertible bond value to changes in both volatility and credit spread. It confirms that credit spread is inversely related to a convertible's fair value and proportional to the volatility of the underlying share price. Both measures seem to have a notable effect on the pricing of the bond and hence we should be wary of inaccurate estimation of these variables when pricing convertible bonds.

In Figure 4.15 on page 74, we show the variation of convertible bond values with differing maturities. We plot the original 'ViroPharma Inc 2% 15 March 2017 USD' as well as hypothetical cases of the bond maturing in 2014 and 2011 all on the same set of axes. At all market levels, the convertible continues to be more than parity as the investor can always postpone conversion until maturity without much risk in order to keep collecting the bond coupons. When the share price is low, the convertible is unlikely to be converted and behaves more like a straight bond. And like an equivalent straight bond, the convertible with the shorter maturity will have a greater value. As the share price increases and conversion becomes more likely, the convertible value increases as a result of the increased value of the conversion privilege. This increase is greater for longer maturity convertibles.

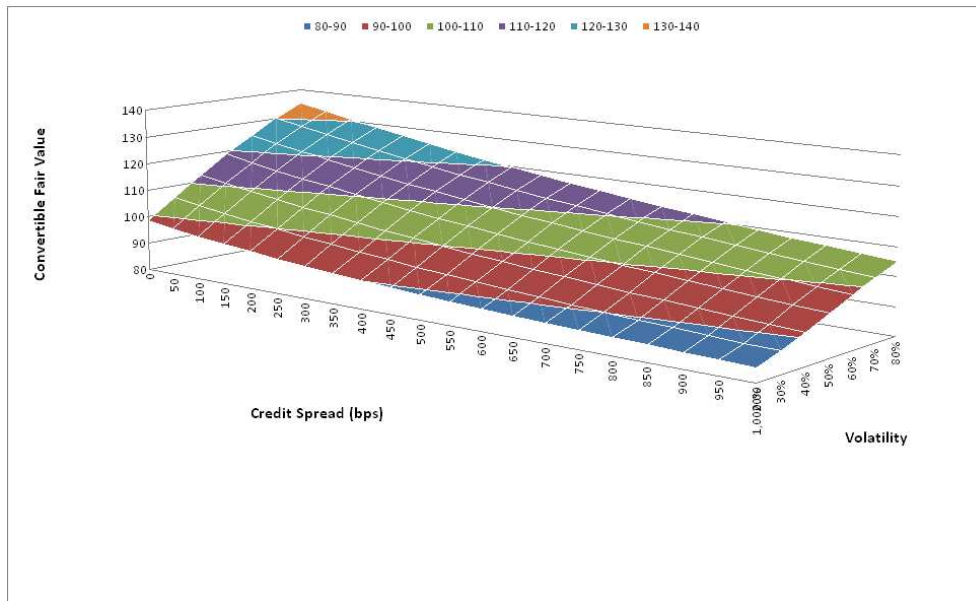


Figure 4.14: Sensitivity of the convertible bond fair value to a change in the volatility and the credit spread.

Figure 4.16 shows the increase in convertible value with stock volatility. At higher volatilities the convertible value increases because the value of the option to exchange the straight bond for underlying stock is greater while at lower volatilities, the shorter term convertibles would have greater value as once again more emphasis is placed on the bond component of the convertible.²

²'ViroPharma Inc 2% 15 March 2017 USD' is calculated with a flat share volatility of 28.5%

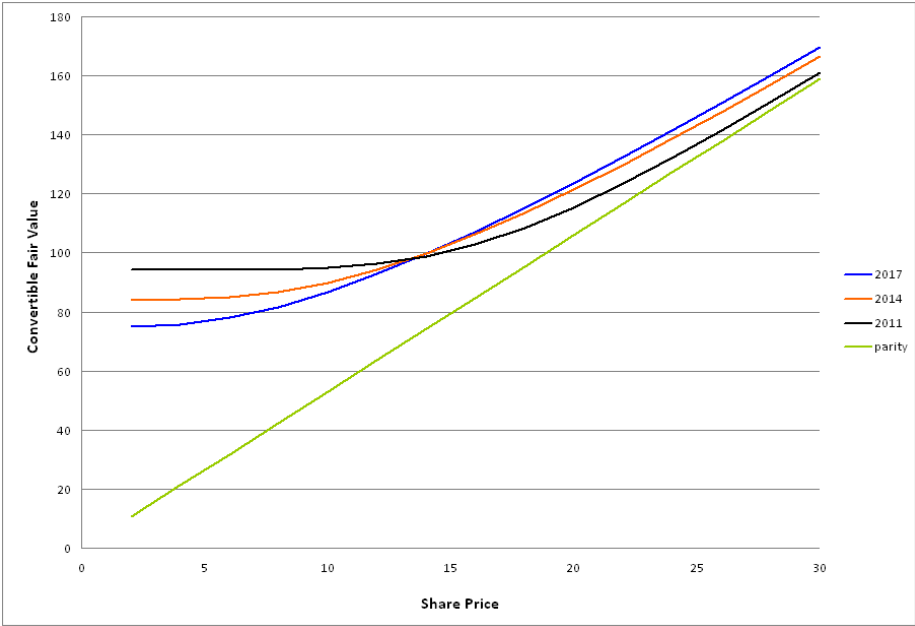


Figure 4.15: Variation of the convertible fair value with share price

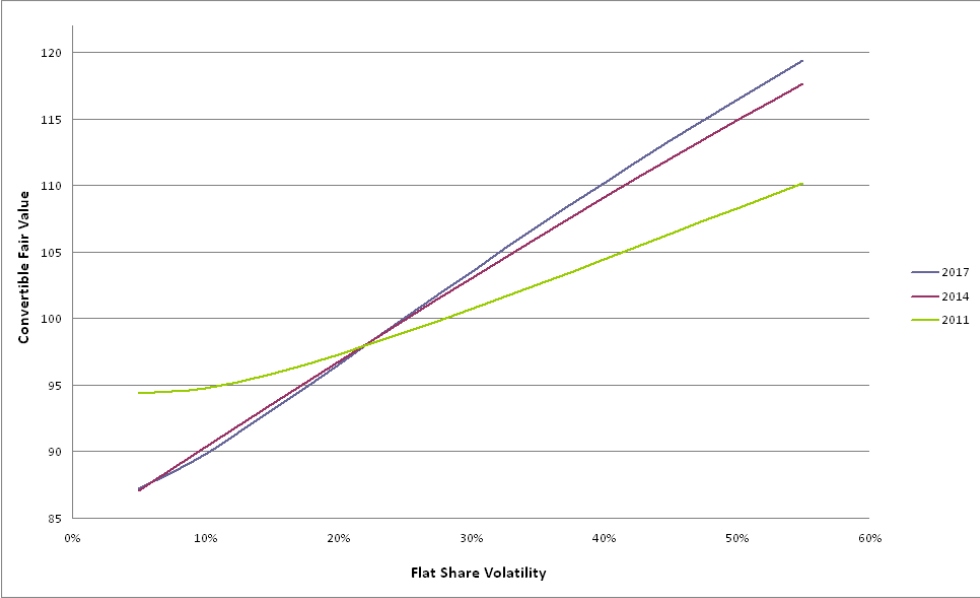


Figure 4.16: Variation of the convertible fair value with volatility

Figure 4.17 below shows the inverse relationship that exists between the convertible fair value and interest rates. Here we assume that while the interest rate changes, the credit spread remains fixed. The rate of decline in value when interest rates rise is lesser for shorter maturity convertibles. Nonetheless, the rate of decline is less for convertible bonds than for equivalent straight bonds because of the cushioning effect of the conversion option. This effect swells as interest rates increase.

Figure 4.18 on page 76 illustrates how the convertible fair value declines with increasing credit spread as this increase lowers the present value of any future coupon and principal payments made to which the investor is entitled. We also see that convertible bonds with longer maturities are more sensitive to increasing credit spreads.³

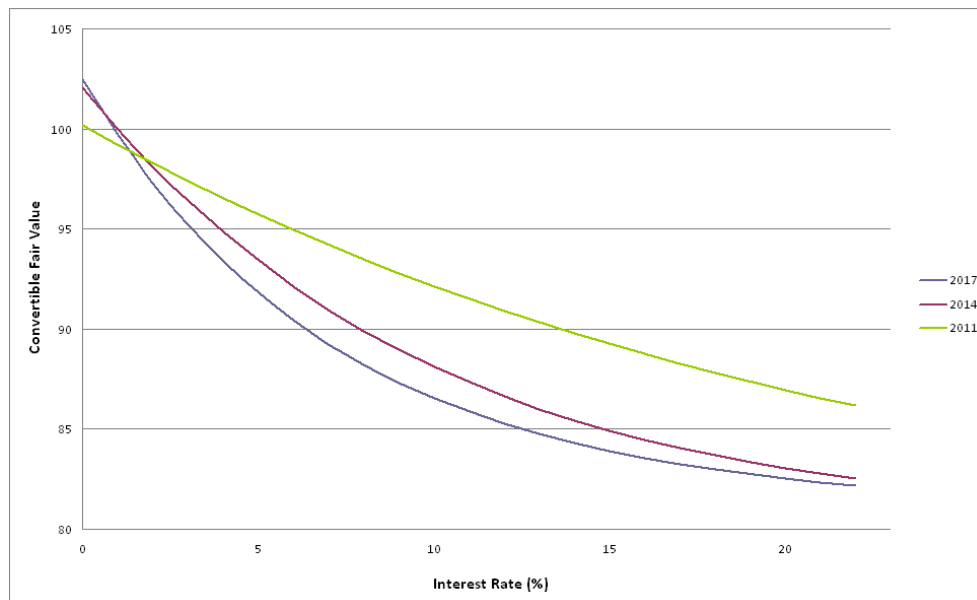


Figure 4.17: Variation of the convertible fair value with interest rate.

On page 76 we also see how a convertible's fair value increases with a greater recovery rate in the case of default (Figure 4.19). This is the expected proportion of the face value of the instrument that is returned to the investor in the event of default. As the recovery rate tends to 100%, the value of the convertible explodes as we tend to a complete arbitrage and unrealistic situation of being fully compensated even if the bond defaults.⁴

³'ViroPharma Inc 2% 15 March 2017 USD' is calculated with a credit spread of 0.69%

⁴'ViroPharma Inc 2% 15 March 2017 USD' is calculated with a recovery rate of 0%

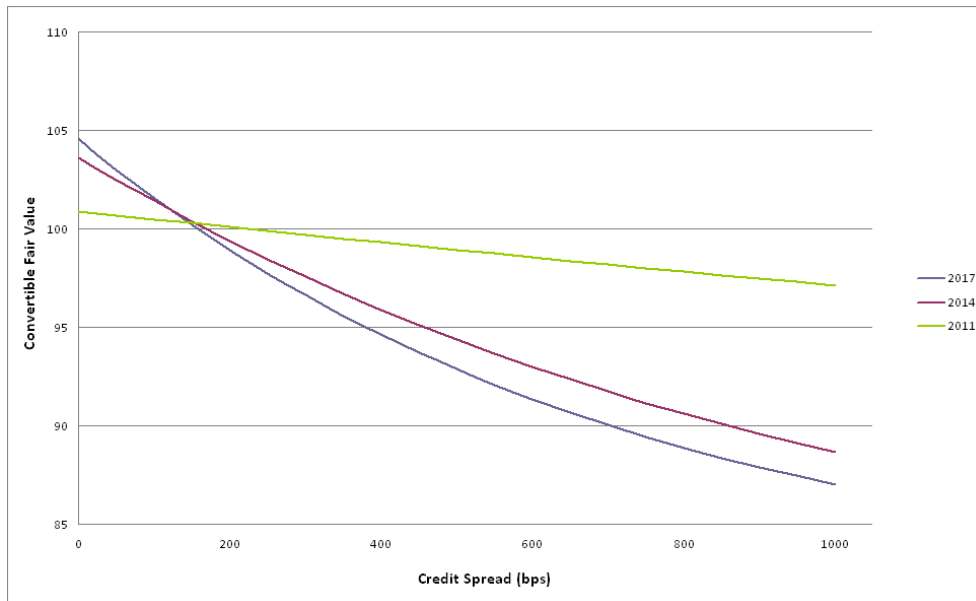


Figure 4.18: Variation of the convertible fair value with credit spread.

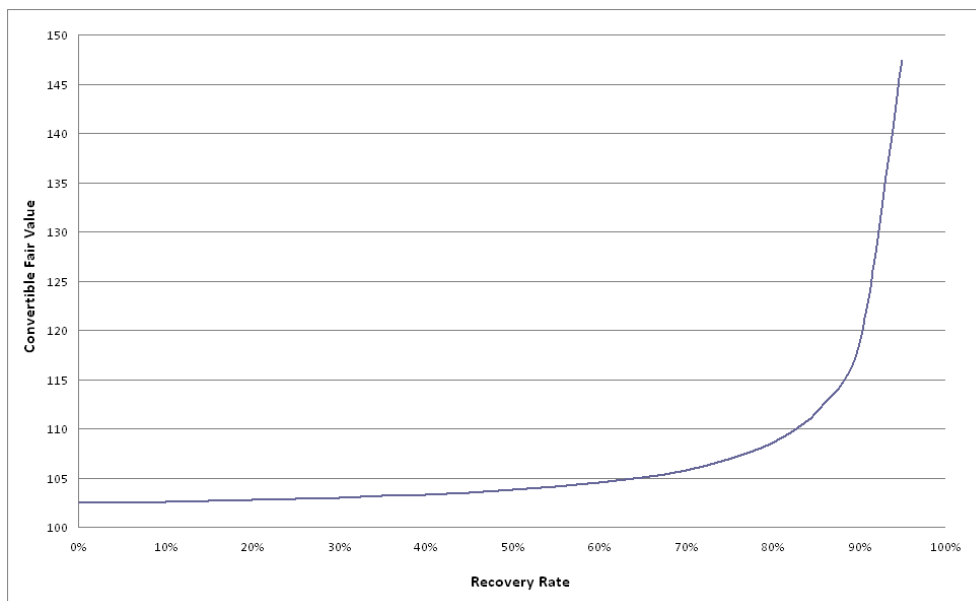


Figure 4.19: Variation of the convertible fair value with different recovery rates.

The last three figures in this section, Figures 4.20 to 4.22, deal with the impact on convertible fair values by plotting the price of the convertible for different equity levels and valuation dates. At low share prices the convertible bond imitates straight debt and trades close to the bond floor while at high share prices the convertible

behaves more like equity and trades very close to parity. Whether or not the holder of the convertible bond will choose to convert in this region depends on the yield advantage. If the share price is high with a large dividend yield then the holder of the convertible will most likely convert to shares. In all other cases, the investor will optimally choose not to convert the bond and will instead receive income through coupon payments. We also see a trend that at relatively high share prices, the value of the convertible decreases over time whereas at low share prices, the opposite is true. In Figure 4.21 we notice three peaks and troughs along the “valuation date” axis at low share prices. The peaks represent the put dates contained within the prospectus of ‘Business Objects 2.25% 01 January 2027 EUR’. Putability increases the value of the bond and once each put date is passed, the value of the convertible drops sharply. Over time the value increases again but will drop at each passing of a put date.

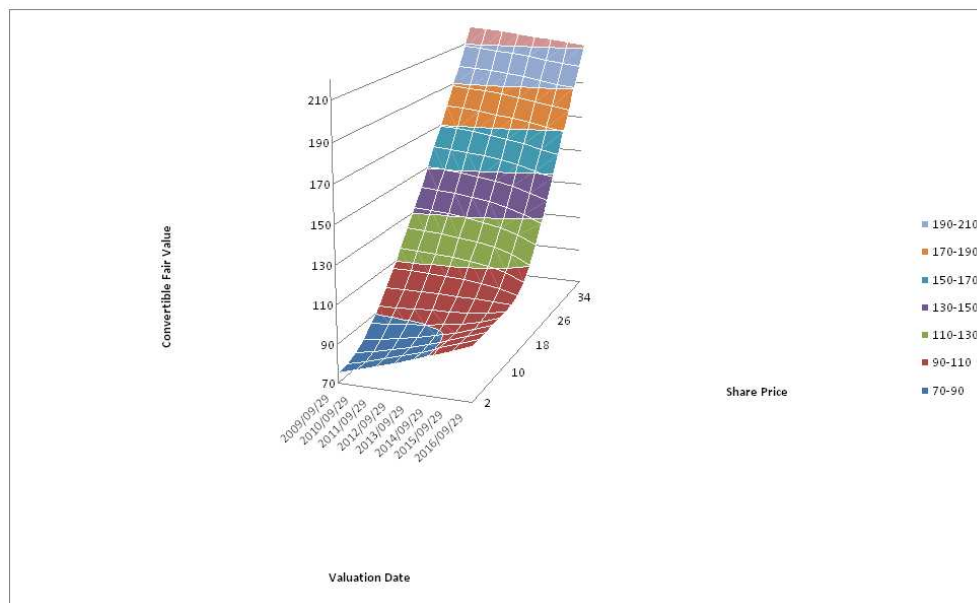


Figure 4.20: Convertible bond values at different valuation dates for ‘ViroPharma Inc 2% 15 March 2017 USD’.

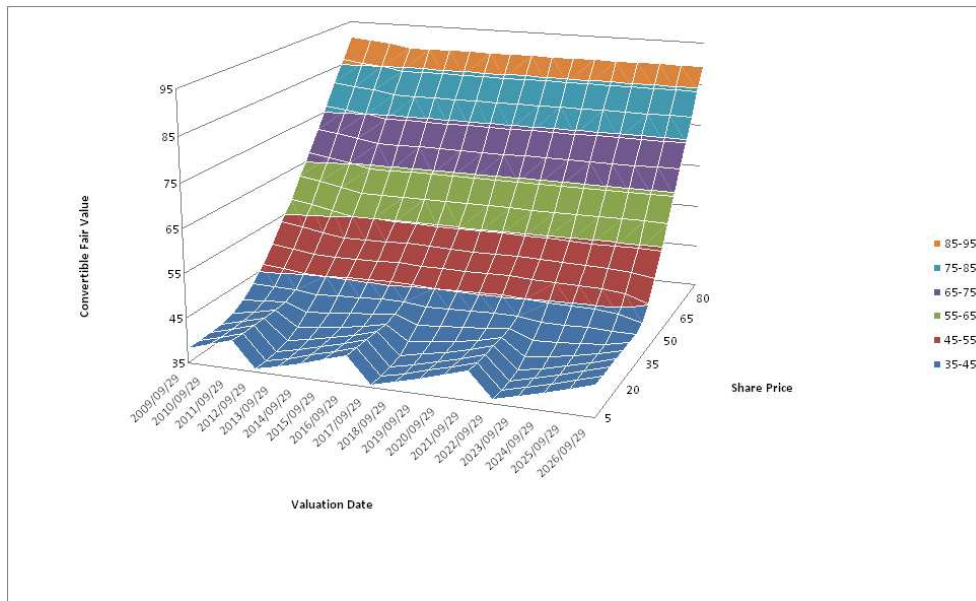


Figure 4.21: Convertible bond values at different valuation dates for ‘Business Objects 2.25% 01 January 2027 EUR’.

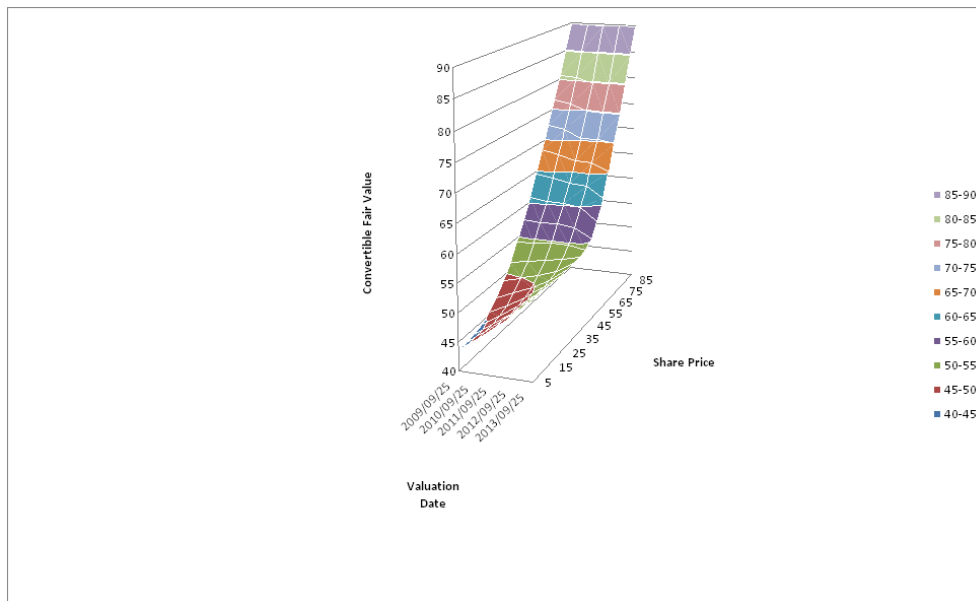


Figure 4.22: Convertible bond values at different valuation dates for ‘Rhodia 0.5% 1 January 2014 EUR’.

4.2 Analysis of the Greeks

Figures A.1 to A.26 in Appendix A (from page 93) exhibit the sensitivities of convertible bond values with respect to certain input parameters. These sensitivities are

numerical derivatives known as the Greeks and we depict these quantities for both ‘ViroPharma Inc 2% 15 March 2017 USD’ which is free of put and call provisions and ‘Business Objects 2.25% 01 January 2027 EUR’ which is subject to certain call and put stipulations.

Let us first consider the Greek measure delta. Delta measures the sensitivity of the equivalent option on one share to a unit increase in the spot share price of the underlying asset and can be thought of as the equity sensitivity of the convertible. Figures A.1 and A.2 show the variation of delta with a change in the share price while figures A.3 and A.4 assess these same values at differing valuation dates with annual intervals. Delta increases with an increasing share price and levels off to a value of 1 for high share prices. In Figure A.4, there are three peaks in the value of delta as we move towards maturity dates for relatively low share prices. These peaks occur leading up to the put dates and decrease after each put date has passed. Delta is an important output but traders are also interested in how delta may change as the share price moves. However, for significant share price moves, delta can be a poor guide to the sensitivity of the convertible bond. In general, the convertible is more equity sensitive in rising markets and less sensitive in falling markets. Gamma is the measure of the intensity of this effect. Figures A.5 and A.6 depict how gamma varies with the spot share price at the date of valuation. Gamma starts at zero and increases to its maximum value. The maximum value of gamma occurs at a share price which is smaller than the conversion price of the convertible bond in at the valuation date in both cases. It then decreases sharply once more and tends to zero as the share price increases. The variation of gamma with valuation date and share price is shown in figures A.7 and A.8. Firstly, at valuation dates closest to maturity, the gamma function peaks at a share price which is almost exactly the conversion price of the convertible⁵. This implies that at maturity, gamma (changes in delta) is greatest when the convertible bond is at the money (or at least close to the money). We also see that gamma increases as the valuation date moves closer to maturity once again proving that the changes in delta as a result of increases in share price are more extreme as we approach maturity date. The three peaks in the surface at low share prices as we move through time are again the result of the put feature built into the instrument. At high share prices, the value of gamma drops sharply to zero. This is a result of the callability of the convertible which commences on 11

⁵‘ViroPharma Inc 2% 15 March 2017 USD’ has a conversion price of \$18.87 while ‘Business Objects 2.25% 01 January 2027 EUR’ conversion price is €42.15

May 2012.

Theta measures the sensitivity of the convertible fair value to a one day decrease in the time to maturity or expiry of the instrument. It is not identical to the term used in the options market as it combines the redemption pull of a bond with the time decay of an option. Theta is expressed as a figure between -1 and 1. A theta of, for example, 0.1 implies a 10% sensitivity to a decrease in the time to maturity. So, should the time to maturity decrease by one day, the price of the equivalent option will increase by R0.10. If you have a negative Theta, the sensitivity is reversed so that if the time to maturity decreases, the price of the equivalent option will go down by the amount indicated⁶. For close to the money convertibles, theta is normally negative as the time decay in the option element outweighs any upward drift in the bond floor. For 'ViroPharma Inc 2% 15 March 2017 USD', when the stock price is very low, theta is close to zero and decreases with an increasing share price until levelling out. As a the convertible in figure A.11 approaches final maturity, we see two opposing effects on the value of the convertible. First, the value of the embedded call option decreases and the value of the convertible decreases. Secondly, the bond floor trends towards the redemption value over time and convertibles trading below redemption value will experience an upward bond floor 'drag to redemption'. At the money convertibles will usually suffer worst from the first of these effects with theta reaching its minimum value at a share price very close to the conversion price. When the convertible is out the money, the drag to redemption of the bond element is the dominant influence. The closer one gets to maturity date, the more noticeable the effects of these consequences. In Figure A.10 on page 97, for low share prices theta has a positive value. This is because of the put features built into the 'Business Objects 2.25% 01 January 2027 EUR' instrument. The investor is more likely to take advantage of the convertible's putability at low share prices and with a decrease in time to maturity, the investor nears closer to a put date thus increasing the convertible fair value. Once all three put dates have passed (the final one being 11 May 2022), the investor no longer has the feature of downside protection and thus from that point onwards the convertible's theta is zero for low share prices.

Figures A.13 to A.16 show the sensitivity of the convertible fair value with respect to movements in spot interest rates, termed Rho. Rho is expressed as a figure be-

⁶On 21 September 2009, 'ViroPharma Inc 2% 15 March 2017 USD' had a theta of -0.00141 and 'Business Objects 2.25% 01 January 2027 EUR' had a theta of -0.00094

tween -1 and 1. A Rho of 0.1 for example implies a 10% sensitivity to an increase in interest rates all along the interest rate curve. Conventionally, rho is expressed as the change in convertible price for one basis point move in interest rates. Increases in the interest rates has a greater negative impact on the value of the bond floor than it does a positive impact on the value of the embedded call option of the convertible and as such, as the share price falls and the bond floor becomes an important component of a convertible's valuation, the sensitivity of the convertible value to changes in interest rates increases.

In order to find the fair value of a convertible bond we have had to make some sort of volatility assumption on the underlying stock. Vega measures the sensitivity of the fair value to a 1% increase in the assumed level of stock volatility for the instrument or underlying asset; whether the data is entered as flat data, term structured or as a volatility surface. As can be seen from figures A.17 and A.18, vega is always positive on standard convertibles and is largest when the convertible is at the money. Changes in the stock volatility assumption may not have any material impact on fair value if the convertible is out of the money or deep in the money. We also learn from figures A.19 and A.20 on page 102 that convertibles are more sensitive to changes in volatility the further away they are from maturity date.

As seen above, there are various ways to account for the probability of the issuing company defaulting in the bond i.e. the company's credit risk. The most straightforward method to account for this is to use a flat credit spread to be added to the risk free yield curve. We define omicron as a measure of the sensitivity of the convertible value to a unit basis point increase in the credit risk data for the instrument or underlying asset. Omicron is expressed as a figure between -1 and 1. For example, an omicron of 0.7 implies a 70% sensitivity to an increase in credit risk and so should the flat credit spread used increase by 1 basis-point, the value of the convertible bond will increase by R0.70. If you have a negative omicron, the sensitivity is reversed; if the credit risk goes up, the price of the equivalent option will go down by the amount indicated. Figures A.21 and A.22 show that the change in convertible bond fair value with respect to the credit spread is less sensitive at lower share prices where the convertible bond is synthesising debt.

We define phi as the sensitivity of the convertible fair value to an increase in the underlying dividend. A phi of 0.05 indicates a 5% sensitivity to an increase in dividend yield. Page 104 depicts the relationship between phi and the underlying share price for both 'ViroPharma Inc 2% 15 March 2017 USD' and 'Business Objects 2.25%

01 January 2027 EUR'. We know that an increase in dividend rates reduces the convertible price from figure A.23 and A.24 we see that the sensitivity to changes in dividend yield increases as the share price increases. This makes sense as the greater the share price, the more the convertible behaves like the underlying share. Since it behaves more like the share, increases in dividend yields would have a greater effect on the convertible and decrease the value of the convertible when the dividend yield increases.

Finally we define ϵ as the measure of the sensitivity of the convertible fair value to a 1% increase in the recovery rate percentage for the instrument. The recovery rate is the expected proportion that is returned to the convertible holder in the event of default and is represented as a proportion of the face value of the instrument. ϵ is expressed as a figure between -1 and 1. So an instrument with an ϵ of 0.2 for example implies that should the recovery rate increase by 1%, the price of the equivalent option will increase by R0.20. If however we had a negative ϵ , the sensitivity is reversed. Figure A.25 shows that the change in convertible fair value with respect to the recovery rate is greatest at low share prices where the convertible is synthesising debt. There is a slight difference when dealing with 'Business Objects 2.25% 01 January 2027 EUR' in figure A.26. ϵ increases until a maximum point when it is near to the money; after which it continues to decrease steadily with an increasing share price as is the case in figure A.25.

4.3 Model Sensitivities and Analysis

Figures 4.23 and 4.24 represent the probability of the convertible defaulting during its lifetime. A default probability of say 0.5 indicates that out of all possible scenarios (conversion, redemption, call, put and default), there is a 50% likelihood that the bond will default. We model the default probability in three dimensions with changing share prices and at different valuation dates. We first consider the default probability of 'ViroPharma Inc 2% 15 March 2017 USD'.

By observing figure 4.23 we deduce that the probability of default decreases as we approach the maturity date of the instrument. Also, when the issuing company's share price is very low, there is a greater chance of default as the company is at risk of being dissolved. This explains the inverse relationship that exists between share price and default probability.

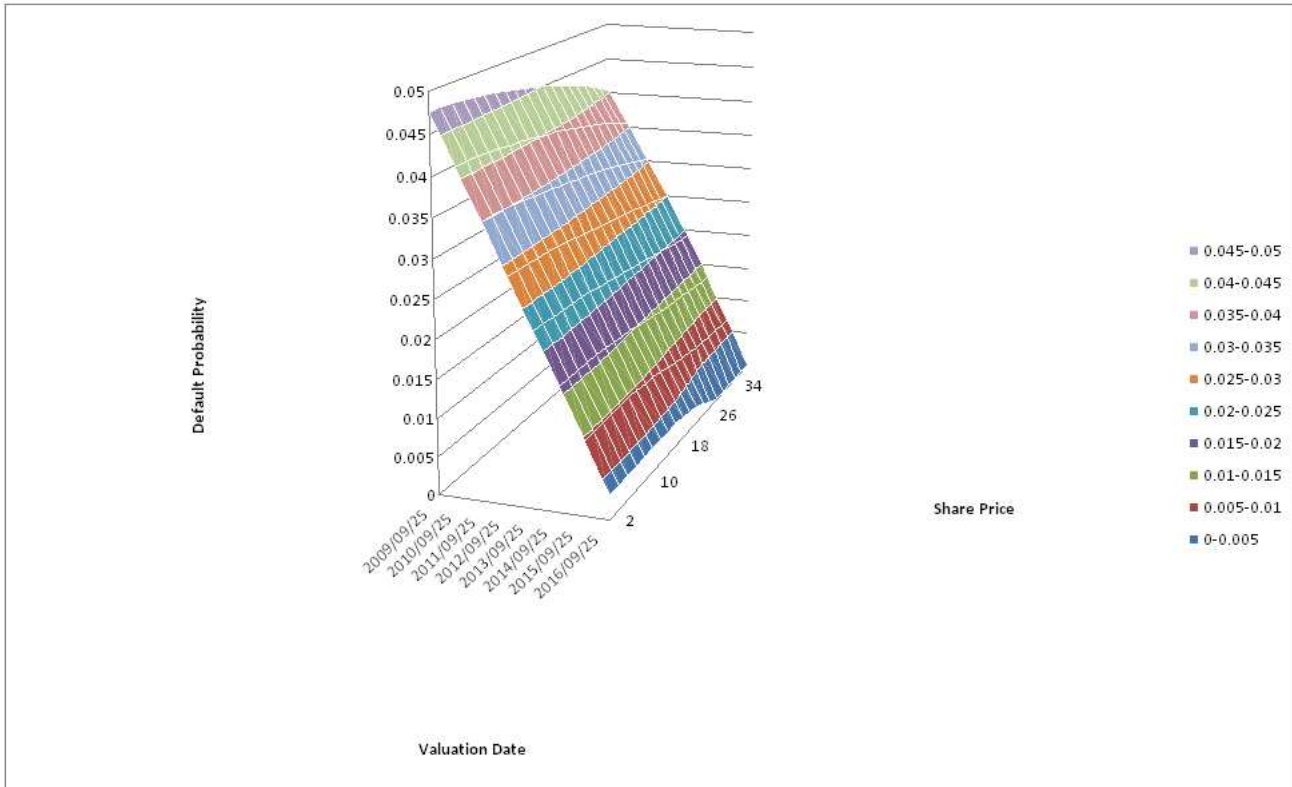


Figure 4.23: Default probability of ‘ViroPharma Inc 2% 15 March 2017 USD’

We now consider ‘Business Objects 2.25% 01 January 2027 EUR’ which is puttable on three distinct dates throughout its lifetime and is callable from 11 May 2012. Once again, there exists an inverse relationship between the share price and the likelihood of default. In fact, with the introduction of a call feature on the convertible bond, it is even less likely that the bond will default at high share prices. This explains why the default probability is actually equal to zero and not slightly bigger than zero for high share prices at valuation dates after the first date at which it is possible for the issuer to call the bond at the corresponding call level. We once again recognise three distinct rises and drops along the curve as we move closer towards maturity date. These peaks followed by big falls occur at low share prices and taper off as the share price increases. The peaks occur at the put dates throughout the life of the convertible and can be explained as follows. At low share prices it is more likely that the convertible bond holder will put the bond if he has the opportunity to do so rather than hold on to the convertible which has less value. Knowing this, the issuer might consider instead to default on the bond. At these low share prices

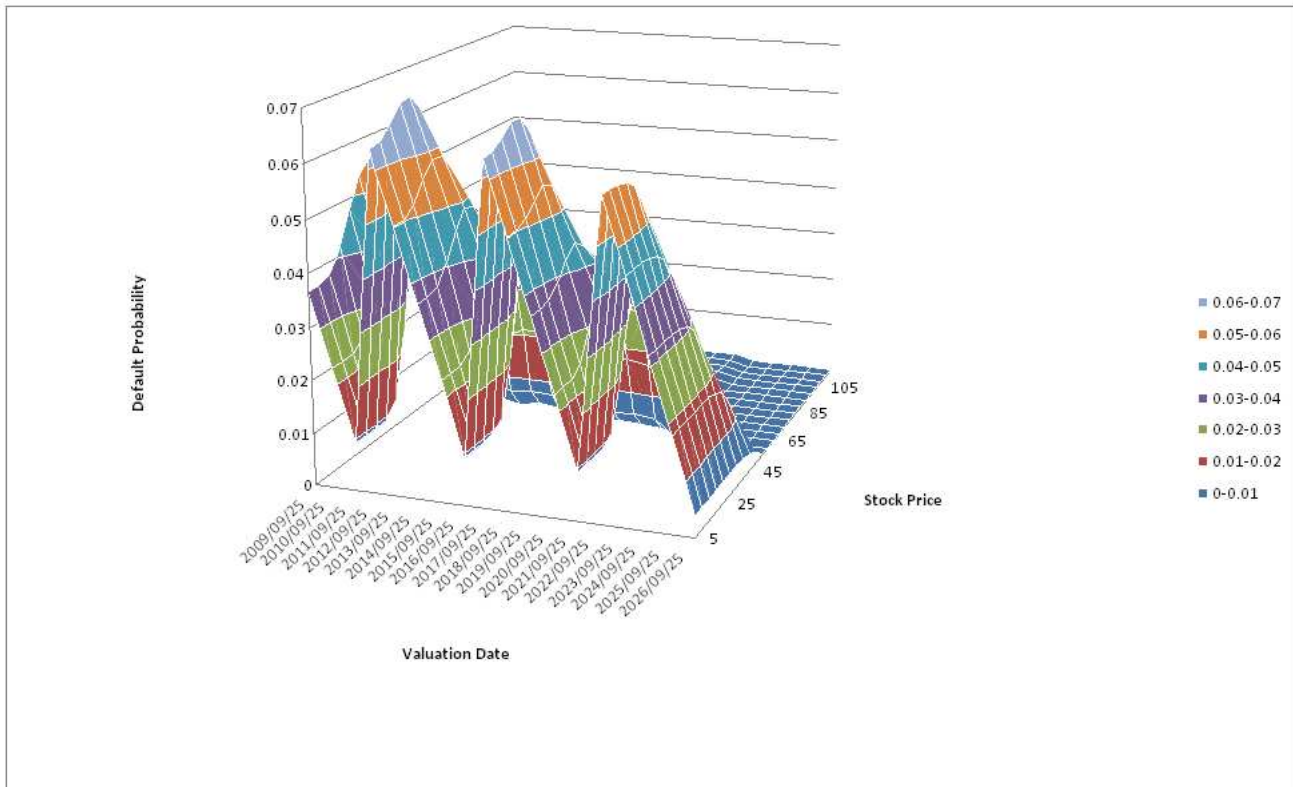


Figure 4.24: Default probability of 'Business Objects 2.25% 01 January 2027 EUR'

where it is inevitable that the convertible holder will exercise his right to put the bond, defaulting becomes comparatively profitable and so defaulting on the bond as we approach each put date will become more attractive.

We now look at the complete opposite spectrum and speak about the possibility that the convertible investor will redeem the convertible bond as he would a normal straight bond. The redemption probability is the probability that the bondholder will not convert (or call or put) on or before the Last conversion date and will therefore hold the bond until the Redemption date. A redemption probability of say 0.05 indicates that out of all possible scenarios (conversion, redemption, call, put and default), there is a 5% likelihood that the life of the bond will expire only when the bond is redeemed.

From figure 4.25 we observe that the probability of redemption decreases with an increasing share price. This is intuitively obvious as the convertible holder is far more likely to convert at high share prices than redeem the bond and receive its redemption value. The degree of the redemption probability also changes with differing

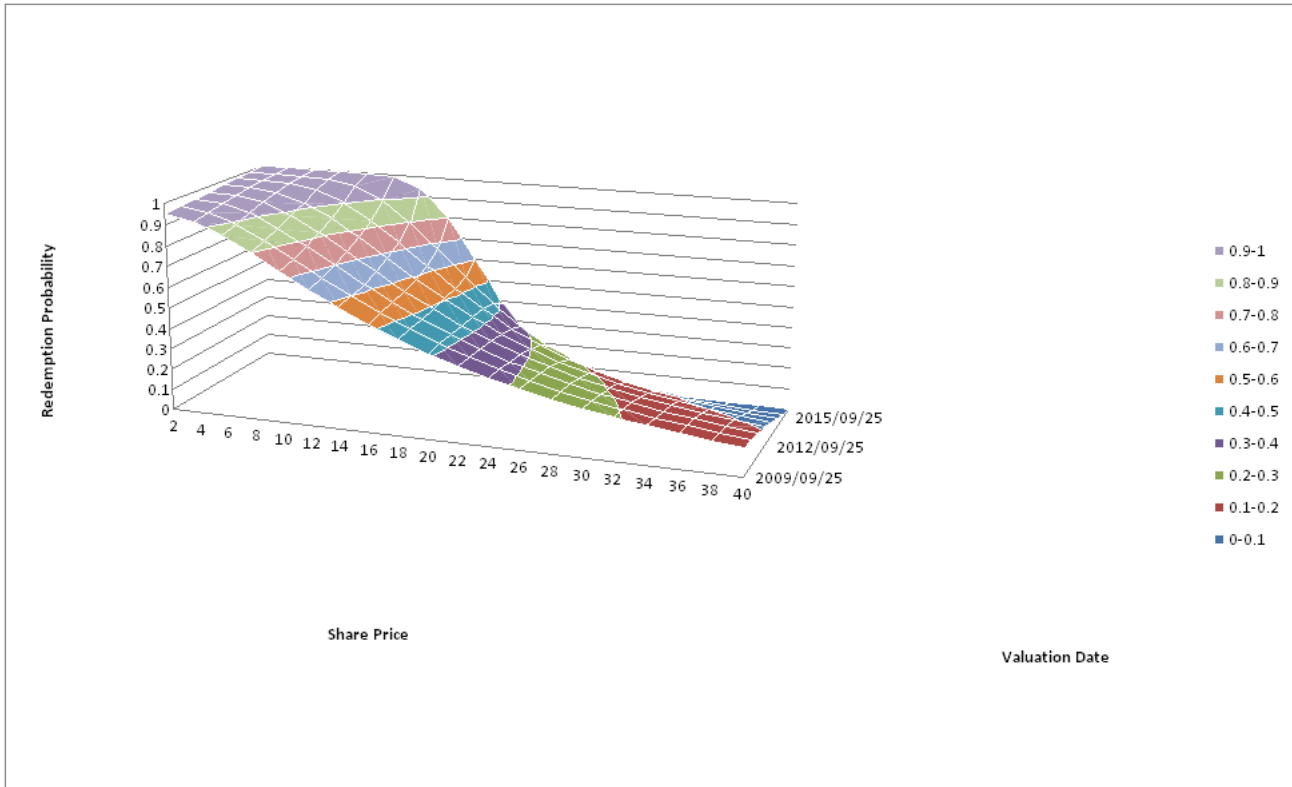


Figure 4.25: Redemption probability of ‘ViroPharma Inc 2% 15 March 2017 USD’

valuation dates and is more extreme the closer the convertible is to its maturity date. When the convertible bond still has some time until maturity, there is much more room for the share price to differ (and differ considerably) to its original value. For this reason, even at extremely low share prices, the convertible is not certain to be redeemed as the share price has time to rally and potentially reach a value that it becomes more profitable to convert the bond into shares. However, at dates close to maturity, even moderately low share prices suggest that the most likely outcome of the convertible would be redemption. The same applies for large share prices. As we move through time to approach the maturity date of the convertible, sizeable share prices will less likely lead to redemption and almost definitely be converted into ordinary shares.

Again our analysis must be modified when we regard the put and call particulars that affect ‘Business Objects 2.25% 01 January 2027 EUR’ in figure 4.26. At low share prices, the ability to put the convertible back to the issuer at three distinct dates makes redemption almost impossible at dates prior to the last put date on

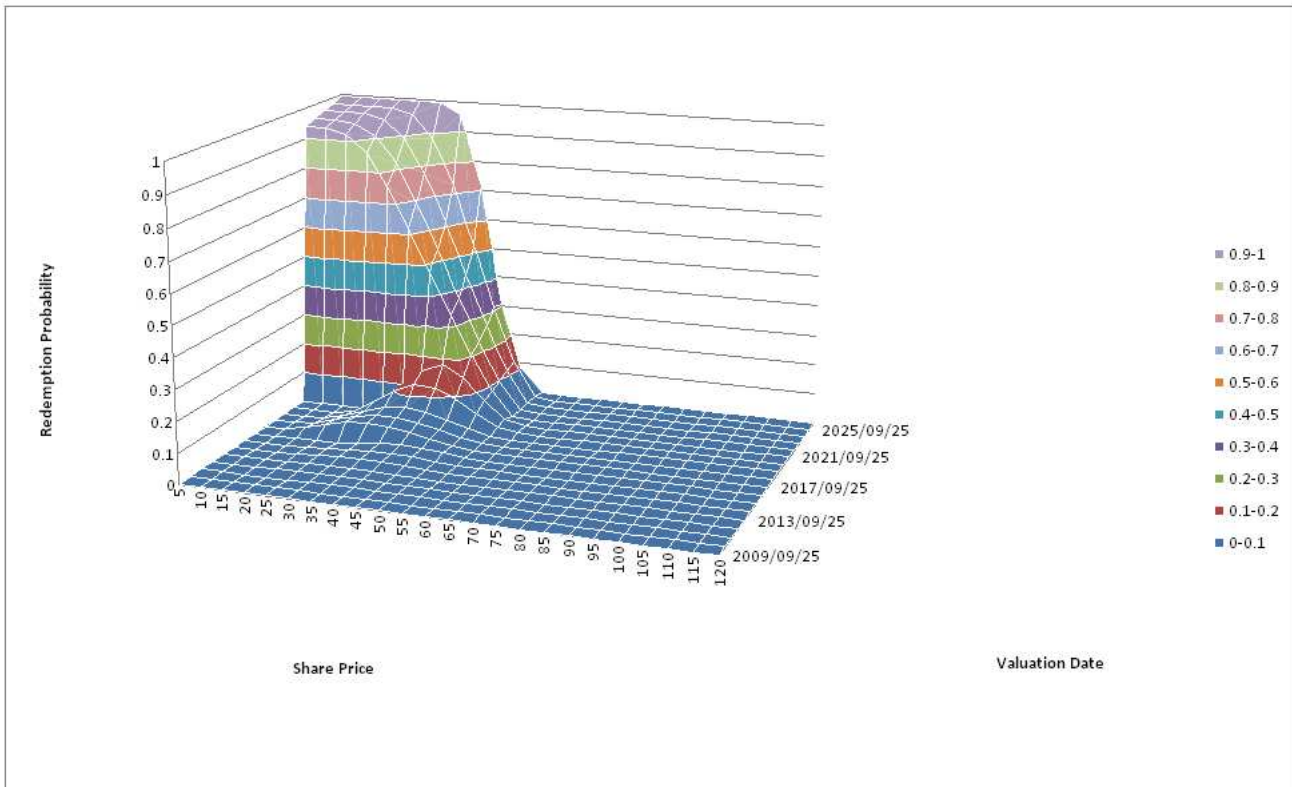


Figure 4.26: Redemption probability of ‘Business Objects 2.25% 01 January 2027 EUR’

11 May 2022. Likewise, at high share prices where redemption is already less likely due to conversion probabilities being greater, we also have the possibility of the convertible being called from 11 May 2012 onwards⁷.

4.3.1 Model Comparisons

It is useful to make some comparisons between different pricing models. As mentioned above, many of the models have different pros and cons and in this section we will focus on two specific models for comparison. The first model is the tree model found in Hull [2005]. This is labeled ‘Trinomial’ in the graphs that follow. The second model is the PDE method by Tavella and Randall [2000], called ‘PDE’. We compare the prices generated using these two approaches with the outputs from a simple binomial pricing model that we have set up.⁸

⁷The bond becomes callable if the trigger is activated. The trigger is 125% of the conversion price i.e. $125\% \times 42.15 = 52.6875$

⁸The VBA code for this pricing model can be found in the appendix.

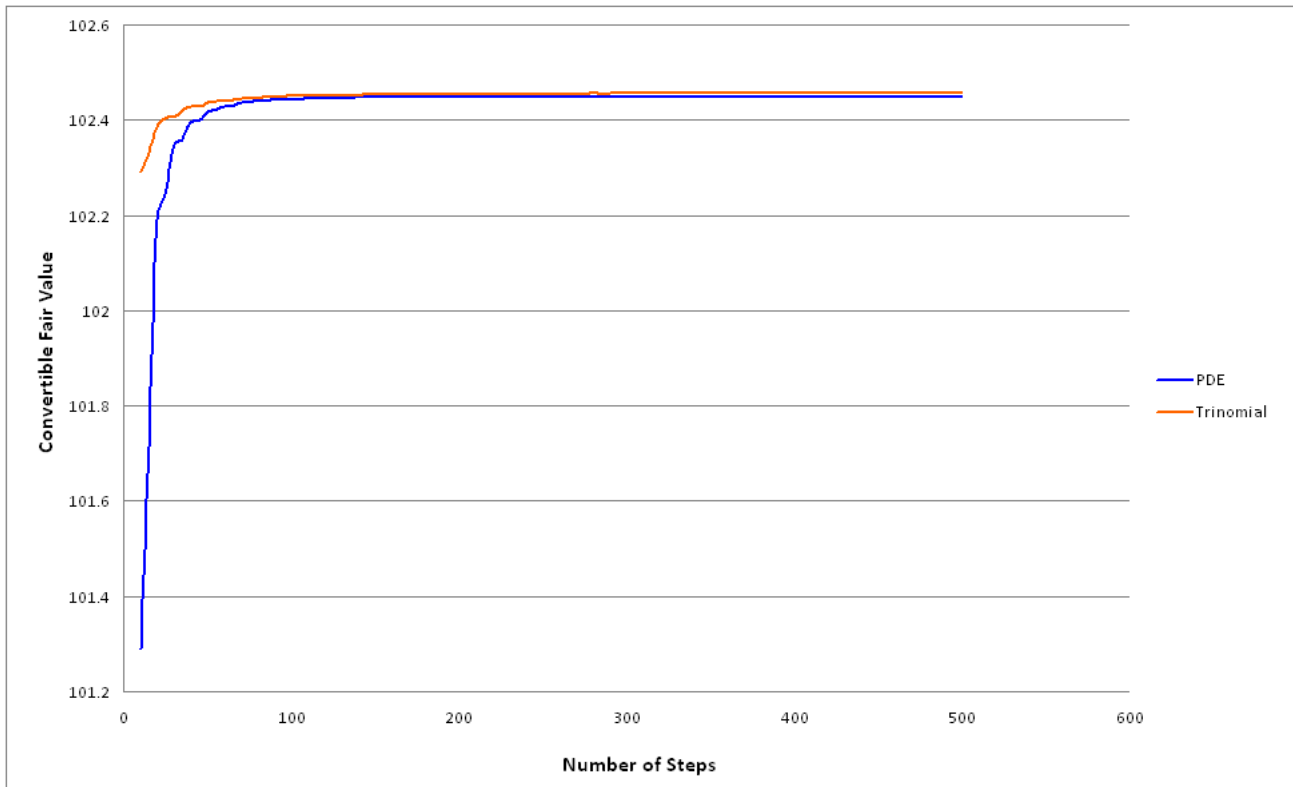


Figure 4.27: Value of ‘ViroPharma Inc 2% 15 March 2017 USD’

Figure 4.27 shows the convertible fair value against the number of steps in either the ‘PDE’ or ‘Trinomial’ model for ‘ViroPharma Inc 2% 15 March 2017 USD’. As the number of steps increases, the two models converge to a value of around 102.43 for both models. It is evident that as the number of time steps increases, the difference in the fair value produced by the two models decreases and they tend towards an accurate convertible price. This price is slightly greater than the value calculated by our simplified binomial model of 101.64237 with 500 time steps which in turn is greater than the price that ‘ViroPharma Inc 2% 15 March 2017 USD’ is trading in the market (on 21 September 2009, ‘ViroPharma Inc 2% 15 March 2017 USD’ was trading at 101.506). This implies that according to the three models used, ‘ViroPharma Inc 2% 15 March 2017 USD’ is believed to be undervalued within the active market. The relatively lower market price may be due to the fact that market practitioners use simpler pricing models that are easy to understand by all market users. This may also explain why our simple pricing model results in a price closer to the market price than to the price generated by the more advanced ‘Trinomial’ and ‘PDE’ models.

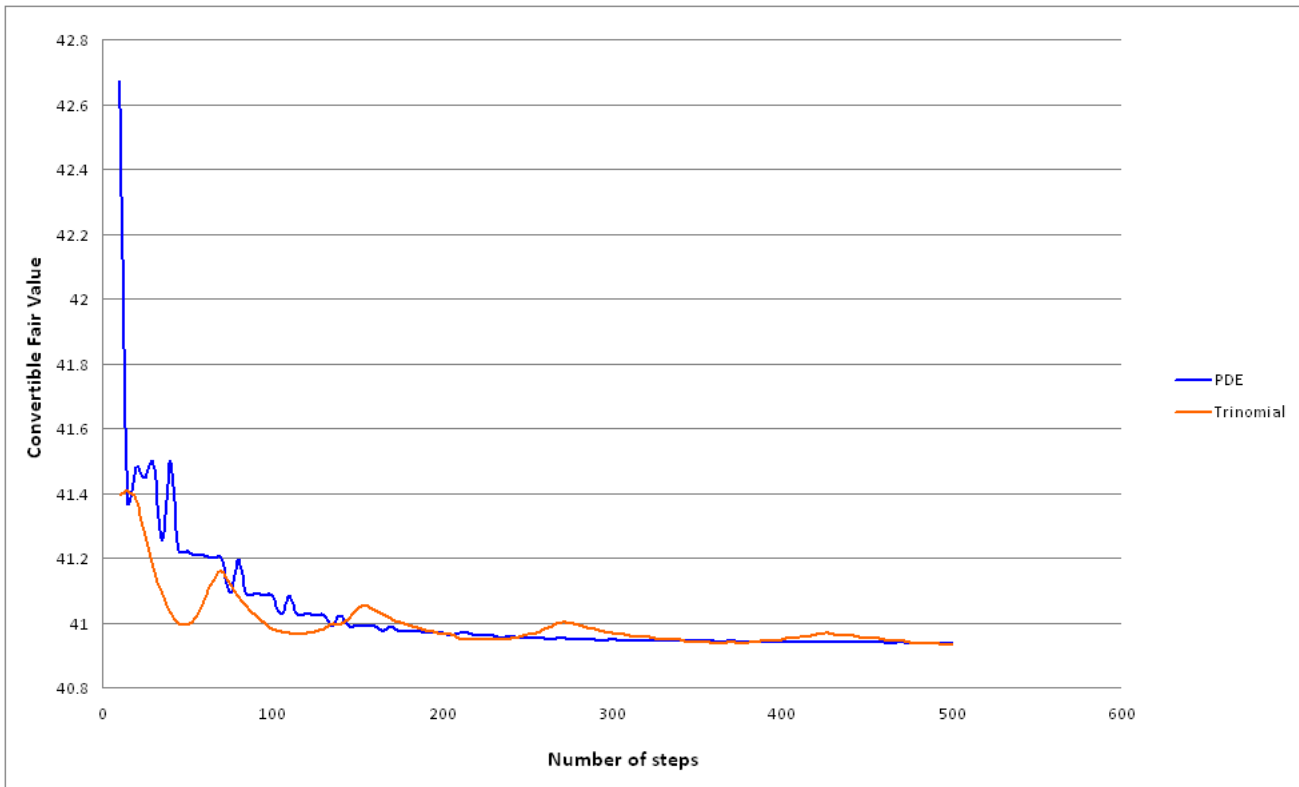


Figure 4.28: Value of ‘Business Objects 2.25% 01 January 2027 EUR’

The analysis of figure 4.28 is slightly more interesting. ‘Business Objects 2.25% 01 January 2027 EUR’ is both callable and putable and the consideration of these features seems to have an affect on both the ‘Trinomial’ and ‘PDE’ models. With an increase in time steps, both models tend towards a price of approximately 40.912. The interesting part is the oscillating behaviour of the pricing model as the time steps increases, especially for the ‘Trinomial’ model. We should be concerned about this phenomenon as this could lead to spurious results in pricing. Suprisingly, this time our two featured models value the convertible bond by a lower amount than the market value. On 21 September 2009, the market values ‘Business Objects 2.25% 01 January 2027 EUR’ at 43.906 and using our simplified pricing model we attain a convertible price of 42.117. The reason for these lower values can be credited to the more accurate treatment of the call and put characteristics in the more intricate ‘Trinomial’ and ‘PDE’ models.

Chapter 5

Convertible bonds in the South African Market

“Education is the most powerful weapon which you can use to change the world.”

Nelson Mandela

On 11 May 2009, Aquarius Platinum, via their manager and underwriter Rand Merchant Bank (RMB) issued a 3 year convertible bond to the value of ZAR 650 000 000. The issue is interesting for a number of reasons. Firstly, the bond will bear a floating interest rate of 3-month JIBAR plus a margin of 3% per annum compounded quarterly in arrears and paid semi-annually in arrears commencing on 30 October 2009. Another rare (but not impossible) feature that the issue contains is a 1 year hard call element requested by the issuer, Aquarius Platinum. This clause stipulates that the issuer may redeem the convertible at any time within the first anniversary of the issue date at 115% of the principal amount together with the accrued interest to date if the volume weighted average price exceeds 128% for more than twenty consecutive dealing days.

This issue confirms the interest shown by South African investors in convertible securities as more convertible bonds continue to be listed on the Main Board of the Johannesburg Stock Exchange (JSE). In the past, this asset class has been relatively under-utilised in the country. Instead, South African companies have issued their convertibles offshore in exchanges such as the London Stock Exchange (LSE) and in particular the Singapore Exchange (SGX) as the latter is known to be relatively lenient with regard to listing regulations. But as with Aquarius Platinum’s convert-

ible bond listing on the JSE under the resource sector, the manager of the bond, RMB, is hoping that this will stimulate the extremely profitable and flexible hybrid securities market within South Africa.

The Aquarius convertible bond issue is attractive to both the issuer and investors. Aquarius will be able to refinance its debt and recapitalise its balance sheet at a far cheaper rate than it would by raising finance from a bank or through a corporate bond issuance. Aquarius plan to use the proceeds of the bond issue to fund existing obligations and re-open other operations such as its Everest mining operation.

Investors in the three year instrument have the ability to convert the bond into a fixed number of Aquarius shares (calculated on a 25% conversion premium to the base share price) at any time after a year after issuance of the bond. If no conversion takes place, investors are repaid their capital outlay, plus any accrued interest, after three years. A private placement of the bond raised the required R650 million. Each bond has a face value of R10 000, a base price of R30.51 and a conversion price of R38.13, fixing the number of shares underlying each bond at 262. It also provides all the standard safety features of a debt security package as well as a twice-yearly interest payment on the bond based on the generous three-month Jibar rate plus 3%.

Barry Martin of RMB Debt Capital Markets summed up what the issue meant to the market when he said the following in a recent press release: 'The success of the private placing illustrates local buy-side demand for convertible bonds. South African corporates which previously had to go offshore to raise convertible bond funding can now do so within the country on the JSE.'

Chapter 6

Conclusion

“It’s more fun to arrive at a conclusion than to justify it.” Malcolm Forbes

Convertible Bonds are complex and innovative financial instruments. Their properties can be tailored to give flexibility to the issuer’s funding and specific risk/return profile to the investor. Additionally these structures can be efficient in tax and regulatory terms for the borrower, while investors enjoy the downside protection of a bond like instrument with healthy equity upside exposure.

In this paper, we have focused on pricing relatively simple convertible bond issues. The main complexities we have included are call and put provisions. The beauty of convertible bonds however is that they may be viewed a mass of gauges and dials, all of which can be tweaked in a variety of manners to create many creative and personalised instruments. Features such as reset clauses and adjustments to conversion price are just two features which may be considered when creating a new convertible issue as was seen with Aquarius Platinum issuing a convertible with a floating coupon rate.

Although we have only dealt with vanilla convertibles in this paper, convertible structures continue to become more complex and innovative and can be structured in many ways to bear a greater resemblance to either debt or equity investments. Zero coupon convertibles pay no coupon and have a current yield of 0% and are usually issued at a large discount. Convertible preferreds are a common structure in the convertible market in the US. The instrument is a preferred stock that pays a fixed dividend and carries rights of conversion into the issuers’s ordinary shares. Manda-

tory structures such as PERCs (preferred equity redemption cumulative stock) and DEC's (dividend enhanced common stock), mandatorily convert into ordinary shares at maturity and therefore create no cash flow problems for the issuer at maturity as redemption will not be required. Another very popular example is exchangeables which are bonds issued by one company that exchange into the shares of another company.

The valuation of convertible bonds is based on derivatives technology, and convertible bonds can be priced using many different techniques. We have highlighted only a handful in this text with the tree methodology being the most popular choice. We must recall however that any valuation model contains assumptions which may not always hold true, and many require calibration which can be complex and problematic. Even our small collection of models may be adjusted in certain ways (by modeling stochastic volatility and interest rates for example) to give new, possibly more accurate pricing models. The key factors within the model in determining fair value are the underlying stock price, its volatility, the risk free interest rate and the issuer's credit spread.

As is seen from our sensitivity analysis in chapter 4, it is vitally important to model the call, put and conversion clauses carefully as these contract features have a profound impact on the convertible fair value especially when the equity is trading close to the call and put prices. Therefore, the start date, end date and prices of these features must be captured accurately within whatever numerical approximation is used. We may also bear in mind the conclusions of Brennan and Schwartz [1980] who found that modeling the interest rate as a stochastic factor rather than a deterministic factor is of secondary importance to modeling the firm value as a stochastic factor ¹ i.e. sometimes simpler models may be as accurate and effective as the more complex models.

Convertibles can offer many opportunities for the investor, but it is important for all the features of the instrument and its issuer to be understood completely before an investment decision is made. A detailed analysis of the future prospects of the borrower is every bit as important as a sophisticated derivatives pricing model.

¹Although, we primarily dealt with modeling the equity price rather than the firm value.

Appendix A

Graphs - Greeks

“There are men who can think no deeper than a fact.” Voltaire

DELTA

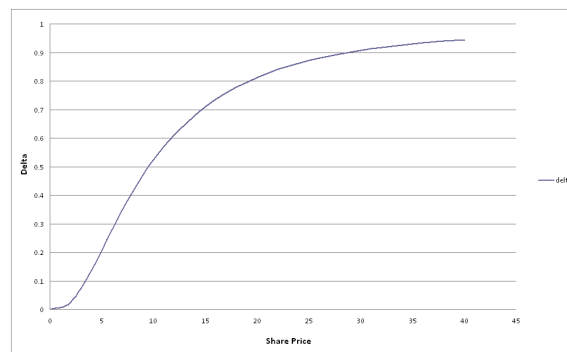


Figure A.1: ‘ViroPharma Inc 2% 15 March 2017 USD’

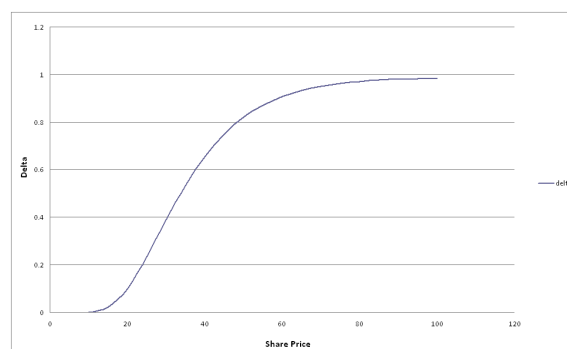


Figure A.2: ‘Business Objects 2.25% 01 January 2027 EUR’

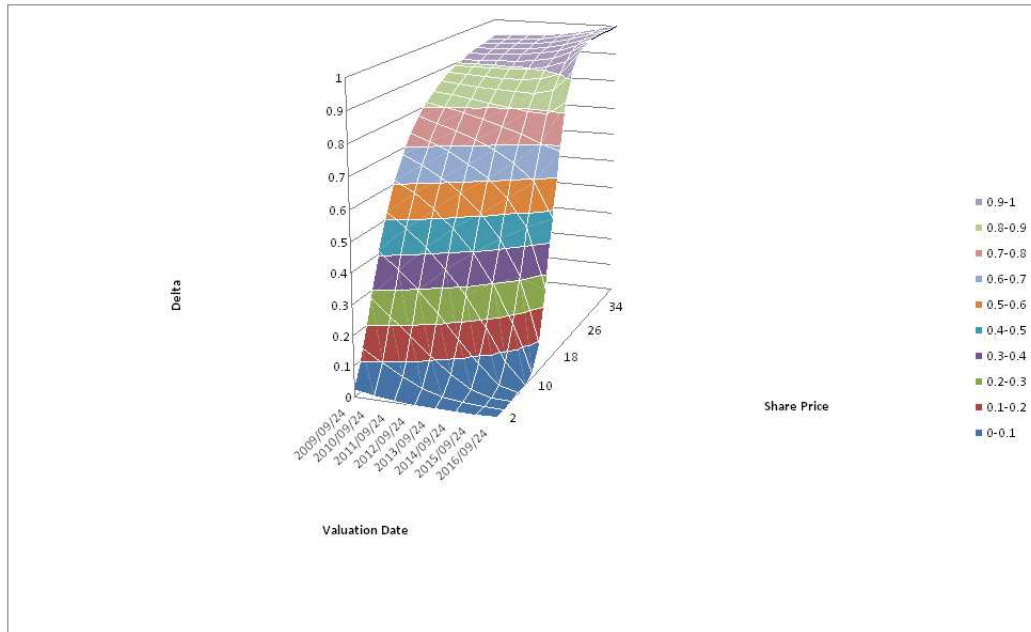


Figure A.3: 'ViroPharma Inc 2% 15 March 2017 USD'

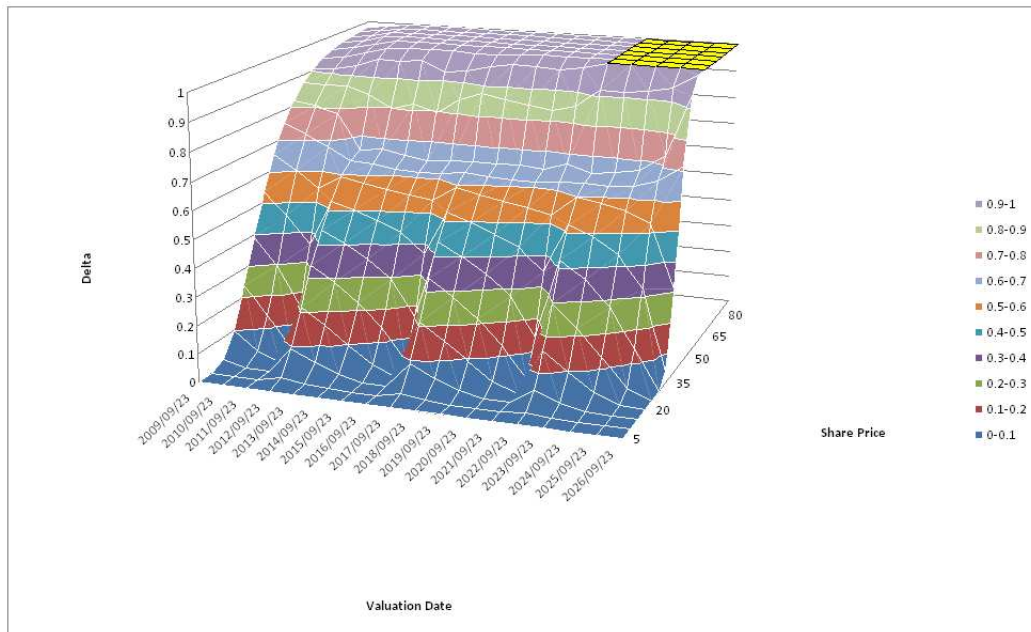


Figure A.4: 'Business Objects 2.25% 01 January 2027 EUR'

GAMMA

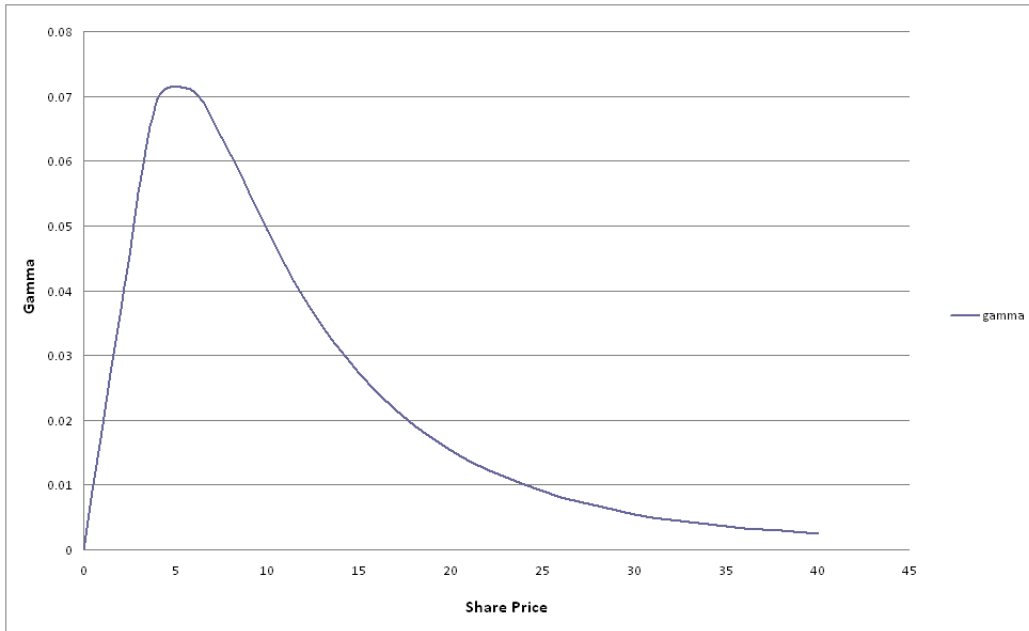


Figure A.5: 'ViroPharma Inc 2% 15 March 2017 USD'

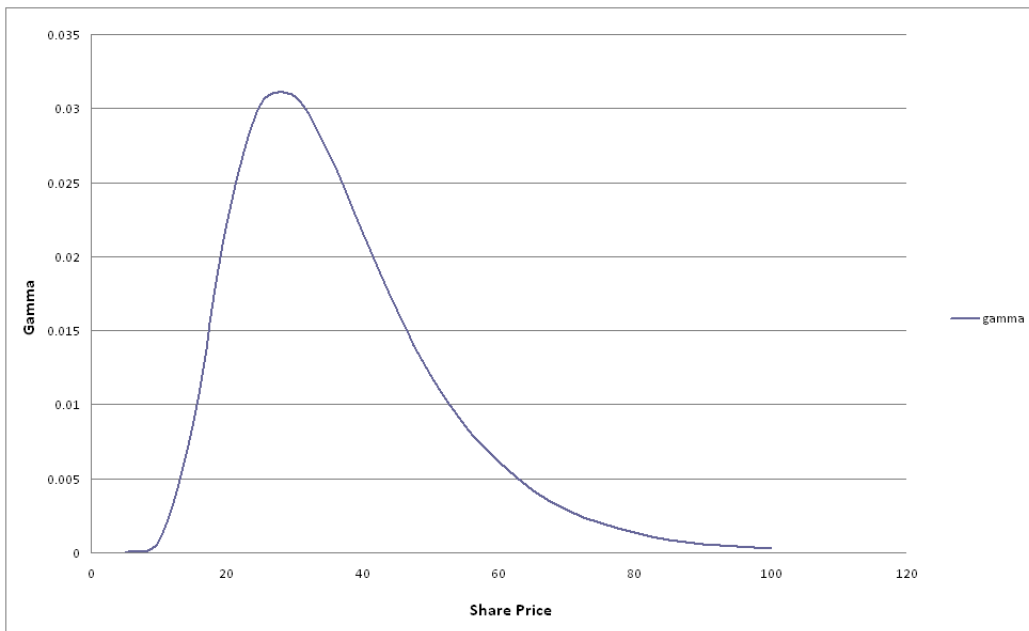


Figure A.6: 'Business Objects 2.25% 01 January 2027 EUR'

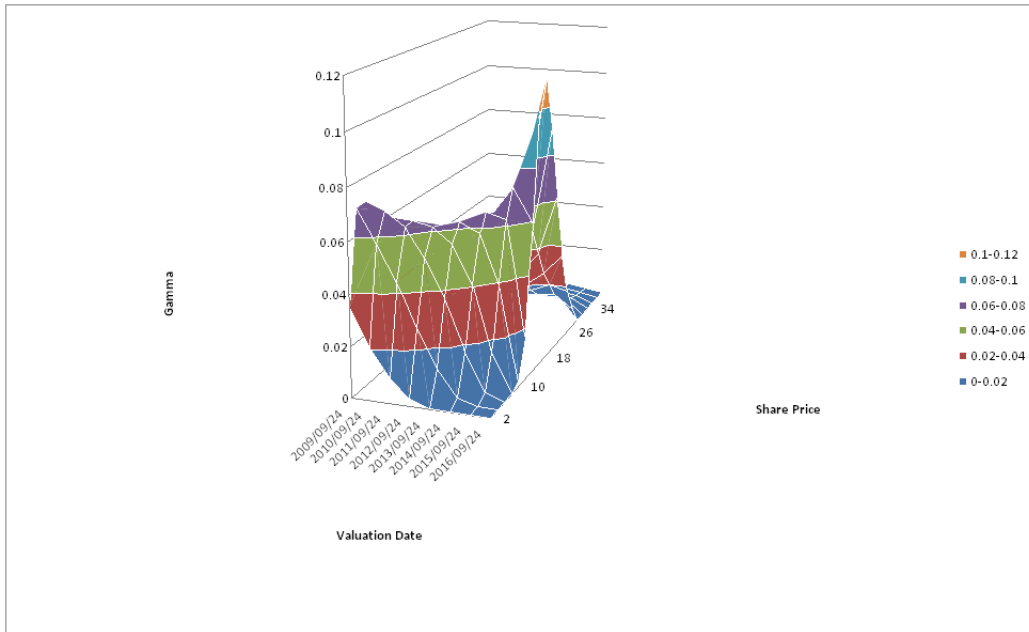


Figure A.7: 'ViroPharma Inc 2% 15 March 2017 USD'

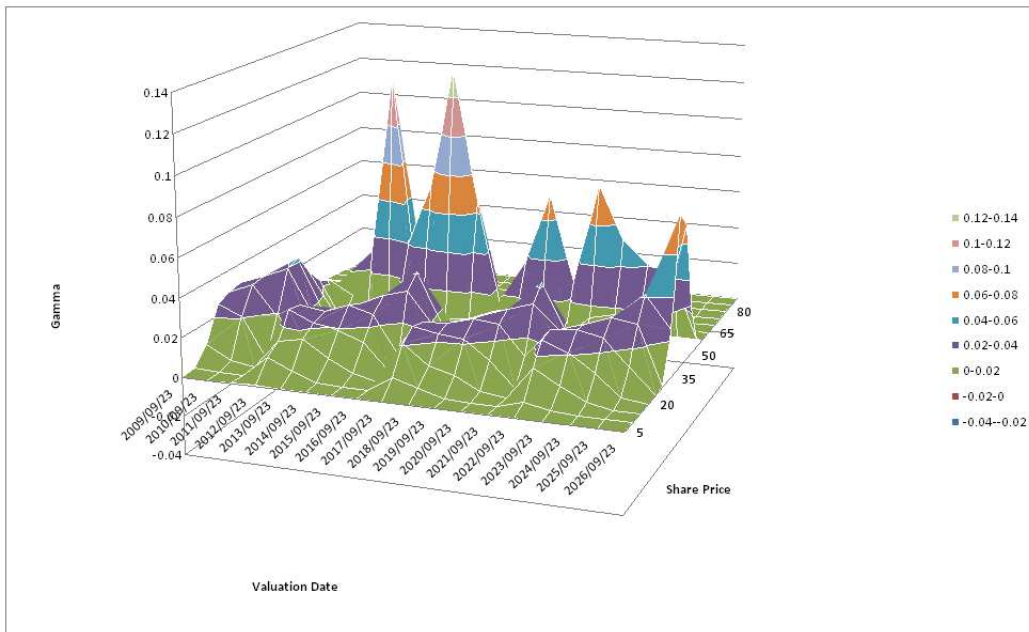


Figure A.8: 'Business Objects 2.25% 01 January 2027 EUR'

THETA

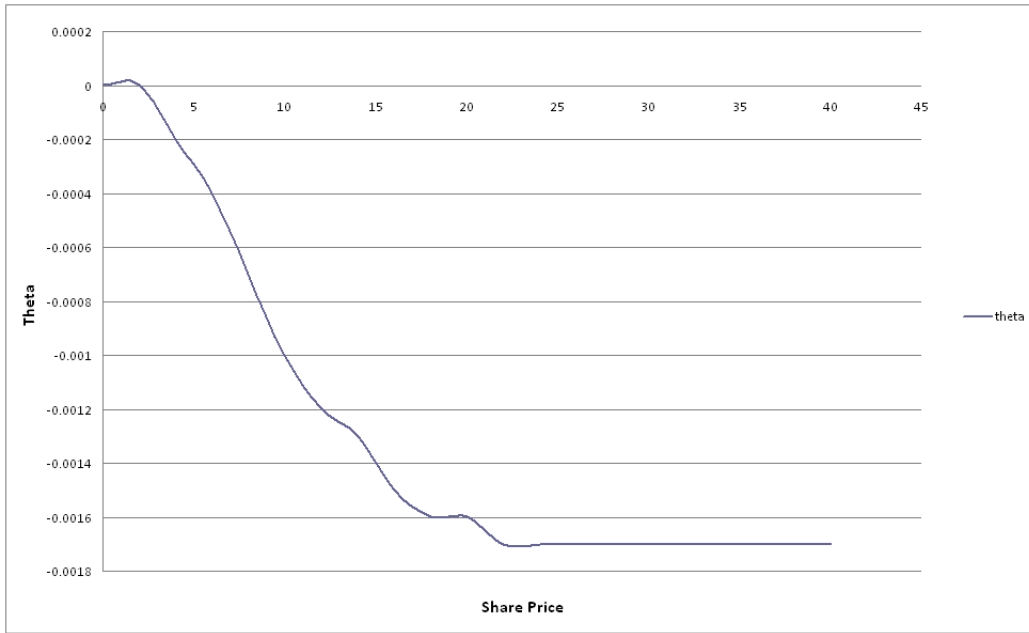


Figure A.9: 'ViroPharma Inc 2% 15 March 2017 USD'

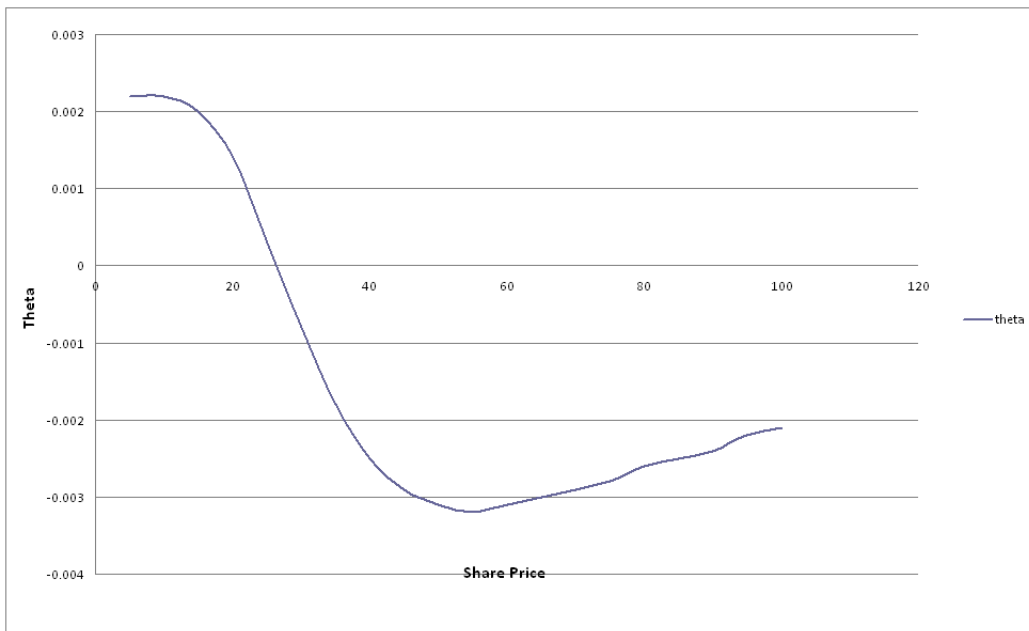


Figure A.10: 'Business Objects 2.25% 01 January 2027 EUR'

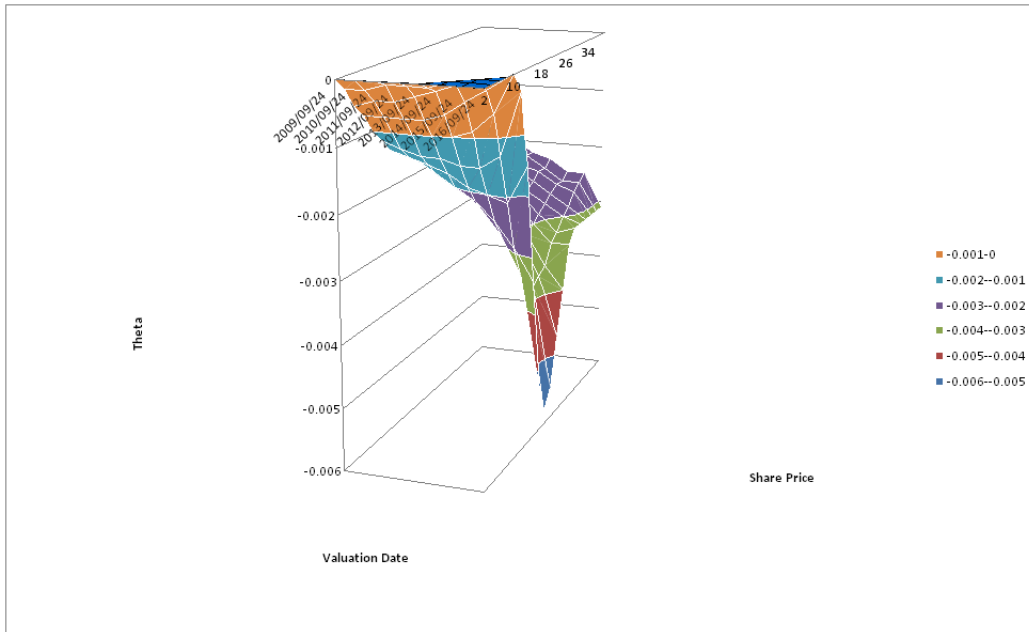


Figure A.11: 'ViroPharma Inc 2% 15 March 2017 USD'

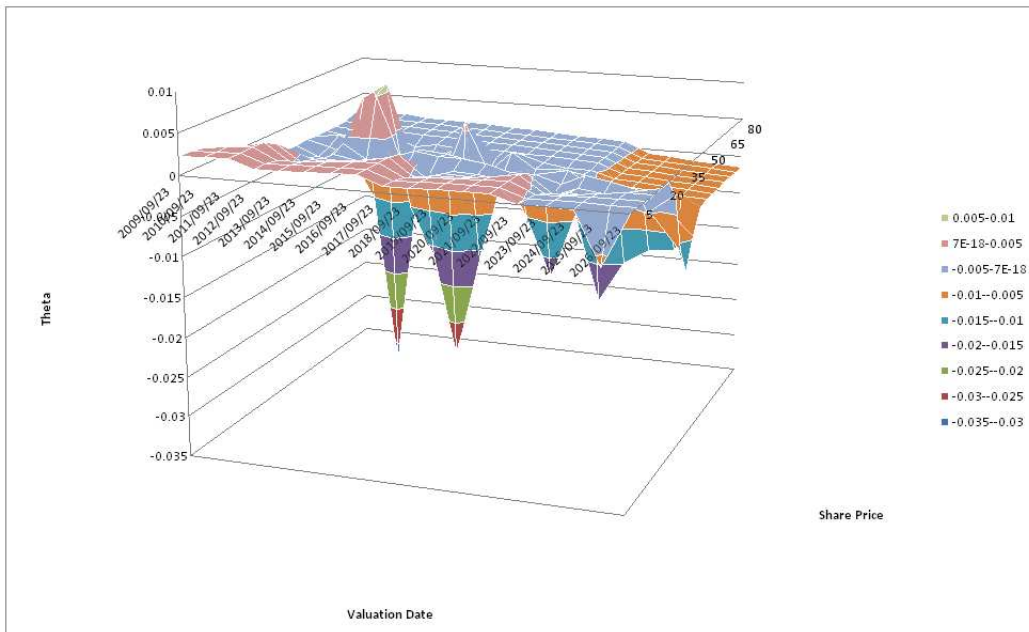


Figure A.12: 'Business Objects 2.25% 01 January 2027 EUR'

RHO

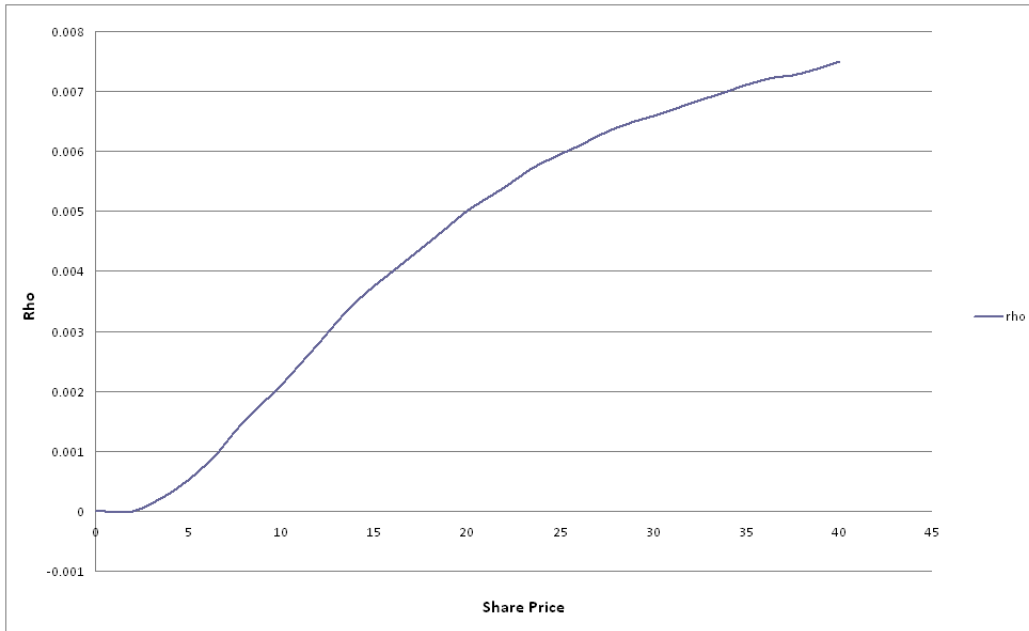


Figure A.13: 'ViroPharma Inc 2% 15 March 2017 USD'

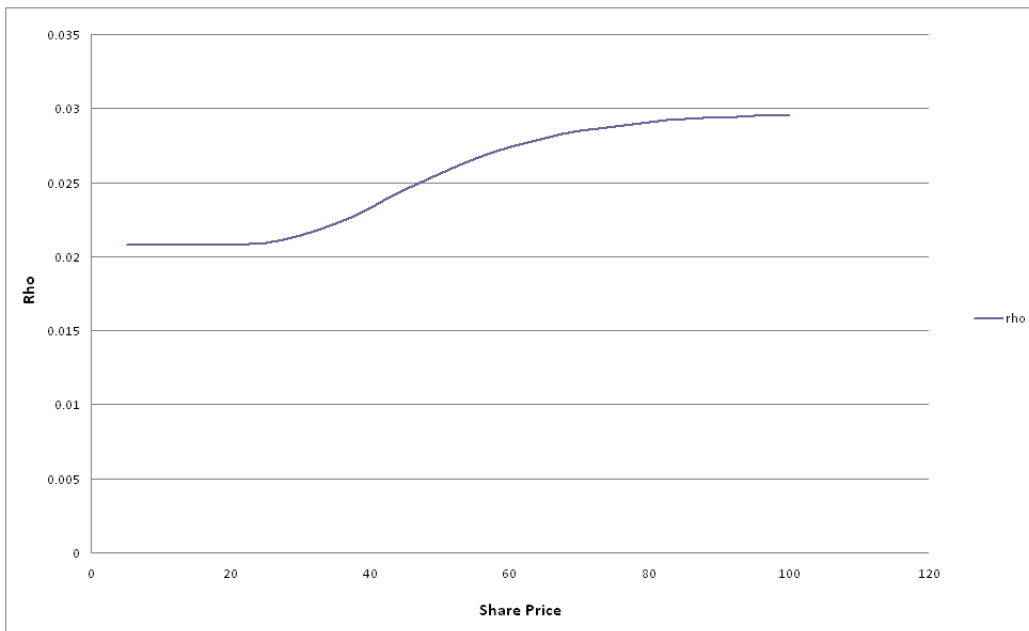


Figure A.14: 'Business Objects 2.25% 01 January 2027 EUR'

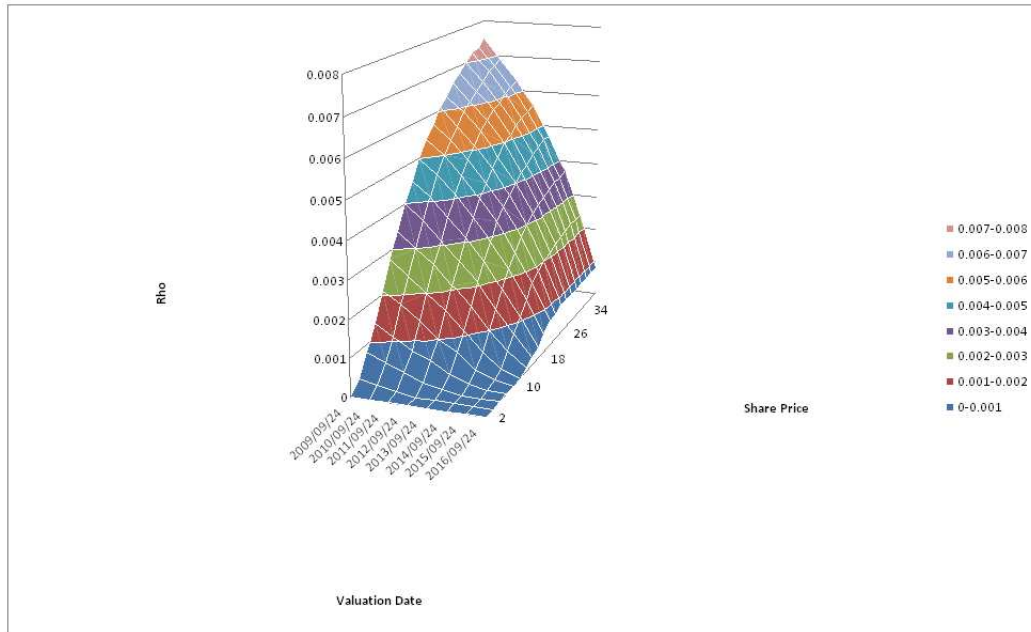


Figure A.15: 'ViroPharma Inc 2% 15 March 2017 USD'

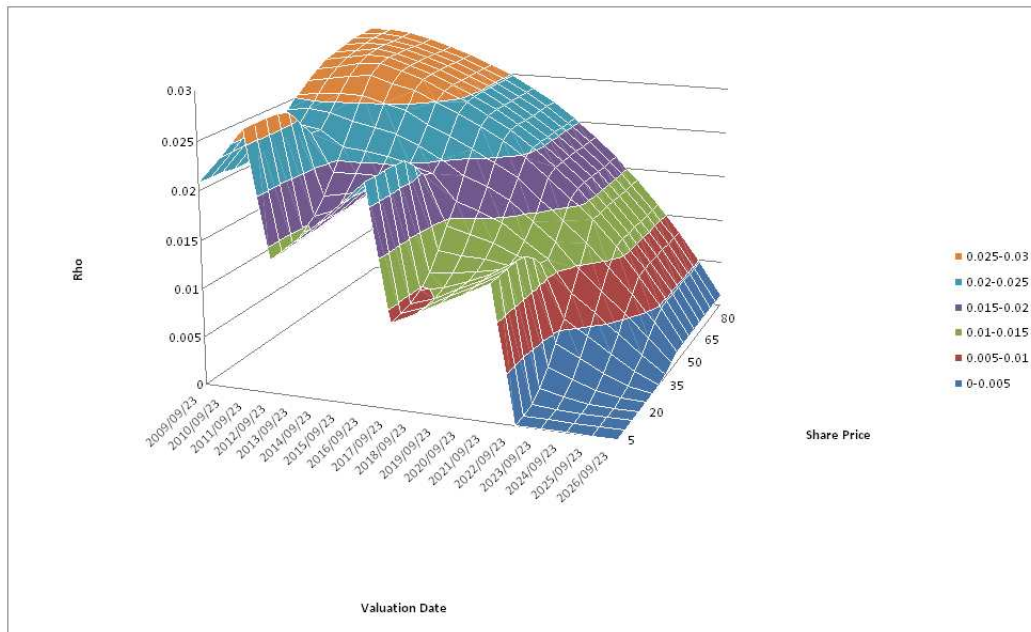


Figure A.16: 'Business Objects 2.25% 01 January 2027 EUR'

VEGA

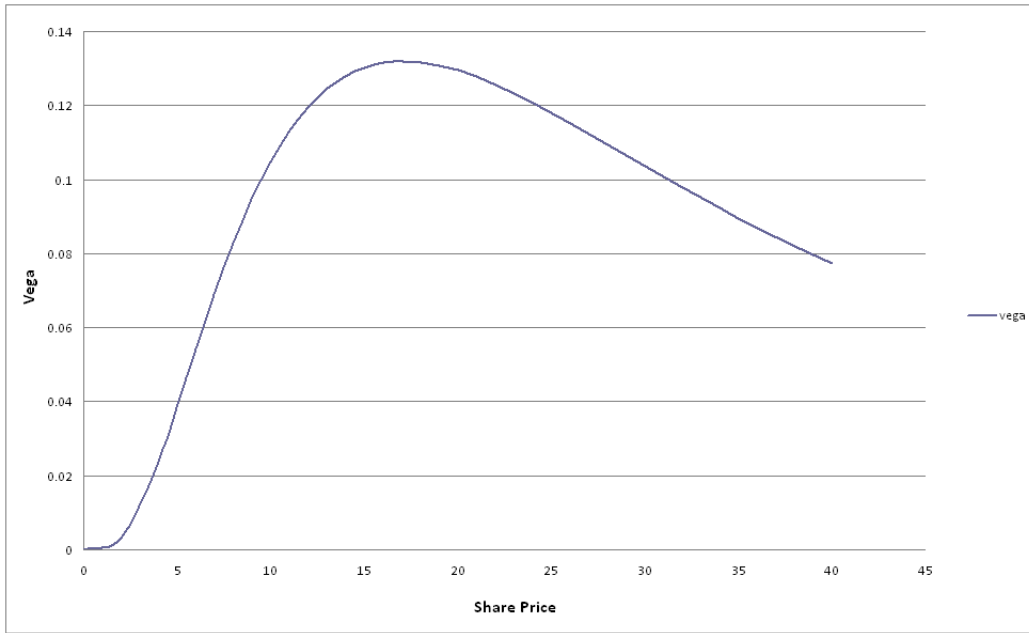


Figure A.17: 'ViroPharma Inc 2% 15 March 2017 USD'

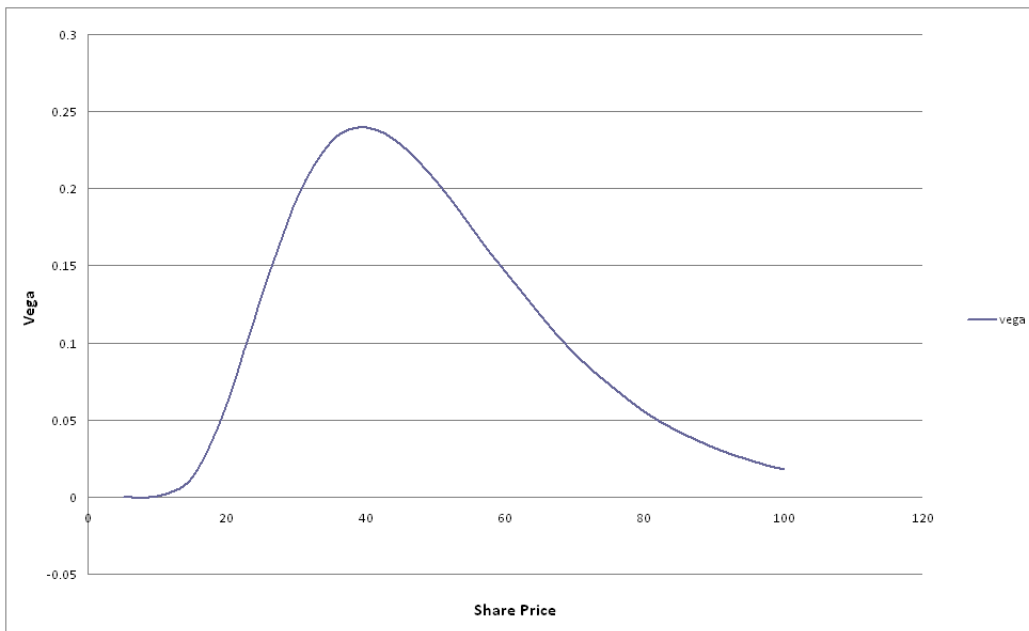


Figure A.18: 'Business Objects 2.25% 01 January 2027 EUR'

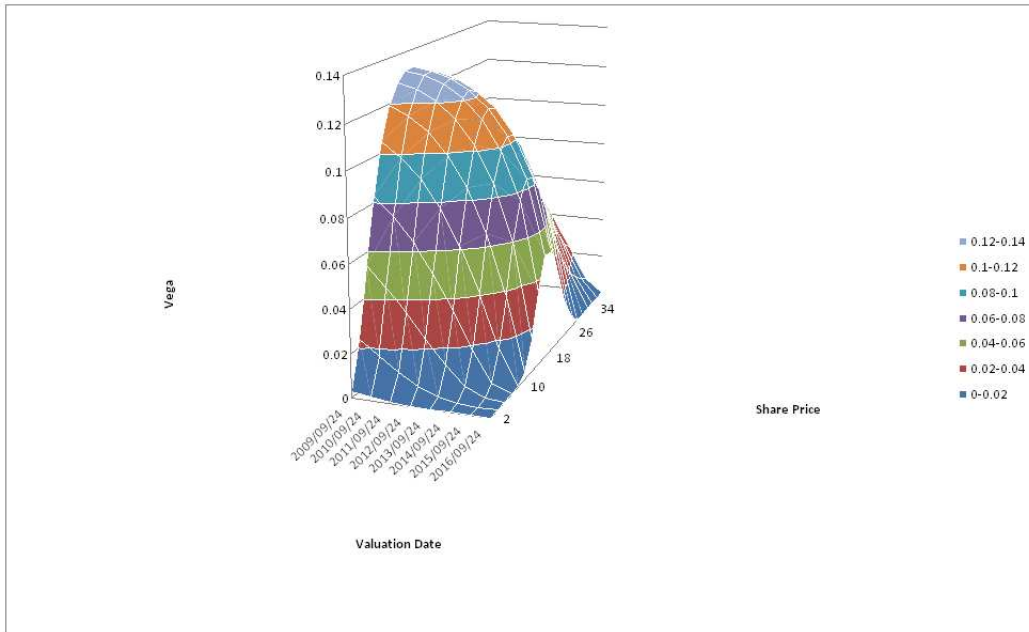


Figure A.19: 'ViroPharma Inc 2% 15 March 2017 USD'

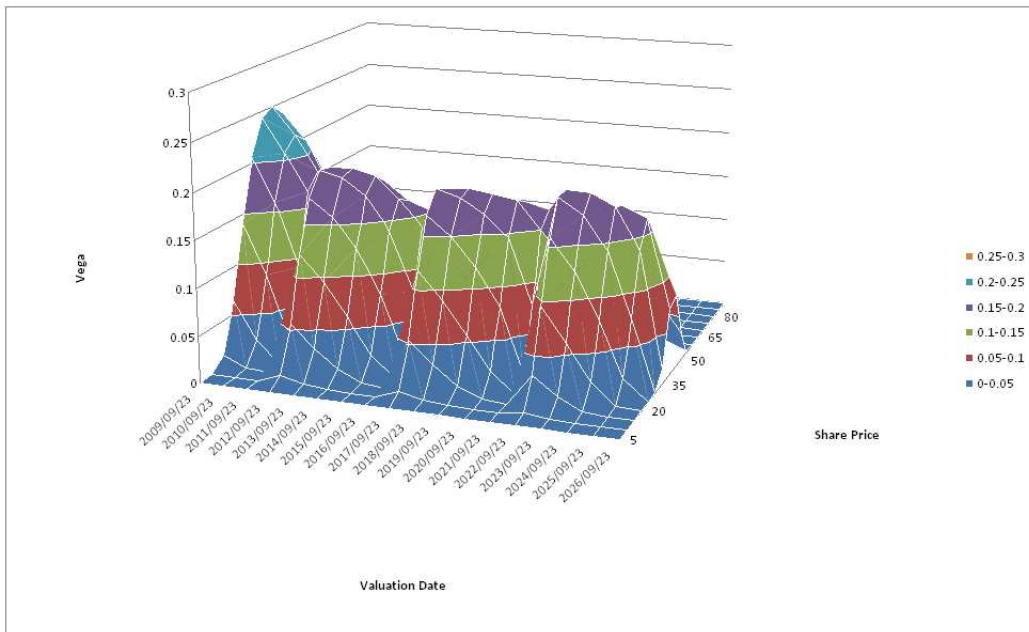


Figure A.20: 'Business Objects 2.25% 01 January 2027 EUR'

OMICRON

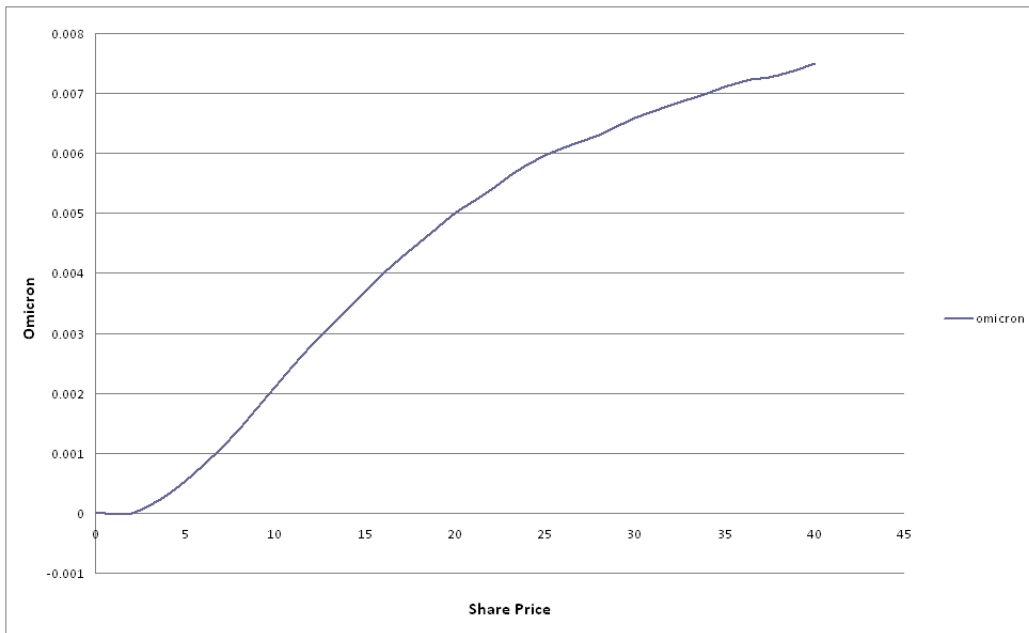


Figure A.21: 'ViroPharma Inc 2% 15 March 2017 USD'

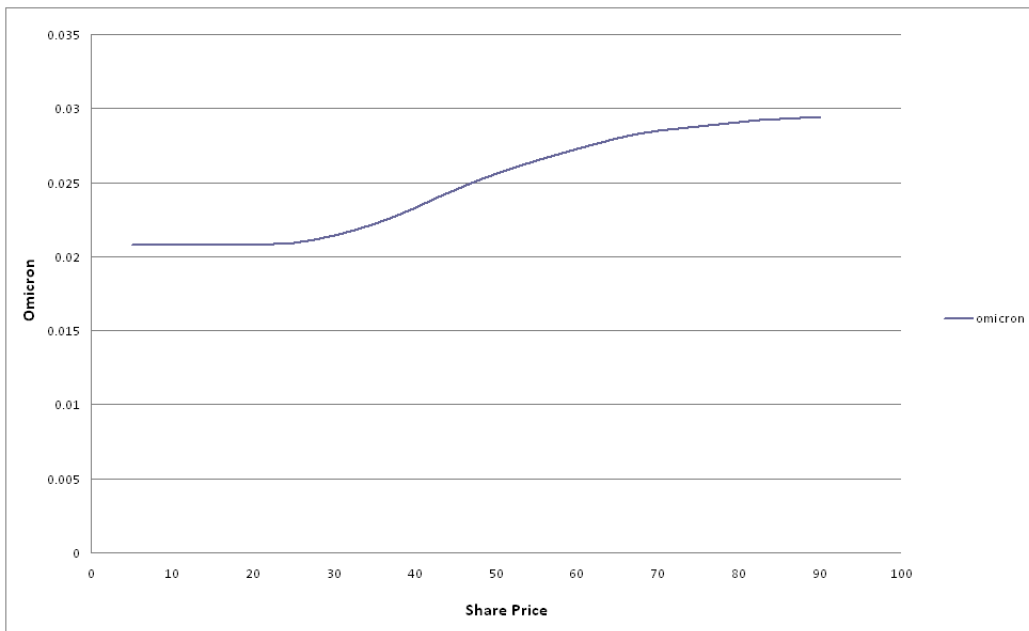


Figure A.22: 'Business Objects 2.25% 01 January 2027 EUR'

PHI

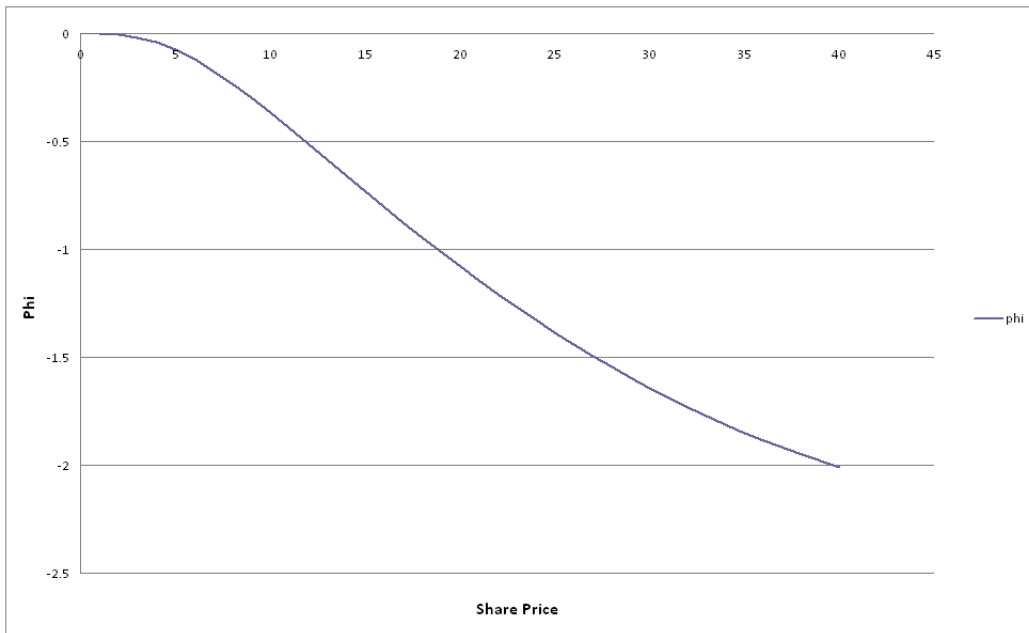


Figure A.23: 'ViroPharma Inc 2% 15 March 2017 USD'

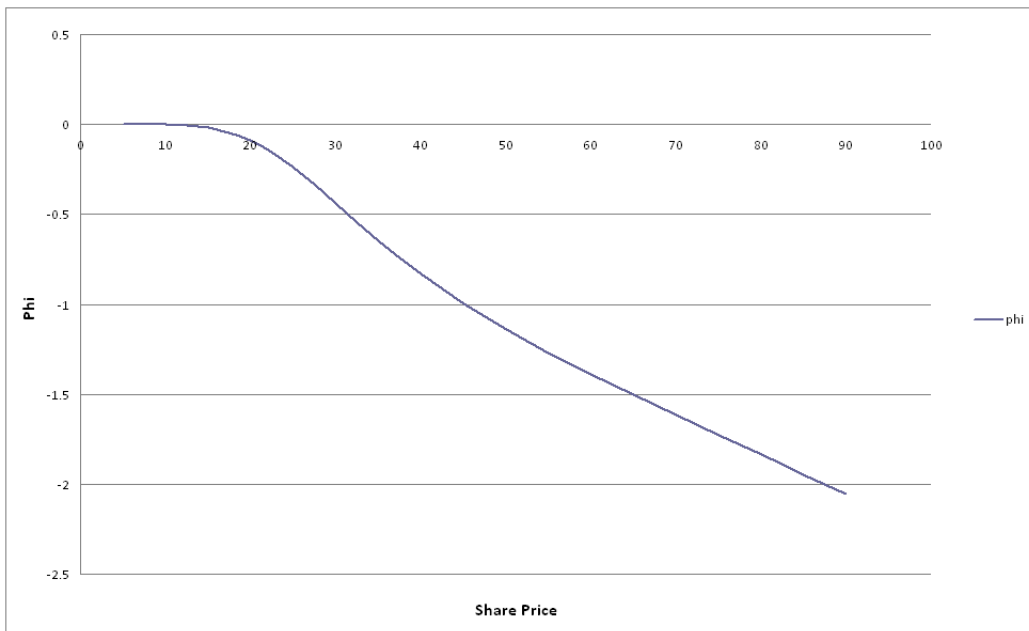


Figure A.24: 'Business Objects 2.25% 01 January 2027 EUR'

UPSILON

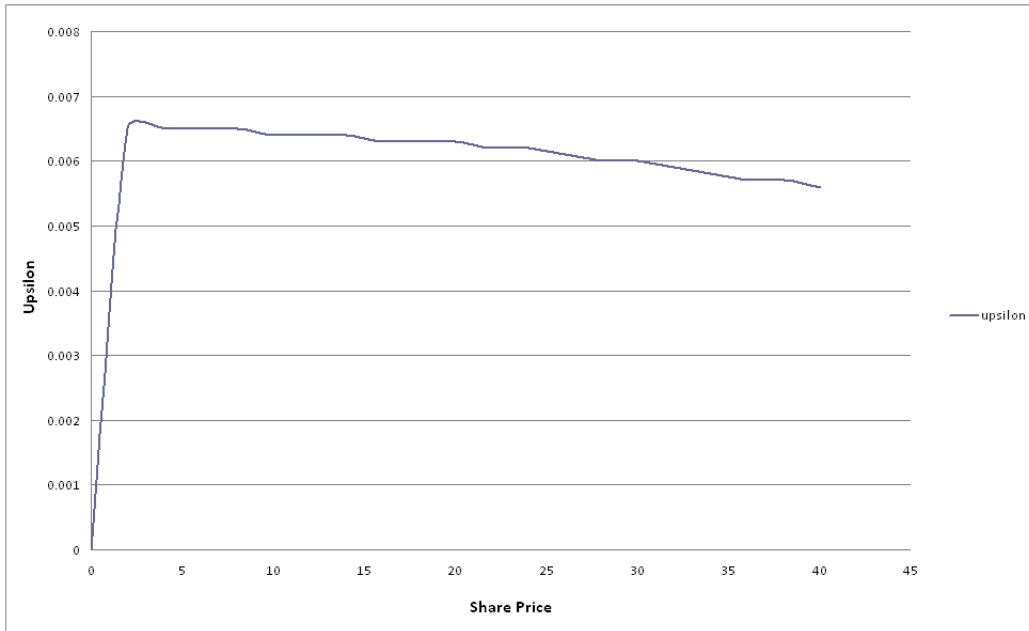


Figure A.25: 'ViroPharma Inc 2% 15 March 2017 USD'

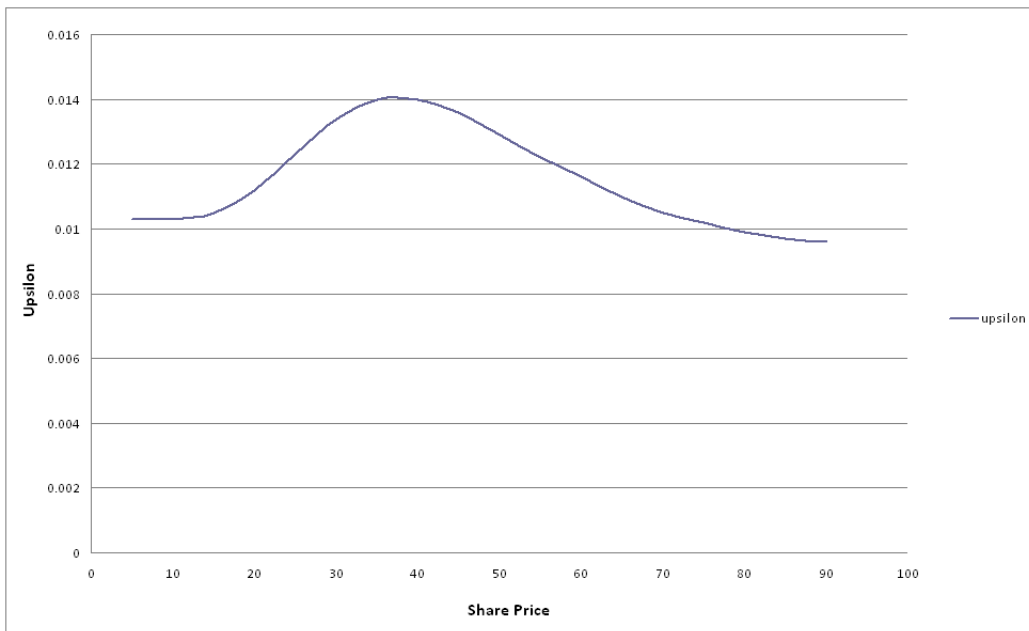


Figure A.26: 'Business Objects 2.25% 01 January 2027 EUR'

Appendix B

Code

Below gives an example of a simple VBA pricing code using a basic binomial tree model. In this model we assume that both the risk free rate and credit spread are constant across time. Once again, we model the stock price dynamics using a binomial tree, which is built in the standard binomial tree model way. The price of a convertible bond, V_i at the end of the binomial tree (time n) is given by:

$$V_n = \max(\text{Redemption Amount, Conversion Value}) + \text{last coupon payment}$$

We then work backwards through the tree considering the following at each time step.

1. Calculate $V_i = (V_{i+1}^u p + V_{i+1}^d (1-p)) / (1+r)$ where r denotes the discount rate.
2. If there is a call feature present and the current step coincides with the call date and the stock price on the current node exceeds the softcall barrier then the following would apply.
Denote $X = \max(\text{call price, conversion value})$, and if $V_i > X$, then set $V_i = X$.
3. If there is a put feature present and the current step coincides with the put date then we set $V_i = \max(V_i, \text{put price})$.
4. Set $V_i = \max(V_i, \text{conversion value})$, and
5. If the coupon payment date falls on the current step then we simply add the coupon payment to the price.

The above procedure is coded in VBA as follows:

Option Base 1

```
Function CBprice(pricedate, matdate, FaceValue, cratio, Coupon, freq, S,
                Div, vol, ir, cs, step, calldate, callprice, softcall,
                putdate, putprice)
```

```
'pricedate: date we are pricing the convertible;
'matdate:    maturity date of the bond;
'FaceValue: par value of the bond;
'cratio:    conversion ratio;
'Coupon:    coupon rate of the bond, in percentage;
'freq:      payment frequency per annum: annual: 1; semmi-annual: 2;
            quarterly: 4; monthly: 12;
'S:         current stock price;
'Div:       dividend yield;
'vol:       return volatility;
'ir:        risk-free rate, assume constant over time;
'cs:        credit spread, assume constant over time;
'step:      number of steps in the binomial tree;
'calldate:  dates on which bond can be called by the investor;
'callprice: call price;
'softcall:  softcall barrier --- the bond can be called only if the
            stock price is above this barrier. 0 if there is no call
            barrier;
'putdate:   dates on which bond can be put back to the ussuer;
'putprice:  put price;
```

```
'define the variables
```

```
Dim Stock(), CB(), Cstep(), Pstep(), CUstep()
```

```
Dim i, j, m As Integer
```

```
ReDim Stock(step + 1, step + 1), CB(step + 1, step + 1)
```

```
Dim t, r, rf As Double
```

```
t = WorksheetFunction.YearFrac(pricedate, matdate)
r = Log(1 + ir + cs)
rf = Log(1 + ir)

'build stock price tree

Dim dt As Double
Dim u As Double
Dim d As Double
Dim a As Double
Dim p As Double
Dim p1 As Double
Dim disf As Double

dt = t / step
u = Exp(vol * Sqr(dt))
d = 1 / u
a = Exp((rf - Div) * dt)
p = (a - d) / (u - d)
p1 = 1 - p
disf = Exp(-r * dt)

Stock(1, 1) = S

For j = 2 To step + 1
Stock(1, j) = Stock(1, j - 1) * u
For i = 2 To j
Stock(i, j) = Stock(i - 1, j) / u * d
Next i
Next j

Dim Callcount As Double
Dim Putcount As Double

'match dates and steps on the tree
```



```
Callcount = calldate.Count
```

```
Putcount = putdate.Count
```

```
'If there is no call feature, select an empty cell when working  
in Excel.
```

```
If IsEmpty(calldate) Then
```

```
ReDim Cstep(1, 1)
```

```
Cstep(1, 1) = -10
```

```
Else
```

```
ReDim Cstep(Callcount + 1, 1)
```

```
    Cstep(1, 1) = -10
```

```
    For i = 2 To Callcount + 1
```

```
        Cstep(i, 1) = Round(WorksheetFunction.YearFrac(pricedate,  
                                                          calldate(i - 1, 1)) / dt, 0) + 1
```

```
    Next i
```

```
End If
```

```
'Similarly, if there is no put feature, click on an empty cell  
in Excel.
```

```
If IsEmpty(putdate) Then
```

```
ReDim Pstep(1, 1)
```

```
Pstep(1, 1) = -10
```

```
Else
```

```
ReDim Pstep(Putcount + 1, 1)
```

```
    Pstep(1, 1) = -10
```

```
    For i = 2 To Putcount + 1
```

```
        Pstep(i, 1) = Round(WorksheetFunction.YearFrac(pricedate,  
                                                          putdate(i - 1, 1)) / dt, 0) + 1
```

```
    Next i
```

```

End If

If Coupon >= 0 Then
  NumC = WorksheetFunction.Ceiling(t * freq, 1)
  ReDim CUstep(NumC, 1)

  For i = 1 To NumC
    CUstep(i, 1) = Round((t - (i - 1) / freq) / dt, 0) + 1
  Next i

Else\\
  ReDim CUstep(1, 1)
  CUstep(1, 1) = 1000000
End If

' calculate the convertible price

For i = 1 To step + 1
  cv = cratio * Stock(i, step + 1) / FaceValue * 100
  CB(i, step + 1) = WorksheetFunction.Max(100, cv) + Coupon / freq
Next i

For m = 1 To step
  j = step + 1 - m
  CUindex = WorksheetFunction.Match(j, CUstep, -1)
  Pindex = WorksheetFunction.Match(j, Pstep)
  Cindex = WorksheetFunction.Match(j, Cstep)

  For i = 1 To j
    sp = Stock(i, j)
    CB(i, j) = disf * (p * CB(i, j + 1) + p1 * CB(i + 1, j + 1))
    cv = cratio * sp / FaceValue * 100

    If j = Cstep(Cindex, 1) And sp >= softcall Then

```

```
X = WorksheetFunction.Max(callprice(Cindex - 1, 1), cv)

If CB(i, j) > X Then
    CB(i, j) = X
End If

End If

If CB(i, j) < cv Then
    CB(i, j) = cv
End If

If j = Pstep(Pindex, 1) Then
    CB(i, j) = WorksheetFunction.Max(CB(i, j), putprice(Pindex - 1, 1))
End If

If j = CUstep(CUindex, 1) Then
    CB(i, j) = CB(i, j) + Coupon / freq
End If

Next i
Next m
CBprice = CB(1, 1)
End Function
```

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