Glava 1

Vektori

U mnogim naukama proučavaju se vektorske i skalarne veličine. Skalarna veličina je određena svojom brojnom vrednošću u izabranom sistemu jedinica. Takve veličine su temperatura, težina tela, površina, zapremina itd.

Vektorska veličina je određena prvcem, smerom i intenzitetom. Takve veličine su na primer brzina, sila, ubrzanje itd.

Vektorske veličine kraće nazivamo **vektorima**.

Oni se mogu predstaviti dužima. Vektor čije su krajnje tačke A i B ima pravac određen pravom AB na kojoj leži ovaj vektor, pri čemu se ta prava naziva nosač vektora. Smer vektora čije su krajnje tačke A i B je određen uređenim parom gde je A početna, a B krajnja tačka vektora. Intenzitet (moduo) se predstavlja dužinom duži AB, tj. duž AB je takva da je njena mera jednaka intenzitetu vektora. Intenzitet je skalarna veličina i uvek je nenegativna.

Vektor je zadat ako mu je zadat pravac, smer i intenzitet.

Definicija 1.1. *Dva vektora su jednaka ako su im jednaki pravac, smer i intenzitet.*

Definicija 1.2. *Vektor je paralelan pravoj, ili nekoj ravni, ako je njegov nosač paralelan sa tom pralom ili sa tom ravni.*

Definicija 1.3. *Vektori istog pravca ili paralelni istoj ravni nazivaju se kolinearnim vektorima.*

Definicija 1.4. *Dva vektora istog pravca, istog intenziteta nazivaju se suprotnim vektorima.*

Definicija 1.5. *Vektori paralelni jednoj ravni nazivaju se komplanarni vektori.*

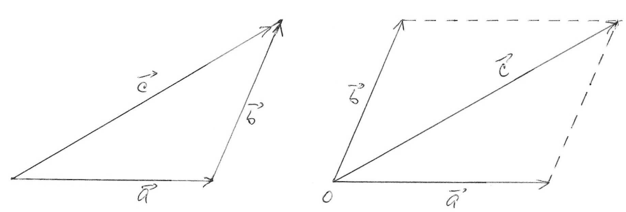
Definicija 1.6. *Vektor čiji je intenzitet jednak jedinici naziva se jedinični vektor.*

Definicija 1.7. *Ort vektora $\vec{a}$ je jedinični vektor istog pravca i smera kao i vektor $\vec{a}$.*

Definicija 1.8. *Nula vektor je vektor čiji intenzitet je jednak nuli.*

1.1 Sabiranje vektora

Neka su data dva vektora $\vec{a}$ i $\vec{b}$ neka je O proizvoljna tačka u prostoru. Ako vektore $\vec{a}$ i $\vec{a}b$ paralelnim pomeranjem dovedemo u položaj da im je O zajednički početak, tada postoje jedinstvene tačke A i B takve da je $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = \vec{b}$. Zbir vektora $\vec{a}$ i $\vec{b}$ u oznaci $\vec{a} + \vec{b}$ je vektor $\vec{c} = \overrightarrow{OC}$ ($\vec{c} = \vec{a} + \vec{b}$) gde je tačka C teme paralelograma $OACB$ suprotno temenu O.



Slika 1.1:

Osobine:

$$(grupa) \begin{cases} \vec{a} + \left(\vec{b} + \vec{c}\right) = \left(\vec{a} + \vec{b}\right) + \vec{c} \\ \vec{a} + \vec{0} = \vec{a} \\ \vec{a} + (-\vec{a}) = \vec{0} \end{cases}$$

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$

1.2 Množenje vektora skalarom

DEFINICIJA 1.9. *Proizvod* $\alpha \cdot \vec{a} = \vec{a} \cdot \alpha$ *proizvoljnog vektora* $\vec{a}$ *i proizvoljnog skalara* α *je vektor za koji važi:*

1. $\vec{a}$ *i* $\alpha\vec{a}$ *su kolinearni vektori*

2. $\vec{a}$ *i* $\alpha\vec{a}$ *su za* $\alpha > 0$ *istog smera, a za* $\alpha < 0$ *suprotnog*

3. $0 \cdot \vec{a} = 0$ *i* $\alpha \cdot \vec{0} = \vec{0}$

4. $|\alpha \cdot \vec{a}| = |\alpha| \cdot |\vec{a}|$

Vektori $\vec{a}$ i $\vec{b}$ su istog pravca (paralelni) ako i samo ako je $\vec{a} = k \cdot \vec{b}$.

$\vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} = k \cdot \vec{b}$

Osobine:

1. $1 \cdot \vec{a} = \vec{a}$
2. $k \cdot \left(\vec{a} + \vec{b}\right) = k \cdot \vec{a} + k \cdot \vec{b}$
3. $(k + k_1) \cdot \vec{a} = k \cdot \vec{a} + k_1 \cdot \vec{a}$
4. $k \cdot (k_1 \cdot \vec{a}) = (k \cdot k_1) \cdot \vec{a}$

$\left|\vec{a} + \vec{b}\right| \leq |\vec{a}| + \left|\vec{b}\right|$ - nejednakost trougla

Zadaci:

1. Ako su $\vec{a}$ i $\vec{b}$ vektori osnovica datog trapeza, a $\vec{m}$ srenje linije, dokazati da je $\vec{m} = \frac{\vec{a}+\vec{b}}{2}$.

Rešenje.

$$\left.\begin{array}{l} \vec{m} = -\vec{f} + \vec{b} + \vec{e} \\ \vec{m} = \vec{f} + \vec{a} - \vec{e} \end{array}\right\} +$$

$2\vec{m} = \vec{a} + \vec{b}$

$\vec{m} = \frac{\vec{a}+\vec{b}}{2}$

$\vec{m} \parallel \vec{a}, \vec{b}$

2. Dokazati da je zbir vektora u pravcu težišne duži trougla jednak 0.

Rešenje.

$$\left.\begin{array}{l} \overrightarrow{AA_1} = \overrightarrow{AC} + \overrightarrow{CA_1} = \overrightarrow{AC} + \frac{1}{2}\overrightarrow{CB} \\ \overrightarrow{CC_1} = \overrightarrow{CB} + \overrightarrow{BC_1} = \overrightarrow{CB} + \frac{1}{2}\overrightarrow{BA} \\ \overrightarrow{BB_1} = \overrightarrow{BA} + \overrightarrow{AB_1} = \overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC} \end{array}\right\} +$$

$$\overrightarrow{AA_1} + \overrightarrow{CC_1} + \overrightarrow{BB_1} = \frac{3}{2}\left(\overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BA}\right) = \vec{0}$$

Domaći. 3. Neka je T težište trougla ABC i O proizvoljna tačka. Dokazati da je $\overrightarrow{OT} = \frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}$.

4. Neka su dati vektori $\vec{a}$ i $\vec{b}$. Pomoću njih odrediti vektor paralelan simetrali ugla između njih.

5. Neka je duž AB podeljena u tački C u razmeri $p : q$ i neka je O proizvoljna tačka. Izraziti vektor $\overrightarrow{OC}$ Preko vektora $\overrightarrow{OA}$ i $\overrightarrow{OB}$.

$$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$$

$$\overrightarrow{AC} = \frac{p}{p+q} \cdot \overrightarrow{AB} = \frac{p}{p+q} \cdot \left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$

$$\overrightarrow{OC} = \overrightarrow{OA} + \frac{p}{p+q} \cdot \left(\overrightarrow{OB} - \overrightarrow{OA}\right) = \frac{q}{p+q} \cdot \overrightarrow{OA} + \frac{p}{p+q} \cdot \overrightarrow{OB}$$

$$\overrightarrow{OC} = \frac{q}{p+q} \cdot \overrightarrow{OA} + \frac{p}{p+q} \cdot \overrightarrow{OB}$$

$$\frac{q}{p+q} + \frac{p}{p+q} = 1$$

Teorema 1.1. *Neka su date tačke A, B i O. Tada je tačka C između tačaka A, B akko $\overrightarrow{OC} = t \cdot \overrightarrow{OB} + (1-t)\,\overrightarrow{OA}$, $0 \leq t \leq 1$.*

6. Dokazati da simetrala ugla u trouglu deli naspramnu stranicu u odnosu krakova $p : q$.

Rešenje.

$$\frac{p}{q} = \frac{\overrightarrow{AB}}{\overrightarrow{AC}}$$

$$\overrightarrow{AD} = t \cdot \overrightarrow{AB} + (1-t)\,\overrightarrow{AC}$$

$$\overrightarrow{AD} = \lambda \left(\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} + \frac{\overrightarrow{AC}}{\left|\overrightarrow{AC}\right|} \right) = \frac{\lambda}{\left|\overrightarrow{AB}\right|}\overrightarrow{AB} + \frac{\lambda}{\left|\overrightarrow{AC}\right|}\overrightarrow{AC} =$$

$$\frac{\left|\overrightarrow{AC}\right|}{\left|\overrightarrow{AB}\right| + \left|\overrightarrow{AC}\right|}\overrightarrow{AB} + \frac{\left|\overrightarrow{AB}\right|}{\left|\overrightarrow{AB}\right| + \left|\overrightarrow{AC}\right|}\overrightarrow{AC}$$

$$\frac{\lambda}{\left|\overrightarrow{AB}\right|} + \frac{\lambda}{\left|\overrightarrow{AC}\right|} = 1$$

$$\lambda = \frac{1}{\frac{1}{\left|\overrightarrow{AB}\right|} + \frac{1}{\left|\overrightarrow{AC}\right|}} = \frac{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}{\left|\overrightarrow{AB}\right| + \left|\overrightarrow{AC}\right|}$$

$$\overrightarrow{AD} = \frac{q}{p+q}\overrightarrow{AB} + \frac{p}{p+q}\overrightarrow{AC}$$

$$\frac{\left|\overrightarrow{AC}\right|}{\left|\overrightarrow{AB}\right| + \left|\overrightarrow{AC}\right|} = \frac{q}{p+q}$$

$$\frac{1}{\frac{\left|\overrightarrow{AB}\right|}{\left|\overrightarrow{AC}\right|} + 1} = \frac{1}{\frac{p}{q} + 1}$$

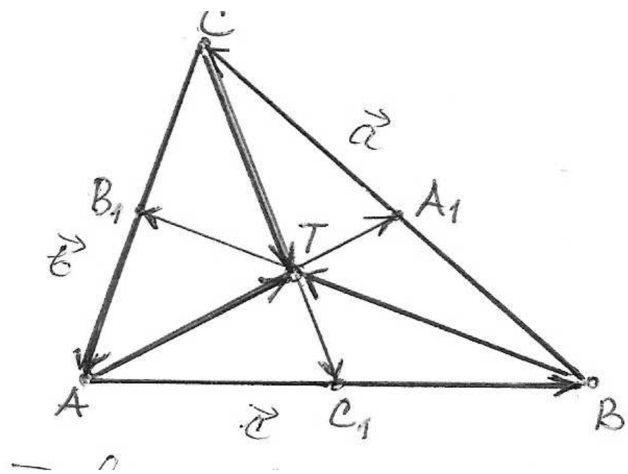
$$\frac{\left|\overrightarrow{AB}\right|}{\left|\overrightarrow{AC}\right|} = \frac{p}{q}$$

Domaći. 7. Odsečci koji spajaju sredine suprotnih ivica tetraedra se uzajamno polove.

8. Neka je T težište ΔABC. Dokazati

$AT^2 + BT^2 + CT^2 = \frac{1}{3}\left(AB^2 + BC^2 + CA^2\right)$

Rešenje.



Slika 1.2:

$$\overrightarrow{AT} + \overrightarrow{BT} + \overrightarrow{CT} = \overrightarrow{0}\,/^2$$

$$\overrightarrow{AT}^2 + \overrightarrow{BT}^2 + \overrightarrow{CT}^2 = -2\left(\overrightarrow{AT}\cdot\overrightarrow{BT} + \overrightarrow{AT}\cdot\overrightarrow{CT} + \overrightarrow{BT}\cdot\overrightarrow{CT}\right)$$

$$\overrightarrow{AB} + \overrightarrow{BT} = \overrightarrow{AT} \Rightarrow \overrightarrow{AB} = \overrightarrow{AT} - \overrightarrow{BT} \Rightarrow \overrightarrow{BA} = \overrightarrow{TA} - \overrightarrow{TB}\,/^2$$

$$\overrightarrow{TA}^2 - 2\cdot\overrightarrow{TA}\cdot\overrightarrow{TB} + \overrightarrow{TB}^2 = \overrightarrow{BA}^2$$

Analogno je,

$$\overrightarrow{CB} = \overrightarrow{TB} - \overrightarrow{TC}\,/^2$$

$\overrightarrow{TB}^2 - 2 \cdot \overrightarrow{TB} \cdot \overrightarrow{TC} + \overrightarrow{TC}^2 = \overrightarrow{CB}^2$

$\overrightarrow{TC}^2 - 2 \cdot \overrightarrow{TC} \cdot \overrightarrow{TA} + \overrightarrow{TA}^2 = \overrightarrow{AC}^2$

$\overrightarrow{TA}^2 + \overrightarrow{TB}^2 + \overrightarrow{TB}^2 + \overrightarrow{TC}^2 + \overrightarrow{TC}^2 + \overrightarrow{TA}^2 - 2\left(\overrightarrow{TA} \cdot \overrightarrow{TB} + \overrightarrow{TA} \cdot \overrightarrow{TC} + \overrightarrow{TB} \cdot \overrightarrow{TC}\right) = \overrightarrow{BA}^2 + \overrightarrow{CB}^2 + \overrightarrow{AC}^2$

$2\left(\overrightarrow{TA}^2 + \overrightarrow{TB}^2 + \overrightarrow{TC}^2\right) - 2\left(\overrightarrow{TA} \cdot \overrightarrow{TB} + \overrightarrow{TA} \cdot \overrightarrow{TC} + \overrightarrow{TB} \cdot \overrightarrow{TC}\right) = \overrightarrow{BA}^2 + \overrightarrow{CB}^2 + \overrightarrow{AC}^2$

$3\left(\overrightarrow{AT}^2 + \overrightarrow{BT}^2 + \overrightarrow{CT}^2\right) = \overrightarrow{AB}^2 + \overrightarrow{AC}^2 + \overrightarrow{BC}^2$

$\left(\overrightarrow{AT}^2 + \overrightarrow{BT}^2 + \overrightarrow{CT}^2\right) = \frac{1}{3}\left(\overrightarrow{AB}^2 + \overrightarrow{AC}^2 + \overrightarrow{BC}^2\right)$

1.3 Skalarni proizvod vektora

DEFINICIJA 1.10. *Skalarni proizvod geometrijskih vektora $\vec{a}$ i $\vec{b}$ je realan broj, u oznaci $\vec{a} \cdot \vec{b}$ koji je jednak proizvodu intenziteta tih vektora i kosinusa ugla između njih, tj.* $\vec{a} \cdot \vec{b} = |\vec{a}| \cdot \left|\vec{b}\right| \cdot \cos \sphericalangle \left(\vec{a}, \vec{b}\right)$.

Osobine.

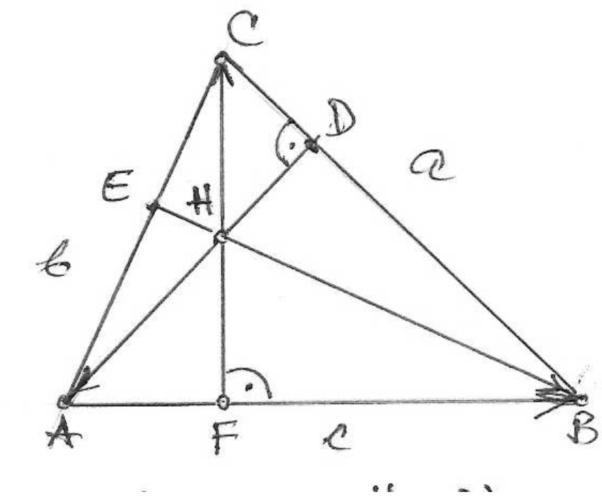
1. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
2. $(\alpha\vec{a}) \cdot \vec{b} = \vec{a} \cdot \left(\alpha\vec{b}\right) = \alpha\left(\vec{a} \cdot \vec{b}\right)$
3. $\left(\vec{a} + \vec{b}\right) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

1. Dokazati da se visine trougla seku u jednoj tački.

Rešenje:

ΔABC

$\overrightarrow{AB} = \overrightarrow{HB} - \overrightarrow{HA}$



Slika 1.3:

$\overrightarrow{HA} + \overrightarrow{AB} = \overrightarrow{HB}$

Kako je $h_c = \overrightarrow{CF} = -\overrightarrow{FC}$ sledi da je skalarni proizvod vekrora $\overrightarrow{AB}$ i $\overrightarrow{HC}$ jednak nuli, tj. $\overrightarrow{HC} \cdot \overrightarrow{AB} = \vec{0}$.

$\overrightarrow{HC} \cdot \left(\overrightarrow{HB} - \overrightarrow{HA}\right) = \vec{0}$..........(1)

Analogno je, $\overrightarrow{HC} - \overrightarrow{HB} = \overrightarrow{BC}$

$\overrightarrow{HA} \perp \overrightarrow{BC}$

$\overrightarrow{HA} \cdot \left(\overrightarrow{HC} - \overrightarrow{HB}\right) = \vec{0}$ ....(2)

Sabiranjem levih i desnih strana jednakosti (1) i (2), dobija se

$\overrightarrow{HC} \cdot \left(\overrightarrow{HB} - \overrightarrow{HA}\right) + \overrightarrow{HA} \cdot \left(\overrightarrow{HC} - \overrightarrow{HB}\right) = 0$

$\overrightarrow{HC} \cdot \overrightarrow{HB} - \overrightarrow{HA} \cdot \overrightarrow{HB} = \vec{0}$

$\left(\overrightarrow{HC} - \overrightarrow{HA}\right) \cdot \overrightarrow{HB} = \vec{0}$

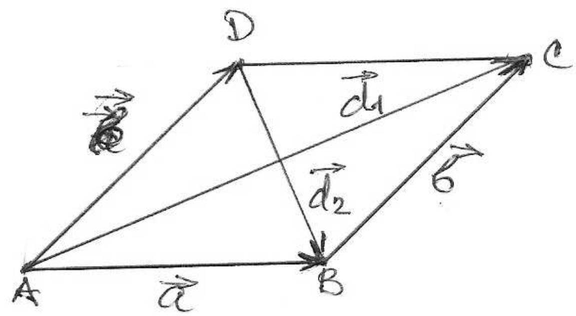
$\overrightarrow{AC} = \overrightarrow{HC} - \overrightarrow{HA}$

$\overrightarrow{HC} \cdot \overrightarrow{HB} = \vec{0} \Rightarrow \overrightarrow{HB} \perp \overrightarrow{AC}$, pa ako je E presečna tačka pravih određenih vektorima $\overrightarrow{AC}$ i $\overrightarrow{HB}$ sledi da je $\overrightarrow{BE} \perp \overrightarrow{AC}$ što znači da visina $\overrightarrow{HB}$ prolazi kroz tačku H.

Visine ΔABC se seku u jednoj tački H.

2. Pokazati da su dijagonale romba normalne.

Rešenje:



Slika 1.4:

$\overrightarrow{AB} = \vec{a}$ $\qquad\qquad$ $\overrightarrow{BC} = \vec{b}$

$\vec{d_1} = \vec{a} + \vec{b}$ $\qquad\qquad$ $\vec{d_2} = \vec{a} - \vec{b}$

$|\vec{a}| = \left|\vec{b}\right|$..................(1)

$$\vec{d_1} \cdot \vec{d_1} = \left(\vec{a} + \vec{b}\right)\left(\vec{a} - \vec{b}\right) = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b} = \vec{a} \cdot \vec{a} - \vec{b} \cdot \vec{c}$$

$$\left|\vec{d_1}\right| \cdot \left|\vec{d_1}\right| \cdot \cos\left(\vec{d_1},\, \vec{d_2}\right) = |\vec{a}| \cdot |\vec{a}| \cdot \cos\left(\vec{a},\, \vec{a}\right) - \left|\vec{b}\right| \cdot \left|\vec{b}\right| \cdot \cos\left(\vec{b},\, \vec{b}\right) \left|\overleftarrow{b}\right| \cdot \left|\vec{b}\right|$$

$$\left|\vec{d_1}\right| \cdot \left|\vec{d_1}\right| \cdot \cos\left(\vec{d_1},\, \vec{d_2}\right) = |\vec{a}| \cdot |\vec{a}| \cdot 1 - \left|\vec{b}\right| \cdot \left|\vec{b}\right| \cdot 1 = |\vec{a}|^2 - \left|\vec{b}\right|^2 \text{............(2)}$$

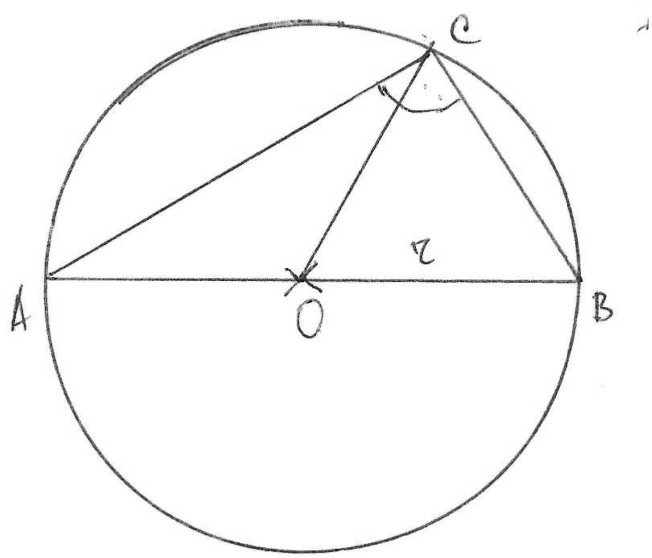
Zamenom (1) u (2), dobija se

$$\left|\vec{d_1}\right| \cdot \left|\vec{d_1}\right| \cdot \cos\left(\vec{d_1},\, \vec{d_2}\right) = |\vec{a}|^2 - |\vec{a}|^2 = 0 / \cdot \frac{1}{\left|\vec{d_1}\right| \cdot \left|\vec{d_1}\right|}$$

$$\cos\left(\vec{d_1},\, \vec{d_2}\right) = 0 \;\Rightarrow\; \left(\vec{d_1},\, \vec{d_2}\right) = \frac{\pi}{2} \;\Rightarrow\; d_1 \perp d_2$$

3. Pokazati da je ugao nad prečnikom prav

Rešenje.



Slika 1.5:

$\vec{a} \perp \vec{b}$

$$\overrightarrow{AC} \perp \overrightarrow{BC} \;\Leftrightarrow\; \left|\overrightarrow{AC}\right| \cdot \left|\overrightarrow{BC}\right| = \vec{0}$$

$$\overrightarrow{AC} \cdot \overrightarrow{BC} \;= \vec{0}$$

$$\overrightarrow{AC} \cdot \overrightarrow{BC} \;=\; \left(\overrightarrow{OA} + \overrightarrow{OC}\right)\left(\overrightarrow{BO} + \overrightarrow{OC}\right) \;=\; -r^2 + \overrightarrow{OA} \cdot \overrightarrow{OC} + \overrightarrow{OC} \cdot \overrightarrow{BO} + r^2 = \overrightarrow{OC}\left(\overrightarrow{OA} + \overrightarrow{BO}\right) = \overrightarrow{OC} \cdot \vec{0} = \vec{0}$$

4. Paralelogram sa jednakim dijagonalama je pravougaonik. Dokazati.

Rešenje.

$\overrightarrow{AB} = \vec{a}, \ \overrightarrow{AD} = \vec{b},$

$\vec{d_1} = \overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\vec{d_2} = \overrightarrow{AB} - \overrightarrow{AD} = \overrightarrow{DB}$

$\vec{d_1} = \overrightarrow{AB} + \overrightarrow{AD} = \vec{a} + \vec{b}$

$\vec{d_2} = \overrightarrow{AB} - \overrightarrow{AD} = \vec{a} - \vec{b}$

$\vec{d_1} \cdot \vec{d_1} = \left|\vec{d_1}\right| \cdot \left|\vec{d_1}\right| \cdot \cos\left(\vec{d_1}, \vec{d_1}\right) = \left|\vec{d_1}\right|^2$

$\vec{d_2} \cdot \vec{d_2} = \left|\vec{d_2}\right| \cdot \left|\vec{d_2}\right| \cdot \cos\left(\vec{d_2}, \vec{d_2}\right) = \left|\vec{d_2}\right|^2$

$\left|\vec{d_1}\right| = \left|\vec{d_2}\right| = d$

$\vec{d_1} \cdot \vec{d_1} = \left(\vec{a} + \vec{b}\right)\left(\vec{a} + \vec{b}\right) = \vec{a} \cdot \vec{a} + 2\vec{a}\vec{b} + \vec{b}\vec{b} = |\vec{a}| \cdot |\vec{a}| \cdot \cos\left(\vec{a}, \vec{a}\right) + 2|\vec{a}| \cdot \left|\vec{b}\right| \cdot \cos\left(\vec{a}, \vec{b}\right) + \left|\vec{b}\right| \cdot \left|\vec{b}\right| \cdot \cos\left(\vec{b}, \vec{b}\right) = |\vec{a}|^2 + 2|\vec{a}| \cdot \left|\vec{b}\right| \cdot \cos\left(\vec{a}, \vec{b}\right) + \left|\vec{b}\right|^2 = \left|\vec{d_1}\right| \cdot \left|\vec{d_1}\right| \cdot \cos\left(\vec{d_1}, \vec{d_1}\right) = \left|\vec{d_1}\right|^2$.........(1)

$\vec{d_2} \cdot \vec{d_2} = \left(\vec{a} - \vec{b}\right)\left(\vec{a} - \vec{b}\right) = \vec{a} \cdot \vec{a} - 2\vec{a}\vec{b} + \vec{b}\vec{b} = |\vec{a}| \cdot |\vec{a}| \cdot \cos\left(\vec{a}, \vec{a}\right) - 2|\vec{a}| \cdot \left|\vec{b}\right| \cdot \cos\left(\vec{a}, \vec{b}\right) + \left|\vec{b}\right| \cdot \left|\vec{b}\right| \cdot \cos\left(\vec{b}, \vec{b}\right) = |\vec{a}|^2 - 2|\vec{a}| \cdot \left|\vec{b}\right| \cdot \cos\left(\vec{a}, \vec{b}\right) + \left|\vec{b}\right|^2 = \left|\vec{d_2}\right| \cdot \left|\vec{d_2}\right| \cdot \cos\left(\vec{d_2}, \vec{d_2}\right) = \left|\vec{d_2}\right|^2$..........(2)

Iz (1) i (2) sledi

$\left[|\vec{a}|^2 + 2|\vec{a}| \cdot \left|\vec{b}\right| \cdot \cos\left(\vec{a}, \vec{b}\right) + \left|\vec{b}\right|^2\right] - \left[|\vec{a}|^2 - 2|\vec{a}| \cdot \left|\vec{b}\right| \cdot \cos\left(\vec{a}, \vec{b}\right) + \left|\vec{b}\right|^2\right] =$

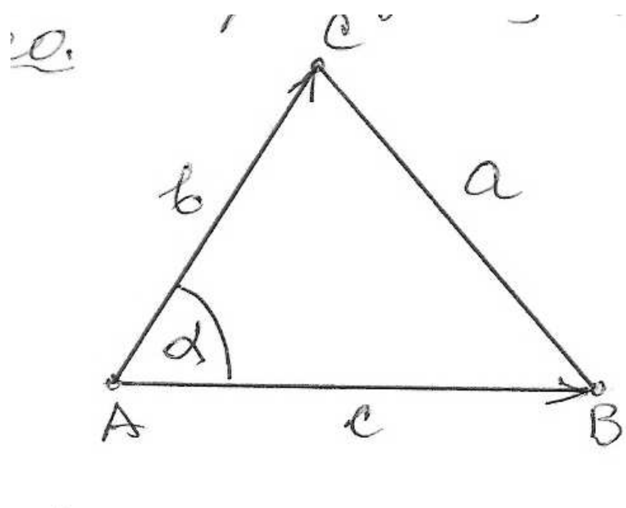
$$\left|\vec{d_1}\right|^2 - \left|\vec{d_2}\right|^2$$

$$4\left|\vec{a}\right| \cdot \left|\vec{b}\right| \cdot \cos\left(\vec{a},\, \vec{b}\right) = 0 / : 4\left|\vec{a}\right| \cdot \left|\vec{b}\right|$$

$$\cos\left(\vec{a},\, \vec{b}\right) = 0 \Rightarrow \left(\vec{a},\, \vec{b}\right) = 90^\circ \Rightarrow \left(\vec{a},\, \vec{b}\right) = \frac{\pi}{2}$$

5. Primenom skalarnog proizvoda vektora dokazati kosinusnu teoremu za ugao.

Rešenje:



Slika 1.6:

$$\overrightarrow{AB} = \vec{c},\ \overrightarrow{BC} = \vec{a},\ \overrightarrow{AC} = \vec{b}$$

$$\sphericalangle\left(\overrightarrow{AB},\, \overrightarrow{AC}\right) = \alpha$$

$$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = \vec{b} - \vec{c}$$

$$\vec{a} = \vec{b} - \vec{c}$$

$$\vec{a} \cdot \vec{a} = \left(\vec{b} - \vec{c}\right)\left(\vec{b} - \vec{c}\right) = \vec{b} \cdot \vec{b} - \vec{b} \cdot \vec{c} - \vec{c} \cdot \vec{b} + \vec{c} \cdot \vec{c} = \vec{b} \cdot \vec{b} - 2\vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{c}$$

$$\left|\vec{a}\right| \cdot \left|\vec{a}\right| \cdot \cos\left(\vec{a},\, \vec{a}\right) = \left|\vec{b}\right| \cdot \left|\vec{b}\right| \cdot \cos\left(\vec{b},\, \vec{b}\right) - 2\left|\vec{b}\right| \cdot \left|\vec{c}\right| \cdot \cos\left(\vec{b},\, \vec{c}\right) +$$
$$\left|\vec{c}\right| \cdot \left|\vec{c}\right| \cdot \cos\left(\vec{c},\, \vec{c}\right)$$

$|\vec{a}|^2 = \left|\vec{b}\right|^2 + |\vec{c}|^2 - 2\left|\vec{b}\right| \cdot |\vec{c}| \cdot \cos\,\alpha$

$a^2 = b^2 + c^2 - 2bc \cdot \cos\,\alpha$

6. Primenom vektora dokazati Pitagorinu teoremu.

Rešenje:

$\overrightarrow{AB} = \vec{a}$, $\overrightarrow{AC} = \vec{b}$

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0} \Rightarrow \overrightarrow{BC} = -\overrightarrow{AB} - \overrightarrow{CA} = \overrightarrow{AC} - \overrightarrow{AB}$........(1)

Iz (1) je $\overrightarrow{BC} \cdot \overrightarrow{BC} = \left(\overrightarrow{AC} - \overrightarrow{AB}\right)\left(\overrightarrow{AC} - \overrightarrow{AB}\right)$

$\overrightarrow{BC} \cdot \overrightarrow{BC} = \overrightarrow{AC} \cdot \overrightarrow{AC} - \overrightarrow{AC} \cdot \overrightarrow{AB} - \overrightarrow{AB} \cdot \overrightarrow{AC} + \overrightarrow{AB} \cdot \overrightarrow{AB}$

$\left|\overrightarrow{BC}\right| \cdot \left|\overrightarrow{BC}\right| \cdot \cos\left(\overrightarrow{BC}, \overrightarrow{BC}\right) = \left|\overrightarrow{AC}\right| \cdot \left|\overrightarrow{AC}\right| \cdot \cos\left(\overrightarrow{AC}, \overrightarrow{AC}\right) - 2\left|\overrightarrow{AB}\right| \cdot \left|\overrightarrow{AC}\right| \cdot \cos\left(\overrightarrow{AB}, \overrightarrow{AC}\right) + \left|\overrightarrow{AB}\right| \cdot \left|\overrightarrow{AB}\right| \cdot \cos\left(\overrightarrow{AB}, \overrightarrow{AB}\right)$

$\left|\overrightarrow{BC}\right|^2 = \left|\overrightarrow{AC}\right|^2 - 2\left|\overrightarrow{AB}\right| \cdot \left|\overrightarrow{AC}\right| \cdot \cos\,\frac{\pi}{2} + \left|\overrightarrow{AB}\right|^2$

$\left|\overrightarrow{BC}\right|^2 = \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{AB}\right|^2$

$c^2 = a^2 + b^2$

7. Neka je u $\Delta\,ABC$ dato $\left|\overrightarrow{AB}\right| = 4$, $\left|\overrightarrow{AC}\right| = 2$, $\alpha = 60°$. Ako tačka D deli stranicu BC u odnosu $1 : 2$, odrediti $\cos\left(\overrightarrow{DA}, \overrightarrow{DC}\right)$.

Rešenje.

$\overrightarrow{DA} = -\overrightarrow{AD} = -\left(\frac{2}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}\right)$

$\overrightarrow{DC} = \frac{2}{3}\overrightarrow{BC} = \frac{2}{3}\left(\overrightarrow{AC} - \overrightarrow{AB}\right)$

$$\left|\overrightarrow{DA}\right|^2 = \left(-\left(\frac{2}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}\right)\right)^2 = \frac{1}{9}\left(4\overrightarrow{AB}^2 + 4\overrightarrow{AB}\cdot\overrightarrow{AC} + \overrightarrow{AC}^2\right) =$$
$$\frac{1}{9}\left(4\left|\overrightarrow{AB}\right|^2 + 4\left|\overrightarrow{AB}\right|\cdot\left|\overrightarrow{AC}\right|\cdot\cos\varepsilon + \left|\overrightarrow{AC}\right|^2\right) = \frac{1}{9}\left(4\cdot 16 + 4\cdot 4\cdot 2\cdot\frac{1}{2} + 16\right) = \frac{84}{9}$$

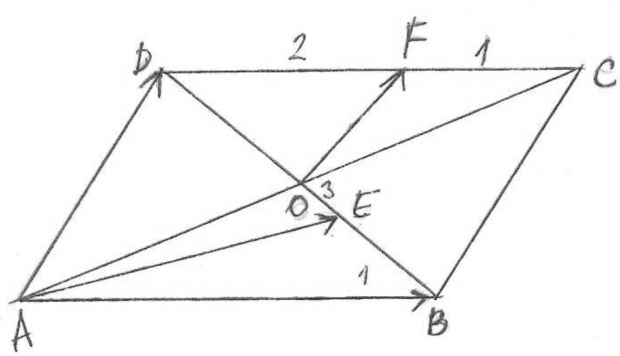
$$\left|\overrightarrow{DC}\right|^2 = \left(\frac{2}{3}\left(\overrightarrow{AC} - \overrightarrow{AB}\right)\right)^2 = \frac{4}{9}\left(\overrightarrow{AC}^2 - 2\overrightarrow{AB}\cdot\overrightarrow{AC} + \overrightarrow{AB}^2\right) =$$
$$\frac{4}{9}\left(\left|\overrightarrow{AC}\right|^2 - 2\left|\overrightarrow{AB}\right|\cdot\left|\overrightarrow{AC}\right|\cdot\cos\alpha + \left|\overrightarrow{AB}\right|^2\right) = \frac{4}{9}\left(4 - 2\cdot 4\cdot 2\cdot\frac{1}{2} + 16\right) = \frac{48}{9}$$

$$\left|\overrightarrow{DA}\right|\cdot\left|\overrightarrow{DC}\right| = \left(-\left(\frac{2}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC}\right)\right)\left(\frac{2}{3}\left(\overrightarrow{AC} - \overrightarrow{AB}\right)\right) = -\frac{2}{9}\left(\overrightarrow{AB}\cdot\overrightarrow{AC} - 2\overrightarrow{AB}^2 + \overrightarrow{AC}\right.$$
$$-\frac{2}{9}\left(4\cdot 2 - 2\cdot 16 + 4\right) = \frac{48}{9}$$

$$\cos\left(\overrightarrow{DA},\, \overrightarrow{DC}\right) = \frac{\overrightarrow{DA}\cdot\overrightarrow{DC}}{\left|\overrightarrow{DA}\right|\cdot\left|\overrightarrow{DC}\right|} = \frac{\frac{48}{9}}{\frac{\sqrt{84}}{3}\cdot\frac{\sqrt{48}}{3}} = \frac{\sqrt{48}}{\sqrt{84}} = \frac{\sqrt{8}}{\sqrt{14}} = \frac{2}{\sqrt{7}}$$

8. Neka je u paralelogramu $ABCD$ dato $\left|\overrightarrow{AB}\right| = 3$, $\left|\overrightarrow{AD}\right| = 2$, $\cos\alpha = \frac{1}{6}$. Neka je tačka O presek dijagonale, tačka F deli DC u odnosu $2:1$, a tačka E deli DB u odnosu $3:1$. Odrediti cos ugla između $\overrightarrow{AE}$ i $\overrightarrow{OF}$.

Rešenje.



Slika 1.7:

$$\overrightarrow{AE} = \frac{1}{4}\overrightarrow{AD} + \frac{3}{4}\overrightarrow{AB}$$

$$\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AD} + \overrightarrow{DF} = -\frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{AB}\right) + \overrightarrow{AD} + \frac{2}{3}\overrightarrow{AB} = \frac{1}{2}\overrightarrow{AD} +$$
$$\frac{1}{6}\overrightarrow{AB}$$

$\left|\overrightarrow{AE}\right|^2 = \left(\frac{1}{4}\overrightarrow{AD} + \frac{3}{4}\overrightarrow{AB}\right)^2 = \frac{1}{16}\overrightarrow{AD}^2 + \frac{6}{16}\overrightarrow{AD}\overrightarrow{AB} + \frac{9}{16}\overrightarrow{AB}^2 =$

$\frac{1}{16}\left|\overrightarrow{AD}\right|^2 + \frac{6}{16}\left|\overrightarrow{AD}\right| \cdot \left|\overrightarrow{AB}\right| \cdot \cos\,\alpha + \frac{9}{16}\left|\overrightarrow{AB}\right|^2 = \frac{1}{16}\left(4 + 6 \cdot 2 \cdot 3 \cdot \frac{1}{6} + 81\right) = \frac{91}{16}$

$\left|\overrightarrow{AE}\right| = \frac{\sqrt{91}}{4}$

$\overrightarrow{OF} = \left(\frac{1}{2}\overrightarrow{AD} + \frac{1}{6}\overrightarrow{AB}\right)^2 = \frac{1}{4}\overrightarrow{AD}^2 + \frac{2}{12}\overrightarrow{AD} \cdot \overrightarrow{AB} + \frac{1}{36}\overrightarrow{AB}^2 =$

$\frac{1}{4}\left(\left|\overrightarrow{AD}\right|^2 + \frac{2}{3}\left|\overrightarrow{AD}\right|\left|\overrightarrow{AB}\right| \cdot \cos\,\alpha + \frac{1}{9}\left|\overrightarrow{AB}\right|^2\right) = \frac{1}{4}\left(4 + \frac{2}{3} + 1\right) = \frac{17}{12}$

$\overrightarrow{OF} = \frac{\sqrt{17}}{\sqrt{12}} = \frac{\sqrt{17}}{2\sqrt{3}}$

$\overrightarrow{OF} \cdot \overrightarrow{AE} = \left(\frac{1}{2}\overrightarrow{AD} + \frac{1}{6}\overrightarrow{AB}\right)\left(\frac{1}{4}\overrightarrow{AD} + \frac{3}{4}\overrightarrow{AB}\right) = \frac{1}{8}\overrightarrow{AD}^2 + \left(\frac{3}{8} + \frac{1}{24}\right)\overrightarrow{AB} \cdot \overrightarrow{AD} + \frac{1}{8}\overrightarrow{AB}^2 =$

$\frac{1}{8}\left(\left|\overrightarrow{AD}\right|^2 + \frac{10}{3}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AD}\right| \cdot \cos\,\alpha + \left|\overrightarrow{AB}\right|^2\right) = \frac{1}{8}\left(4 + \frac{10}{3} \cdot 3 \cdot 2 \cdot \frac{1}{6} + 9\right) = \frac{1}{8}\left(4 + \frac{10}{3} + 9\right) = \frac{1}{8} \cdot \frac{49}{3} = \frac{49}{24}$

$\cos\left(\overrightarrow{OF},\, \overrightarrow{AE}\right) = \frac{\overrightarrow{OF} \cdot \overrightarrow{AE}}{\left|\overrightarrow{OF}\right| \cdot \left|\overrightarrow{AE}\right|} = \frac{\frac{49}{24}}{\frac{\sqrt{17}}{2\sqrt{3}} \cdot \frac{\sqrt{91}}{4}} = \frac{49 \cdot 2\sqrt{3} \cdot 4}{24\sqrt{17} \cdot \sqrt{91}} = \frac{49 \cdot \sqrt{3}}{3\sqrt{17} \cdot \sqrt{91}}$

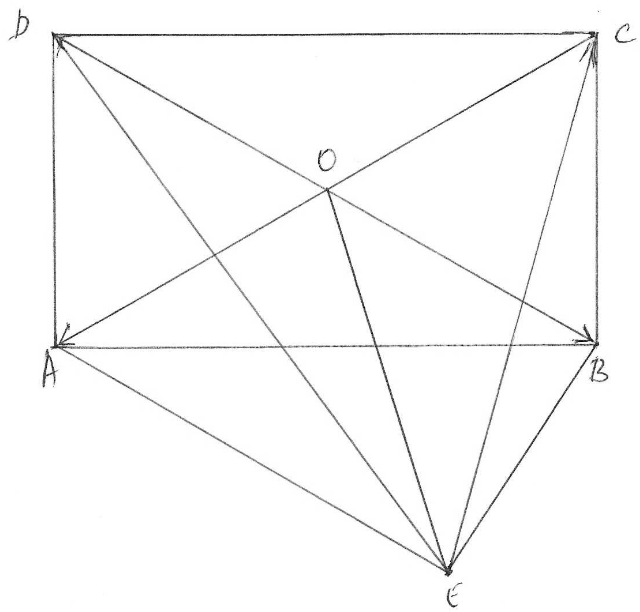
9. Dat je pravougaonik $ABCD$ i tačka E. Dokazati:

a) $\overrightarrow{ED} \cdot \overrightarrow{EB} = \overrightarrow{EA} \cdot \overrightarrow{EC}$

b) $\overrightarrow{EA}^2 + \overrightarrow{EC}^2 = \overrightarrow{EB}^2 + \overrightarrow{ED}^2$

Rešenje.

a) $\overrightarrow{ED} \cdot \overrightarrow{EB} = \left(\overrightarrow{EO} + \overrightarrow{OD}\right)\left(\overrightarrow{EO} + \overrightarrow{OB}\right) = \left|\overrightarrow{EO}\right|^2 + \overrightarrow{EO}\left(\overrightarrow{OB} + \overrightarrow{OD}\right) - \left|\overrightarrow{OD}\right|^2 = \left|\overrightarrow{EO}\right|^2 - \left|\overrightarrow{OD}\right|^2 ...(1)$



Slika 1.8:

$\overrightarrow{EA}\cdot\overrightarrow{EC} = \left(\overrightarrow{EO} + \overrightarrow{OA}\right)\left(\overrightarrow{EO} + \overrightarrow{OC}\right) = \left|\overrightarrow{EO}\right|^2 + \overrightarrow{EO}\left(\overrightarrow{OC} + \overrightarrow{OA}\right) - \left|\overrightarrow{AO}\right|^2 = \left|\overrightarrow{EO}\right|^2 - \left|\overrightarrow{AO}\right|^2 ...(2)$

$(\overrightarrow{AO} = \overrightarrow{OD})$

Iz (1) i (2) sledi $\overrightarrow{ED}\cdot\overrightarrow{EB} = \overrightarrow{EA}\cdot\overrightarrow{EC}$

b) $\left|\overrightarrow{AC}\right|^2 = \left|\overrightarrow{DB}\right|^2$

$$\left|\overrightarrow{EC} - \overrightarrow{EA}\right|^2 = \left|\overrightarrow{EB} - \overrightarrow{ED}\right|^2$$

$$\left(\overrightarrow{EC} - \overrightarrow{EA}\right)^2 = \left(\overrightarrow{EB} - \overrightarrow{ED}\right)^2$$

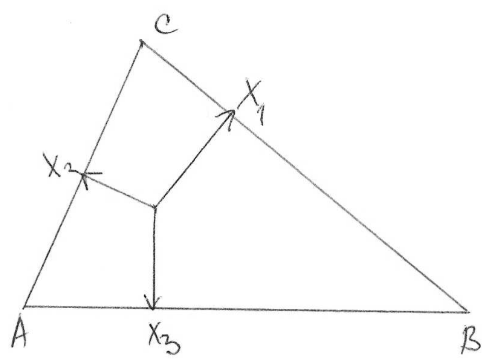
$$\left|\overrightarrow{EC}\right|^2 - 2\overrightarrow{EC}\cdot\overrightarrow{EA} + \left|\overrightarrow{EA}\right|^2 = \left|\overrightarrow{EB}\right|^2 - 2\overrightarrow{EB}\cdot\overrightarrow{ED} + \left|\overrightarrow{ED}\right|^2$$

$$\left|\overrightarrow{EC}\right|^2 + \left|\overrightarrow{EA}\right|^2 = \left|\overrightarrow{EB}\right|^2 + \left|\overrightarrow{ED}\right|^2$$

$$\overrightarrow{EA}^2 + \overrightarrow{EC}^2 = \overrightarrow{EB}^2 + \overrightarrow{ED}^2$$

10. Neka je dat jednakostranični ΔABC i njemu tačka X čija su odstojanja od stranica trougla jednaka t_1, t_2, t_3. Ako su X_1, X_2, X_3 podnožja normala iz tačke X na stranice odrediti koeficijente k_1, k_2, k_3, takve da važi:

Rešenje.



Slika 1.9:

$$k_1\left(\overrightarrow{XX}_1\right) + k_2\left(\overrightarrow{XX}_2\right) + k_3\left(\overrightarrow{XX}_3\right) = \vec{0}$$

$$\frac{a\cdot\overrightarrow{XX}_1}{t_1} + \frac{a\cdot\overrightarrow{XX}_2}{t_2} + \frac{a\cdot\overrightarrow{XX}_3}{t_3} = \vec{0}$$

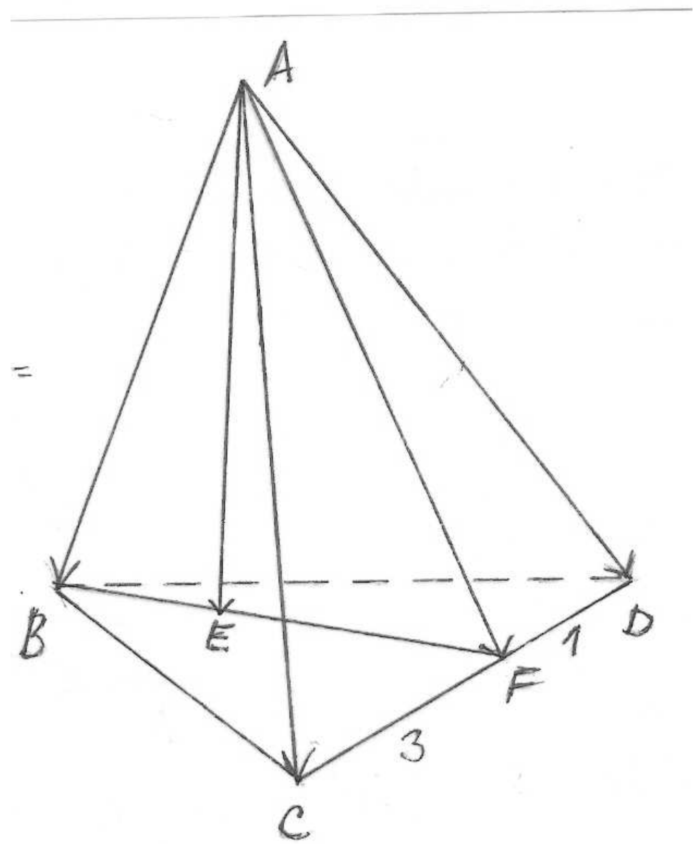
$k_1 = \frac{1}{t_1}$, $k_2 = \frac{1}{t_2}$, $k_3 = \frac{1}{t_3}$

11. Neka je u tetraedru $ABCD$ dato $\left|\overrightarrow{AB}\right| = 1$, $\left|\overrightarrow{AC}\right| = 2$, $\left|\overrightarrow{AD}\right| = 3$, $\cos\left(\overrightarrow{AB}, \overrightarrow{AC}\right) = \frac{1}{2}$, $\cos\left(\overrightarrow{AC}, \overrightarrow{AD}\right) = \frac{1}{6}$, $\cos\left(\overrightarrow{AB}, \overrightarrow{AD}\right) = \frac{1}{3}$. Neka tačka F deli CD u odnosu $3:1$, a tačka E deli BF u odnosu $2:3$. Odrediti ugao između $\overrightarrow{AE}$ i $\overrightarrow{AF}$.

Rešenje:

$$\overrightarrow{AE} = \frac{3}{5}\overrightarrow{AB} + \frac{2}{5}\overrightarrow{AF}$$

$$\overrightarrow{AF} = \frac{1}{4}\overrightarrow{AC} + \frac{3}{4}\overrightarrow{AD}$$



Slika 1.10:

$$\overrightarrow{AE} = \frac{3}{5}\overrightarrow{AB} + \frac{2}{5}\left(\frac{1}{4}\overrightarrow{AC} + \frac{3}{4}\overrightarrow{AD}\right) = \frac{3}{5}\overrightarrow{AB} + \frac{1}{10}\overrightarrow{AC} + \frac{3}{10}\overrightarrow{AD}$$

$\left|\overrightarrow{AE}\right|^2 = \left(\frac{3}{5}\overrightarrow{AB} + \frac{1}{10}\overrightarrow{AC} + \frac{3}{10}\overrightarrow{AD}\right)^2 = \frac{9}{25}\overrightarrow{AB}^2 + \frac{1}{100}\overrightarrow{AC}^2 + \frac{9}{100}\overrightarrow{AD}^2 + \frac{6}{50}\overrightarrow{AB}\cdot\overrightarrow{AC} + \frac{18}{50}\overrightarrow{AB}\cdot\overrightarrow{AD} + \frac{6}{100}\overrightarrow{AC}\cdot\overrightarrow{AD} =$

$\frac{9}{25}\left|\overrightarrow{AB}\right|^2 + \frac{1}{100}\left|\overrightarrow{AC}\right|^2 + \frac{9}{100}\left|\overrightarrow{AD}\right|^2 + \frac{6}{50}\left|\overrightarrow{AB}\right|\cdot\left|\overrightarrow{AC}\right|\cdot\cos\left(\overrightarrow{AB},\,\overrightarrow{AC}\right) + \frac{18}{50}\left|\overrightarrow{AB}\right|\cdot\left|\overrightarrow{AD}\right|\cdot\cos\left(\overrightarrow{AB},\,\overrightarrow{AD}\right) + \frac{6}{100}\left|\overrightarrow{AC}\right|\cdot\left|\overrightarrow{AD}\right|\cdot\cos\left(\overrightarrow{AC},\,\overrightarrow{AD}\right) =$

$\frac{9}{25}\cdot 1 + \frac{1}{100}\cdot 4 + \frac{9}{100}\cdot 9 + \frac{6}{50}\cdot 1\cdot 2\cdot\frac{1}{2} + \frac{18}{50}\cdot 1\cdot 3\cdot\frac{1}{3} + \frac{6}{100}2\cdot 3\cdot\frac{1}{6} =$

$\frac{9}{25} + \frac{1}{25} + \frac{81}{100} + \frac{6}{50} + \frac{18}{50} + \frac{6}{100} = \frac{36+4+81+12+36+6}{100} = \frac{175}{100} = \frac{7}{4}$

$\left|\overrightarrow{AE}\right| = \frac{\sqrt{7}}{2}$

$$\left|\overrightarrow{AF}\right|^2 = \left(\frac{1}{4}\overrightarrow{AC} + \frac{3}{4}\overrightarrow{AD}\right)^2 = \frac{1}{16}\overrightarrow{AC}^2 + \frac{6}{16}\overrightarrow{AC}\cdot\overrightarrow{AD} + \frac{9}{16}\overrightarrow{AD}^2 =$$

$$\frac{1}{16}\left|\overrightarrow{AC}\right|^2 + \frac{6}{16}\left|\overrightarrow{AC}\right|\cdot\left|\overrightarrow{AD}\right|\cdot\cos\left(\overrightarrow{AC},\, \overrightarrow{AD}\right) + \frac{9}{16}\left|\overrightarrow{AD}\right|^2 =$$

$$\frac{1}{16}\cdot 4 + \frac{6}{16}\cdot 2\cdot 3\cdot\frac{1}{6} + \frac{9}{16}\cdot 9 = \frac{1}{4} + \frac{3}{8} + \frac{81}{16} = \frac{4+6+81}{100} = \frac{91}{16}$$

$$\left|\overrightarrow{AF}\right| = \frac{\sqrt{91}}{4}$$

$$\overrightarrow{AE}\cdot\overrightarrow{AF} = \left(\frac{3}{5}\overrightarrow{AB} + \frac{1}{10}\overrightarrow{AC} + \frac{3}{10}\overrightarrow{AD}\right)\left(\frac{1}{4}\overrightarrow{AC} + \frac{3}{4}\overrightarrow{AD}\right) =$$

$$= \frac{3}{20}\overrightarrow{AB}\cdot\overrightarrow{AC} + \frac{9}{20}\overrightarrow{AB}\cdot\overrightarrow{AD} + \frac{1}{40}\overrightarrow{AC}^2 + \frac{3}{40}\overrightarrow{AC}\cdot\overrightarrow{AD} + \frac{3}{40}\overrightarrow{AD}\cdot\overrightarrow{AC} + \frac{9}{40}\overrightarrow{AD}^2 =$$

$$\frac{3}{20}\left|\overrightarrow{AB}\right|\cdot\left|\overrightarrow{AC}\right|\cdot\cos\left(\overrightarrow{AB},\, \overrightarrow{AC}\right) + \frac{9}{20}\left|\overrightarrow{AB}\right|\cdot\left|\overrightarrow{AD}\right|\cdot\cos\left(\overrightarrow{AB},\, \overrightarrow{AD}\right) +$$
$$\frac{1}{40}\left|\overrightarrow{AC}\right|^2 + \frac{3}{40}\left|\overrightarrow{AC}\right|\cdot\left|\overrightarrow{AD}\right|\cdot\cos\left(\overrightarrow{AC},\, \overrightarrow{AD}\right) + \frac{3}{40}\left|\overrightarrow{AD}\right|\cdot\left|\overrightarrow{AC}\right|\cdot\cos\left(\overrightarrow{AD},\, \overrightarrow{AC}\right) +$$
$$\frac{9}{40}\left|\overrightarrow{AD}\right|^2 =$$

$$\frac{3}{20}\cdot 1\cdot 2\cdot\frac{1}{2} + \frac{9}{20}\cdot 1\cdot 3\cdot\frac{1}{3} + \frac{1}{40}\cdot 4 + \frac{3}{40}\cdot 2\cdot 3\cdot\frac{1}{6} + \frac{3}{40}\cdot 3\cdot 2\cdot\frac{1}{6} + \frac{9}{40}\cdot 9 =$$

$$\frac{3}{20} + \frac{9}{20} + \frac{1}{10} + \frac{3}{40} + \frac{3}{40} + \frac{9}{40} = \frac{6+18+4+3+9}{40} = \frac{43}{40}$$

$$\cos\left(\overrightarrow{AE},\, \overrightarrow{AF}\right) = \frac{\overrightarrow{AE}\cdot\overrightarrow{AF}}{\left|\overrightarrow{AE}\right|\cdot\left|\overrightarrow{AF}\right|} = \frac{\frac{43}{40}}{\frac{\sqrt{7}}{2}\cdot\frac{\sqrt{91}}{4}} = \frac{43\cdot 2\cdot 4}{40\sqrt{7}\cdot\sqrt{91}} = \frac{344}{40\sqrt{637}}$$

12. Neka su $\vec{u}$ i $\vec{v}$ vektori različiti od $\vec{0}$, i takvi da je vektor $2\vec{u} - \vec{v}$ normalan na vektor $\vec{u} + \vec{v}$ i $\vec{u} - 2\vec{v}$ normalan na vektor $2\vec{u} + \vec{v}$. Odrediti ugao između vektora $\vec{u}$ i $\vec{v}$.

Rešenje.

$$\left.\begin{array}{l}(2\vec{u} - \vec{v})(\vec{u} + \vec{v}) = \vec{0}\\ (\vec{u} - 2\vec{v})(2\vec{u} + \vec{v}) = \vec{0}\end{array}\right\}$$

$$\left.\begin{array}{l}2\vec{u}^2+2\vec{u}\vec{v}-\vec{u}\vec{v}-\vec{v}^2=\vec{0}\\ 2\vec{u}^2+2\vec{u}\vec{v}-4\vec{u}\vec{v}-2\vec{v}^2=\vec{0}\end{array}\right\}$$

$$\left.\begin{array}{l}2\vec{u}^2+\vec{u}\vec{v}-\vec{v}^2=\vec{0}\\ 2\vec{u}^2-3\vec{u}\vec{v}-2\vec{v}^2=\vec{0}\end{array}\right\}$$

$$\left.\begin{array}{l}2\left|\vec{u}\right|^2+\left|\vec{u}\right|\left|\vec{v}\right|\cdot\cos\,\alpha-\left|\vec{v}\right|^2=\vec{0}/:\left|\vec{v}\right|^2\\ 2\left|\vec{u}\right|^2-3\left|\vec{u}\right|\left|\vec{v}\right|\cdot\cos\,\alpha-2\left|\vec{v}\right|^2=\vec{0}/:\left|\vec{v}\right|^2\end{array}\right\}$$

$$\left.\begin{array}{l}2\left(\frac{|\vec{u}|}{|\vec{v}|}\right)^2+\frac{|\vec{u}|}{|\vec{v}|}\cdot\cos\,\alpha-1=\vec{0}\\ 2\left(\frac{|\vec{u}|}{|\vec{v}|}\right)^2-3\frac{|\vec{u}|}{|\vec{v}|}\cdot\cos\,\alpha-2=\vec{0}/\cdot(-1)\end{array}\right\}$$

$$4\frac{|\vec{u}|}{|\vec{v}|}\cdot\cos\,\alpha+1=\vec{0}$$

$$\frac{|\vec{u}|}{|\vec{v}|}\cdot\cos\,\alpha=\frac{1}{4}$$

$$2\left(\frac{|\vec{u}|}{|\vec{v}|}\right)^2-\frac{1}{4}-1=0$$

$$2\left(\frac{|\vec{u}|}{|\vec{v}|}\right)^2=\frac{5}{4}$$

$$\left(\frac{|\vec{u}|}{|\vec{v}|}\right)^2=\frac{5}{8}$$

$$\cos\,\alpha=-\frac{1}{4}\cdot\frac{|\vec{u}|}{|\vec{v}|}=-\frac{1}{4}\cdot\frac{\sqrt{8}}{\sqrt{5}}=-\frac{2\sqrt{2}}{4\sqrt{5}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=-\frac{1}{\sqrt{10}}$$

13. Neka je u $\Delta\,ABC$ dato $\left|\overrightarrow{AB}\right|=4$, $\left|\overrightarrow{AC}\right|=2$, $\alpha=60^\circ$. Ako tačka D deli stranicu BC u odnosu $1:2$, odrediti $\cos\left(\overrightarrow{DA},\,\overrightarrow{DC}\right)$.

Rešenje.

$$\overrightarrow{DA}=-\overrightarrow{AD}=-\left(\frac{2}{3}\overrightarrow{AB}+\frac{1}{3}\overrightarrow{AC}\right)$$

$$\overrightarrow{DC}=\frac{2}{3}\overrightarrow{BC}=\frac{2}{3}\left(\overrightarrow{AC}-\overrightarrow{AB}\right)$$

$$\left|\overrightarrow{DA}\right|^2=\left(-\left(\frac{2}{3}\overrightarrow{AB}+\frac{1}{3}\overrightarrow{AC}\right)\right)^2=\frac{1}{9}\left(4\overrightarrow{AB}^2+4\overrightarrow{AB}\cdot\overrightarrow{AC}+\overrightarrow{AC}^2\right)=$$
$$\frac{1}{9}\left(4\left|\overrightarrow{AB}\right|^2+4\left|\overrightarrow{AB}\right|\cdot\left|\overrightarrow{AC}\right|\cdot\cos\varepsilon+\left|\overrightarrow{AC}\right|^2\right)=\frac{1}{9}\left(4\cdot 16+4\cdot 4\cdot 2\cdot\frac{1}{2}+16\right)=\frac{84}{9}$$

$$\left|\overrightarrow{DC}\right|^2=\left(\frac{2}{3}\left(\overrightarrow{AC}-\overrightarrow{AB}\right)\right)^2=\frac{4}{9}\left(\overrightarrow{AC}^2-2\overrightarrow{AB}\cdot\overrightarrow{AC}+\overrightarrow{AB}^2\right)=$$
$$\frac{4}{9}\left(\left|\overrightarrow{AC}\right|^2-2\left|\overrightarrow{AB}\right|\cdot\left|\overrightarrow{AC}\right|\cdot\cos\alpha+\left|\overrightarrow{AB}\right|^2\right)=\frac{4}{9}\left(4-2\cdot 4\cdot 2\cdot\frac{1}{2}+16\right)=\frac{48}{9}$$

$$\left|\overrightarrow{DA}\right|\cdot\left|\overrightarrow{DC}\right|=\left(-\left(\frac{2}{3}\overrightarrow{AB}+\frac{1}{3}\overrightarrow{AC}\right)\right)\left(\frac{2}{3}\left(\overrightarrow{AC}-\overrightarrow{AB}\right)\right)=-\frac{2}{9}\left(\overrightarrow{AB}\cdot\overrightarrow{AC}-2\overrightarrow{AB}^2+\overrightarrow{AC}\right.$$
$$-\frac{2}{9}\left(4\cdot 2-2\cdot 16+4\right)=\frac{48}{9}$$

$$\cos\left(\overrightarrow{DA},\,\overrightarrow{DC}\right)=\frac{\overrightarrow{DA}\cdot\overrightarrow{DC}}{\left|\overrightarrow{DA}\right|\cdot\left|\overrightarrow{DC}\right|}=\frac{\frac{48}{9}}{\frac{\sqrt{84}}{3}\cdot\frac{\sqrt{48}}{3}}=\frac{\sqrt{48}}{\sqrt{84}}=\frac{\sqrt{8}}{\sqrt{14}}=\frac{2}{\sqrt{7}}$$

14. Neka su A, B, C, D proizvoljne četiri tačke u prostoru. Dokazati da je

$$\overrightarrow{AB}\cdot\overrightarrow{CD}+\overrightarrow{AC}\cdot\overrightarrow{DB}+\overrightarrow{AD}\cdot\overrightarrow{BC}=\vec{0}$$

Rešenje:

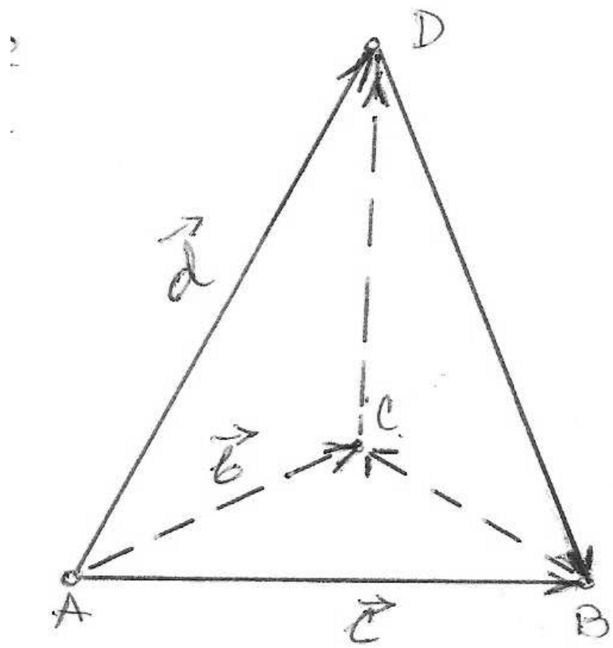
$\overrightarrow{AB}=\vec{c}$, $\overrightarrow{AC}=\vec{b}$, $\overrightarrow{AD}=\vec{d}$

iz $\Delta ABC\ \Rightarrow\ \overrightarrow{BC}=\overrightarrow{AC}-\overrightarrow{AB}=\vec{b}-\vec{c}$

iz $\Delta ABD\ \Rightarrow\ \overrightarrow{DB}=\overrightarrow{AB}-\overrightarrow{AD}=\vec{c}-\vec{d}$

iz $\Delta ACD\ \Rightarrow\ \overrightarrow{CD}=\overrightarrow{AD}-\overrightarrow{AC}=\vec{d}-\vec{b}$

$$\overrightarrow{AB}\cdot\overrightarrow{CD}+\overrightarrow{AC}\cdot\overrightarrow{DB}+\overrightarrow{AD}\cdot\overrightarrow{BC}=\vec{c}\left(\vec{d}-\vec{b}\right)+\vec{b}\left(\vec{c}-\vec{d}\right)+\vec{d}\left(\vec{b}-\vec{c}\right)=$$
$$\vec{c}\cdot\vec{d}-\vec{c}\cdot\vec{b}+\vec{b}\cdot\vec{c}-\vec{b}\cdot\vec{d}+\vec{d}\cdot\vec{b}-\vec{d}\cdot\vec{c}=\vec{c}\cdot\vec{d}-\vec{b}\cdot\vec{c}+\vec{b}\cdot\vec{c}-\vec{d}\cdot\vec{b}+\vec{d}\cdot\vec{b}-\vec{c}\cdot\vec{d}=$$
$$\left(\vec{c}\cdot\vec{d}-\vec{c}\cdot\vec{d}\right)+\left(\vec{b}\cdot\vec{c}-\vec{b}\cdot\vec{c}\right)+\left(\vec{b}\cdot\vec{d}-\vec{b}\cdot\vec{d}\right)=\vec{0}$$



Slika 1.11:

$$\overrightarrow{AB}\cdot\overrightarrow{CD}+\overrightarrow{AC}\cdot\overrightarrow{DB}+\overrightarrow{AD}\cdot\overrightarrow{BC}=\vec{0}$$

15. Odrediti ugao između naspramnih ivica tetraedra.

Rešenje:

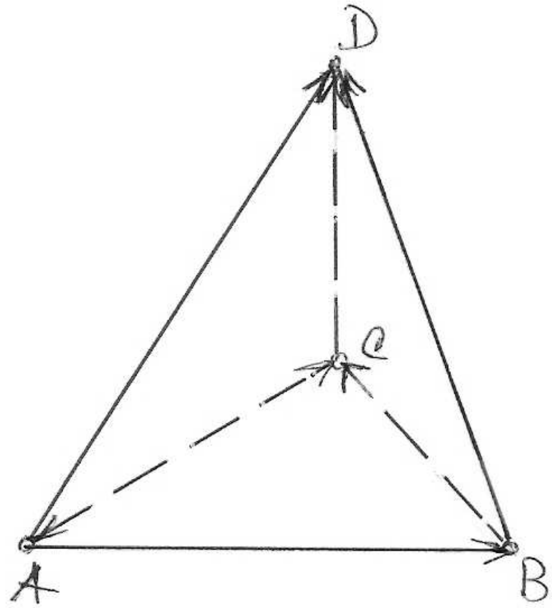
$ABCD$ - tetraedar sa osnovom ΔABC. Određuje se ugao između bočnih ivica AB i CD (BC i AD; CA i BD;)

$$\overrightarrow{BC}=\overrightarrow{AC}-\overrightarrow{AB}/\cdot\overrightarrow{AD}$$

$$\overrightarrow{CD}=\overrightarrow{AD}-\overrightarrow{AC}/\cdot\overrightarrow{AB}$$

$$\overrightarrow{BD}=\overrightarrow{AD}-\overrightarrow{AB}/\cdot\overrightarrow{CA}$$

$$\left.\begin{array}{l}\overrightarrow{BC}\cdot\overrightarrow{AD}=\left(\overrightarrow{AC}-\overrightarrow{AB}\right)\cdot\overrightarrow{AD}=\overrightarrow{AC}\cdot\overrightarrow{AD}-\overrightarrow{AB}\cdot\overrightarrow{AD}\\ \overrightarrow{CD}\cdot\overrightarrow{AB}=\left(\overrightarrow{AD}-\overrightarrow{AC}\right)\cdot\overrightarrow{AB}=\overrightarrow{AD}\cdot\overrightarrow{AB}-\overrightarrow{AC}\cdot\overrightarrow{AB}\\ \overrightarrow{BD}\cdot\overrightarrow{CA}=\left(\overrightarrow{AD}-\overrightarrow{AB}\right)\cdot\overrightarrow{CA}=\overrightarrow{AD}\cdot\overrightarrow{CA}-\overrightarrow{AB}\cdot\overrightarrow{CA}\end{array}\right\}+$$



Slika 1.12:

$$\overrightarrow{BC} \cdot \overrightarrow{AD} + \overrightarrow{CD} \cdot \overrightarrow{AB} + \overrightarrow{BD} \cdot \overrightarrow{CA} = \left(\overrightarrow{AC} \cdot \overrightarrow{AD} - \overrightarrow{AB} \cdot \overrightarrow{AD}\right) + \left(\overrightarrow{AD} \cdot \overrightarrow{AB} - \overrightarrow{AC} \cdot A\right.$$
$$\left(\overrightarrow{AD} \cdot \overrightarrow{CA} - \overrightarrow{AB} \cdot \overrightarrow{CA}\right) = \left(\overrightarrow{AC} \cdot \overrightarrow{AD} - \overrightarrow{AC} \cdot \overrightarrow{AD}\right) + \left(\overrightarrow{AB} \cdot \overrightarrow{AD} - \overrightarrow{AB} \cdot \overrightarrow{AD}\right) +$$
$$\left(\overrightarrow{AB} \cdot \overrightarrow{AC} - \overrightarrow{AB} \cdot \overrightarrow{AC}\right) = \vec{0}$$

$$\overrightarrow{BC} \cdot \overrightarrow{AD} = \left|\overrightarrow{BC}\right| \cdot \left|\overrightarrow{AD}\right| \cdot \cos\left(\overrightarrow{BC}, \overrightarrow{AD}\right)$$

$$\overrightarrow{CD} \cdot \overrightarrow{AB} = \left|\overrightarrow{CD}\right| \cdot \left|\overrightarrow{AB}\right| \cdot \cos\left(\overrightarrow{CD}, \overrightarrow{AB}\right)$$

$$\overrightarrow{BD} \cdot \overrightarrow{CA} = \left|\overrightarrow{BD}\right| \cdot \left|\overrightarrow{CA}\right| \cdot \cos\left(\overrightarrow{BD}, \overrightarrow{CA}\right)$$

$$\overrightarrow{BC} \cdot \overrightarrow{AD} + \overrightarrow{CD} \cdot \overrightarrow{AB} + \overrightarrow{BD} \cdot \overrightarrow{CA} = \left|\overrightarrow{BC}\right| \cdot \left|\overrightarrow{AD}\right| \cdot \cos\left(\overrightarrow{BC}, \overrightarrow{AD}\right) +$$
$$\left|\overrightarrow{CD}\right| \cdot \left|\overrightarrow{AB}\right| \cdot \cos\left(\overrightarrow{CD}, \overrightarrow{AB}\right) + \left|\overrightarrow{BD}\right| \cdot \left|\overrightarrow{CA}\right| \cdot \cos\left(\overrightarrow{BD}, \overrightarrow{CA}\right) = \vec{0}$$

Kako je $\left|\overrightarrow{BC}\right| \cdot \left|\overrightarrow{AD}\right| > 0$, $\left|\overrightarrow{CD}\right| \cdot \left|\overrightarrow{AB}\right| > 0$, $\left|\overrightarrow{BD}\right| \cdot \left|\overrightarrow{CA}\right| > 0$, (jer vektori $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{BD}$, $\overrightarrow{CD}$ nisu nulti vektori), onda je

$\cos\left(\overrightarrow{BC}, \overrightarrow{AD}\right) = \cos\frac{\pi}{2} = 0$

$\cos\left(\overrightarrow{CD}, \overrightarrow{AB}\right) = \cos\frac{\pi}{2} = 0$

$\cos\left(\overrightarrow{BD}, \overrightarrow{CA}\right) = \cos\frac{\pi}{2} = 0$, što znači da su naspramne ivice tetraedra normalne.

16. Ako je u teraedru $ABCD$ $\overrightarrow{AB}\perp\overrightarrow{CD}$, dokazati da je

$AC^2 - AD^2 = BC^2 - BD^2$

Rešenje:

$\overrightarrow{AB}\perp\overrightarrow{CD}$

$\overrightarrow{AB}\perp\overrightarrow{DC}$

$(\vec{a}^2 = \vec{a}\cdot\vec{a}\cdot\cos(\vec{a}, \vec{a}) = |\vec{a}|^2\cdot\cos 0^\circ = a^2\cdot 1 = a^2)$

$AC^2 - AD^2 = \left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{AD}\right|^2 = \overrightarrow{AC}^2 - \overrightarrow{AD}^2 = \left(\overrightarrow{AC} - \overrightarrow{AD}\right)\left(\overrightarrow{AC} + \overrightarrow{AD}\right) = \left(\overrightarrow{AC} + \overrightarrow{AD}\right)\left(\overrightarrow{AC} - \overrightarrow{AD}\right) = \left(\overrightarrow{AC} + \overrightarrow{AD}\right)\cdot\overrightarrow{DC}$

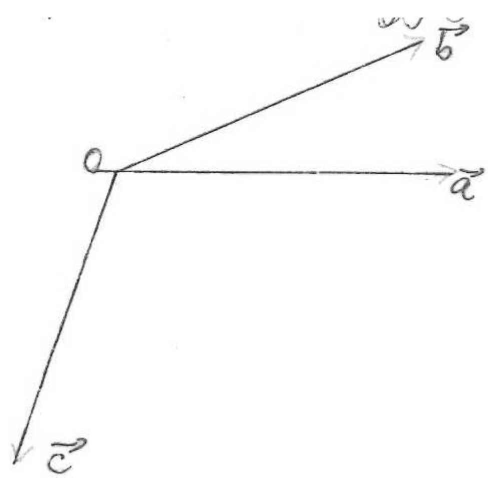
Iz $\Delta ABC \Rightarrow \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ iz $\Delta ABD \Rightarrow \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$

$$
\begin{aligned}
&AC^2 - AD^2 = \left[\left(\overrightarrow{AB} + \overrightarrow{BC}\right) + \left(\overrightarrow{AB} + \overrightarrow{BD}\right)\right]\cdot\overrightarrow{DC} = \\
&\left[2\overrightarrow{AB} + \left(\overrightarrow{BC} + \overrightarrow{BD}\right)\right]\cdot\overrightarrow{DC} = 2\overrightarrow{AB}\cdot\overrightarrow{DC} + \left(\overrightarrow{BC} + \overrightarrow{BD}\right)\cdot\overrightarrow{DC} = \\
&\vec{0} + \left(\overrightarrow{BC} + \overrightarrow{BD}\right)\cdot\overrightarrow{DC} = \left(\overrightarrow{BC} + \overrightarrow{BD}\right)\cdot\overrightarrow{DC} = \left(\overrightarrow{BC} + \overrightarrow{BD}\right)\left(\overrightarrow{BC} - \overrightarrow{BD}\right) = \\
&\overrightarrow{BC}\cdot\overrightarrow{BC} + \overrightarrow{BD}\cdot\overrightarrow{BC} - \overrightarrow{BC}\cdot\overrightarrow{BD} - \overrightarrow{BD}\cdot\overrightarrow{BD} = BC^2 - BD^2 = \\
&\left|\overrightarrow{BC}\right|\cdot\left|\overrightarrow{BC}\right|\cdot\cos\left(\overrightarrow{BC}, \overrightarrow{BC}\right) - \left|\overrightarrow{BD}\right|\cdot\left|\overrightarrow{BD}\right|\cdot\cos\left(\overrightarrow{BD}, \overrightarrow{BD}\right) = \\
&\left|\overrightarrow{BC}\right|^2\cdot cos\,0^\circ - \left|\overrightarrow{BD}\right|^2\cdot cos\,0^\circ = \left|\overrightarrow{BC}\right|^2 - \left|\overrightarrow{BD}\right|^2 = BC^2 - BD^2
\end{aligned}
$$

1.4 Vektorski proizvod vektora

Definicija 1.11. *Tri nekomplanarna vektora $\vec{a}$, $\vec{b}$ i $\vec{c}$ sa zajedničkim početkom obrazuju desni trijedar ako se rotacija vektora $\vec{a}$ prema vektoru $\vec{b}$, najkraćim putem, posmatra sa kraja vektora $\vec{c}$, vrši suprotno kretanju kazaljke na časovniku.*

Slično se definiše levi trijedar, koji obrazuju tri nekomplanarna vektora $\vec{a}$, $\vec{b}$ i $\vec{c}$ sa zajedničkim početkom.



Slika 1.13:

Definicija 1.12. *Ako je n_0 jedinični vektor normalan na ravan koji obrazuju vektori $\vec{a}$, $\vec{b}$, pri čemu $\vec{a}$, $\vec{b}$ i n_0 obrazuju desni trijedar, onda se vektor $|\vec{a}| \left|\vec{b}\right| \sin \sphericalangle \left(\vec{a}, \vec{b}\right) \vec{n}_0$ naziva vektorski proizvod vektora $\vec{a}$ i $\vec{b}$.*

$\left|\vec{a} \times \vec{b}\right| |\vec{a}| \left|\vec{b}\right| \sin \sphericalangle \left(\vec{a}, \vec{b}\right)$

Osobine vektorskog proizvoda

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ - antikomutativnost
2. $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b} \Leftrightarrow \vec{a} = k\vec{b}$
3. $k\left(\vec{a} \times \vec{b}\right) = k\vec{a} \times \vec{b} = \vec{a} \times k\vec{b}$ - homogenost
4. $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$\vec{a} \times \sum_{i=1}^{n} \vec{a}_i = \sum_{i=1}^{n} \vec{a} \times \vec{a}_i$

Površina paralelograma konstruisanog nad vektorima $\vec{a}$, $\vec{b}$ brojno je jednaka intenzitetu vektorskog proizvoda tih vektora.

$P = \left|\vec{a} \times \vec{b}\right|$

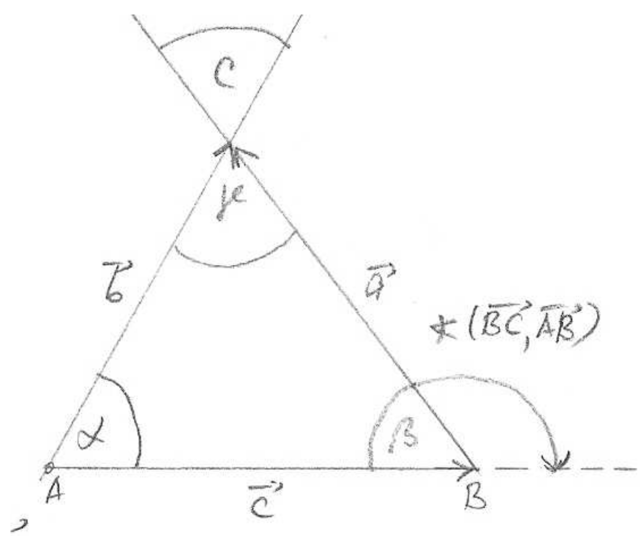
Brzina $\vec{V}$ ma koje tačke M krutog tela koje rotira brzinom ω oko date ose jednaka je

$\vec{V} = \vec{\omega} \times \vec{r}$,

gde je $\vec{r}$ vektor položaja tačke M, a osa rotacije prolazi kroz koordinatni početak.

1. Koristeći vektorski proizvod dokazati sinusnu teoremu za trougao u ravni.

Rešenje:



Slika 1.14:

$\overrightarrow{AB} = \vec{c}$, $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{AC} = \vec{b}$

$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = \vec{b} - \vec{c}$

$\overrightarrow{BC} \times \overrightarrow{BC} = \left|\overrightarrow{BC}\right| \cdot \left|\overrightarrow{BC}\right| \cdot \sin\left(\overrightarrow{BC}, \overrightarrow{BC}\right) = \left|\overrightarrow{BC}\right| \cdot \left|\overrightarrow{BC}\right| \cdot \sin 0^\circ = \vec{0}$

$$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} / \times \overrightarrow{BC}$$

$$\overrightarrow{BC} \times \overrightarrow{BC} = \overrightarrow{BC} \times \left(\overrightarrow{AC} - \overrightarrow{AB}\right) = \overrightarrow{BC} \times \overrightarrow{AC} - \overrightarrow{BC} \times \overrightarrow{AB} = \vec{0} \Rightarrow$$

$$\overrightarrow{BC} \times \overrightarrow{AC} = \overrightarrow{BC} \times \overrightarrow{AB}$$

$$\left|\overrightarrow{BC}\right| \cdot \left|\overrightarrow{AC}\right| \cdot \sin\left(\overrightarrow{BC}, \overrightarrow{AC}\right) = \left|\overrightarrow{BC}\right| \cdot \left|\overrightarrow{AB}\right| \cdot \sin\left(\overrightarrow{BC}, \overrightarrow{AB}\right)$$

$$\left|\overrightarrow{BC}\right| \cdot \left|\overrightarrow{AC}\right| \cdot \sin\gamma = \left|\overrightarrow{BC}\right| \cdot \left|\overrightarrow{AB}\right| \cdot \sin(\pi - \beta)$$

$$\left|\overrightarrow{BC}\right| \cdot \left|\overrightarrow{AC}\right| \cdot \sin\gamma = \left|\overrightarrow{BC}\right| \cdot \left|\overrightarrow{AB}\right| \cdot \sin\beta / \cdot \frac{1}{\left|\overrightarrow{BC}\right| \cdot \left|\overrightarrow{AC}\right| \cdot \left|\overrightarrow{AB}\right|}$$

$$\frac{\sin\gamma}{\left|\overrightarrow{AB}\right|} = \frac{\sin\beta}{\left|\overrightarrow{AC}\right|}$$

$$\frac{\sin\alpha}{c} = \frac{\sin\gamma}{b}$$

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} / \times \overrightarrow{AC}$$

$$\overrightarrow{AC} \times \overrightarrow{AC} = \overrightarrow{AC} \times \left(\overrightarrow{AB} + \overrightarrow{BC}\right) = \overrightarrow{AC} \times \overrightarrow{AB} + \overrightarrow{AC} \times \overrightarrow{BC} = \vec{0} \Rightarrow$$

$$\overrightarrow{AC} \times \overrightarrow{AC} = -\overrightarrow{AC} \times \overrightarrow{BC}$$

$$\left|\overrightarrow{AC}\right| \cdot \left|\overrightarrow{AB}\right| \cdot (-\sin\alpha) = -\left|\overrightarrow{AC}\right| \cdot \left|\overrightarrow{BC}\right| \cdot \sin\gamma / \cdot - \frac{1}{\left|\overrightarrow{AC}\right| \cdot \left|\overrightarrow{AB}\right| \cdot \left|\overrightarrow{BC}\right|}$$

$$\frac{\sin\alpha}{\left|\overrightarrow{BC}\right|} = \frac{\sin\gamma}{\left|\overrightarrow{AB}\right|}$$

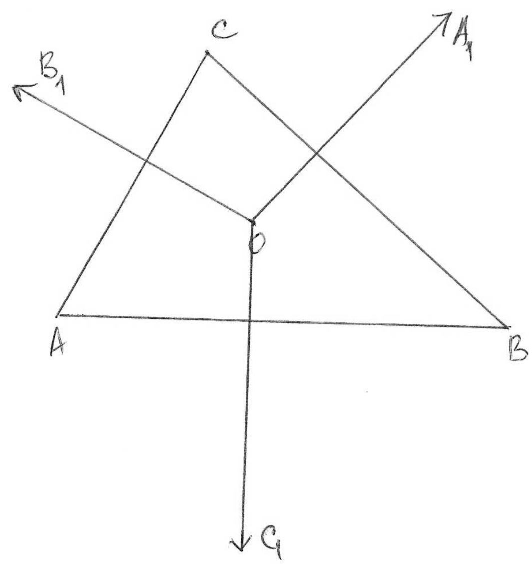
$$\frac{\sin\alpha}{c} = \frac{\sin\gamma}{a}$$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

2. Neka je dat $\Delta\, ABC$ i tačka O u njemu. Neka su vektori $\overrightarrow{OA}_1$, $\overrightarrow{OB}_1$, $\overrightarrow{OC}_1$ normalni na odgovarajuće stranice i imaju intenzitete jednake njihovim dužinama. Dokazati da je $\overrightarrow{OA}_1 + \overrightarrow{OB}_1 + \overrightarrow{OC}_1 = \vec{0}$.

Rešenje.



Slika 1.15:

$\overrightarrow{OA}_1 + \overrightarrow{OB}_1 + \overrightarrow{OC}_1 = \vec{a}$

$\vec{k} \times \vec{a}$ $\vec{k}$ je jedinični vektor normalan na ravan trougla

$\vec{k} \times \left(\overrightarrow{OA}_1 + \overrightarrow{OB}_1 + \overrightarrow{OC}_1\right) = \vec{k} \times \overrightarrow{OA}_1 + \vec{k} \times \overrightarrow{OB}_1 + \vec{k} \times \overrightarrow{OC}_1 = \overrightarrow{BC} + \overrightarrow{BA} + \overrightarrow{CA} = \vec{0}$

$\vec{k} \times \vec{a} = \vec{0}$

$\left|\vec{k} \times \vec{a}\right| = \vec{0}$

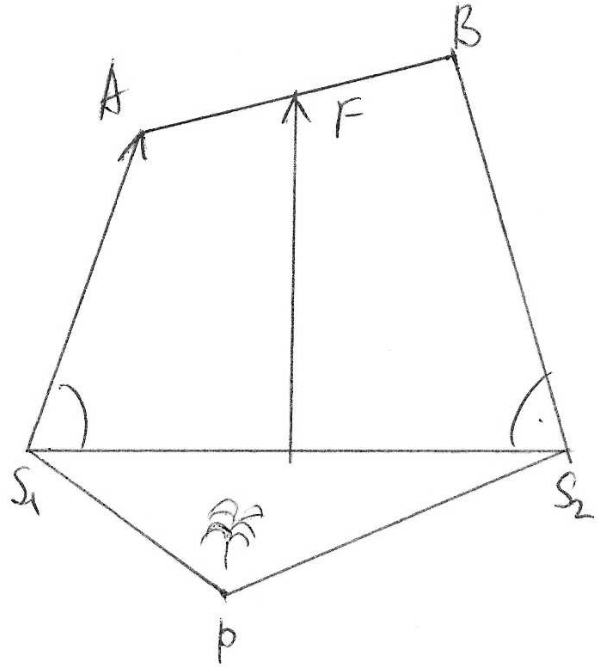
$\left|\vec{k}\right| \cdot |\vec{a}| \cdot \sin 90^0 = \vec{0}$

$|\vec{a}| = \vec{0}$

$\vec{a} = \vec{0}$

3. Na pustom ostrvu se nalaze palma i dve stene. Gusari su zakopali blago na mestu koje su odredili na sledeći način: položaj palme su rotirali oko stena u suprotnim smerovima za 90° i zatim su blago zakopali na sredini između tako dobijenih tačaka. Kada su došli iduće godine da otkopaju blago videli su da je neko isčupao palmu. Kako da gusari pronađu blago?

Rešenje.



Slika 1.16:

$\overrightarrow{AB}$ - A pomera u B

$\overrightarrow{BC}$ - B pomera u C

$\overrightarrow{AB} + \overrightarrow{BC}$ - A pomera u B, i B pomera u $C = \overrightarrow{AC}$

$\overrightarrow{PA} = \overrightarrow{PS}_1 + \overrightarrow{S_1A} = \overrightarrow{PS}_1 - k \times \overrightarrow{PS_1}$

$\overrightarrow{PB} = \overrightarrow{PS}_2 + \overrightarrow{S_2B} = \overrightarrow{PS}_2 + k \times \overrightarrow{PS_2}$

$\overrightarrow{\text{PF}} = \frac{1}{2}\left(\overrightarrow{PA} + \overrightarrow{PB}\right) = \frac{1}{2}\left(\overrightarrow{PS_1} + \overrightarrow{PS_2} + k \times \overrightarrow{PS_2} - k \times \overrightarrow{PS_1}\right) = \frac{1}{2}\left(\overrightarrow{PS_1} + \overrightarrow{PS_2}\right) + \frac{1}{2}k \times \left(\overrightarrow{PS_2} - \overrightarrow{PS_1}\right) = \frac{1}{2}\left(\overrightarrow{PS_1} + \overrightarrow{PS_2}\right) + \frac{1}{2}k \times \overrightarrow{S_1S_2}$

4. Za koju vrednost paramerta k će vektori $\vec{p} = k\vec{a} + 5\vec{b}$ i $\vec{q} = 3\vec{a} - \vec{b}$ biti kolinearni, ako vektori $\vec{a}$ i $\vec{b}$ to nisu

Rešenje.

$\vec{p} \parallel \vec{q} \Leftrightarrow \vec{p} \times \vec{q} = \vec{0}$

$\vec{p} \times \vec{q} = \left(k\vec{a} + 5\vec{b}\right) \times \left(3\vec{a} - \vec{b}\right) = 3k\,(\vec{a} \times \vec{a}) - k\left(\vec{a} \times \vec{b}\right) + 15\left(\vec{b} \times \vec{a}\right) - 5\left(\vec{b} \times \vec{b}\right) =$

$-k\left(\vec{a} \times \vec{b}\right) - 15\left(\vec{a} \times \vec{b}\right) = \left(\vec{a} \times \vec{b}\right)(-k - 15)$

$\vec{p} \times \vec{q} = \vec{0} \Rightarrow -k - 15 = 0 \Rightarrow k = -15$

5. Odrediti površinu paralelograma čije su stranice vektori $\vec{a} = \vec{m} - 2\vec{n}$ i $\vec{b} = \vec{n} - 2\vec{m}$, gde su $\vec{m}$ i $\vec{n}$ jedinični vektori, a ugao između $\vec{m}$ i $\vec{n}$ je $\frac{\pi}{6}$.

Rešenje;

$\vec{a} = \vec{m} - 2\vec{n}\ \vec{b} = \vec{n} - 2\vec{m}$

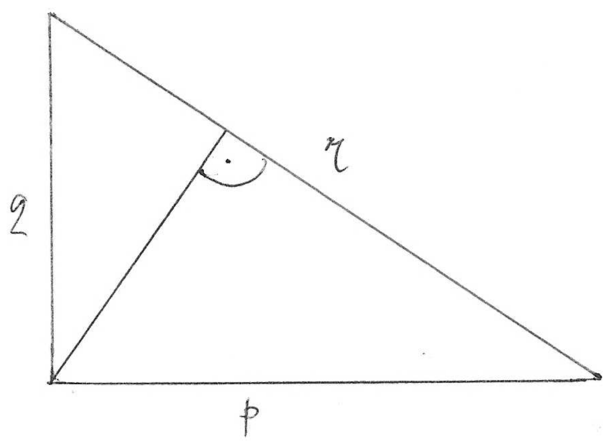
$(\vec{m}, \vec{n}) = \frac{\pi}{6}$

$|\vec{m}| = |\vec{n}| = 1$

$P = \left|\vec{a} \times \vec{b}\right| = |(\vec{m} - 2\vec{n}) \times (\vec{n} - 2\vec{m})| = |\vec{m} \times \vec{n} + 4\vec{n} \times \vec{m}| = |3\vec{n} \times \vec{m}| = 3 \cdot 1 \cdot 1 \cdot \frac{1}{2} = \frac{3}{2}$

6. Dve stranice trougla su $\vec{p} = 2\vec{a} + 3\vec{b}$ i $\vec{q} = \vec{a} - 4\vec{b}$, gde su $\vec{a}$ i $\vec{b}$ normalni ortovi. Izračunati visinu prema trećoj stranici trougla.

Rešenje.



Slika 1.17:

$\vec{p} = 2\vec{a} + 3\vec{b}$ $\qquad\qquad$ $\vec{q} = \vec{a} - 4\vec{b}$

$|\vec{a}| = \left|\vec{b}\right| = 1$

$\vec{a} \perp \vec{b}$

$P = \frac{1}{2}\,|\vec{p} \times \vec{q}| = \frac{1}{2}\left|\left(2\vec{a} + 3\vec{b}\right) \times \left(\vec{a} - 4\vec{b}\right)\right| = \frac{1}{2}\left|2\,(\vec{a} \times \vec{a}) - 8\left(\vec{a} \times \vec{b}\right) + 3\left(\vec{b} \times \vec{a}\right) - 1\right.$ $\frac{1}{2}\left|-8\left(\vec{a} \times \vec{b}\right) + 3\left(\vec{b} \times \vec{a}\right)\right| = \frac{1}{2}\left|-11\left(\vec{a} \times \vec{b}\right)\right| = \frac{1}{2} \cdot 11\left|\vec{a} \times \vec{b}\right| = \frac{11}{2}\,|\vec{a}| \cdot \left|\vec{b}\right| \cdot \sin\left(\vec{a}, \vec{b}\right) = \frac{11}{2}$

$\vec{r} = \vec{p} - \vec{q} = 2\vec{a} + 3\vec{b} - \vec{a} + 4\vec{b} = \vec{a} + 7\vec{b}$

$\vec{r}^2 = \left(\vec{a} + 7\vec{b}\right) = |\vec{a}|^2 + 49\left|\vec{b}\right|^2 + 14\,|\vec{a}| \cdot \left|\vec{b}\right| = 1 + 49 \cdot 1 = 50$

$|\vec{r}| = \sqrt{50} = 5\sqrt{2}$

$P = \frac{r \cdot h}{2}$

$\frac{11}{2} = \frac{5\sqrt{2}}{2} \cdot h$

$h = \frac{11}{5\sqrt{2}}$

7. Primenom vektorskog proizvoda izvesti Heronov obrazac za izračunavanje površine trougla.

Rešenje.

$P = \frac{1}{2} \left| \vec{c} \times \vec{b} \right|$

$P = \frac{1}{2} \cdot \vec{c} \cdot \vec{b} \cdot \sin\,\alpha$

$$P^2 = \frac{1}{4} \cdot \vec{c}^2 \cdot \vec{b}^2 \cdot \sin^2\alpha = \frac{1}{4} \cdot \vec{c}^2 \cdot \vec{b}^2 \cdot \left(1 - \cos^2\alpha\right) =$$
$$\frac{1}{4}\left(\vec{c}^2 \cdot \vec{b}^2 - \vec{c}^2 \cdot \vec{b}^2 \cdot \cos^2\alpha\right) =$$
$$\frac{1}{4}\left(\vec{c} \cdot \vec{b} - \vec{c} \cdot \vec{b} \cdot \cos\alpha\right)\left(\vec{c} \cdot \vec{b} + \vec{c} \cdot \vec{b} \cdot \cos\alpha\right)$$

$\vec{a}^2 = \vec{b} + \vec{c} - 2\vec{b} \cdot \vec{c} \cdot \cos\,\alpha$

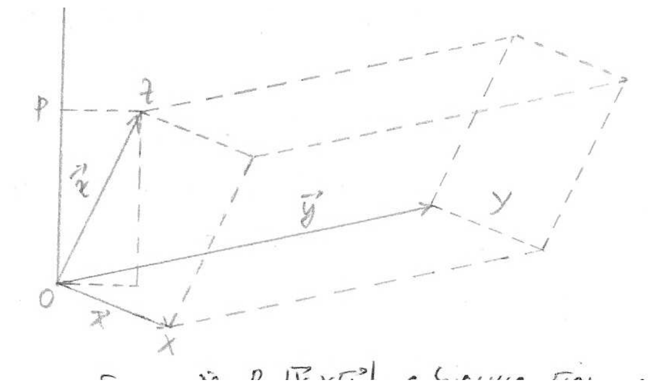
$\vec{a}^2 = \vec{b} + \vec{c} - 2\vec{b} \cdot \vec{c} \cdot \cos\,\alpha = \frac{\vec{b}^2 + \vec{c}^2 - \vec{a}^2}{2}$

$$P^2 = \frac{1}{4}\left(\vec{c} \cdot \vec{b} - \frac{\vec{b}^2 + \vec{c}^2 - \vec{a}^2}{2}\right)\left(\vec{c} \cdot \vec{b} + \frac{\vec{b}^2 + \vec{c}^2 - \vec{a}^2}{2}\right) =$$
$$\frac{1}{16}\left(\vec{a}^2 - \left(\vec{b} - \vec{c}\right)^2\right)\left(\left(\vec{b} + \vec{c}\right)^2 - a^2\right) =$$
$$\frac{1}{16}\left(\vec{a}^2 - \left(\vec{b} - \vec{c}\right)^2\right)\left(\left(\vec{b} + \vec{c}\right)^2 - a^2\right)$$
$$\frac{1}{16}\left(\left(\vec{a} - \vec{b} + \vec{c}\right)\left(\vec{a} + \vec{b} - \vec{c}\right)\left(\vec{b} + \vec{c} - \vec{a}\right)\left(\vec{b} + \vec{c} + \vec{a}\right)\right)$$

$$P = \frac{1}{4}\sqrt{\left(\vec{a} - \vec{b} + \vec{c}\right)\left(\vec{a} + \vec{b} - \vec{c}\right)\left(\vec{b} + \vec{c} - \vec{a}\right)\left(\vec{b} + \vec{c} + \vec{a}\right)}$$

1.5 Mešoviti proizvod vektora

Definicija 1.13. *Broj, odnosno skalar* $\left[\vec{a},\, \vec{b},\, c\right] = \left(\vec{a} \times \vec{b}\right) \cdot \vec{c}$ *naziva se mešoviti proizvod vektora* $\vec{a}$, $\vec{b}$ *i* $\vec{c}$.



Slika 1.18:

Kada vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ obrazuju desni trijedar onda je mešovit proizvod $\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}$ jednak zapremini paralelopipeda konstruisanog nad vektorima $\vec{a}$, $\vec{b}$ i $\vec{c}$.

Površina bazisa je $B = \left|\vec{a} \times \vec{b}\right|$

visina paralelograma je jednaka skalarnoj projekciji vektora $\vec{c}$ na vektor $\vec{a} \times \vec{b}$ pa je

$V = B \cdot H$

$V = \left|\vec{a} \times \vec{b}\right| \cdot \frac{\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}}{\left|\vec{a} \times \vec{b}\right|} = \left(\vec{a} \times \vec{b}\right) \cdot \vec{c}$

$V = \left(\vec{a} \times \vec{b}\right) \cdot \vec{c}$

Tri vektora $\vec{a}$, $\vec{b}$ i $\vec{c}$ su komplanarna (linearno zavisna) ako i samo ako je njihov mešoviti proizvod jednak nuli, tj.

$\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} = \vec{0} \Leftrightarrow \vec{a} \neq \vec{0},\, \vec{b} \neq \vec{0},\, c \neq \vec{0}$

Osobine mešovitog proizvoda

1. $\left[\vec{a},\, \vec{b},\, \vec{c}\right] = \left[\vec{c},\, \vec{a},\, \vec{b}\right] = \left[\vec{b},\, \vec{c},\, \vec{a}\right]$ - mešovit proizvod se ne menja pri cikličnoj permutaciji argumenata

2. $\left[\vec{a},\, \vec{b},\, \vec{c}\right] = -\left[\vec{b},\, \vec{a},\, \vec{c}\right]$ - mešovit proizvod menja znak ako dva argumenta zamene mesta

3. $\left[\alpha\vec{a},\, \vec{b},\, \vec{c}\right] = \alpha\left[\vec{a},\, \vec{b},\, \vec{c}\right]$ - homogenost

4. $\left[\vec{a} + \vec{a}_1,\, \vec{b},\, \vec{c}\right] = \left[\vec{a},\, \vec{b},\, \vec{c}\right] + \left[\vec{a}_1,\, \vec{b},\, \vec{c}\right]$ - aditivnost

1. Dokazati da su vektori $\vec{a}$, $\vec{b}$ i $\vec{c}$ komplanarni ako važi

$$\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{0}$$

$$\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{0}/\cdot\vec{a}$$

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{a}+\left(\vec{b}\times\vec{c}\right)\cdot\vec{a}+(\vec{c}\times\vec{a})\cdot\vec{a}=\vec{0}$$

$$\left(\vec{b}\times\vec{c}\right)\cdot\vec{a}$$

$$\left[\vec{b},\, \vec{c},\, \vec{a}\right]=\vec{0}$$

2. Neka su dati vektori $\vec{V}_1 = \vec{a}+\vec{b}+\vec{c}, \vec{V}_2 = \vec{a}-2\vec{b}+2\vec{c}, \vec{V}_3 = 4\vec{a}+\vec{b}+5\vec{c}$. Pokazati da su komplanarni.

$$\vec{V}_1 = \vec{a}+\vec{b}+\vec{c}$$

$$\vec{V}_2 = \vec{a}-2\vec{b}+2\vec{c}$$

$$\vec{V}_3 = 4\vec{a}+\vec{b}+5\vec{c}$$

$$\left[\vec{V}_1,\, \vec{V}_2,\, \vec{V}_3\right]=\vec{0}$$

$$\left(\vec{V}_1\times\vec{V}_2\right)\cdot\vec{V}_3=\vec{0}$$

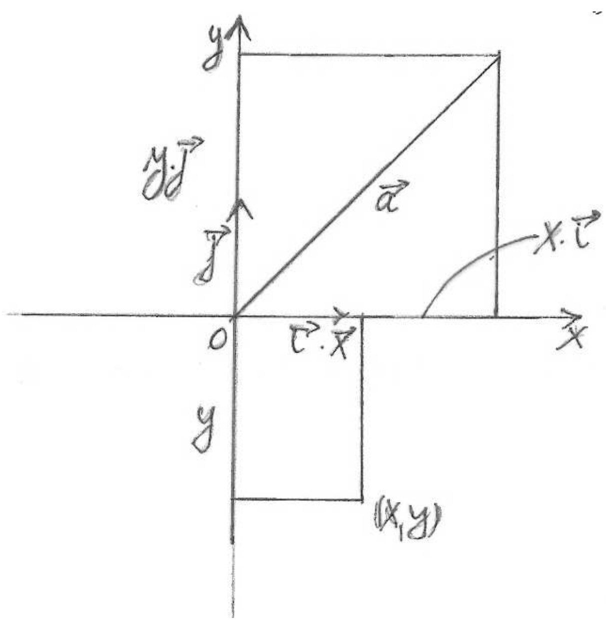
$$\vec{V}_1\times\vec{V}_2=\left(\vec{a}+\vec{b}+\vec{c}\right)\times\left(\vec{a}-2\vec{b}+2\vec{c}\right)=(\vec{a}\times\vec{a})-2\left(\vec{a}\times\vec{b}\right)+2\left(\vec{a}\times\vec{c}\right)+\left(\vec{b}\times\vec{c}\right)$$
$$=-3\left(\vec{a}\times\vec{b}\right)+(\vec{a}\times\vec{c})+4\left(\vec{b}\times\vec{c}\right)$$

$$\left(\vec{V_1} \times \vec{V_2}\right) \cdot \vec{V_3} = \left[-3\left(\vec{a} \times \vec{b}\right) + (\vec{a} \times \vec{c}) + 4\left(\vec{b} \times \vec{c}\right)\right]\left(4\vec{a} + \vec{b} + 5\vec{c}\right)$$
$$= -12\left(\vec{a} \times \vec{b}\right) \cdot \vec{a} - 3\left(\vec{a} \times \vec{b}\right) \cdot \vec{b} - 15\left(\vec{a} \times \vec{b}\right) \cdot \vec{c} + 4\,(\vec{a} \times \vec{c}) \cdot \vec{a} + (\vec{a} \times \vec{c}) \cdot \vec{b} + 5\,(\vec{a} \times$$
$$= -15\left[\vec{a},\, \vec{b},\, \vec{c}\right] + \left[\vec{a},\, \vec{c},\, \vec{b}\right] + 16\left[\vec{b},\, \vec{c},\, \vec{a}\right] = -15\left[\vec{a},\, \vec{b},\, \vec{c}\right] - \left[\vec{a},\, \vec{b},\, \vec{c}\right] + 16\left[\vec{a},\, \vec{b},\, \vec{c}\right] =$$

1.6 Vektori i koordinate

$\vec{a} = x \cdot \vec{i} + y \cdot \vec{j} = (x,\, y)$

Koordinate nekog vektora su koordinate njegovog vrha, pri čemu se početak tog vektora nalazi u koordinatnom početku.



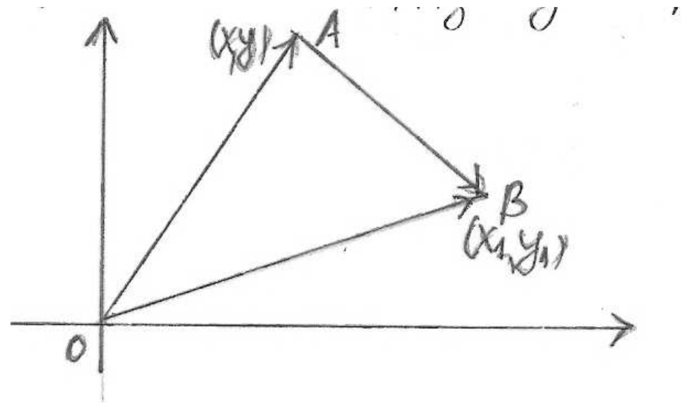
Slika 1.19:

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = x_1 \cdot \vec{i} + y_1 \cdot \vec{j} - \left(x \cdot \vec{i} + y \cdot \vec{j}\right) = (x_1 - x) \cdot \vec{i} + (y_1 - y) \cdot \vec{j} = (x_1 - x,\, y_1 - y)$

Koordinate vektora u ravni ili u prostoru dobijaju se tako što od koordinata vrha oduzmemo koordinate početka.

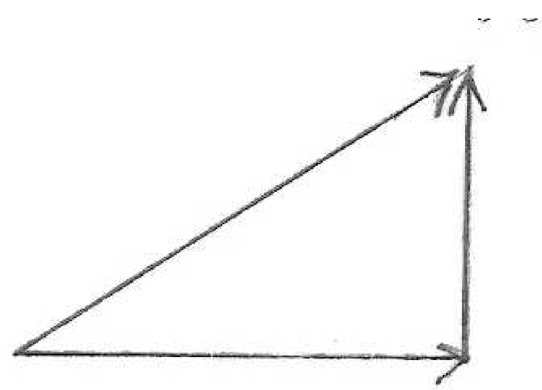
$A\,(5,\, 2)\,;\; B\,(0,\, 3)$

$\overrightarrow{AB} = (0 - 5,\, 3 - 2) = (-5,\, 1)$



Slika 1.20:

1.7 Operacije sa vektorima zadatim koordinatama



Slika 1.21:

$\vec{a} = (x,\, y)$

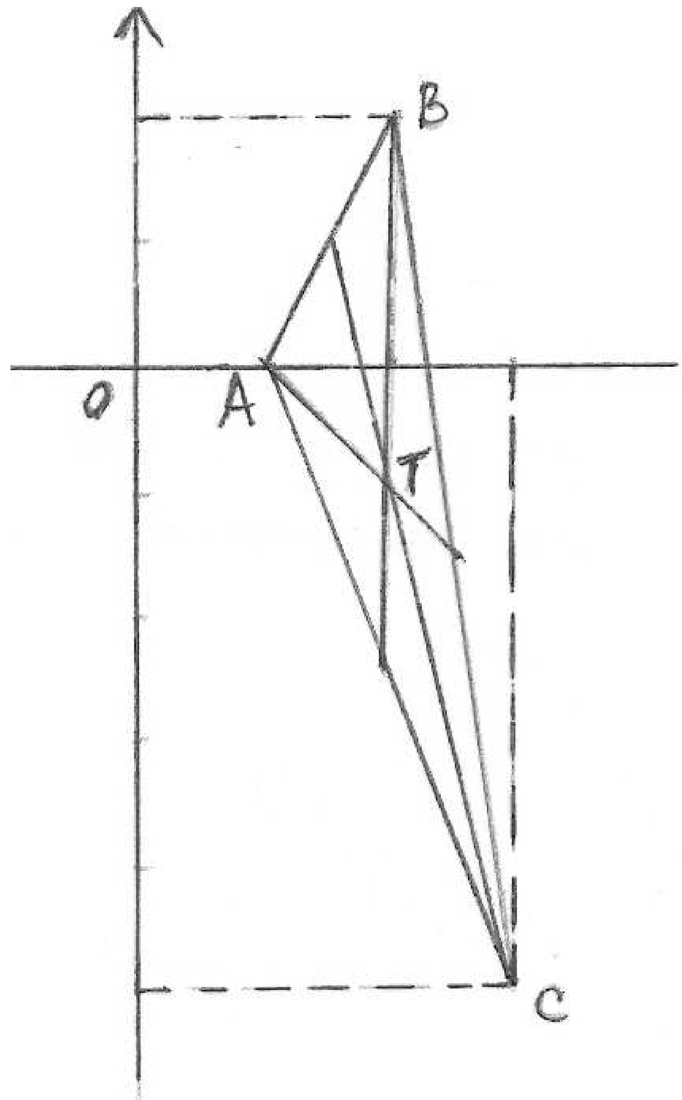
$\vec{b} = (x_1,\, y_1)$

$\vec{a} + \vec{b} = (x + x_1,\, y + y_1)$

$k \cdot \vec{a} = (kx,\, ky)$

1. Neka je dat trougao $A\,(1,\,0)\,;\ B\,(2,\,2)\,;\ C\,(3,\,-5)\,.$ Odrediti vektore granica kao i težište trougla.

$\overrightarrow{AB} = (1,\, 2)$



Slika 1.22:

$\overrightarrow{BC} = (1,\,-7)$

$\overrightarrow{CA} = (-2,\,5)$

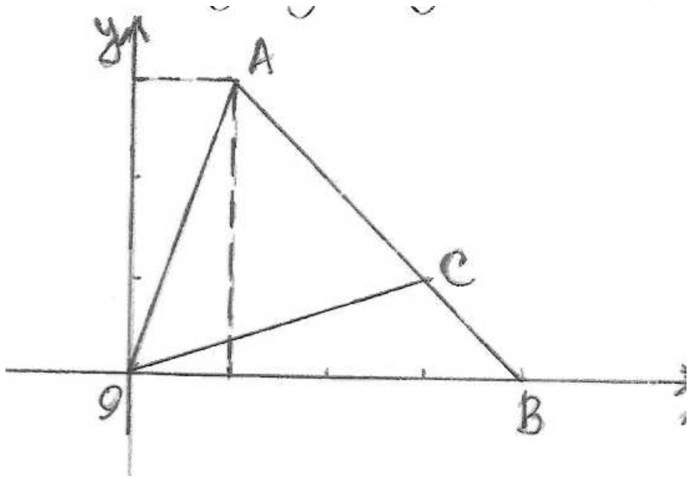
$\overrightarrow{OT} = \frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}$

$\overrightarrow{OT} = \frac{(1,0)+(2,2)+(3,-5)}{3}$

$\overrightarrow{OT} = \frac{1}{3}\,(6,\,-3)$

$\overrightarrow{OT} = (2,\,-1)$

2. Neka je data duž sa krajevima $A(1,\, 3)\,;\; B(4,\, 0)$. Odrediti tačku na ovoj duži koja je deli u odnosu $3:2$.



Slika 1.23:

$\overrightarrow{OC} = (x,\, y)$

$\overrightarrow{OC} = \frac{2}{5}\overrightarrow{OA} + \frac{3}{5}\overrightarrow{OB} = \frac{2}{5}(1,\, 3) + \frac{3}{5}(4,\, 0) = \left(\frac{14}{5},\, \frac{6}{5}\right)$

1. Neka su date tačke $A(-1,\, 3)\,;\; B(4,\, 2)\,;\; C(3,\, -3)$. Odrediti četvrto teme paralelograma $ABCD$.

$\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{BC}$

$(x-4,\, y-2) = (-5,\, 1) + (-1,\, -5) = (-6,\, -4)$

$$\left.\begin{array}{l} x-4=-6 \\ y-2=-4 \end{array}\right\}$$

$$\left.\begin{array}{l} x=-2 \\ y=-2 \end{array}\right\}$$

$D(-2,\, -2)$

Pokazati sa su tačke $A(-4,\, -3)\,;\; B(-5,\, 0)\,;\; C(5,\, 6)\,;\; D(1,\, 0)$ temena trapeza.

$\overrightarrow{BC} \parallel \overrightarrow{AD}$

$\overrightarrow{BC} = (10,\, 6)$

$\overrightarrow{AD} = (5,\, 3)$

$\overrightarrow{BC} = k \cdot \overrightarrow{AD}$

$k = 2$

$\overrightarrow{BC} = 2 \cdot \overrightarrow{AD}$

1.8 Skalarni proizvod u koordinatama

Data su dva vektora:
$\vec{a} = (a_1,\, a_2,\, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$
$\vec{b} = (b_1,\, b_2,\, b_3) = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$
$\vec{a} \cdot \vec{b} = \left(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}\right)\left(b_1\vec{i} + b_2\vec{j} + b_3\vec{k}\right) = (a_1b_1 + a_2b_2 + a_3b_3)$

Primer. Odrediti ugao između vektora $\vec{a} = (3,\, -1)$ i $\vec{b} = (4,\, 2)$.

$\vec{a} = (3,\, -1)$
$\vec{b} = (4,\, 2)$

$\vec{a} \cdot \vec{b} = (12 - 2) = 10$

$|\vec{a}|^2 = \vec{a} \cdot \vec{a} = 9 + 1 = 10$

$|\vec{a}| = \sqrt{10}$

$\left|\vec{b}\right|^2 = 20$

$\left|\vec{b}\right| = 2\sqrt{5}$

$\cos\,\alpha = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot\left|\vec{b}\right|} = \frac{10}{\sqrt{10}\cdot 2\sqrt{5}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\alpha = 45^\circ$

Formula za rastojanje između dve tačke $A\,(x,\,y)$ i $B\,(x_1,\,y_1)$:

$d\,(A,\,B) = \left|\overrightarrow{AB}\right| = \sqrt{(x_1 - x)^2 + (y_1 - y)^2}$

$\overrightarrow{AB} = (x_1 - x,\, y_1 - y)$

1. Izračunati dužinu duži AB, $A = (2,\,1)$, $B = (3,\,4)$.

Rešenje.

$|AB| = \left|\overrightarrow{AB}\right| = |(1,\,3)| = \sqrt{10}$

2. Data su dva temena paralelograma $A = (-3,\,-5)$, $B = (1,\,-2)$ i presek dijagonala $O = (-1,\,-1)$. Odrediti koordinate ostalih temena i pokazati da je dati paralelogram romb.

Rešenje.

$A = (-3,\,-5)$

$B = (1,\,-2)$

$O = (-1,\,-1)$

$D = (x,\,y)$

$\overrightarrow{BD} = 2\overrightarrow{BO}$

$(x - 1,\, y + 2) = 2\,(-2,\,1) = (-4,\,2)$

$$\begin{cases} x-1=-4 \\ y+2=2 \end{cases}$$

$$\begin{cases} x=-3 \\ y=0 \end{cases}$$

$$\overrightarrow{AC}=2\overrightarrow{AO}$$

$$(x+3,\ y+5)=2\,(2,\ 4)=(4,\ 8)$$

$$\begin{cases} x+3=4 \\ y+5=8 \end{cases}$$

$$\begin{cases} x=1 \\ y=3 \end{cases}$$

$$C=(1,\ 3)$$

$$D=(-3,\ 0)$$

$$\left|\overrightarrow{AB}\right|=|(4,\ 3)|=\sqrt{4^2+3^2}=5$$

$$\left|\overrightarrow{AD}\right|=|(0,\ 5)|=\sqrt{0^2+5^2}=5$$

$$\overrightarrow{OA}\perp\overrightarrow{OB}$$

$$\overrightarrow{OA}=(-2,\ -4)$$

$$\overrightarrow{OB}=(2,\ -1)$$

$$\overrightarrow{OA}\cdot\overrightarrow{OB}=-2\cdot 2+(-4)\cdot(-1)=0$$

Domaći 3. Dokazati da su vektori $\vec{a}=(10,\ -5,\ 10)\,,\ \vec{b}=(-11,\ -2,\ 10)\,,$ $\vec{c}=(-2,\ -14,\ ?)\,,$ ivice kocke.

4. Data su temena trougla $A = (-1, -2, 4)$, $B = (-4, -2, 0)$, $C = (3, -2, 1)$. Odrediti uglove α i β.

Rešenje.

$$\overrightarrow{AB} = (-3, 0, -4)$$

$$\overrightarrow{AC} = (4, 0, -3)$$

$$\left|\overrightarrow{AB}\right| = 5$$

$$\left|\overrightarrow{AC}\right| = 5$$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = (-12 + 0 + 12) = 0$$

$$\cos\alpha = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right| \cdot \left|\overrightarrow{AC}\right|} = 0$$

$$\alpha = 90^\circ$$

$$\alpha = \beta = 45^\circ$$

1.9 Vektorski proizvod u koordinatama

Data su dva vektora:

$\vec{a} = (a_1, a_2, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = (b_1, b_2, b_3) = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$$

Osobine:

1. Antikomutativnost

$$\vec{b}\times\vec{a}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ b_1 & b_2 & b_3\\ a_1 & a_2 & a_3\end{vmatrix}=-\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=-\left(\vec{a}\times\vec{b}\right)$$

2. Homogenost

$$k\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ ka_1 & ka_2 & ka_3\\ b_1 & b_2 & b_3\end{vmatrix}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=k\left(\vec{a}\times\vec{b}\right)$$

3. Aditivnost

$$\left(\vec{a}+\vec{b}\right)\times\vec{c}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1+b_1 & a_2+b_2 & a_3+b_3\\ c_1 & c_2 & c_3\end{vmatrix}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}+\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=$$
$$\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$$

1. Odrediti površinu trougla određenog tačkama $A=(6,\,3,\,1)$, $B=(3,\,6,\,1)$, $C=(1,\,3,\,6)$.

Rešenje.

$$\overrightarrow{AB}=(-3,\,3,\,0)$$

$$\overrightarrow{AC}=(-5,\,0,\,5)$$

$$P=\tfrac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -3 & 3 & 0\\ -5 & 0 & 5\end{vmatrix}=(15,\,15,\,15)=15\,(1,\,1,\,1)$$

$$P=\tfrac{1}{2}\left|15\,(1,\,1,\,1)\right|=\tfrac{15}{2}\left|(1,\,1,\,1)\right|=\tfrac{15\sqrt{3}}{2}$$

$$P_{\Delta ABC}=\tfrac{15\sqrt{3}}{2}$$

Domaći

2. $A=(1,\,2,\,1)$, $B=(4,\,3,\,3)$, $C=(3,\,0,\,5)$.

3. $A = (1,\, -1,\, 2)\,,\ B = (5,\, -6,\, 2)\,,\ C = (1,\, 3,\, -1)\,.$ Naći visinu i dužinu iz temena B.

4. Izvesti formulu za površinu trougla u ravni preko koordinata njegovih temena.

Rešenje.

$A\,(x_1,\, x_2)\,;\ B\,(y_1,\, y_2)\,;\ C\,(z_1,\, z_2)$

$\overrightarrow{AC} = (z_1 - x_1,\, z_2 - x_2)$

$\overrightarrow{AB} = (y_1 - x_1,\, y_2 - x_2)$

$P = \left|\overrightarrow{AC} \times \overrightarrow{AB}\right|$

$$P = \frac{1}{2}\left\|\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ z_1 - x_1 & z_2 - x_2 & 0 \\ y_1 - x_1 & y_2 - x_2 & 0 \end{vmatrix}\right\| = \frac{1}{2}\left\|\begin{vmatrix} z_1 - x_1 & z_2 - x_2 \\ y_1 - x_1 & y_2 - x_2 \end{vmatrix}\right\| =$$
$$\frac{1}{2}\left|z_1y_2 - z_1x_2 - x_1y_2 + x_1x_2 - z_2y_1 + z_2x_1 + x_2y_1 - x_1x_2\right| =$$
$$\frac{1}{2}\left\|\begin{vmatrix} z_1 & z_2 & 1 \\ y_1 & y_2 & 1 \\ x_1 & x_2 & 1 \end{vmatrix}\right\|$$

1.10 Mešoviti proizvod u koordinatama

$\vec{a} = (a_1,\, a_2,\, a_3) = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$
$\vec{b} = (b_1,\, b_2,\, b_3) = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$
$\vec{c} = (c_1,\, c_2,\, c_3) = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix},\, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix},\, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right)$$

$$\left[\vec{a},\,\vec{b},\,\vec{c}\right]=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=c_1\begin{vmatrix}a_2&a_3\\b_2&b_3\end{vmatrix}-c_2\begin{vmatrix}a_1&a_3\\b_1&b_3\end{vmatrix}+c_3\begin{vmatrix}a_1&a_2\\b_1&b_2\end{vmatrix}=$$
$$\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}$$

$$V=\left\|\begin{matrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{matrix}\right\|$$

1. Za koju vrednost parametra m tačke $A=(m,\,2,\,-1)$, $B=(0,\,m,\,5)$, $C=(-1,\,2,\,m)$ i $D=(2,\,1,\,3)$ pripadaju istoj ravni.

$$\overrightarrow{DA}=(m-2,\,1,\,-4)$$

$$\overrightarrow{DB}=(-2,\,m-1,\,2)$$

$$\overrightarrow{DC}=(-3,\,1,\,m-3)$$

$$\left[\overrightarrow{DA},\,\overrightarrow{DB},\,\overrightarrow{DC}\right]=0$$

$$\begin{vmatrix}m-2&1&-4\\-2&m-1&2\\-3&1&m-3\end{vmatrix}=0$$

$$\begin{aligned}&(m-2)(m-1)(m-3)-6+8-12(m-1)-2(m-2)+2(m-3)=\\&\left(m^2-3m+2\right)(m-3)+2-12m+12-2m+4+2m-6=\\&m^3-3m^2+2m-3m^2+9m-6+2-12m+12-2m+4+2m-6=\\&m^2(m-6)-(m-6)=\left(m^2-1\right)(m-6)=(m-1)(m+1)(m-6)\end{aligned}$$

$$(m-1)(m+1)(m-6)=0$$

$$m\in\{1,\,-1,\,6\}$$

2. Odrediti zapreminu tetraedra čija su temena $A=(2,\,-3,\,5)$, $B=(0,\,2,\,1)$, $C=(-2,\,-2,\,3)$, $D=(3,\,2,\,4)$.

$\overrightarrow{AD} = (1,\, 5,\, -1)$

$\overrightarrow{AC} = (-4,\, 1,\, -2)$

$\overrightarrow{AB} = (-2,\, 5,\, -4)$

$V_P = \left\| \begin{array}{ccc} 1 & 5 & -1 \\ -4 & 1 & -2 \\ -2 & 5 & -4 \end{array} \right\| = |-4+20+20-2+10-80| = |-36| =$ 36

$V_t = \frac{1}{6}V_P$

$V_t = \frac{1}{6} \cdot 36$

$V_t = \frac{1}{6}$

3. Zapremina tetraedra je 5. Tri njegova temena su $A = (2,\, 1,\, -1)$, $B = (3,\, 0,\, 1)$, $C = (2,\, -1,\, 3)$. Naći četvrto teme, ako se zna da je ono na y osi.

Rešenje.

$A = (2,\, 1,\, -1)$, $B = (3,\, 0,\, 1)$, $C = (2,\, -1,\, 3)$. $D = (0,\, y,\, 0)$.

$\overrightarrow{AB} = (1,\, -1,\, 2)$

$\overrightarrow{AC} = (0,\, -2,\, 4)$

$\overrightarrow{AD} = (-2,\, y-1,\, 1)$

$$\left| \begin{array}{ccc} 1 & -1 & 2 \\ 0 & -2 & 4 \\ -2 & y-1 & 1 \end{array} \right| = -2+8-8-4\,(y-1) = -4y+2$$

$V_t = 5$

$V_P = 6 \cdot 5 = 30$

$|-4y+2| = 5 \cdot 6 = 30$

1. $-4y+2 = 30$

$-4y = 28$

$y = -7$

$D = (0,\, -7,\, 0)$

2. $-4y+2 = -30$

$-4y = -32$

$y = 8$

$D = (0,\, 8,\, 0)$

4. Odrediti vektor $\vec{r}$ koji je normalan na vektore $\vec{a} = (4,\, -2,\, -3)$, $\vec{b} = (0,\, 1,\, 3)$, sa osom O_y gradi tup ugao i $|\vec{r}| = 26$.

Rešenje.

$\vec{a} = (4,\, -2,\, -3)$

$\vec{b} = (0,\, 1,\, 3)$

$|\vec{r}| = 26$

$\vec{r} = (x,\, y,\, z)$

1. $\vec{r} \perp \vec{a} \wedge \vec{r} \perp \vec{b} \Rightarrow \vec{r} = \lambda \left(\vec{a} \times \vec{b} \right)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -2 & -3 \\ 0 & 1 & 3 \end{vmatrix} = \vec{i}(-6+3) - 12\vec{j} + 4\vec{k} = (-3,\, -12,\, 4)$$

$\vec{r} = \lambda\,(-3,\,-12,\,4)$

2. $\sphericalangle\,(\vec{r},\,O_y) > \frac{\pi}{2}$

$\sphericalangle\,(\vec{r},\,j) > \frac{\pi}{2},\, j \in O_y$

$\vec{j} = (0,\,1,\,0)$

$\vec{r} \cdot \vec{j} < 0$

$\lambda\,(-3,\,-12,\,4) \cdot (0,\,1,\,0) = -12\lambda < 0 \;\Rightarrow\; \lambda > 0$

3. $|\vec{r}| = 26$

$|\vec{r}| = |\lambda| \cdot \sqrt{9 + 144 + 16} = 13\,|\lambda|$

$13\,|\lambda| = 26$

$|\lambda| = 2$

$\lambda = -2$ hspace8mm $\lambda = 2$

$\lambda = 2$

$\vec{r} = 2\,(-3,\,-12,\,4) = (-6,\,-24,\,8)$

4. Dati su vektori $\vec{a} = (2,\,4,\,2)\,,\, \vec{b} = (1,\,1,\,2)\,,\, \vec{c} = (1,\,-2,\,3)\,.$ Naći vektor $\vec{d}\,\left(\vec{d}\perp\vec{a},\,\vec{d}\perp\vec{c}\right)$ koji sa vektorom $\vec{b}$ gradi oštar ugao. Zapremina paralelopipeda određenog vektorima $\vec{b}$, $\vec{c}$ i $\vec{d}$ je 140.

Rešenje.

1. $\vec{d}\perp\vec{a},\vec{c}$

$\vec{d}\perp\vec{a} \,\wedge\, \vec{d}\perp\vec{c} \;\Rightarrow\; \vec{d} = \lambda\,(\vec{a} \times \vec{c})$

$$\vec{a} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 4 & 2 \\ 1 & -2 & 3 \end{vmatrix} = \vec{i}\,(12+4) - \vec{j}\,(6-2) + \vec{k}\,(-4-4) = 16\vec{i} - 4\vec{j} - 8\vec{k}$$

$$\vec{d} = \lambda\,(16,\,-4,\,-8)$$

$$\vec{d} = 4\lambda\,(4,\,-1,\,-2)$$

$$\vec{a} \times \vec{c} \parallel \vec{d}$$

$$\tfrac{1}{4}\,(\vec{a} \times \vec{c}) \parallel \vec{d}$$

$$(4,\,-1,\,-2) \parallel \vec{d}$$

2. $\sphericalangle\left(\vec{b},\,\vec{d}\right) < \frac{\pi}{2}$

$$\vec{d} \cdot \vec{b} > 0$$

$$4\lambda\,(4,\,-1,\,-2)\,(1,\,1,\,2) > 0 \;\Rightarrow\; -\lambda > 0 \;\Rightarrow\; \lambda < 0$$

$$\left| 4\lambda \begin{vmatrix} 1 & 1 & 2 \\ 1 & -2 & 3 \\ 4 & -1 & -2 \end{vmatrix} \right| = 140$$

$$|4\lambda\,(4+12-2+16+3+2)| = 35$$

$$140\,|\lambda| = 140$$

$$|\lambda| = 1$$

$$\lambda = -1,\; \lambda = 1$$

$$\lambda < 0 \;\Rightarrow\; \lambda = -1$$

$$\vec{d} = (-16,\,4,\,8)$$

5. Odrediti vektor $\vec{d}$ koji je noemalan na vektore $\vec{a} = (4,\ 1,\ 1)$, $\vec{b} = (6,\ 0,\ 2)$, sa vektorom $\vec{c} = (1,\ 2,\ 3)$ gradi oštar ugao, a sa vektorima $\vec{c}$ i $\vec{b}$ obrazuje paralelopiped zapremine 24.

Rešenje.

$\vec{a} = (4,\ 1,\ 1)$

$\vec{b} = (6,\ 0,\ 2)$

$\vec{d} \perp \vec{a} \ \wedge \ \vec{d} \perp \vec{b} \ \Rightarrow \ \vec{d} = \lambda \left(\vec{a} \times \vec{b} \right)$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 1 & 1 \\ 6 & 0 & 2 \end{vmatrix} = 2\vec{i} - \vec{j}\,(8-6) - 6\vec{k} = 2\vec{i} - 2\vec{j} - 6\vec{k}$$

$\vec{a} \times \vec{b} = (2,\ -2,\ -6)$

$\left(\bar{d},\ \vec{c} \right) < \frac{\pi}{2}$

$\bar{d} \cdot \vec{c} > 0$

$\lambda\,(1,\ -1,\ -3)\,(1,\ 2,\ 3) > 0 \ \Rightarrow \ \lambda\,(1,\ -2,\ -9) > 0 \ \Rightarrow \ -10\lambda > 0 \ \Rightarrow \ \lambda < 0$

$$\left| \lambda \begin{vmatrix} 1 & 2 & 3 \\ 6 & 0 & 2 \\ 1 & -1 & -3 \end{vmatrix} \right| = 24$$

$|\lambda\,(4 - 18 + 2 + 36)| = 24$

$24\,|\lambda| = 24$

$|\lambda| = 1$

$\lambda = -1,\ \lambda = 1$

$\lambda < 0 \ \Rightarrow \ \lambda = -1$

$\vec{d} = (-1,\, 1,\, 3)$

Domaći. 6. Odrediti vektor $\vec{d}$ koji je noemalan na vektor $\vec{a} = (8,\, -15,\, 3)$ sa osom O_x gradi oštar ugao i $\left|\vec{d}\right| = 51$.

Rešenje.

$\vec{d} = (45,\, 24,\, 0)$

7. Dati su vektori $\vec{a} = (2\lambda,\, 1,\, 1-\lambda)\,,\, \vec{b} = (-1,\, 3,\, 0)\,,\, \vec{c} = (5,\, -1,\, 8)\,.$

1. Odrediti λ tako da vektor $\vec{a}$ zaklapa jednake uglove sa $\vec{b}$ i $\vec{c}$. Za tako određeno λ odrediti ugao vektora $\vec{c}$ prema ravni određene vektorima $\vec{b}$ i $\vec{a}$.

2. Za tako određeno λ odrediti zapreminu piramide, kao i visinu koja odgovara jednoj od strana piramide.

Rešenje.

1. $\cos\left(\vec{a},\, \vec{b}\right) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot\left|\vec{b}\right|}$

$\cos\,(\vec{a},\, \vec{c}) = \frac{\vec{a}\cdot\vec{c}}{|\vec{a}|\cdot|\vec{c}|}$

$\cos\left(\vec{a},\, \vec{b}\right) = \cos\,(\vec{a},\, \vec{c}) \Longrightarrow$

$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot\left|\vec{b}\right|} = \frac{\vec{a}\cdot\vec{c}}{|\vec{a}|\cdot|\vec{c}|} \Longrightarrow$

$\frac{(2\lambda,1,1-\lambda)(-1,3,0)}{\sqrt{4\lambda^2+1+(1-\lambda)^2}\cdot\sqrt{1+9}} = \frac{(2\lambda,1,1-\lambda)(5,-1,8)}{\sqrt{4\lambda^2+1+(1-\lambda)^2}\cdot\sqrt{25+1+64}} \Rightarrow$

$\frac{-2\lambda+3}{\sqrt{5\lambda^2+-2\lambda+2}\cdot\sqrt{10}} = \frac{10\lambda-1+8-8\lambda}{\sqrt{5\lambda^2+-2\lambda+2}\cdot\sqrt{90}} / \cdot 3\sqrt{10}\cdot\sqrt{5\lambda^2+-2\lambda+2} \Rightarrow$

$-6\lambda+9 = 2\lambda+7 \Rightarrow -8\lambda = -2 \Rightarrow \lambda = \frac{1}{4}$

$\cos\left(\frac{\pi}{2}-\varphi\right) = \frac{\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)}{|\vec{c}|\cdot\left|\vec{a}\times\vec{b}\right|}$

$\cos\left(\frac{\pi}{2}-\varphi\right) = \frac{(5,-1,8)\cdot\left(-\frac{9}{4},-\frac{3}{4},\frac{5}{12}\right)}{\sqrt{90}\cdot\sqrt{\frac{81}{16}+\frac{9}{16}+\frac{25}{144}}} = \frac{-\frac{45}{4}+\frac{3}{4}+20}{\sqrt{90}\cdot\sqrt{\frac{130}{16}}} = \frac{38}{3\sqrt{10}\cdot\sqrt{190}} = \frac{38}{30\sqrt{19}} \cdot \frac{\sqrt{19}}{\sqrt{19}} = \frac{38\sqrt{19}}{30\cdot 19} = \frac{\sqrt{19}}{15}$

$\frac{\pi}{2}-\varphi = arc\cos\left(\frac{\sqrt{19}}{15}\right) \Rightarrow \varphi = \frac{\pi}{2} - arc\cos\left(\frac{\sqrt{19}}{15}\right) \Rightarrow \varphi = arc\sin\left(\frac{\sqrt{19}}{15}\right)$

2. $V = \frac{1}{6}\begin{vmatrix} \frac{1}{2} & 1 & \frac{3}{4} \\ -1 & 3 & 0 \\ 5 & -1 & 8 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} \frac{11}{2} & 0 & \frac{35}{4} \\ 14 & 0 & 24 \\ 5 & -1 & 8 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} \frac{11}{2} & \frac{35}{4} \\ 14 & 24 \end{vmatrix} = \frac{1}{6}\left(132-\frac{245}{2}\right) = \frac{19}{12}$

$V = \frac{B\cdot H}{3}$

$H = \frac{3\cdot V}{B}$

$B = \left|\vec{a}\times\vec{b}\right| = \left|-\frac{9}{4}\vec{i}-\frac{3}{4}\vec{j}+\frac{5}{4}\vec{k}\right| = \frac{\sqrt{190}}{4}$

$H = \frac{3V}{\left|\vec{a}\times\vec{b}\right|} = \frac{3\cdot\frac{19}{12}}{\frac{\sqrt{190}}{4}} = \frac{19}{\sqrt{190}} = \frac{19}{\sqrt{10}\cdot\sqrt{19}}\cdot\frac{\sqrt{19}}{\sqrt{19}} = \frac{\sqrt{19}}{\sqrt{10}} = \sqrt{1,9}$

8. Dokazati $\left(\vec{a}\times\vec{b}\right)\times\vec{c} = (\vec{a}\cdot\vec{c})\cdot\vec{b} - \left(\vec{b}\cdot\vec{c}\right)\cdot\vec{a}$

$\vec{a} = (a_1,\, a_2,\, a_3)$

$\vec{b} = (b_1,\, b_2,\, b_3)$

$\vec{c} = (c_1,\, c_2,\, c_3)$

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right)$

$$\left(\vec{a}\times\vec{b}\right)\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$\left(\begin{vmatrix} -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ c_2 & c_3 \end{vmatrix}, -\begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \\ c_1 & c_3 \end{vmatrix}, \begin{vmatrix} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} & -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \\ c_1 & c_2 \end{vmatrix} \right)$$

$$\vec{a}\cdot\vec{c} = a_1c_1 + a_2c_2 + a_3c_3$$

$$\vec{b}\cdot\vec{c} = b_1c_1 + b_2c_2 + b_3c_3$$

$$\begin{aligned} &(\vec{a}\cdot\vec{c})\cdot\vec{b} - \left(\vec{b}\cdot\vec{c}\right)\cdot\vec{a} = (a_1c_1 + a_2c_2 + a_3c_3)\,(b_1,\, b_2,\, b_3) - \\ &(b_1c_1 + b_2c_2 + b_3c_3)\,(a_1,\, a_2,\, a_3) = (a_1c_1 + a_2c_2 + a_3c_3)\cdot b_1 - \\ &(b_1c_1 + b_2c_2 + b_3c_3)\cdot a_1,\, (a_1c_1 + a_2c_2 + a_3c_3)\cdot b_2 - (b_1c_1 + b_2c_2 + b_3c_3)\cdot a_2, \\ &(a_1c_1 + a_2c_2 + a_3c_3)\cdot b_3 - (b_1c_1 + b_2c_2 + b_3c_3)\cdot a_3 \end{aligned}$$

9. Dati su vektori $\vec{a} = (8,\, 4,\, 1)\,;\ \vec{b} = (2,\, -2,\, 1)\,;\ \vec{c} = (1,\, 1,\, 9)\,$. Odrediti projekciju vektora *vecc* na ravan određenu vektorima $\vec{a}$ i $\vec{b}$.

Rešenje.

$(\vec{x} - \vec{c})\perp\vec{a},\, \vec{b}$

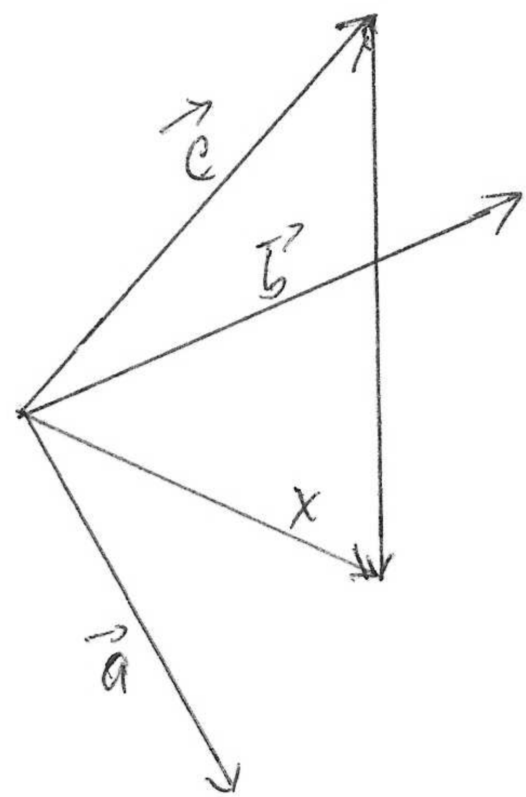
$$\begin{cases} (\vec{x} - \vec{c})\cdot\vec{a} = \vec{0} \\ (\vec{x} - \vec{c})\cdot\vec{b} = \vec{0} \end{cases}$$

$$\vec{x} = \alpha\vec{a} + \beta\vec{b} = (8\alpha + 2\beta,\, 4\alpha - 2\beta,\, \alpha + \beta)$$

$$\begin{cases} (8\alpha + 2\beta - 1,\, 4\alpha - 2\beta - 1,\, \alpha + \beta - 9)\cdot(8,\, 4,\, 1) = \vec{0} \\ (8\alpha + 2\beta - 1,\, 4\alpha - 2\beta - 1,\, \alpha + \beta - 9)\cdot(2,\, 2,\, 1) = \vec{0} \end{cases}$$

$$\begin{cases} 81\alpha + 9\beta - 21 = \vec{0} \\ 9\alpha + 9\beta - 9 = \vec{0} \end{cases}$$

$$\begin{cases} \alpha = \frac{1}{6} \\ \beta = \frac{5}{6} \end{cases}$$



Slika 1.24:

$\vec{x} = \frac{1}{6}\vec{a} + \frac{5}{6}\vec{b}$

1.11 Prava u ravni

Skup tačaka u ravni je prava akko je definisana jednačinom $Ax + By + C = 0$.

$(x_0,\, y_0) \in p$

$(A,\, B) \perp p$

$(x,\, y) \in p \Leftrightarrow (x - x_0,\, y - y_0) \perp (A,\, B) \Leftrightarrow$

$A\,(x-x_0)+B\,(y-y_0)=\vec{0}$

$Ax+By+C=0,\;\; C=-x_0A-y_0B$

$Ax+By+C=0$

$$\begin{cases} Ax_0+B_0y+C=0 \\ Ax+By+C=0 \end{cases}$$

$A\,(x-x_0)+B\,(y-y_0)=0$

$(x-x_0,\, y-y_0)\perp(A,\, B)$

$p:\; Ax+By+C=0$ - implicitni oblik jednačine prave

$(A,\, B)$ - vektor položaja prave p.

$(x_0,\, y_0)\in p$

$A\,(x-x_0)+B\,(y-y_0)=0$ - jednačina prave kroz tačku $(x_0,\, y_0)$ koja je normalna na vektor $(A,\, B)$.

Vektor položaja neke prave nije jedinstven, ali su svi vektori položaja međusobno kolinearni.

1. Odrediti jednačinu prave koja je paralelna sa pravom $2x-6y+5=0$ i prolazi kroz tačku $(1,\, 1)$.

Rešenje.

$A=2 \qquad B=-6$

$p:\; 2\,(x-1)+(-6)\,(y-1)=0$

$p:\; 2x-6y+4=0$

$p:\; x-3y+2=0$

2. Odrediti parametar m tako da prava $3x+5y-1=0$ bude paralelna, odnosno normalna na pravu $-4x+my-2=0$.

Rešenje.

a) $3x+5y-1=0$

$-4x+my-2=0$

$\frac{3}{-4}=\frac{5}{m}\Rightarrow m=-\frac{20}{3}$

b) $(3,\,5)\perp(-4,\,m)$

$(3,\,5)\cdot(-4,\,m)=0$

$-12+5m=0\Rightarrow m=\frac{12}{5}$

3. Odrediti jednačinu prave koja sadrži tačku $(-1,\,5)$ i na koordinatnim osama odseca odsečke jednake dužine.

Rešenje.

$A\,(x-x_0)+B\,(y-y_0)=0$

$(x+1)+(y-5)=0$

$x+y-4=0$

4. Odrediti tačku simetričnu sa tačkom $(3,\,3)$ u odnosu na pravu $2x+y-4=0$.

$q:\;A\,(x-3)-2A\,(y-3)=0/:A$

$q:\;1\,(x-3)-2\,(y-3)=0$

$x-2y+3=0$

$B\,(x_1,\,y_1)$

$$\left.\begin{array}{l}2x_1+y_1-4=0\\x_1-2y_1+3=0\end{array}\right\}$$

$$\left.\begin{array}{l} x_1 = 1 \\ y_1 = 2 \end{array}\right\}$$

$B = \frac{A+A'}{2}$

$2\,(1,\,2) = (3,\,3) + A'$

$2\,(1,\,2) - (3,\,3) = A'$

$A' = (-1,\,1)$

4. Odrediti jednačinu simetrale duži čiji su krajevi $A\,(2,\,3)$ i $B\,(-1,\,5)$.

I način:

$C = \frac{A+B}{2} \in s$

$\overrightarrow{AB} \perp s$

$C = \left(\frac{1}{2},\,4\right)$

$\overrightarrow{AB} = (-3,\,2) \perp s$

$s:\ -3\left(x - \frac{1}{2}\right) + 2\,(y-4) = 0$

$-3x + 2y - \frac{13}{2} = 0$

$-6x + 4y - 13 = 0$

II način:

skup tačaka u ravni jednako udaljen od temena-simetrala

$M\,(x,\,y) \in s \ \Leftrightarrow\ d\,(A,\,M) = d\,(B,\,M)$

$\sqrt{(x-2)^2 + (y-3)^2} = \sqrt{(x+1)^2 + (y-5)^2} /^2$

$x^2 - 4x + 4 + y^2 - 6y + 9 = x^2 + 2x + 1 + y^2 - 10y + 25$

$-6x + 4y - 13 = 0$

5. Odrediti težište, ortocentar i centar opisanog kruga u ΔABC : $A(2,\ 1),\ B(-2,\ 2),\ C(3,\ -1)$

$\overrightarrow{OT} = \frac{1}{3}\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\right)$

$T = \frac{1}{3}(A + B + C)$

$T = \frac{(2,1)+(-2,2)+(3,-1)}{3}$

$T = \left(1,\ \frac{2}{3}\right)$

$\overrightarrow{BC} = (5,\ -3)$

$h_a:\ 5(x-2) - 3(y-1) = 0$

$h_a:\ 5x - 3y - 7 = 0$

$\overrightarrow{AC} = (1,\ -2)$

$h_b:\ (x+2) - 2(y-2) = 0$

$h_b:\ x - 2y + 6 = 0$

$$\left.\begin{array}{r} 5x - 3y - 7 = 0 \\ x - 2y + 6 = 0 \end{array}\right\}$$

$$\left.\begin{array}{l} x = \frac{32}{7} \\ y = \frac{37}{7} \end{array}\right\}$$

$H\left(\frac{32}{7},\ \frac{37}{7}\right)$

$\overrightarrow{BC} = (5,\ -3)$

$$A' = \left(\frac{3-2}{2},\, \frac{-1+2}{2}\right) = \left(\frac{1}{2},\, \frac{1}{2}\right)$$

$$s_a:\; 5\left(x - \frac{1}{2}\right) - 3\left(y - \frac{1}{2}\right) = 0$$

$$s_a:\; 5x - 3y - 1 = 0$$

$$B' = \left(\frac{3+2}{2},\, \frac{-1+1}{2}\right) = \left(\frac{5}{2},\, 0\right)$$

$$\overrightarrow{AC} = (1,\, -2)$$

$$s_b:\; 1\left(x - \frac{5}{2}\right) - 2\,(y - 0) = 0$$

$$s_b:\; x - 2y - \frac{5}{2} = 0$$

$$\left.\begin{array}{l} 5x - 3y - 1 = 0 \\ x - 2y - \frac{5}{2} = 0 \end{array}\right\}$$

$$\left.\begin{array}{l} x = -\frac{11}{14} \\ y = -\frac{23}{14} \end{array}\right\}$$

$$O\left(-\frac{11}{14}, -\frac{23}{14}\right)$$

$$T = \left(1,\, \frac{2}{3}\right)$$

$$H\left(\frac{32}{7},\, \frac{37}{7}\right)$$

$$O\left(-\frac{11}{14}, -\frac{23}{14}\right)$$

$$y - \frac{2}{3} = \frac{\frac{37}{7} - \frac{2}{3}}{\frac{32}{7} - 1}\;(x - 1)$$

$$y - \frac{2}{3} = \frac{\frac{111-14}{21}}{\frac{32-7}{7}}\;(x - 1)$$

$$y - \frac{2}{3} = \frac{97}{3 \cdot 25}\;(x - 1)$$

$$y - \frac{2}{3} = \frac{97}{75}\;(x - 1)$$

$$y = \frac{97}{75}x - \frac{47}{75}$$

$-\frac{23}{14}=\frac{97}{75}\left(-\frac{11}{14}\right)-\frac{47}{75}$

$-\frac{23}{14}=-\frac{97\cdot 11}{75\cdot 14}-\frac{47}{75}$

$-\frac{23}{14}=-\frac{1067}{1050}-\frac{47}{75}$

$-\frac{23}{14}=-\frac{1725}{1050}$

$1=1$

Domaći. 6. Odrediti težište, ortocentar i centar opisanog kruga u $\Delta ABC:\ A\,(3,\,2)\,,\ B\,(4,\,5)\,,\ C\,(4,\,4)\,.$

1.12 Pramen pravih

Definicija 1.14. *Pod pramenom pravih podrazumevamo skup svih tačaka koje prolaze kroz datu tačku.*

Ako je pramen određen pravama $\begin{cases} Ax+By+C=0 \\ A_1x+B_1y+C_1=0 \end{cases}$ onda opšti element pramena ima oblik: $Ax+By+C+\alpha\,(A_1x+B_1y+C_1)=0$

1. Odrediti jednačinu prave koja prolazi kroz presek pravih $\begin{cases} 2x+y-6=0 \\ \mathrm{x}-3\mathrm{y}+4=0 \end{cases}$ i sadrži tačku $(0,\,3)\,.$

Rešenje.

$2x+y-6+\alpha\,(\mathrm{x}-3\mathrm{y}+4)=0$

$2\cdot 0+3-6+\alpha\,(0-3\cdot 3+4)=0$

$-3-5\alpha=0$

$\alpha=-\frac{3}{5}$

$2x + y - 6 + \left(-\frac{3}{5}\right)(\mathrm{x} - 3\mathrm{y} + 4) = 0$

$10x + 5y - 30 - 3(\mathrm{x} - 3\mathrm{y} + 4) = 0$

$p:\ 7x + 14y - 42 = 0 / :7$

$p:\ x + 2y - 6 = 0$

2. Odrediti jednačinu prave koja prolazi kroz presek pravih
$\begin{cases} 2x + y - 6 = 0 \\ \mathrm{x} - 3\mathrm{y} + 4 = 0 \end{cases}$ i

a) paralelna je sa pravom $3x - y + 1 = 0$

b) normalna je na pravu $2x - 3y + 1 = 0$

Rešenje.

a) $\begin{cases} 2x + y - 6 = 0 \\ \mathrm{x} - 3\mathrm{y} + 4 = 0 \end{cases}$

$2x + y - 6 + \alpha(\mathrm{x} - 3\mathrm{y} + 4) = 0$

$2x + y - 6 + \alpha x - 3\alpha y + 4\alpha = 0$

$(2 + \alpha)x + (1 - 3\alpha)y + (-6 + 4\alpha) = 0\ (2 + \alpha,\ 1 - 3\alpha) \parallel (3,\ -1)$

$\frac{2+\alpha}{3} = \frac{1-3\alpha}{-1} / \cdot (-3)$

$-2 - \alpha = 3 - 9\alpha$

$\alpha = \frac{5}{8}$

$p:\ 2x + y - 6 + \frac{5}{8}(\mathrm{x} - 3\mathrm{y} + 4) = 0$

$16x + 8y - 48 + 5x - 15y + 20 = 0$

$21x - 7y - 28 = 0 / :7$

$3x-y-4=0$

b) $(2+\alpha)\,x+(1-3\alpha)\,y+(-6+4\alpha)=0$

$(2+\alpha,\,1-3\alpha)\cdot(2,\,-3)=0$

$2\,(2+\alpha)-3\,(\,1-3\alpha)=0$

$4+2\alpha-3+9\alpha=0$

$\alpha=-\frac{1}{11}$

$2x+y-6+\left(-\frac{1}{11}\right)(x-3y+4)=0$

$2x+y-6+\left(-\frac{1}{11}\right)(x-3y+4)=0$

$22x+11y-66-(x-3y+4)=0$

$21x+14y-70=0$

$7x+2y-10=0$

3. Odrediti jednačinu prave koja sadrži presečnu tačku pravih $2x+7y-8=0$ i $3x+2y+5=0$ i sa pravom $2x+3y-7=0$ gradi ugao od $45°$.

Rešenje.

$2x+7y-8+\alpha\,(3\text{x}+2\text{y}+5)=0$

$(2+3\alpha)\,x+(7+2\alpha)\,y+(5\alpha-8)=0$

$(7+2\alpha)\,y=(8-5\alpha)-(2+3\alpha)\,x$

$y=\frac{-(2+3\alpha)x+8-5\alpha}{7+2\alpha}$

$y=-\frac{2+3\alpha}{7+2\alpha}x+\frac{8-5\alpha}{7+2\alpha}$

$2x + 3y - 7 = 0$

$3y = 7 - 2x$

$y = \frac{7-2x}{3}$

$y = -\frac{2x}{3} + \frac{7}{3}$

$k_1 = -\frac{2+3\alpha}{7+2\alpha}$

$k_2 = -\frac{2}{3}$

$tg\,\alpha\ =\ \left|\frac{k_2-k_1}{1+k_1k_2}\right|$

$tg\,45^\circ\ =\ \left|\frac{-\frac{2}{3}+\frac{2+3\alpha}{7+2\alpha}}{1+\left(\frac{2}{3}\right)\left(\frac{2+3\alpha}{7+2\alpha}\right)}\right|$

$1 = \left|\frac{\frac{-14-4\alpha+6+9\alpha}{3(7+2\alpha)}}{\frac{21+6\alpha+4+6\alpha}{3(7+2\alpha)}}\right|$

$\left|\frac{5\alpha-8}{12\alpha+25}\right| = 1$

1. $5\alpha - 8 = 12\alpha + 25$

$-7\alpha = 33$

$\alpha = -\frac{33}{7}$

$2x + 7y - 8 - \frac{33}{7}\,(3x + 2y + 5) = 0$

$14x + 49y - 56 - 99x - 66y - 165 = 0$

$-85x - 17y + 109 = 0$

$85x + 17y - 109 = 0$

2. $-5\alpha + 8 = 12\alpha + 25$

$-17\alpha = 17$

$\alpha = -1$

$2x + 7y - 8 - (3x + 2y + 5) = 0$

$2x + 7y - 8 - 3x - 2y - 5 = 0$

$-x + 5y - 13 = 0$

$x - 5y + 13 = 0$

1.13 Odstojanje tačke od prave

$p:\ Ax + By + C = 0$

$\vec{a} = (x_0 - x,\ y_0 - y)$

$\vec{a} \cdot \vec{n} = |\vec{n}| \cdot d$

(jer je $d = \vec{a} \cdot \cos(x,\ y)$)

$d = \frac{|\vec{a}\cdot\vec{n}|}{|\vec{n}|} = \frac{|\vec{a}\cdot\vec{n}|}{\sqrt{A^2+B^2}} = \frac{|A(x_0-x)+B(y_0-y)|}{\sqrt{A^2+B^2}} = \frac{|-Ax-By+Ax_0+By_0|}{\sqrt{A^2+B^2}} = \frac{|-Ax-By-C|}{\sqrt{A^2+B^2}} = \frac{|Ax+By+C|}{\sqrt{A^2+B^2}}$

$d = \frac{|Ax+By+C|}{\sqrt{A^2+B^2}}$

$(x_0,\ y_0) \in p$

$Ax_0 + By_0 + C = 0$

$Ax_0 + By_0 = -C$

1. Odrediti odstojanje tačke $(2,\ 3)$ od prave $3x - y + 5 = 0$.

Rešenje.

$(2,\ 3)$

$3x - y + 5 = 0$

$d=\frac{3\cdot 2-3+5}{\sqrt{3^2+(-1)^2}}=\frac{8}{\sqrt{10}}$

2. Odrediti jednačine simetrala uglova koje grade prave $x+y+2=0$ i $x+7y+3=0$.

Rešenje.

$$\begin{cases} p:\ x+y+2=0 \\ \text{q}:\ \text{x}+7\text{y}+3=0 \end{cases}$$

$dp=dq$

$\frac{|x+y+2|}{\sqrt{1^2+1^2}}=\frac{|x+7y+3|}{\sqrt{1^2+7^2}}$

$\frac{|x+y+2|}{\sqrt{2}}=\frac{|x+7y+3|}{5\sqrt{2}}$

$5\,|x+y+2|=|x+7y+3|$

1. $5\,(x+y+2)=(x+7y+3)$

$4x-2y+7=0$

2. $5\,(x+y+2)=-\,(x+7y+3)$

$6x+12y+13=0$

3. Odrediti odstojanje između pravih $4x-3y+15=0$ i $8x-6y+25=0$.

Rešenje.

One su paralelne, jer je $4:8=(-3):(-6)$

$(0,\,5)\in p$

$p:4x-3y+15=0$

$d = \frac{|8\cdot 0 - 6\cdot 5 + 25|}{\sqrt{8^2+6^2}} = \frac{5}{10} = \frac{1}{2}$

4. Odrediti jednačinu prave koja je 2 puta bliža pravoj $4x - 3y + 15 = 0$ nego pravoj $8x - 6y + 25 = 0$.

Rešenje.

$2d_1 = d_2$

$2\frac{|4x-3y+15|}{\sqrt{4^2+3^2}} = \frac{|8x-6y+25|}{\sqrt{8^2+6^2}}$

$2\frac{|4x-3y+15|}{5} = \frac{|8x-6y+25|}{10}$

$4\,|4x - 3y + 15| = |8x - 6y + 25|$

1. $4\,(4x - 3y + 15) = 8x - 6y + 25$

$16x - 12y + 60 = 8x - 6y + 25$

$8x - 6y + 35 = 0$

2. $16x - 12y + 60 = -\,(8x - 6y + 25)$

$2x - 18y + 85 = 0$

5. Ako su $A\,(4,\,-5)$ i $B\,(2,\,9)$ dva temena trougla ABC odrediti geometrijsko mesto tačaka C tako da je $P_{\Delta ABC} = 50$.

Rešenje. $\overrightarrow{AB} = (-2,\,14) \parallel (-1,\,7)$

$AB:\; 7\,(x-4) + (y+5) = 0$

$7x + y - 23 = 0$

$P = \frac{|AB|\cdot h_c}{2}$

$P = \frac{\sqrt{1^2+7^2}\cdot h_c}{2} = 50 \;\Rightarrow\; 5\sqrt{2}\cdot h_c = 50 \;\Rightarrow\; h_c = \frac{10}{\sqrt{2}}$

$\frac{|7x+y-23|}{\sqrt{7^2+1^2}} = \frac{10}{\sqrt{2}}$

$|7x + y - 23| = 50$

1. $7x + y - 23 = 50$

$7x + y - 73 = 0$

2. $7x + y - 23 = -50$

$7x + y + 27 = 0$

6. Odrediti jednačinu prave koja prolazi kroz presek pravih $\begin{cases} 2x - y - 1 = 0 \\ 3\text{x} + \text{y} - 4 = 0 \end{cases}$ a od tačke $(2, -1)$ je udaljena za $d = \frac{3}{\sqrt{5}}$.
Rešenje.

$2x - y - 1 + \alpha\,(3x + y - 4) = 0$

$p:\ (2 + 3\alpha)\,x + (\alpha - 1)\,y + (-4\alpha - 1) = 0$

$d = \frac{3}{\sqrt{5}}$

$\frac{|(2+3\alpha)\cdot 2+(\alpha-1)\cdot(-1)+(-4\alpha-1)|}{\sqrt{(2+3\alpha)^2+(\alpha-1)^2}} = \frac{3}{\sqrt{5}} /^2$

$5\,(4 + 6\alpha - \alpha + 1 - 4\alpha - 1) = 9\left(4 + 12\alpha + 9\alpha^2 + \alpha^2 - 2\alpha + 1\right)$
$17\alpha^2 + 10\alpha - 7 = 0$

$\alpha_{1,2} = \frac{-10\pm\sqrt{100+476}}{34}$

$\alpha_{1,2} = \frac{-10\pm 24}{34}$

$\alpha_1 = -1$

$\alpha_2 = \frac{7}{17}$

$\alpha_1 = -1 \Rightarrow p: -x - 2y + 3 = 0, \Rightarrow p: x + 2y - 3 = 0$

$\alpha_2 = \frac{7}{17} \Rightarrow p: 11x - 2y - 9 = 0$

7. Na pravoj $p: x - 2y + 8 = 0$ odrediti tačku jednako udaljenu od tačke (8, 3) i prave $3x + 4y - 11 = 0$.

Rešenje.

$p: x - 2y + 8 = 0$

$A(8, 3)$

$q: 3x + 4y - 11 = 0$

$d_1 = d_2$

$A(x, y)$

$$\begin{cases} \sqrt{(x-8)^2 + (y-3)^2} = \frac{|3x+4y-11|}{5} \\ \text{x} - 2\text{y} + 8 = 0 \end{cases}$$

$\text{x} = 2\text{y} - 8$

$\sqrt{(2y-16)^2 + (y-3)^2} = \frac{|10y-35|}{5} /^2$

$4y^2 - 64y + 256 + y^2 - 6y + 9 = \frac{100y^2 - 700y + 1225}{25}$

$125y^2 - 1750y + 6625 = 100y^2 - 700y + 1225$

$25y^2 - 1050y + 5400 = 0/ : 25$

$y^2 - 42y + 216 = 0$

$y_{1,2} = \frac{42 \pm \sqrt{1764 - 864}}{2}$

$y_{1,2} = \frac{42 \pm 30}{2}$

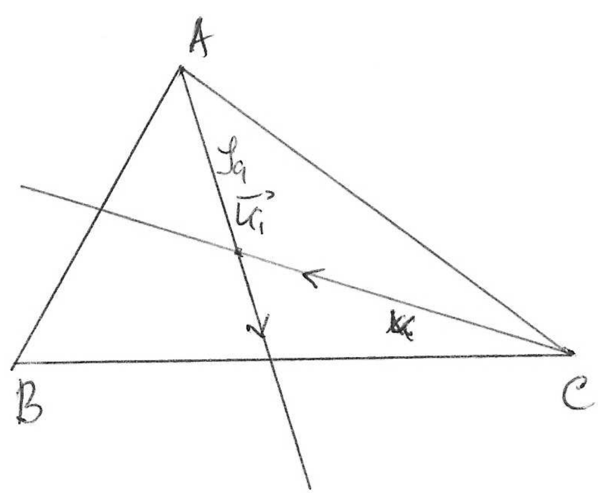
$$y = \begin{cases} 36 \\ 6 \end{cases}$$

$$x = \begin{cases} 64 \\ 4 \end{cases}$$

$A_1\,(64,\,36)\,,\ A_2\,(4,\,6)$

8. Odrediti centar upisanog kruga i njegov poluprečnik u trouglu čija su temena $A\,(-3,\,5)\,,\ B\,(5,\,-3)\,,\ C\,(4,\,4)\,.$

Rešenje.



Slika 1.25:

$\overrightarrow{CA} = (-7,\,1)$

$\overrightarrow{CB} = (1,\,-7)$

$\overrightarrow{AB} = (8,\,-8) = 8\,(1,\,-1)$

$\vec{k} = \dfrac{\overrightarrow{CA}}{\left|\overrightarrow{CA}\right|} + \dfrac{\overrightarrow{CB}}{\left|\overrightarrow{CB}\right|} = \frac{1}{\sqrt{50}}\left((-7,\,1) + (1,\,-7)\right) = \frac{1}{5\sqrt{2}}\,(-6,\,-6) = -\frac{6}{5\sqrt{2}}\,(1,\,1)$

$(1,\ 1) \perp (1,\ -1)$

$(1,\ -1)$

$(x-4)-(y-4)=0$

$x-y=0$

$\overrightarrow{AB}=(8,\ -8)$

$\overrightarrow{AC}=(7,\ -1)$

$\vec{k}_1=\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}+\frac{\overrightarrow{AC}}{\left|\overrightarrow{AC}\right|}=\frac{(8,-8)}{8\sqrt{2}}+\frac{(7,-1)}{5\sqrt{2}}=\frac{(1,-1)}{\sqrt{2}}+\frac{(7,-1)}{5\sqrt{2}}=\frac{1}{\sqrt{2}}\left(1+\frac{7}{5},\ -1-\frac{1}{5}\right)=$
$\frac{1}{\sqrt{2}}\left(\frac{12}{5},\ -\frac{6}{5}\right) \parallel (2,\ -1)$

$(2,\ -1) \perp (1,\ 2)$

$(x+3)+2\,(y-5)=0$

$x+2y-7=0$

$$\begin{cases} x+2y-7=0 \\ \mathrm{x-y=0} \end{cases}$$

$x=y$

$3y-7=0$

$$\begin{cases} x=\frac{7}{3} \\ y=\frac{7}{3} \end{cases}$$

$O\left(\frac{7}{3},\ \frac{7}{3}\right)$

$AB:\ (x+3)+(y-5)=0$

$x+y-2=0$

$r = \frac{\left|\frac{7}{3}+\frac{7}{3}-2\right|}{\sqrt{2}} = \frac{8}{3\sqrt{2}}$

1.14 Ugao između dve prave

Ugao između dve prave je oštar ugao koje one zaklapaju

$\sphericalangle\left(\vec{n}_1,\, \vec{n}_2\right) = \alpha_1$

$\cos\, \alpha = |\cos\, \alpha_1|$

$\cos\, \alpha = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|}$

1. Odrediti ugao između pravih $3x - y + 5 = 0$ i $2x + y - 7 = 0$.

Rešenje.

$\vec{n}_1 = (3,\, -1)$

$\vec{n}_2 = (2,\, 1)$

$\cos\, \alpha = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{5}{\sqrt{10} \cdot \sqrt{5}} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}} \Rightarrow \alpha = \frac{\pi}{4}$

2. Odrediti jednačinu prave koja prolazi kroz presek pravih $\begin{cases} 2x + 3y - 3 = 0 \\ \text{x} + \text{y} - 1 = 0 \end{cases}$ i sa pravom $3x + 3y - 1 = 0$ gradi ugao $\cos\, \alpha = \frac{5}{\sqrt{34}}$.

Rešenje.

$2x + 3y - 3 + \alpha\,(\text{x} + \text{y} - 1) = 0$

$(2 + \alpha)\, x + (3 + \alpha)\, y + (-3 - \alpha) = 0$

$\vec{n}_1 = (2 + \alpha,\, 3 + \alpha)$

$\vec{n}_2 = (3,\, 3)$

$\cos\, \alpha = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{|(2+\alpha, 3+\alpha)(3, 3)|}{\sqrt{(2+\alpha)^2 + (3+\alpha)^2} \cdot \sqrt{3^2+3^2}} = \frac{|15+6\alpha|}{3\sqrt{2} \cdot \sqrt{2\alpha^2+10\alpha+13}} = \frac{5}{\sqrt{34}} /^2$

$34\left(36\alpha^2+180\alpha+225\right)=25\cdot 18\left(2\alpha^2+10\alpha+13\right)$

$9\alpha^2-45\alpha+50=0$

$\alpha_{1,2}=\frac{45\pm15}{18}=\begin{cases}\frac{10}{3}\\ \frac{5}{3}\end{cases}$

$\left(2+\frac{10}{3}\right)x+\left(3+\frac{10}{3}\right)y+\left(-3-\frac{10}{3}\right)=0$

$16x+19y-19=0$

$\left(2+\frac{5}{3}\right)x+\left(3+\frac{5}{3}\right)y+\left(-3-\frac{5}{3}\right)=0$

$11x+14y-14=0$

Domaći.

3. Odrediti ravan koja sadrži presečnu tačku pravih $5x-4y-6=0$ i $x-y-1=0$, a sa pravom $2x-y+3=0$ gradi ugao $45°$.

(Rešenje: $x-3y+1=0$)

Eksplicitni oblik jednačine prave: $y=kx+n$

n je odsečak na y osi, k je koeficijent pravca $k=tg\,\alpha$

$p_1:\ y=k_1x+n_1$

$p_2:\ y=k_2x+n_2$

$\varphi=\sphericalangle\left(p_1,\,p_2\right)$

$\varphi=\varphi_2-\varphi_1$

Uzima se $\varphi=\left|\varphi_2-\varphi_1\right|$, da bi se izbegao slučaj $\varphi_2<\varphi_1$

$tg\,\varphi=\left|tg\left(\varphi_2-\varphi_1\right)\right|$

$tg\,\varphi = \frac{tg\,\varphi_2 - tg\,\varphi_1}{1 + tg\,\varphi_2 \cdot tg\,\varphi_1}$

$tg\,\varphi = \frac{|k_2 - k_1|}{|1 + k_1 \cdot k_2|}$ - ugao između pravih

$p_1 \perp p_2 \Leftrightarrow \varphi = \frac{\pi}{2} \Leftrightarrow tg\,\varphi = \infty \Leftrightarrow 1 + k_1 \cdot k_2 = 0 \Leftrightarrow k_1 = -\frac{1}{k_2}$

$p_1 \parallel p_2 \Leftrightarrow k_1 = k_2$

Parametarski oblik jednačine prave:

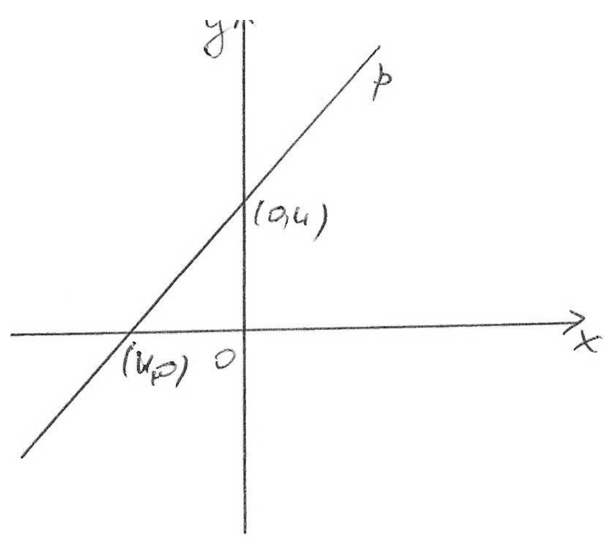
$(x,\, y) \in p \Leftrightarrow (A,\, B) \parallel (x - x_0,\, y - y_0)$

$\Leftrightarrow \frac{x - x_0}{A} = \frac{y - y_0}{B} = t$

$\Leftrightarrow \left.\begin{array}{c} x = A \cdot t + x_0 \\ y = B \cdot t + y_0 \end{array}\right\}$ - Parametarski oblik jednačine prave

$p \parallel (A,\, B)$

Segmentni oblik jednačine prave: -



Slika 1.26:

$\frac{x}{n} + \frac{y}{m} = 1$

$Ax + By + C = 0$

$Ax+By=-C$

$\frac{x}{-\frac{C}{A}}+\frac{y}{-\frac{C}{B}}=1$

1. Odrediti jednačinu prave koja sadrži presek pravih $2x+y+1=0$ i $x-y+2=0$ i na koordinatnim osama odseca odsečke jednake po apsolutnoj vrednosti.

Rešenje.

$2x+y+1+\alpha\left(\mathrm{x}-\mathrm{y}+2\right)=0$

$\left(2+\alpha\right)x+\left(1-\alpha\right)y+1+2\alpha=0$

$\left|-\frac{C}{A}\right|=\left|-\frac{C}{B}\right|$

$\left|\frac{1+2\alpha}{2+\alpha}\right|=\left|\frac{1+2\alpha}{1-\alpha}\right|$

1. $\frac{1+2\alpha}{2+\alpha}=\frac{1+2\alpha}{1-\alpha}$

$\left(1+2\alpha\right)\left(1-\alpha\right)=\left(1+2\alpha\right)\left(2+\alpha\right)$

$\left(1+2\alpha\right)\left(1-\alpha-2-\alpha\right)=0$

$-\left(1+2\alpha\right)^2=0$

$\alpha=-\frac{1}{2}$

2. $\frac{1+2\alpha}{2+\alpha}=\frac{1+2\alpha}{\alpha-1}$

$\left(1+2\alpha\right)\left(\alpha-1\right)=\left(1+2\alpha\right)\left(2+\alpha\right)$

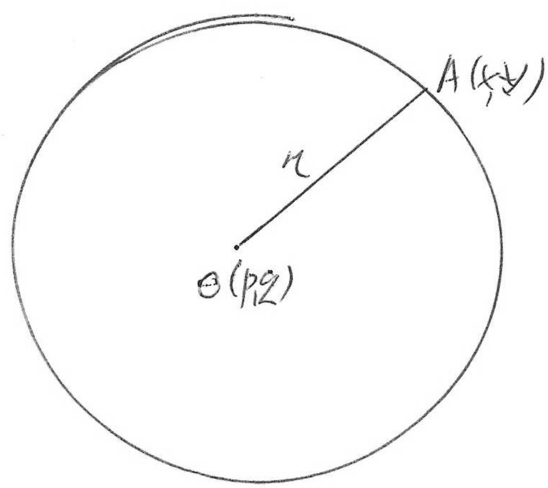
$\left(1+2\alpha\right)\left(\alpha-1-2-\alpha\right)=0$

$-3\left(1+2\alpha\right)=0\;\Rightarrow\;\alpha=-\frac{1}{2}$

$\left(2-\frac{1}{2}\right)x+\left(1+\frac{1}{2}\right)y+0=0$

$\frac{3}{2}x+\frac{3}{2}y=0$

1.15 Kružnica



Slika 1.27:

$d\left(A,\,O\right)=r$

$\sqrt{\left(x-p\right)^2+\left(y-q\right)^2}=r/^2$

$\left(x-p\right)^2+\left(y-q\right)^2=r^2$ - jednačina kružnice

$$\begin{cases} x=x\left(t\right) \\ y=y\left(t\right) \end{cases}$$

$$\begin{cases} x=r\cdot\cos\,t \\ y=r\cdot\sin\,t \end{cases}$$

$t=\varphi$

Pr. Odrediti jednačinu kružnice kojoj je duž AB prečnik, pri čemu je $A\left(1,\,1\right)$ i $A\left(5,\,3\right)$.

$O\left(3,\,2\right)$

$r=\left(A,\,O\right)=\sqrt{\left(3-1\right)^2+\left(2-1\right)^2}=\sqrt{5}/^2$

$K : (x-3)^2 + (y-2)^2 = 5$

Pr. Naći jednačinu kružnice koja prolazi kroz tačku $(3, -6)$ i koncentrična je sa kružnicom $x^2 + y^2 + 6x - 4y - 62 = 0$.

$x^2 + y^2 + 6x - 4y - 62 = 0$

$(x+3)^2 - 9 + (y-2)^2 - 4 - 62 = 0$

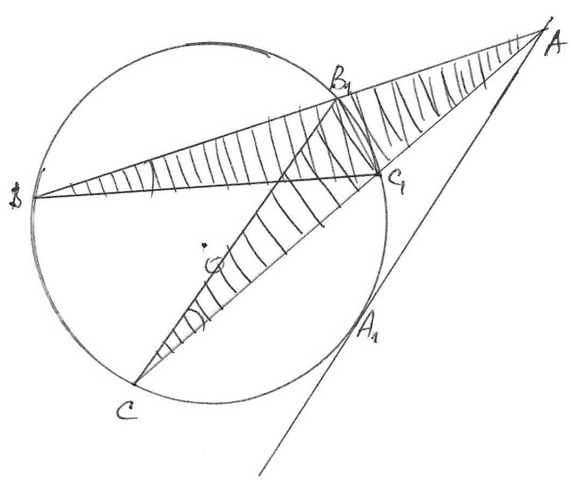
$(x+3)^2 + (y-2)^2 = 75$

$O\,(-3,\, 2)$

$r = (A,\, O) = \sqrt{(3+3)^2 + (-6-2)^2} = \sqrt{100} = 10$

$(x+3)^2 + (y-2)^2 = 0$

-Potencija tačke u odnosu na krug



Slika 1.28:

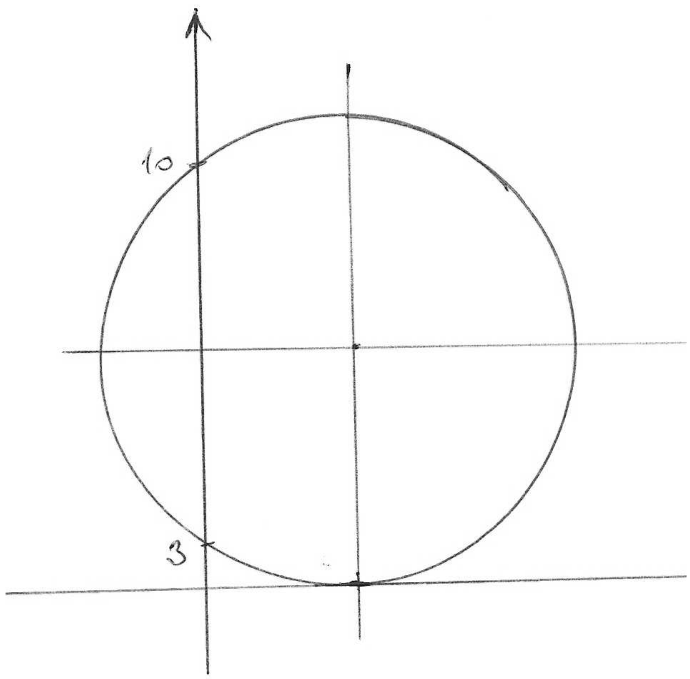
TEOREMA 1.2. *Neka prava p koja sadrži tačku A seče krug u tačkama A_1 i A_2. Tada proizvod $AA_1 \cdot AA_2$ ne zavisi od prave p. Proizvod $AA_1 \cdot AA_2$ naziva se* ***pontecijom*** *tačke A u odnosu na krug K. Ako je t*

dužina tangente duži povučene iz tačke A na krug K tada je pontencija tačke A jednaka t^2.

$\Delta AA_2A_1' \sim \Delta AA_1A_2'$ jer je $\sphericalangle A$ je zajednički, $\sphericalangle A_2 = \sphericalangle A_2'$ (kao periferijski ugao nad istom tetivom)

$\frac{AA_2}{AA_2'} = \frac{AA_1'}{AA_1}$

$AA_2 \cdot AA_1 = AA_1' \cdot AA_2'$



Slika 1.29:

Pr. Odrediti jednačinu kruga koji seče y-osu u tačkama (0, 12) i (0, 3) i koji dodiruje x-osu.

$$(x-p)^2 + (y-q)^2 = r^2$$

$$(x-p)^2 + (y-7,5)^2 = 7,5^2$$

$$x = 0,\ y = 3$$

$$p^2 = 36$$

$$p = \pm 6$$

$K:\ (x-6)^2+(y-7,5)^2=7,5^2$

1.16 Uslov dodira prave i kruga

$k\,(O,\,r)$

$k:\ (x-p)^2+(y-q)^2=r^2$

$p:\ Ax+By+C=0$

$\frac{|Ap+Bq+C|}{\sqrt{A^2+B^2}}=r$ - uslov dodira prave i kruga

DEFINICIJA 1.15. *Ugao između dve krive u zajedničkoj tački A je ugao između njihovih tangenti u toj tački.*

TEOREMA 1.3. *Prava $y=kx+n$ je tangenta na krug $x^2+y^2=r^2$, ako je $r^2\left(1+k^2\right)=n^2$, a kruga $(x-a)^2+(y-b)^2=r^2$ ako je $r^2\left(1+k^2\right)=(ka-b+n)^2$.*

TEOREMA 1.4. *Ako je $M\,(x_1,\,y_1)$ neka tačka kruga $(x-a)^2+(y-b)^2=r^2$ jednačina tangente kruga u toj tački glasi $(x-a)\,(x_1-a)+(y-b)\,(y_1-b)=r^2$.*

1. Odrediti jednačinu tangente kroz tačku $A\,(-4,\,3)$ na kružnicu $x^2+y^2-2x+4y=0$.

$x^2-2\cdot x+1-1+y^2+4y+4-4=0$

$(x-1)^2+(y+2)^2=5$

$t:\ Ax+By+C=0$

$A\in t\ \Rightarrow\ -4A+3B+C=0$

$C=4A-3B$

$\frac{|1\cdot A-2\cdot B+C|}{\sqrt{A^2+B^2}}=\sqrt{5}$

$\frac{|A-2B+4A-3B|}{\sqrt{A^2+B^2}}=\sqrt{5}/\cdot\sqrt{A^2+B^2}$

$|A-2B+4A-3B|=\sqrt{5}\sqrt{A^2+B^2}/^2$

$25\left(A^2-2AB+B^2\right)=5\left(A^2+B^2\right)$

$25A^2-50AB+25B^2=5A^2+5B^2$

$4A^2-10AB+4B^2=0/:B$

$4\left(\frac{A}{B}\right)^2-10\left(\frac{A}{B}\right)+4=0/:2$

$2\left(\frac{A}{B}\right)^2-5\left(\frac{A}{B}\right)+2=0$

$\left(\frac{A}{B}\right)_{1,2}=\frac{5\pm\sqrt{25-16}}{4}=\frac{5\pm3}{4}$

$\left(\frac{A}{B}\right)_1=\frac{1}{2}$

$\left(\frac{A}{B}\right)_2=\frac{8}{4}=2$

$C=4A-3B$

$\left(\frac{C}{B}\right)=4\frac{A}{B}-3$

$\left(\frac{C}{B}\right)_1=4\cdot\frac{1}{2}-3=-1$

$\left(\frac{C}{B}\right)_2=4\cdot2-3=5$

$t:\ Ax+By+C=0/:B$

$\frac{A}{B}x+y+\frac{C}{B}=0$

$\frac{1}{2}x+y-1=0/\cdot2$

$t_1:\ x+2y-2=0$

$t_2:\ 2x+y+5=0$

1. Pod kojim uglom se seku prava $x-3y-5=0$ i krug $x^2+y^2=5$.

$$\begin{cases} x^2+y^2=5 \\ x-3y-5=0\ \Rightarrow\ x=3y+5 \end{cases}$$

$(3y+5)^2+y^2=5$

$9y^2+30y+25+y^2=5$

$10y^2+30y+20=0/:10$

$y^2+3y+2=0$

$y_{1,2}=\frac{-3\pm\sqrt{9-8}}{2}=\frac{-3\pm1}{2}$

$$\begin{cases} y_1=-2 \\ y_2=-1 \end{cases}$$

$$\begin{cases} x_1=-1 \\ x_2=2 \end{cases}$$

$A\,(2,\,-1)\,,\qquad B\,(-1,\,-2)$

$t_A:\ (x-0)\,(2-0)+(y-0)\,(-1-0)=5$

$t_A:\ 2x-y-5=0\ \Rightarrow\ y=2x-5\ \Rightarrow\ k_t=2$

$x-3y-5=0\ \Rightarrow\ 3y=x-5\ \Rightarrow\ y=\frac{1}{3}\,x-\frac{5}{3}\ \Rightarrow\ k=\frac{1}{3}$

$tg\,\alpha=\left|\frac{k_t-k}{1+k_t\cdot k}\right|=\left|\frac{2-\frac{1}{3}}{1+2\cdot\frac{1}{3}}\right|=\left|\frac{\frac{5}{3}}{\frac{5}{3}}\right|=1$

$tg\,\alpha=1$

$\alpha = 45^\circ$

$\alpha = \frac{\pi}{4}$

2. Odrediti jednačinu tangente na krug $(x-3)^2 + (y-1)^2 = 4$ u tački $A(1, 1)$.

$(1-3)(x-3) + (1-1)(y-1) = 4$

$-2(x-3) = 4$

$-2x + 6 = 4 / \cdot (-1)$

$2x - 6 = -4$

$2x = 2$

$x = 1$

3.Naći jednačine tangenti kruga $x^2 + y^2 - 10x - 12y + 36 = 0$ koje su paralelne pravoj $4x - 3y + 10 = 0$.

Rešenje. $(A, B) = (4, -3)$

$Ax + By + C = 0$

$4x - 3y + 10 = 0$

$x^2 - 10x + 25 - 25 + y^2 - 12y + 36 = 0$

$(x-5)^2 + (y-6)^2 = 25$

$p = 5,\ q = 6$

$\frac{|4 \cdot 5 - 3 \cdot 6 + C|}{\sqrt{4^2 + 3^2}} = 5 / \cdot 5$

$|2 + C| = 25$

1. $2 + C = 25$

$C = 23$

$t_1 : 4x - 3y + 23 = 0$

2. $-2 - C = 25$

$C = -27$

$t_2 : 4x - 3y - 27 = 0$

4. Napisati jednačinu kruga koji prolazi kroz tačke $A(-2, 9)$; $B(-4, 5)$; $C(5, 8)$. Odrediti ugao koji tetiva AB zaklapa sa njim kao i tangente na krug iz tačke $D(8, 4)$.

Rešenje.

$k : (x - p)^2 + (y - q)^2 = r^2$

$$\begin{cases} A \in k \Rightarrow (-2 - p)^2 + (9 - q)^2 = r^2 \\ B \in k \Rightarrow (-4 - p)^2 + (5 - q)^2 = r^2 \\ C \in k \Rightarrow (5 - p)^2 + (8 - q)^2 = r^2 \end{cases}$$

$$\begin{cases} p^2 + 4p + 4 + 81 - 18q + q^2 = r^2 \\ p^2 + 8p + 16 + q^2 - 10q + 25 = r^2 \\ p^2 - 10p + 25 + q^2 - 16q + 64 = r^2 \end{cases}$$

$$\begin{cases} p^2 + 4p + q^2 - 18q + 85 = p^2 + 8p + q^2 - 10q + 41 \\ p^2 + 4p + q^2 - 18q + 85 = p^2 - 10p + q^2 - 16q + 89 \end{cases}$$

$$\begin{cases} -4p - 8q + 44 = 0 / : (-4) \\ 14p - 2q - 4 = 0 / : 2 \end{cases}$$

$$\begin{cases} p + 2q - 11 = 0 \\ 7p - q - 2 = 0 \end{cases}$$

$q = 7p - 2$

$p + 14p - 4 - 11 = 0 \Rightarrow 15p = 15 \Rightarrow p = 1$

$q = 5$

$O\,(1,\,5)$

$r = d\,(O,\,A) = \sqrt{(1+2)^2 + (5-9)^2} = \sqrt{9+16} = \sqrt{25} = 5$

$k:\ (x-1)^2 + (9-5)^2 = 25$

$t_B:\ (x-1)\,(-4-1) + (y-5)\,(5-5) = 25$

$-5\,(x-1) = 25 / \cdot (-5)$

$x - 1 = -5$

$t_B:\ x = -4$

$AB:\ y - 9 = \frac{5-9}{-4+2}\,(x+2)$

$AB:\ y - 9 = 2\,(x+2)$

$AB:\ y = 2x + 13$

$AB:\ 2x - y + 13 = 0$

$\cos\varphi = \frac{2\cdot 1}{\sqrt{1}\cdot\sqrt{5}} = \frac{2}{\sqrt{5}}$

$D\,(8,\,4)$

$t:\ Ax + By + C = 0$

$8A + 4B + C = 0$

$C = -8A - 4B$

$\frac{|A+5B+C|}{\sqrt{A^2+B^2}} = 5/ \cdot \sqrt{A^2+B^2}$

$5\sqrt{A^2+B^2} = |A+5B-8A-4B| /^2$

$25\left(A^2+B^2\right) = B^2 - 14AB + 49A^2$

$-24A^2 + 14AB + 24B^2 = 0/ : \left(-2B^2\right)$

$12\left(\frac{A}{B}\right)^2 - 7\left(\frac{A}{B}\right) - 12 = 0$

$\left(\frac{A}{B}\right)_{1,2} = \frac{7\pm\sqrt{49+576}}{24} = \frac{7\pm25}{24}$

$\left(\frac{A}{B}\right)_1 = \frac{4}{3}$

$\left(\frac{A}{B}\right)_2 = -\frac{3}{4}$

$C = -8A - 4B/ : B$

$\frac{C}{B} = -8\frac{A}{B} - 4$

$\left(\frac{C}{B}\right)_1 = -8 \cdot \frac{4}{3} - 4 = -\frac{44}{3}$

$\left(\frac{C}{B}\right)_2 = -8 \cdot \frac{-3}{3} - 4 = 2$

$t:\ 8A + 4B + C = 0/ : B$

$t:\ 8\left(\frac{A}{B}\right) + 4 + \frac{C}{B} = 0$

$Ax + By + C = 0$

$t_1:\ \frac{4}{3}x + y - \frac{44}{3} = 0/ \cdot 3$

$4x + 3y - 44 = 0$

$t_2:\ -\frac{3}{4}x + y + 2 = 0/ \cdot (-4)$

$3x - 4y - 8 = 0$

1.17 Elipsa

Elipsa je geometrijsko mesto tačaka u ravni sa osobinom da je zbir rastojanja ma koje tačke tog skupa od dve fiksne tačke F_1F_2 (žiže elipse) te ravni ima konstantnu vrednost.

F_1F_2 - velika osa elipse

simetrala duži F_1F_2 - mala osa elipse

$M\,(x,\,y)$ - proizvoljna tačka elipse

$d\,(M,\,F_1) + d\,(M,\,F_2) = 2a = d$

$F_1 = (-c,\,0)\,,\ F_2 = (c,\,0)\,,\ 0 < c < a,\ |F_1F_2| = 2c$

$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a/^2$

$(x+c)^2 + y^2 + (x-c)^2 + y^2 = 4a^2$

$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$ (smena $a^2 - c^2 = b^2$)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ - jednačina elipse

Veličina $e = \frac{c}{a}$ naziva se ekscentricitet elipse.

$c = \sqrt{a^2 - b^2}$

Prave $x = \frac{a}{e}$ i $x = -\frac{a}{e}$, tj. kada se zameni $x = \frac{a^2}{c}$ i $x = -\frac{a^2}{c}$ nazivaju se direktrise elipse.

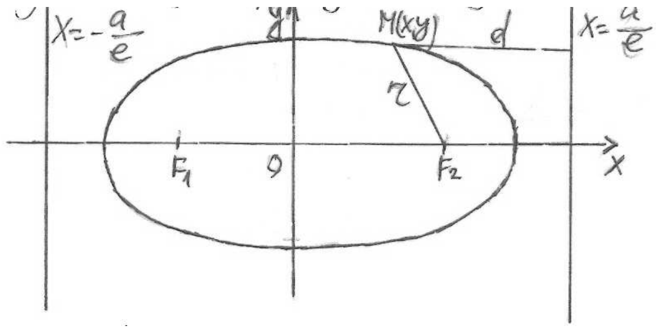
Direktrisa $x = \frac{a}{e}$ odgovara žiži $F_2 = (c,\,0)\,,$ a direktrisa $x = -\frac{a}{e}$ odgovara žiči $F_1 = (-c,\,0)\,.$

Količnik rastojanja od proizvoljne tačke elipse $M = (x,\,y)$ do žiže i rastojanja od te tačke do odgovarajuće direktrise je konstantan i jednak je ekscentricitetu e.

Parametarske jednačine elipse:

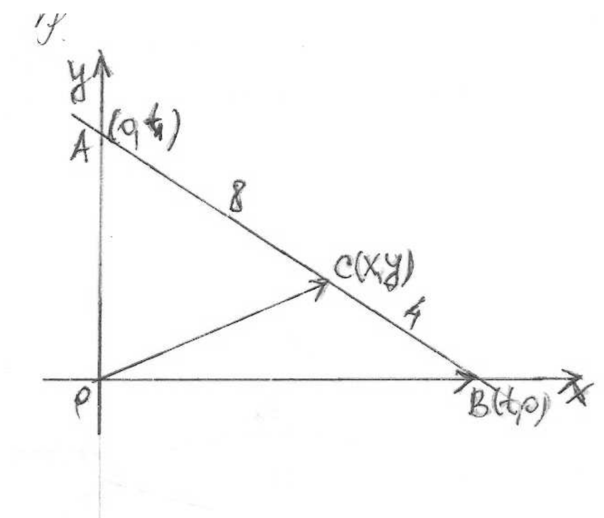
$$\begin{cases} x = a \cdot \cos\, t \\ y = b \cdot \sin\, t \end{cases}.$$

Krug je elipsa kod koje je $b = a$, tj. $c = 0$.



Slika 1.30:

1. Duž AB klizi krajem A po osi O_y, a krajem B po osi O_x. Ako je $AB = 12$ odrediti geometrijsko mesto tačaka koje duž AB deli u odnosu $2:1$.



Slika 1.31:

$$\overrightarrow{OC} = \frac{2}{3}\overrightarrow{OB} + \frac{1}{3}\overrightarrow{OA} = \frac{2}{3}(t,\,0) + \frac{1}{3}(0,\,t_1) = \left(\frac{2}{3}t,\,\frac{1}{3}t_1\right) = (x,\,y)$$

$$x = \frac{2}{3}t$$

$$y = \frac{1}{3}t_1$$

$$t = \frac{3}{2}x$$

$$t_1 = 3y$$

$$t^2 + t_1^2 = \left(\frac{3}{2}x\right)^2 + (3y)^2 = \frac{9}{4}x^2 + 9y^2$$

$t^2 + t_1^2 = 12^2$

$\frac{9}{4}x^2 + 9y^2 = 144/ : 9$

$\frac{1}{4}x^2 + y^2 = 16/ : 16$

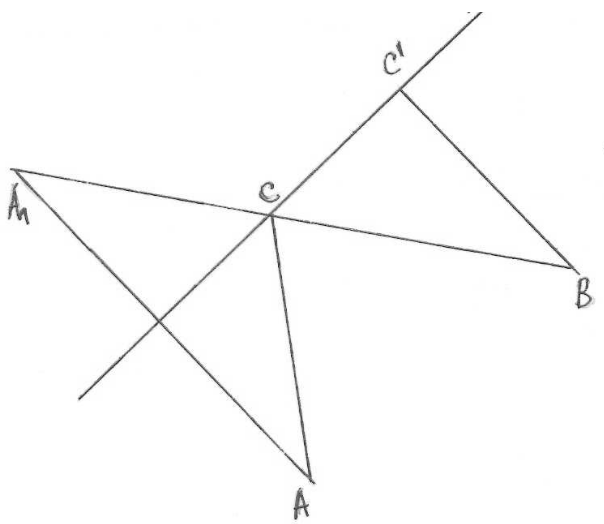
$\frac{x^2}{64} + \frac{y^2}{16} = 1$

1.18 Optičko svojstvo elipse

Problem:

Neka je data prava p i tačke A i B sa iste strane prave p na kojoj treba da se nalazi tačka C tako da zbir $AB + CB$ bude najmanji mogući.

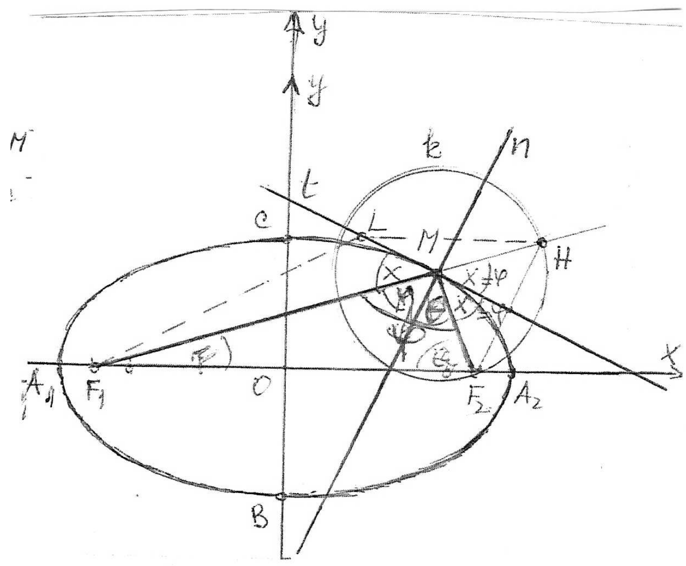
- Tačka C se dobija tako što se tačka simetrično preslika u odnosu na pravu p i pri tome se dobije tačka A_1. Tačka CO koju tražimo će se nalaziti u preseku prave p i prave AB.



Slika 1.32:

$AC + CB$ - minimalno

$AC' + C'B > AC + CB$



Slika 1.33:

$$A_1C' = C'A$$

$$A_1C' + C'B > A_1B = A_1C + CB$$

$$A_1C = CA$$

$$A_1C' + C'B > CA + CB$$

$$AC' + C'B > CA + CB$$

$$AC' + C'B > AC + BC$$

Teorema 1.5. *Svetlosni zrak koji prolazi iz jedne žiže date elipse posle odbijanja od nje proći će kroz drugu žižu ili drugim rečima tangenta na elipsu u njenoj proizvoljnoj tački gradi jednake uglove sa dužima koje tu tačku spajaju sa žižama.*

Opišimo krug k oko proizvoljne tačke M na elipsi, koji prolazi kroz F_2, i neka produžena duž F_1M, preko tačke M, preseca krug k u tački H.

Kako je M središte kruga k $MF_2 = MH \Rightarrow \Delta F_2MH$ je jednakokraki, i simetrala t kroz M je tangenta elipse u tački M.

$F_1H = F_1M + MH = F_1M + MF_2 = A_1A_2$

Uzmimo neku drugu tačku $L \in t$, $L \neq M$. Kako je t simetrala stranice F_2H, $\Delta F_2MH \Rightarrow LF_2 = LH \Rightarrow LF_2 + LF_1 = LF_1 + LH > F_1H \Rightarrow LF_1 + LF_2 > A_1A_2$. Svaka tačka $L \neq M$, prave t je izvan elipse, a to znači da je prava t tangenta elipse u tački M.

Iz $\Delta F_2MH \Rightarrow MF_2 = MH$ Kako je t simetrala $\sphericalangle F_2MH \Rightarrow \sphericalangle(F_2M, t) = \sphericalangle(t, MH) = \sphericalangle(t, MF_1) \Rightarrow \sphericalangle(t, MF_1) = \sphericalangle(t, MF_2)$.

1.19 Tangenta na elipsu

Tangenta je prava koja sa elipsom ima tačno jednu zajedničku tačku.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$Ax + By + C = 0 \Rightarrow y = -\frac{C+Ax}{B}$

zamenom u jednačini elipse dobija se

$A^2a^2 + B^2b^2 - C^2 = 0$

$A^2a^2 + B^2b^2 = C^2$ uslov dodira prave i elipse

1. Odrediti jednačinu elipse koja dodiruje prave $3x - 2y - 20 = 0$ i $x + 6y - 20 = 0$.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\left.\begin{array}{l} 9a^2 + 4b^2 = 400 \\ a^2 + 36b^2 = 400 / \cdot (-9) \end{array}\right\}$$

$-320b^2 = -3200$

$b^2 = 10$

$a^2 = 400 - 360$

$a^2 = 40$

$\frac{x^2}{40} + \frac{y^2}{10} = 1$

2. Odrediti tangente na elipsu $x^2 + 4y^2 = 20$ koje su paralelne, a zatim i one koje su normalne na ravan $2x - 2y - 13 = 0$.

$x^2 + 4y^2 = 20 / : 20$

$\frac{x^2}{20} + \frac{y^2}{5} = 1$

$p:\ 2x - 2y - 13 = 0$

$t:\ Ax + By + C = 0$

$t \parallel p \Rightarrow (A,\ B) = (2,\ -2)$

$t:\ 2x - 2y + C = 0$

$$\left.\begin{array}{l} 2x - 2y + C = 0 \\ \frac{x^2}{20} + \frac{y^2}{5} = 1 \end{array}\right\}$$

$4 \cdot 20 + 4 \cdot 5 = C^2$

$100 = C^2$

$C = \pm 10$

$t:\ 2x - 2y \pm 10 = 0 / : 2$

$t:\ x - y \pm 5 = 0$

$t \perp \Rightarrow (A,\ B) \cdot (2,\ -2) = 0$

$2A - 2B = 0$

$B = A$

$t:\ Ax+By+C=0$

$20A^2+5A^2=C^2$

$25A^2=C^2$

$C=\pm 5A$

$t:\ Ax+Ay\pm 5A=0/:A$

$t:\ x+y\pm 5=0$

3. Odrediti tangente na elipsu $2x^2+3y^2=21$ u tački $A\,(3,\,1)$.

$2x^2+3y^2=21/:21$

$\frac{x^2}{\frac{21}{2}}+\frac{y^2}{7}=1$

$t:\ Ax+By+C=0$

$A\in t\ \Rightarrow\ 3A+B+C=0\ \Rightarrow\ C=-\left(3A+B\right)$

$\frac{21}{2}A^2+\frac{21}{3}B^2=C^2$

$\frac{x^2}{\frac{21}{2}}+\frac{y^2}{\frac{21}{3}}=1$

$\frac{21}{2}A^2+\frac{21}{3}B^2=9A^2+6AB+B^2$

$\frac{3}{2}A^2+\frac{18}{3}B^2-6AB=0/:\frac{3}{2}B^2$

$\left(\frac{A}{B}\right)^2-4\left(\frac{A}{B}\right)+4=0$

$\left(\frac{A}{B}\right)_{1,2}=\frac{4\pm\sqrt{16-16}}{2}$

$\left(\frac{A}{B}\right)_{1,2}=2$

$3A + B + C = 0 / : B$

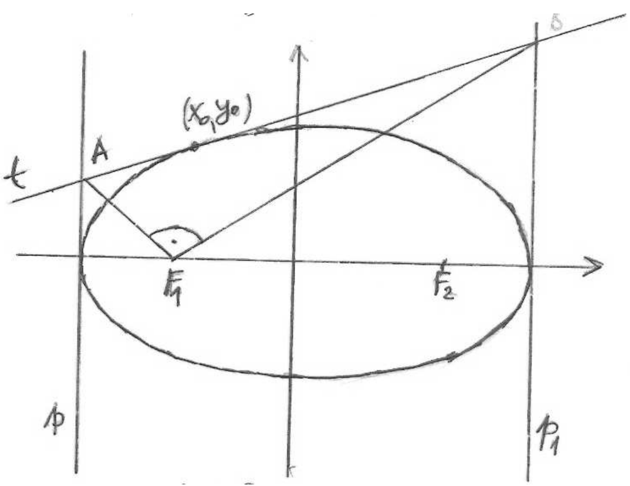
$3\frac{A}{B} + 1 + \frac{C}{B} = 0$

$3 \cdot 2 + 1 + \frac{C}{B} = 0 \Rightarrow \frac{C}{B} = -7$

$t_{1,2} : 2 \cdot 3x + 3 \cdot 1y = 21 / : 3$

$t_{1,2} : 2x + y - 7 = 0$

4. Neka su p, p_1 i t prave koje su tangente na elipsu $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, pri čemu su p i p_1 tangente u krajnjim tačkama velike ose. Ako su A i B presečne tačke ovih tangenti sa tangentom t pokazati da se duž AB iz proizvoljne žiže vidi pod pravim uglom.



Slika 1.34:

$\overrightarrow{F_1A} \perp \overrightarrow{F_1B} \Leftrightarrow \overrightarrow{F_1A} \cdot \overrightarrow{F_1B} = \vec{0}$

$t : \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$

$p : x = -a$

$p_1 : x = a$

$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$

$x = -a$

$-\frac{x_0}{a} + \frac{y_0 y}{b^2} = 1$

$y = \frac{ab^2 + b^2 x_0}{a y_0}$

$A\left(-a,\ \frac{ab^2 + b^2 x_0}{a y_0}\right)$

$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$

$x = a$

$y = \frac{b^2 x_0 - ab^2}{-a y_0}$

$y = \frac{ab^2 - b^2 x_0}{a y_0}$

$B\left(a,\ \frac{ab^2 - b^2 x_0}{a y_0}\right)$

$\overrightarrow{F_1 A} = \left(-a + c,\ \frac{ab^2 + b^2 x_0}{a y_0}\right)$

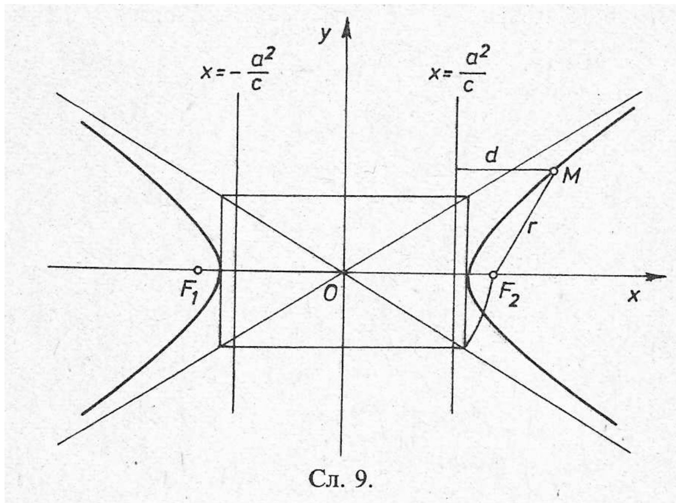
$\overrightarrow{F_1 B} = \left(a + c,\ \frac{ab^2 - b^2 x_0}{a y_0}\right)$

1.20 Hiperbola

Definicija 1.16. *Hiperbola je geometrijsko mesto tačaka kod kojih je razlika rastojanja od dve fiksne tačke po modulu konstantna i jednaka* $2a > 0$.

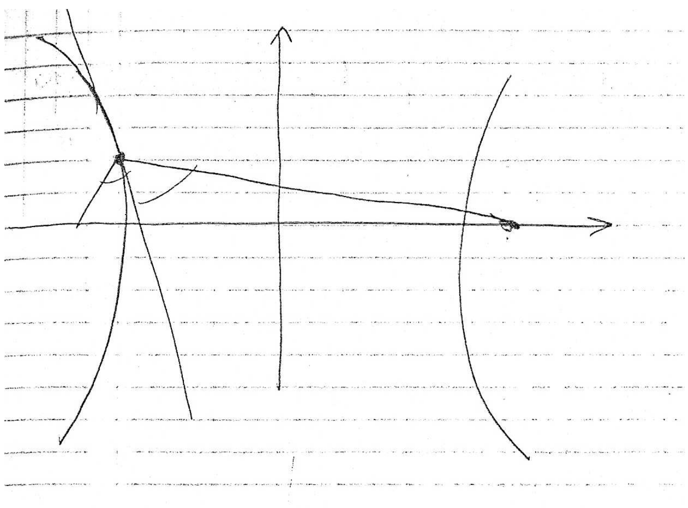
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ - jednačina hiperbole

$A^2 a^2 - B^2 b^2 = c^2$ - uslov dodira prave i hiperbole



Slika 1.35:

1.21 Optičko svojstvo hiperbole



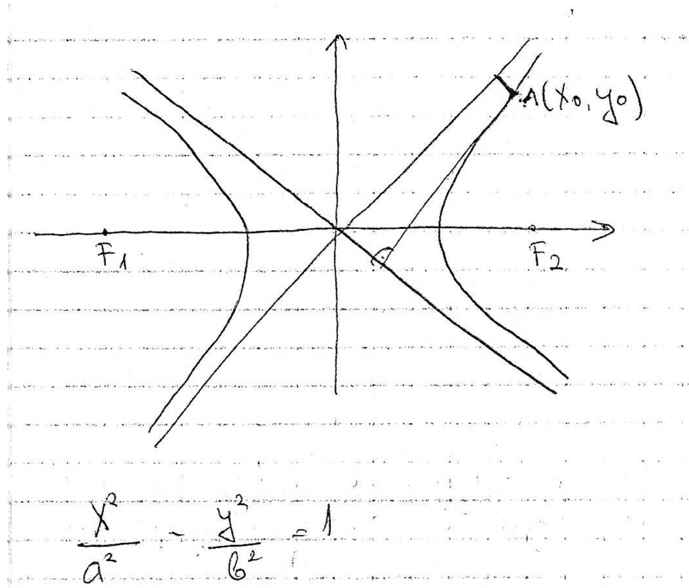
Slika 1.36:

Tangenta hiperbole u nekoj njenoj tački je simertala ugla koja se dobija kada se ta tačka kao teme spoji sa žižama.

1. Dokazati da je proizvod rastojanja tačke na hiperboliod njenih asimptota konstantan za datu hiperbolu.

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$a_1 : y = \frac{b}{a}x \Leftrightarrow 0 = \frac{b}{a}x - y$



Slika 1.37:

$a_2:\ y=-\frac{b}{a}x \Leftrightarrow 0=\frac{b}{a}x+y$

$$d(A,\,a_1)\cdot d(A,\,a_2)=\frac{\left|\frac{b}{a}x_0-y_0\right|}{\sqrt{\left(\frac{b}{a}\right)^2+(-1)^2}}\cdot\frac{\left|\frac{b}{a}x_0+y_0\right|}{\sqrt{\left(\frac{b}{a}\right)^2+1^2}}=\frac{\left|\frac{b^2}{a^2}x_0^2-y_0^2\right|}{\frac{b^2+a^2}{a^2}}=$$

$$=\frac{\left|\frac{b^2x_0^2-a^2y_0^2}{a^2}\right|}{\frac{b^2+a^2}{a^2}}=\frac{b^2a^2\left|\frac{x_0^2}{a^2}-\frac{y_0^2}{b^2}\right|}{b^2+a^2}=\frac{b^2a^2}{b^2+a^2}=const$$

2. Dokazati da je deo tangente hiperbole koji se nalazi između njenih asimptota prepolovljen dodirnom tačkom.

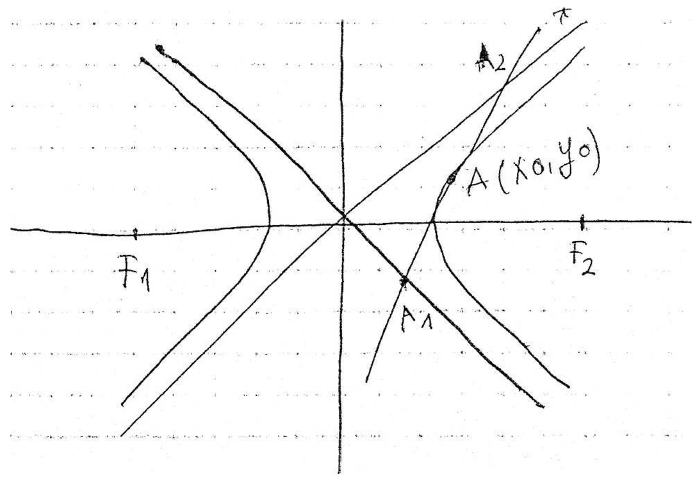
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

$t:\ \frac{x_0x}{a^2}-\frac{y_0y}{b^2}=1$

$$\left.\begin{array}{l} y=-\frac{b}{a}x \\ \frac{x_0x}{a^2}-\frac{y_0y}{b^2}=1 \end{array}\right\}\text{koor. tačke } A_1$$

$x=\frac{a^2b}{bx_0+ay_0}=x_1$

$$\left.\begin{array}{l} y=\frac{b}{a}x \\ \frac{x_0x}{a^2}-\frac{y_0y}{b^2}=1 \end{array}\right\}$$



Slika 1.38:

$x = -\frac{a^2b}{ay_0 - bx_0} = \frac{a^2b}{bx_0 - ay_0} = x_2$

$x_1 + x_2 = \frac{1}{2}\left(\frac{a^2b}{bx_0 + ay_0} + \frac{a^2b}{bx_0 - ay_0}\right) = \frac{a^2b}{2} \cdot \frac{bx_0 - ay_0 + bx_0 + ay_0}{(bx_0 + ay_0)(bx_0 - ay_0)} =$
$\frac{a^2b}{2} \cdot \frac{2bx_0}{b^2x_0^2 - a^2y_0^2} = \frac{a^2bx_0}{b^2x_0^2 - a^2y_0^2} = \frac{x_0}{\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}} = x_0$

$\Rightarrow$ x_0 je sredina duži x_1x_2.

1.22 Parabola

Definicija 1.17. *Parabola je geometrijsko mesto tačaka kod kojih je zbir rastojanja od date tačke (žiže) do date prave (direktrise) jednako.*

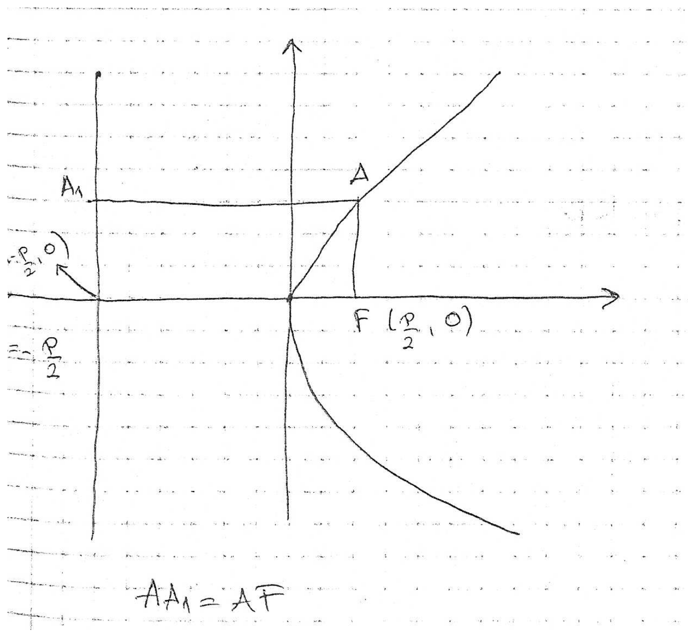
$Ax + By + C = 0$

$y^2 = 2px$

$B^2p - 2AC = 0$ - uslov dodira prave i parabole

1. Iz tačke $(-2, 2)$ povući tangente na parabolu $y^2 = 16x$

$Ax + By + C = 0/ : B$



Slika 1.39:

$\frac{A}{B}x + y + \frac{C}{B} = 0$

$-2A + 2B + C = 0$

$C = 2A - 2B$

$B^2 \cdot 8 - 2AC = 0$

$4B^2 - A\left(2A - 2B\right) = 0$

$2B^2 - A^2 + AB = 0 / : B^2$

$2 - \frac{A^2}{B^2} + \frac{A}{B} = 0$

$\left(\frac{A}{B}\right)^2 - \frac{A}{B} - 2 = 0$

$\left(\frac{A}{B}\right)_{1,2} = \frac{1 \pm \sqrt{1+8}}{2}$

$\left(\frac{A}{B}\right)_{1,2} = \frac{1 \pm 3}{2}$

$\frac{A}{B} = \begin{cases} 2 \\ -1 \end{cases}$

$\frac{C}{B} = 2\frac{A}{B} - 2$

$\frac{C}{B} = \begin{cases} 2 \\ -4 \end{cases}$

1. $2x + y + 2 = 0$

2. $-x + y - 4 = 0$

$x - y + 4 = 0$

1.23 Tačka na paraboli

$A(x_0, y_0)$ - tačka na paraboli $y^2 = 2px = px + px$

$y_0 y = px_0 + px = p(x_0 + x)$

$y_0 y = p(x_0 + x)$ - jednačina tangente na paraboli u tački (x_0, y_0)

1. Naći jednačinu normale parabole $y^2 = 12x$ u njenoj tački $(x_0, -6)$.

$(-6)^2 = 12x_0$

$x_0 = 3$

$(3, -6)$

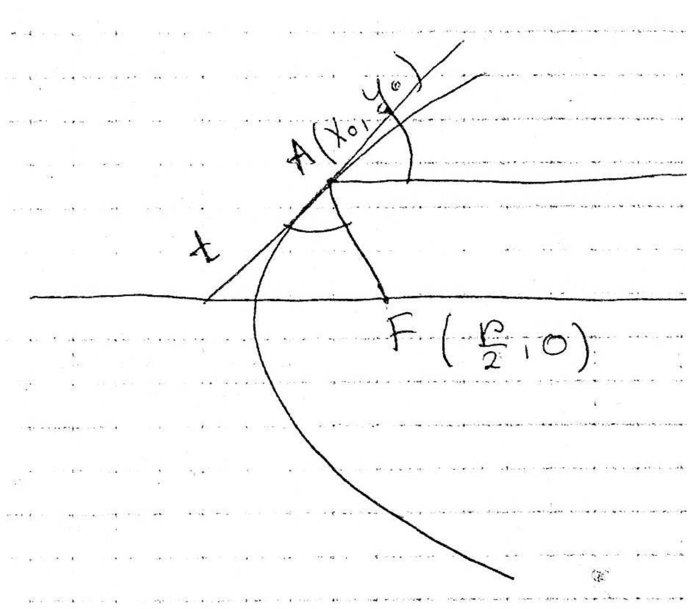
$t: -6y = 6(x + 3)$

$x + y + 3 = 0$

1.24 Optičko svojstvo parabole

Svetlosni zrak koji kreće iz žiže, posle odbijanja od parabole nastavlja kretanje paralelno sa osom parabole.

Drugim rečima: Tangenta parabole u tački A gradi jednake uglove sa pravom AF i sa pravom koja prolazi kroz tačku A i paralelna je sa osom parabole.



Slika 1.40:

$$y^2 = 2px$$

$$t:\ y_0 y = p\left(x_0 + x\right)$$

$$0 = px - y_0 y + p x_0$$

$$AF:\ y_0\left(x - \frac{p}{2}\right) + \left(\frac{p}{2} - x_0\right) y = 0$$

$$\overrightarrow{AF}:\ \left(y_0,\ \frac{p}{2} - x_0\right)$$

$$s:\ y = y_0$$

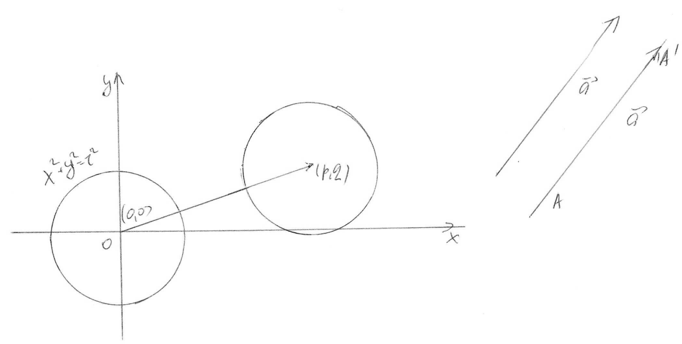
$$y - y_0 = 0$$

$$\cos\ (t,\ AF) = \frac{\left|p y_0 - y_0 \frac{p}{2} + x_0 y_0\right|}{\sqrt{p^2+y_0^2}\sqrt{y_0^2+\left(\frac{p}{2}-x_0\right)^2}} = \frac{|y_0|\left|x_0+\frac{p}{2}\right|}{\sqrt{p^2+y_0^2}\left|x_0+\frac{p}{2}\right|} = \frac{|y_0|}{\sqrt{p^2+y_0^2}}$$

$\cos\sphericalangle(t,\, s) = \frac{|(p,-y_0)(0,1)|}{\sqrt{p^2+y_0^2}\cdot 1} = \frac{|y_0|}{\sqrt{p^2+y_0^2}}$

$\cos\sphericalangle(t,\, AF) = \cos\sphericalangle(t,\, s)$

1.25 Translacija u koordinatnom sistemu



Slika 1.41:

Ako je u ravni ili prostoru zadat skup tačaka jednačinom $F(x,\, y,\, z) = 0$, tada skup tačaka koji je dobijen od ovog skupa tačaka, translacijom za dati vektor $(p,\, q,\, r)$ ima jednačinu $F(x-p,\, y-q,\, z-r) = 0$.

1. Napisati jednačinu prave koja se dobija translacijom prave $2x - 3y + 1 = 0$ za vektor $(-1,\, 2)$.

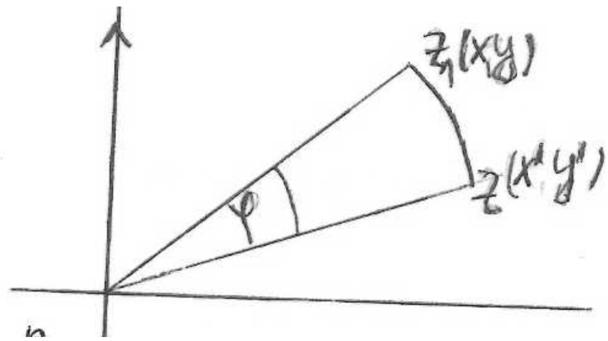
$2(x+1) - 3(y-2) + 1 = 0$

$2x - 3y + 9 = 0$

2. $x^2 + x^3 y + y^2 x = 0$ za $(p,\, q)$

$(x-p)^2 + (x-p)^3 (y-q) + (y-q)^2 (x-p) = 0.$

1.26 Rotacija u koordinatnom sistemu

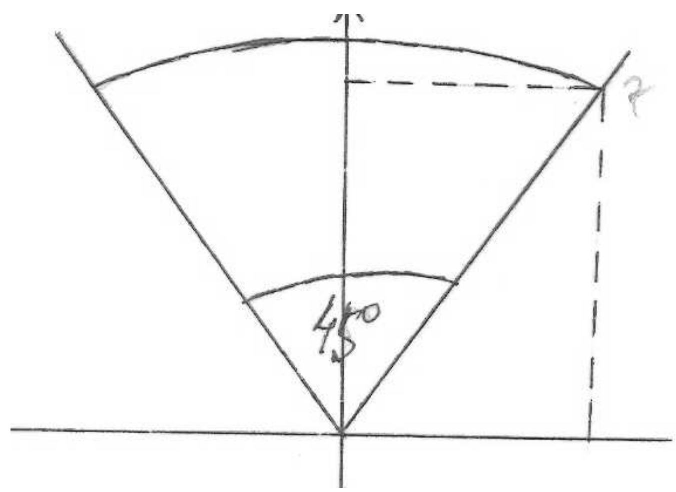


Slika 1.42:

$z = \cos\varphi + i\,\sin\varphi = e^{i\varphi}$

$z_1 = z\,(\cos\varphi + i\,\sin\varphi)$

Ako tačku z u kompleksnoj ravni hoćemo da rotiramo oko koordinatnog početka, onda z treba pomnožiti sa $\cos\varphi + i\,\sin\varphi$, gde je φ traženi ugao rotacije.



Slika 1.43:

$z = 2 + 3i\;45^\circ$

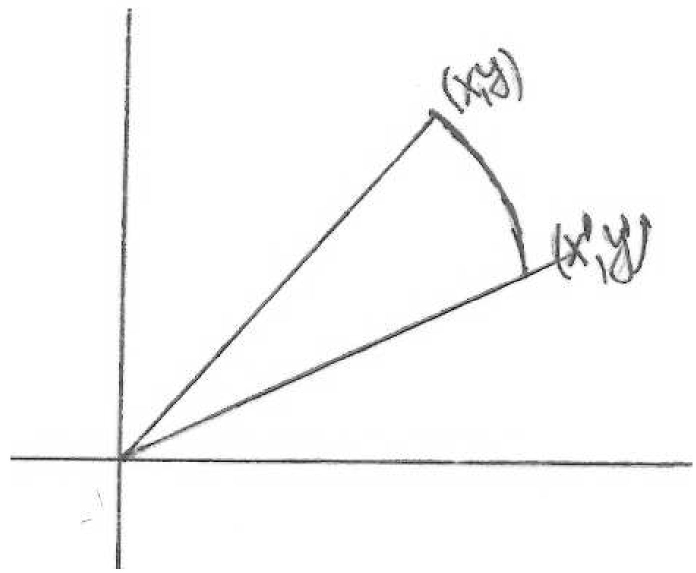
$z_1 = z\,(\cos 45^\circ + i\,\sin 45^\circ)$

$z_1 = (2+3i)\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right)$

$z_1 = \sqrt{2} + i\sqrt{2} + \frac{3\sqrt{2}}{2}i - \frac{3\sqrt{2}}{2}$

$z_1 = -\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}i$

$(x,\,y) = (x',\,y')\,(\cos\varphi,\,\sin\varphi)$



Slika 1.44:

$(x,\, y) = (x' \cos\, \varphi - y' \sin\, \varphi,\, x' \sin\, \varphi + y' \cos\, \varphi)$

$\begin{cases} x = x' \cos\, \varphi - y' \sin\, \varphi \\ y = x' \sin\, \varphi + y' \cos\, \varphi \end{cases}$

Formule za rotaciju tačke $(x',\, y')$ za ugao φ.

1.27 Opšta jednačina krivih drugog reda

Opšta jednačina krivih drugog reda je oblika:

$Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$

Koriste se formule za rotaciju

$\begin{cases} x = x' \cos\, \alpha - y' \sin\, \alpha \\ y = x' \sin\, \alpha + y' \cos\, \alpha \end{cases}$

$A\,(x' \cos\, \alpha - y' \sin\, \alpha)^2 + 2B\,(x' \cos\, \alpha - y' \sin\, \alpha)\,(x' \sin\, \alpha + y' \cos\, \alpha) + C\,(x' \sin\, \alpha + y' \cos\, \alpha)^2 + D\,(x' \cos\, \alpha - y' \sin\, \alpha) + E\,(x' \sin\, \alpha + y' \cos\, \alpha) + F = 0$

$-2A\, \cos\, \alpha \sin\, \alpha + 2B\, \left(\cos^2 \alpha - \sin^2 \alpha\right) + 2C\, \sin\, \alpha \cos \alpha = 0$ - koeficijent uz $x'y'$.

$-A\sin 2\alpha + 2B\cos 2\alpha + C\sin 2\alpha = 0$

$(C-A)\sin 2\alpha = -2B\cos 2\alpha \ \frac{C-A}{2B} = \frac{-\cos 2\alpha}{\sin 2\alpha} \Rightarrow \frac{A-C}{2B} = ctg\, 2\alpha$

Zadaci

1. Odrediti šta predstavlja skup tačaka zadat jednačinom

$5x^2 + 4xy + 8y^2 + 8x + 14y + 5 = 0.$

Rešenje.

$\frac{A-C}{2B} = ctg\, 2\alpha \Rightarrow \frac{5-8}{4} = ctg\, 2\alpha \Rightarrow tg\, 2\alpha = -\frac{4}{3};\ tg\, 2\alpha = \frac{2\, tg\, \alpha}{1-tg^2\alpha}$

$-\frac{4}{3} = \frac{2\, tg\, \alpha}{1-tg^2\alpha}$

$-4 + 4\, tg^2\alpha = 6\, tg\, \alpha$

$2\, tg^2\alpha - 3\, tg\, \alpha - 2 = 0$

$tg\, \alpha_{1/2} = \frac{3\pm\sqrt{9+16}}{4} = \frac{3\pm 5}{4} = \begin{cases} -\frac{1}{2} \\ 2 \end{cases}$

$\left.\begin{array}{l} \sin^2\alpha = \frac{tg^2\alpha}{1+tg^2\alpha} = \frac{4}{1+4} = \frac{4}{5} \\ \cos^2\alpha = \frac{1}{1+tg^2\alpha} = \frac{1}{1+4} = \frac{1}{5} \end{array}\right\} \Rightarrow \sin\alpha = \frac{2}{\sqrt{5}};\ \cos\alpha = \frac{1}{\sqrt{5}}.$

$x = x'\frac{1}{\sqrt{5}} - y'\frac{2}{\sqrt{5}}$

$y = x'\frac{2}{\sqrt{5}} + y'\frac{1}{\sqrt{5}}$

$5\left(x'\frac{1}{\sqrt{5}} - y'\frac{2}{\sqrt{5}}\right)^2 + 4\left(x'\frac{1}{\sqrt{5}} - y'\frac{2}{\sqrt{5}}\right)\left(x'\frac{2}{\sqrt{5}} + y'\frac{1}{\sqrt{5}}\right) + 8\left(x'\frac{2}{\sqrt{5}} + y'\frac{1}{\sqrt{5}}\right)^2 +$
$8\left(x'\frac{1}{\sqrt{5}} - y'\frac{2}{\sqrt{5}}\right) + 14\left(x'\frac{1}{\sqrt{5}} - y'\frac{2}{\sqrt{5}}\right) + 5 = 0$

$5\left(\frac{1}{5}x'^2 - \frac{4}{5}x'y' + \frac{4}{5}y'^2\right) + 4\left(\frac{2}{5}x'^2 + \frac{1}{5}x'y' - \frac{4}{5}x'y' - \frac{2}{5}y'^2\right) + 8\left(\frac{4}{5}x'^2 + \frac{4}{5}x'y' + \frac{1}{5}y'^2\right) +$
$\frac{8}{\sqrt{5}}x' - \frac{16}{\sqrt{5}}y' + \frac{28}{\sqrt{5}}x' + \frac{14}{\sqrt{5}}y' + 5 = 0$

$9x'^2 + 4y'^2 + \frac{36}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y' + 5 = 0$

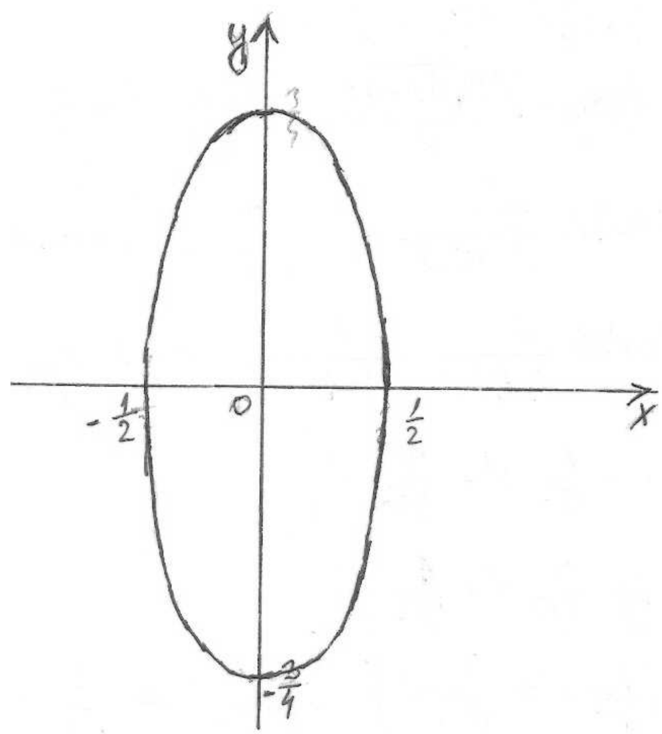
$9\left(x' + \frac{2}{\sqrt{5}}\right)^2 + 4\left(y' - \frac{1}{4\sqrt{5}}\right)^2 = \frac{36}{5} + \frac{4}{16\cdot 5} - 5$

$$9\left(x' + \frac{2}{\sqrt{5}}\right)^2 + 4\left(y' - \frac{1}{4\sqrt{5}}\right)^2 = \frac{45}{20}$$

$$9\left(x' + \frac{2}{\sqrt{5}}\right)^2 + 4\left(y' - \frac{1}{4\sqrt{5}}\right)^2 = \frac{9}{4}$$

$$\frac{\left(x'+\frac{2}{\sqrt{5}}\right)^2}{\frac{1}{4}} + \frac{\left(y'-\frac{1}{4\sqrt{5}}\right)^2}{\frac{9}{16}} = 1$$

$$\frac{\left(x'+\frac{2}{\sqrt{5}}\right)^2}{\left(\frac{1}{2}\right)^2} + \frac{\left(y'-\frac{1}{4\sqrt{5}}\right)^2}{\left(\frac{3}{4}\right)^2} = 1$$



Slika 1.45:

2. Šta predstavlja kriva zadata jednačinom $3x^2 - 10xy + 3y^2 - 16x + 24 = 0$?

Rešenje.

$$\frac{A-C}{2B} = ctg\,2\alpha \Rightarrow \frac{3-3}{-10} = ctg\,2\alpha \Rightarrow ctg\,2\alpha = 0 \Rightarrow 2\alpha = \frac{\pi}{2} \Rightarrow \alpha = 45^\circ$$

$$x = x'\frac{\sqrt{2}}{2} - y'\frac{\sqrt{2}}{2}$$

$y = x'\frac{\sqrt{2}}{2} + y'\frac{\sqrt{2}}{2}$

$3\left(x'\frac{\sqrt{2}}{2} - y'\frac{\sqrt{2}}{2}\right)^2 - 10\left(x'\frac{\sqrt{2}}{2} - y'\frac{\sqrt{2}}{2}\right)\left(x'\frac{\sqrt{2}}{2} + y'\frac{\sqrt{2}}{2}\right) + 3\left(x'\frac{\sqrt{2}}{2} + y'\frac{\sqrt{2}}{2}\right)^2 - 16\left(x'\frac{\sqrt{2}}{2} - y'\frac{\sqrt{2}}{2}\right) + 24 = 0$

$3\left(\frac{1}{2}x'^2 - x'y' + \frac{1}{2}y'^2\right) - 10\left(\frac{1}{2}x'^2 - \frac{1}{2}y'^2\right) + 3\left(\frac{1}{2}x'^2 + x'y' + \frac{1}{2}y'^2\right) - 8\sqrt{2}\,x' + 8\sqrt{2}y' + 24 = 0$

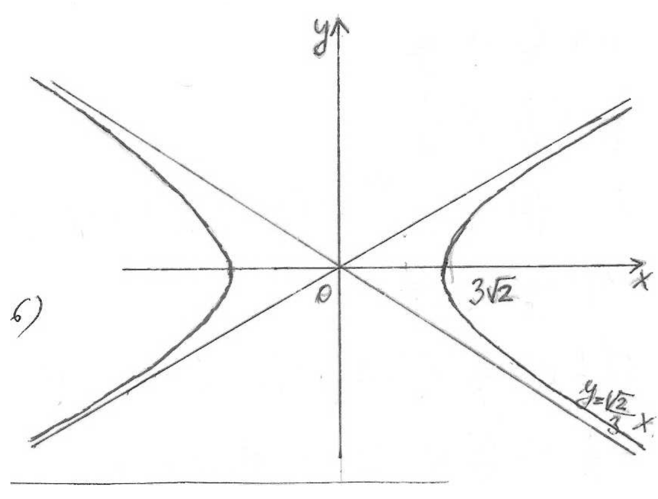
$-2x'^2 + 8y'^2 - 8\sqrt{2}\,x' + 8\sqrt{2}y' + 24 = 0$

$-2\left(x'^2 + 4\sqrt{2}\,x'\right) + 8\left(y'^2 + \frac{2\sqrt{2}}{2}\,y'\right) + 24 = 0$

$-2\left(x' + 2\sqrt{2}\right)^2 + 8\left(y' + \frac{\sqrt{2}}{2}\right)^2 + 24 = -16 + 4$

$-2\left(x' + 2\sqrt{2}\right)^2 + 8\left(y' + \frac{\sqrt{2}}{2}\right)^2 + 24 = -36$

$\frac{\left(x'+2\sqrt{2}\right)^2}{18} - \frac{\left(y'+\frac{\sqrt{2}}{2}\right)^2}{4} = 1$



Slika 1.46:

3. $6xy + 8y^2 - 12x - 26y + 11 = 0.$

$\frac{A-C}{2B} = ctg\,2\alpha \Rightarrow \frac{-8}{6} = ctg\,2\alpha \Rightarrow tg\,2\alpha = -\frac{3}{4}$

$-\frac{3}{4} = \frac{2\,tg\,\alpha}{1-tg^2\alpha}$

$-3 + 3\,tg^2\alpha = 8\,tg\,\alpha$

$3\,tg^2\alpha - 8\,tg\,\alpha - 3 = 0$

$tg\,\alpha_{1/2} = \frac{8\pm\sqrt{64+36}}{6} = \frac{8\pm 10}{6} = \begin{cases} -\frac{1}{3} \\ 3 \end{cases}$

$tg\,\alpha = 3$

$\left.\begin{array}{l} \sin^2\alpha = \frac{tg^2\alpha}{1+tg^2\alpha} = \frac{9}{1+9} = \frac{9}{10} \\ \cos^2\alpha = \frac{1}{1+tg^2\alpha} = \frac{1}{1+9} = \frac{1}{10} \end{array}\right\} \Rightarrow \sin\,\alpha = \frac{3}{\sqrt{10}};\ \cos\,\alpha = \frac{1}{\sqrt{10}}.$

$x = x'\frac{1}{\sqrt{10}} - y'\frac{3}{\sqrt{10}}$

$y = x'\frac{3}{\sqrt{10}} + y'\frac{1}{\sqrt{10}}$

$6\left(x'\frac{1}{\sqrt{10}} - y'\frac{3}{10}\right)\left(x'\frac{3}{\sqrt{10}} + y'\frac{1}{\sqrt{10}}\right) + 8\left(x'\frac{3}{\sqrt{10}} + y'\frac{1}{\sqrt{10}}\right)^2 - 12\left(x'\frac{1}{\sqrt{10}} - y'\frac{3}{10}\right) - 26\left(x'\frac{3}{\sqrt{10}} + y'\frac{1}{\sqrt{10}}\right) + 11 = 0$

$6\left(\frac{3}{10}x'^2 - \frac{8}{10}x'y' - \frac{3}{10}y'^2\right) + 8\left(\frac{9}{10}x'^2 + \frac{6}{10}x'y' + \frac{1}{10}y'^2\right) - \frac{12}{\sqrt{10}}x' + \frac{36}{\sqrt{10}}y' - \frac{78}{\sqrt{10}}x' - \frac{26}{\sqrt{10}}y' + 11 = 0$

$\frac{18}{10}x'^2 - \frac{48}{10}x'y' - \frac{18}{10}y'^2 + \frac{72}{10}x'^2 + \frac{48}{10}x'y' + \frac{8}{10}y'^2 - \frac{12}{\sqrt{10}}x' + \frac{36}{\sqrt{10}}y' - \frac{78}{\sqrt{10}}x' - \frac{26}{\sqrt{10}}y' + 11 = 0$

$9x'^2 - y'^2 - \frac{90}{\sqrt{10}}\,x' + \frac{10}{\sqrt{10}}y' + 11 = 0$

$9\left(x'^2 - \frac{10}{\sqrt{10}}x'\right) - \left(y'^2 - \frac{10}{\sqrt{10}}y'\right) + 11 = 0$

$9\left(x' - \frac{5}{\sqrt{10}}\right)^2 - \left(y' - \frac{5}{\sqrt{10}}\right)^2 + 11 = 20$

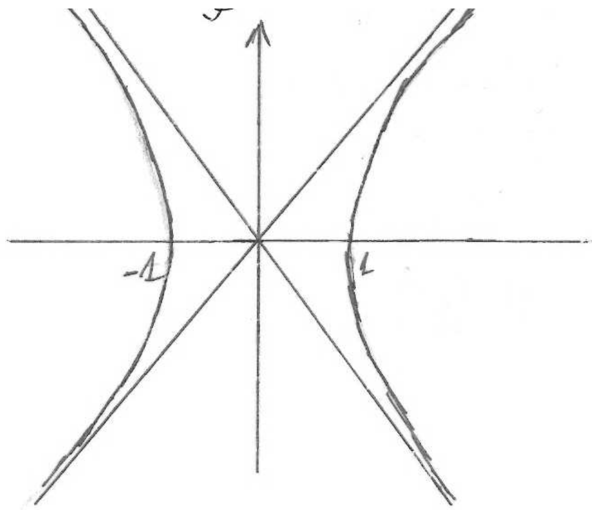
$$9\left(x'-\frac{5}{\sqrt{10}}\right)^2-\left(y'-\frac{5}{\sqrt{10}}\right)^2=9$$

$$\frac{\left(x'-\frac{5}{\sqrt{10}}\right)^2}{1}-\frac{\left(y'-\frac{5}{\sqrt{10}}\right)^2}{9}=1$$

$$\frac{\left(x'-\frac{5}{\sqrt{10}}\cdot\frac{10}{\sqrt{10}}\right)^2}{1}-\frac{\left(y'-\frac{5}{\sqrt{10}}\cdot\frac{10}{\sqrt{10}}\right)^2}{9}=1$$

$$\frac{\left(x'-\frac{\sqrt{10}}{2}\right)^2}{1}-\frac{\left(y'-\frac{\sqrt{10}}{2}\right)^2}{9}=1$$

$$X^2-\frac{Y^2}{9}=1$$



Slika 1.47:

4. $4x^2-4xy+y^2+4x-2y+1=0.$
$\frac{A-C}{2B}=ctg\,2\alpha \Rightarrow \frac{4-1}{-4}=ctg\,2\alpha \Rightarrow tg\,2\alpha=-\frac{4}{3};$

$$tg\,2\alpha=\frac{2\,tg\,\alpha}{1-tg^2\alpha}$$

$$-\frac{4}{3}=\frac{2\,tg\,\alpha}{1-tg^2\alpha}$$

$$-4+4\,tg^2\alpha=6\,tg\,\alpha$$

$$2\,tg^2\alpha-3\,tg\,\alpha-2=0$$

$$tg\,\alpha_{1/2} = \frac{3\pm\sqrt{9+16}}{4} = \frac{3\pm5}{4} = \begin{cases} -\frac{1}{2} \\ 2 \end{cases}$$

$$\left.\begin{array}{l} \sin^2\alpha = \frac{tg^2\alpha}{1+tg^2\alpha} = \frac{4}{1+4} = \frac{4}{5} \\ \cos^2\alpha = \frac{1}{1+tg^2\alpha} = \frac{1}{1+4} = \frac{1}{5} \end{array}\right\} \Rightarrow \sin\,\alpha = \frac{2}{\sqrt{5}};\ \cos\,\alpha = \frac{1}{\sqrt{5}}.$$

$$x = x'\frac{1}{\sqrt{5}} - y'\frac{2}{\sqrt{5}}$$

$$y = x'\frac{2}{\sqrt{5}} + y'\frac{1}{\sqrt{5}}$$

$$4\cdot\frac{1}{5}\left(x' - 2y'\right)^2 - 4\cdot\frac{1}{5}\left(x' - 2y'\right)\left(2x' + y'\right) + \frac{1}{5}\left(2x' + y'\right)^2 + 4\cdot\frac{1}{\sqrt{5}}\left(x' - 2y'\right) - 2\cdot\frac{1}{\sqrt{5}}\left(2x' + y'\right) + 1 = 0$$

$$\frac{4}{5}\left(x'^2 - 4x'y' + 4y'^2\right) - \frac{4}{5}\left(2x'^2 + x'y' - 4x'y' - 2y'^2\right) + \frac{1}{5}\left(4x'^2 + 4x'y' + y'^2\right) + \frac{4}{\sqrt{5}}x' - \frac{8}{\sqrt{5}}y' - \frac{4}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y' + 1 = 0$$

$$\frac{4}{5}x'^2 - \frac{16}{5}x'y' + \frac{16}{5}y'^2 - \frac{8}{5}x'^2 + \frac{12}{5}x'y' + \frac{8}{5}y'^2 + \frac{4}{5}x'^2 + \frac{4}{5}x'y' + \frac{1}{5}y'^2 + \frac{4}{\sqrt{5}}x' - \frac{8}{\sqrt{5}}y' - \frac{4}{\sqrt{5}}x' - \frac{2}{\sqrt{5}}y' + 1 = 0$$

$$5y'^2 - \frac{10}{\sqrt{5}}y' + 1 = 0$$

$$5y'^2 - \frac{10}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}}y' + 1 = 0$$

$$5y'^2 - 2\sqrt{5}y' + 1 = 0$$

$$\left(5y' - 1\right)^2 = 0$$

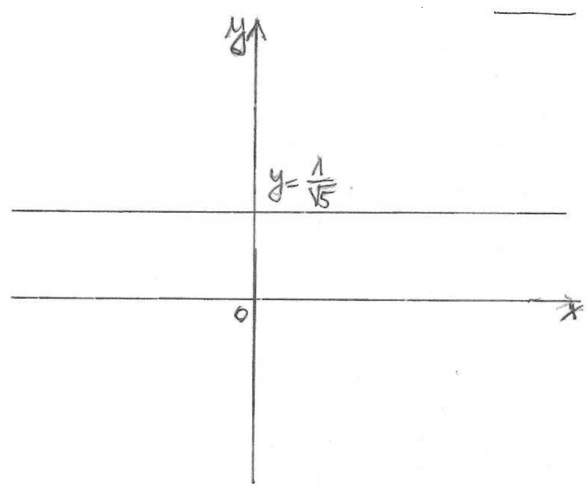
$$5y' - 1 = 0$$

$$y' = \frac{1}{\sqrt{5}}$$

5. $x^2 - 2xy + y^2 - 10x - 6y + 25 = 0.$

$$\frac{A-C}{2B} = ctg\,2\alpha \Rightarrow \frac{1-1}{-2} = ctg\,2\alpha \Rightarrow ctg\,2\alpha = 0 \Rightarrow 2\alpha = \frac{\pi}{2} \Rightarrow \alpha = 45^\circ$$

$$x = x'\frac{\sqrt{2}}{2} - y'\frac{\sqrt{2}}{2}$$



Slika 1.48:

$y = x'\frac{\sqrt{2}}{2} + y'\frac{\sqrt{2}}{2}$

$\frac{2}{4}\left(x'-y'\right)^2 - 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\left(x'-y'\right)\left(x'+y'\right) + \frac{2}{4}\left(x'+y'\right)^2 - 10\cdot\frac{\sqrt{2}}{2}\left(x'-y'\right) - 6\cdot\frac{\sqrt{2}}{2}\left(x'+y'\right) + 25 = 0$

$\frac{1}{2}\left(x'^2 - 2x'y' + y'^2\right) - \left(x'^2 - y'^2\right) + \frac{1}{2}\left(x'^2 + 2x'y' + y'^2\right) - 5\sqrt{2}\left(x'-y'\right) - 3\sqrt{2}\left(x'+y'\right) + 25 = 0$

$\frac{1}{2}x'^2 - x'y' + \frac{1}{2}y'^2 - x'^2 + y'^2 + \frac{1}{2}x'^2 + x'y' + \frac{1}{2}y'^2 - 5\sqrt{2}x' + 5\sqrt{2}y' - 3\sqrt{2}x' - 3\sqrt{2}y' + 25 = 0$

$2y'^2 - 8\sqrt{2}x' + 2\sqrt{2}y' + 25 = 0$

$2\left(y'^2 + \sqrt{2}y'\right) - 8\sqrt{2}x' + 25 = 0$

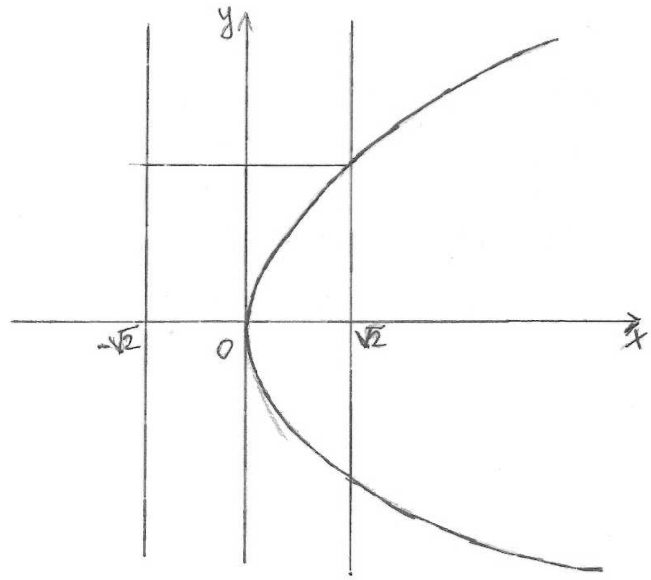
$2\left(y'^2 + \frac{2}{\sqrt{2}}y'\right) - 8\sqrt{2}x' + 25 = 0$

$2\left(y' + \frac{1}{\sqrt{2}}\right)^2 - 8\sqrt{2}x' + 25 = 1$

$\left(y' + \frac{1}{\sqrt{2}}\right)^2 = 4\sqrt{2}x' - 12$

$$\left(y' + \frac{1}{\sqrt{2}}\right)^2 = 4\sqrt{2}\left(x' - \frac{3}{\sqrt{2}}\right)$$

$$Y^2 = 4\sqrt{2}X$$



Slika 1.49:

6. $34x^2 + 24xy + 41y^2 - 25 = 0.$

$$\frac{A-C}{2B} = ctg\,2\alpha \Rightarrow \frac{34-41}{24} = ctg\,2\alpha \Rightarrow tg\,2\alpha = -\frac{24}{7};$$

$$tg\,2\alpha = \frac{2\,tg\,\alpha}{1-tg^2\alpha}$$

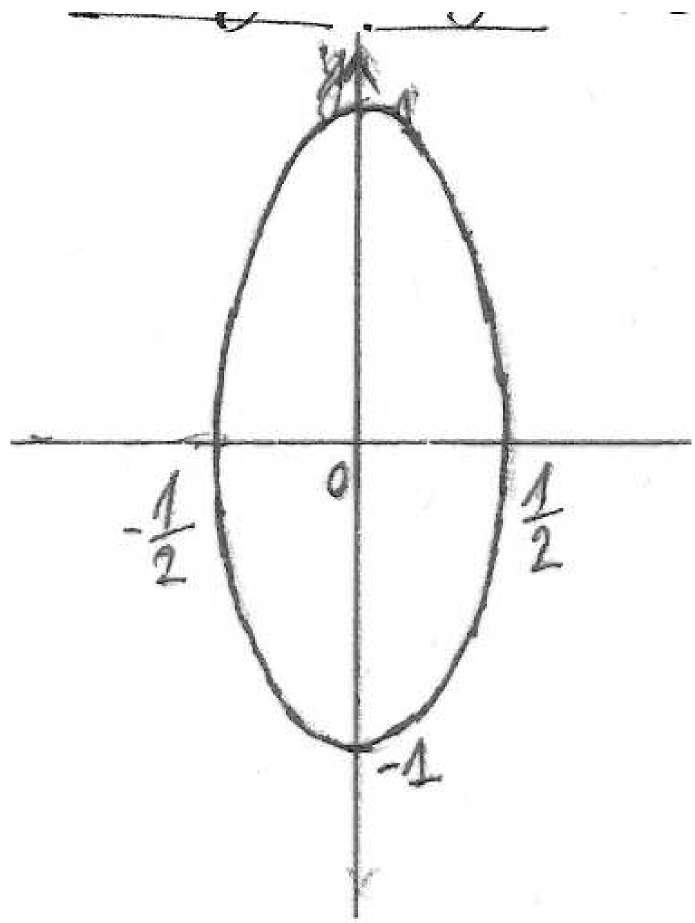
$$-\frac{24}{7} = \frac{2\,tg\,\alpha}{1-tg^2\alpha}$$

$$-24 + 24\,tg^2\alpha = 14\,tg\,\alpha$$

$$12\,tg^2\alpha - 7\,tg\,\alpha - 12 = 0$$

$$tg\,\alpha_{1/2} = \frac{7 \pm \sqrt{49+576}}{24} = \frac{7 \pm 25}{24} = \begin{cases} -\frac{3}{4} \\ \frac{4}{3} \end{cases}$$

$$\left.\begin{array}{l} \sin^2\alpha = \frac{tg^2\alpha}{1+tg^2\alpha} = \frac{\frac{16}{9}}{1+\frac{16}{9}} = \frac{\frac{16}{9}}{\frac{25}{9}} \\ \cos^2\alpha = \frac{1}{1+tg^2\alpha} = \frac{1}{1+\frac{16}{9}} = \frac{1}{\frac{25}{9}} = \frac{9}{25} \end{array}\right\} \Rightarrow \sin\,\alpha = \frac{4}{5};\ \cos\,\alpha = \frac{3}{5}.$$



Slika 1.50:

$x = x'\frac{3}{5} - y'\frac{4}{5}$

$y = x'\frac{4}{5} + y'\frac{3}{5}$

$34\cdot\frac{1}{25}\left(3x' - 4y'\right)^2 + 24\cdot\frac{1}{25}\left(3x' - 4y'\right)\left(4x' + 3y'\right) + 41\cdot\frac{1}{25}\left(4x' + 3y'\right)^2 - 25 = 0$

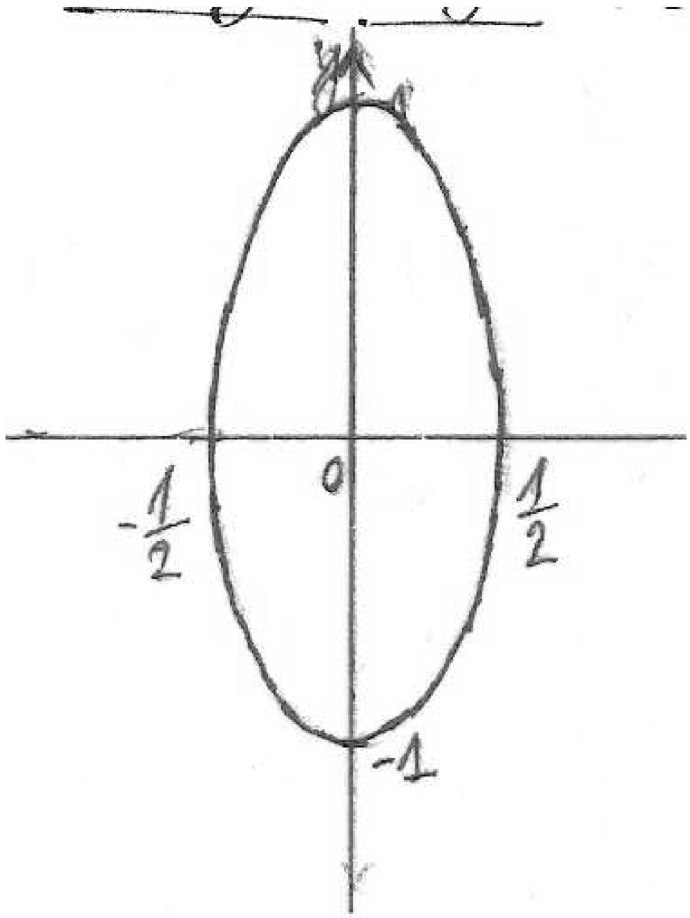
$\frac{34}{25}\left(9x'^2 - 24x'y' + 16y'^2\right) + \frac{24}{25}\left(12x'^2 + 9x'y' - 16x'y' - 12y'^2\right) + \frac{41}{25}\left(16x'^2 + 24x'y' + 9y\right.$
$25 = 0$

$306x'^2 - 816x'y' + 544y'^2 + 288x'^2 - 168x'y' - 288y'^2 + 656x'^2 + 984x'y' + 369y'^2 - 625 = 0$

$1250x'^2 + 625y'^2 - 625 = 0$

$2x'^2 + y'^2 = 1$

$\frac{x'^2}{\frac{1}{2}} + y'^2 = 1$



Slika 1.51:

1.28 Ravan u prostoru

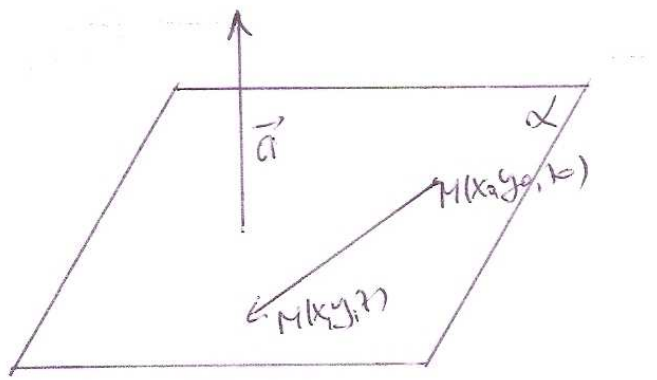
$\vec{a} = (A,\, B,\, C)$-vektor položaja ravni α

$\alpha \perp \vec{a}$

$A \in \alpha \;\Leftrightarrow\; \overrightarrow{MA} \perp \vec{a} \;\Leftrightarrow\; \overrightarrow{MA} \cdot \vec{a} = 0$

$\Leftrightarrow\; (x - x_0,\, y - y_0,\, z - z_0)\,(A,\, B,\, C) = 0$

$\Leftrightarrow\; A\,(x - x_0) + B\,(y - y_0) + C\,(z - z_0) = 0$



Slika 1.52:

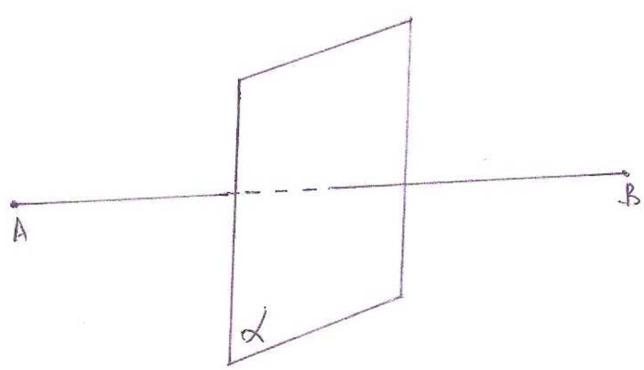
$A\,(x-x_0)+B\,(y-y_0)+C\,(z-z_0)=0$ - jednačina ravni kroz tačku $(x_0,\,y_0,\,z_0)$, normalna na $\vec{a}$.

$Ax+By+Cz+D=0$

$D=-Ax_0-By_0-Cz_0$

$\frac{x}{m}+\frac{y}{n}+\frac{z}{k}=1$ - segmentni oblik jednačine ravni

1. Odrediti jednačinu ravni koja je simetralna ravan za duž AB : $A\,(2,\,3,\,-1)\,;\,B\,(0,\,5,\,3)\,.$



Slika 1.53:

$C=\frac{A+B}{2}$

$C\,(1,\,4,\,1)$

$\overrightarrow{AB}\perp\alpha$

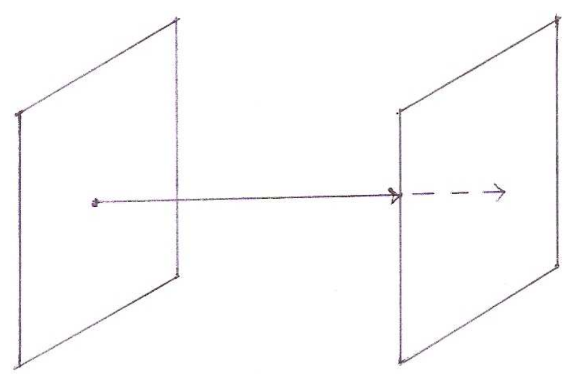
$\overrightarrow{AB}=(-2,\,2,\,4)$

$\alpha:\;-2\,(x-1)+2\,(y-4)+4\,(z-1)=0$

$\alpha:\;-x+y+2z-5=0/\cdot(-1)$

$\alpha:\;x-y-2z+5=0$

2. Odrediti parametar a tako da ravan $2x+y+az+5=0$ bude:



Slika 1.54:

a) paralelan sa ravni $4x+2y+z+10=0$

b) normalna na ravni $4x+2y+z+10=0$

Rešenje.

a. $(2,\,1,\,a)\parallel(4,\,2,\,1)$

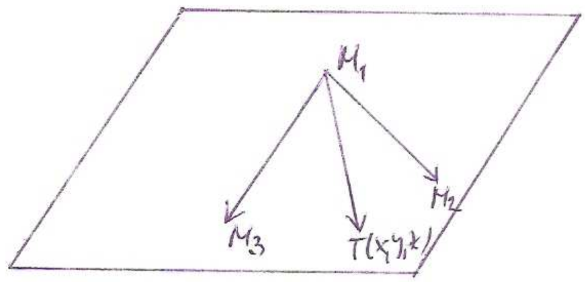
$\frac{2}{4}=\frac{1}{2}=\frac{a}{1}\;\Rightarrow\;a=\frac{1}{2}$

b. $(2,\,1,\,a)\perp(4,\,2,\,1)$

$(2,\,1,\,a)\cdot(4,\,2,\,1)=0$

$8 + 2 + a = 0 \Rightarrow a = -10$

3. Odrediti jednačinu ravni koja prolazi kroz tačke $M_1(1, 2, 1)$, $M_2(5, 7, 3)$, $M_3(6, 4, 5)$.



Slika 1.55:

I način:

$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

$\overrightarrow{M_1M_2} = (4, 5, 2)$

$\overrightarrow{M_1M_3} = (5, 2, 4)$

$$\overrightarrow{M_1M_2} \times \overrightarrow{M_1M_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 5 & 2 \\ 5 & 2 & 4 \end{vmatrix} = (16, -6, -17)$$

$M_1(1, 2, 1)$

$16(x - 1) - 6(y - 2) - 17(z - 1) = 0$

II način:

$\alpha : 16x - 6y - 17z + 13 = 0$

$\overrightarrow{M_1M_2} = (4, 5, 2)$

$\overrightarrow{M_1M_3} = (5,\ 2,\ 4)$

$\overrightarrow{M_1T} = (x-1,\ y-2,\ z-1)$

$\left(\overrightarrow{M_1T} \times \overrightarrow{M_1M_3}\right) \cdot \overrightarrow{M_1M_3} = \begin{vmatrix} x-1 & y-2 & z-1 \\ 4 & 5 & 2 \\ 5 & 2 & 4 \end{vmatrix} = 16\,(x-1) -$
$6\,(y-2) - 17\,(z-1) = 0$

$\alpha:\ 16x - 6y - 17z + 13 = 0$

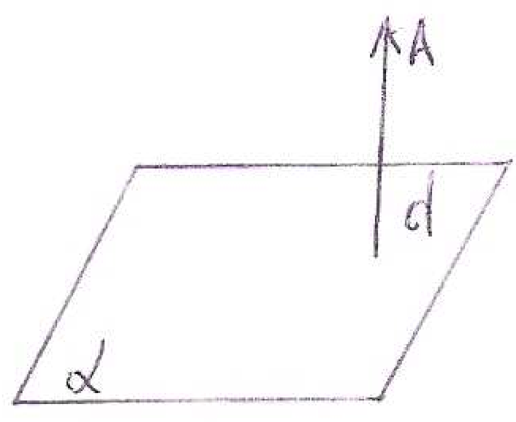
4. Odrediti ugao koji obrazuju ravni $x-2y+2z-8=0$ i $x+z-6=0$

$\cos\ \alpha = \frac{|(1,-2,2)\cdot(1,0,1)|}{\sqrt{1^2+(-2)^2+2^2}\cdot\sqrt{1^2+1^2}}$

$\cos\ \alpha = \frac{3}{3\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\alpha = 45^\circ$

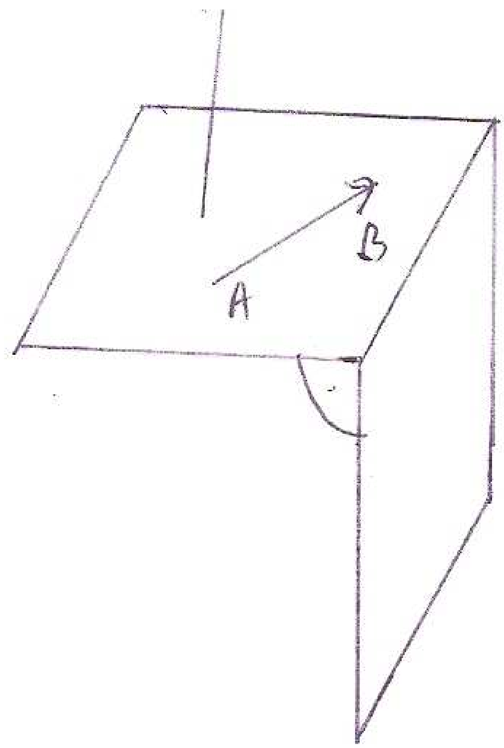
5. Odrediti odstojanje tačke $(2,\ 1,\ 3)$ od ravni $2x+3y-z+1=0$.



Slika 1.56:

$d = \frac{|2\cdot2+3\cdot1-1\cdot3|}{\sqrt{2^2+3^2+(-1)^2}} = \frac{4}{\sqrt{14}}$

6. Odrediti jednačinu ravni koja prolazi kroz tačke $A(2,\ 3,\ -1)$, $B(1,\ 5,\ 3)$, i normalna je na ravan $3x - y + 3z + 15 = 0$.



Slika 1.57:

$\vec{a}$-vektor položaja tražene ravni

$\overrightarrow{AB} = (-1, 2, 4)$

$\vec{a} \perp \overrightarrow{AB}$

$$\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 4 \\ 3 & -1 & 3 \end{vmatrix} = (10,\ -15,\ -5) \parallel (2,\ 3,\ -1)$$

$2(x-2) + 3(y-3) - 1(z+1) = 0$

$2x + 3y - z - 14 = 0$

7. Odrediti jednačinu ravni koja pripada pramenu određenom ran-

ima $\alpha_1 : x - y + 2z + 3 = 0$ i $\alpha_2 : 4x + y + z + 1 = 0$, a koja je normalna na ravan $\alpha_3 : 2x - 2y + 3z + 2 = 0$.

$$x - y + 2z + 3 + \lambda\,(4x + y + z + 1) = 0$$

$$(1 + 4\lambda)\,x + (\lambda - 1)\,y + (2 + \lambda)\,z + \lambda = 0$$

$$(1 + 4\lambda,\ \lambda - 1,\ 2 + \lambda) \perp (2,\ -2,\ 3)$$

$$(1 + 4\lambda,\ \lambda - 1,\ 2 + \lambda) \cdot (2,\ -2,\ 3) = 0$$

$$2 + 8\lambda - 2\lambda + 2 + 6 + 3\lambda = 0$$

$$10 + 9\lambda = 0$$

$$\lambda = -\tfrac{10}{9}$$

$$\alpha :\ x - y + 2z + 3 - \tfrac{10}{9}\,(4x + y + z + 1) = 0$$

$$\alpha :\ 9x - 9y + 18z + 27 - 40x - 10y - 10z - 10 = 0$$

$$\alpha :\ -31x - 19y + 8z + 17 = 0$$

$$\alpha :\ 31x + 19y - 8z - 17 = 0$$

8. Odrediti jednačinu ravni koja pripada pramenu ravni određenog ravnima $x + 3y - 5 = 0$ i $x - y - 2z + 4 = 0$ i koja sa ravni $3x + y + z + 1 = 0$ gradi ugao $\cos\,\alpha = \frac{1}{3}$.

$$x + 3y - 5 + \lambda\,(x - y - 2z + 4) = 0$$

$$(1 + \lambda)\,x + (3 - \lambda)\,y + (-2\lambda)\,z + (4\lambda - 5) = 0$$

$$\cos\,\alpha = \frac{|(1+\lambda, 3-\lambda, -2\lambda)\cdot(1,1,1)|}{\sqrt{(1+\lambda)^2 + (3-\lambda)^2 + (-2\lambda)^2}\cdot\sqrt{1^2+1^2+1^2}}$$

$$\cos\,\alpha = \frac{|4-2\lambda|}{\sqrt{6\lambda^2 - 4\lambda + 10}\cdot\sqrt{3}}$$

$\frac{|4-2\lambda|}{\sqrt{6\lambda^2-4\lambda+10}\cdot\sqrt{3}} = \frac{1}{3}/^2$

$9\left(16-16\lambda+4\lambda^2\right) = 3\left(6\lambda^2-4\lambda+10\right)$

$48-48\lambda+12\lambda^2 = 6\lambda^2-4\lambda+10$

$6\lambda^2-44\lambda+38 = 0/:2$

$3\lambda^2-22\lambda+19 = 0$

$\lambda_{1/2} = \begin{cases} 1 \\ \frac{19}{3} \end{cases}$

$\lambda = 1 \Rightarrow x+3y-5+x-y-2z+4 = 0$

$2x+2y-2z-1 = 0$

$\lambda = \frac{19}{3} \Rightarrow x+3y-5+\frac{19}{3}(x-y-2z+4) = 0$

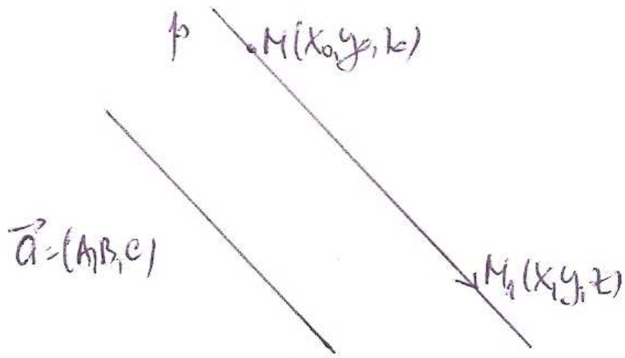
$3x+9y-15+19x-19y-38z+76 = 0$

$22x-10y-38z+61 = 0$

1.29 Prava u prostoru

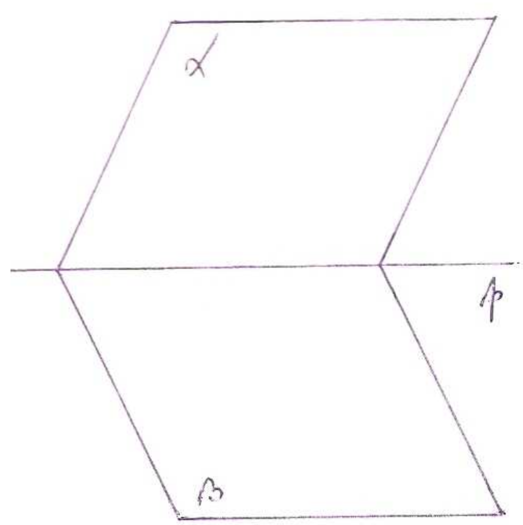
$\vec{a} = (A, B, C)$
$M_1 \in p \Leftrightarrow \overrightarrow{MM_1} \parallel \vec{a}$
$\Leftrightarrow (x-x_0,\ y-y_0,\ z-z_0) \parallel (A,\ B,\ C)$
$\Leftrightarrow \frac{x-x_0}{A} = \frac{y-y_0}{B} = \frac{z-z_0}{C}$
$\Leftrightarrow \frac{x-x_0}{A} = \frac{y-y_0}{B} = \frac{z-z_0}{C} = t$ - kanonska formula jednačine prave

$\begin{cases} x = At + x_0 \\ y = Bt + y_0 \\ z = Ct + z_0 \end{cases}$ - parametarske jednačine prave



Slika 1.58:

1. Odrediti jednačinu prave koja je presek ravni $\begin{cases} \alpha : \ 2x - 3y + z - 5 = 0 \\ \beta : \ 3x + y - 2z + 4 = 0 \end{cases}$



Slika 1.59:

I način: $\begin{cases} 2x - 3y + z - 5 = 0 \\ 3x + y - 2z + 4 = 0 \end{cases}$

$$\begin{cases} y = 0 \\ 2x + z - 5 = 0 \\ 3x - 2z + 4 = 0 \end{cases}$$

$$\begin{cases} y = 0 \\ 2x + z - 5 = 0 \\ 7x - 14 = 0 \end{cases}$$

$$\begin{cases} y = 0 \\ 2x + z - 5 = 0 \\ x = 2 \end{cases}$$

$$\begin{cases} x = 2 \\ z = 1 \end{cases}$$

$A\,(2,\ 0,\ 1)$

$$\begin{cases} x = 0 \\ -3y + z - 5 = 0 \\ y - 2z - 4 = 0 \end{cases}$$

$$\begin{cases} x = 0 \\ -3y + z - 5 = 0 \\ -5y - 14 = 0 \end{cases}$$

$$\begin{cases} x = 0 \\ -3y + z - 5 = 0 \\ y = -\frac{14}{5} \end{cases}$$

$$\begin{cases} y = -\frac{14}{5} \\ z = -\frac{17}{5} \end{cases}$$

$B\left(0, -\frac{14}{5}, -\frac{17}{5}\right)$

$A,\ B \in p$

$\overrightarrow{AB} = \left(-2,\ -\frac{14}{5},\ -\frac{22}{5}\right) \parallel (5,\ 7,\ 11)$

$\frac{x-2}{5} = \frac{y-0}{7} = \frac{z-1}{11}$

II način: Vektor $(A,\ B,\ C)$ može da se dobije kao vektorski proizvod vektora položaja datih ravni:

$$(2,\,-3,\,1)\times(3,\,1,\,-2)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -3 & 1\\ 3 & 1 & -2\end{vmatrix}=(5,\,7,\,11)$$

2. Odrediti parametar a, tako da su prave $\begin{cases} p_1: \frac{x-1}{2}=\frac{y+2}{6}=\frac{z}{a}\\ p_2: \frac{x}{1}=\frac{y+1}{3}=\frac{z+2}{-1}\end{cases}$

a) normalne; b) paralelne.

Rešenje.

a) $(1,\,3,\,-1)\cdot(2,\,6,\,a)=0$

$2+18-a=0$

$a=20$

b) $\frac{2}{1}=\frac{6}{3}=\frac{a}{-1}\;\Rightarrow\; a=-2$

3. Odrediti tačku koja je simetrična tački $M\,(4,\,13,\,10)$ u odnosu na pravu $\frac{x-1}{2}=\frac{y-2}{4}=\frac{z-3}{5}$

$\alpha\perp p\bigwedge M\in\alpha$

$(2,\,4,\,5)\perp\alpha$

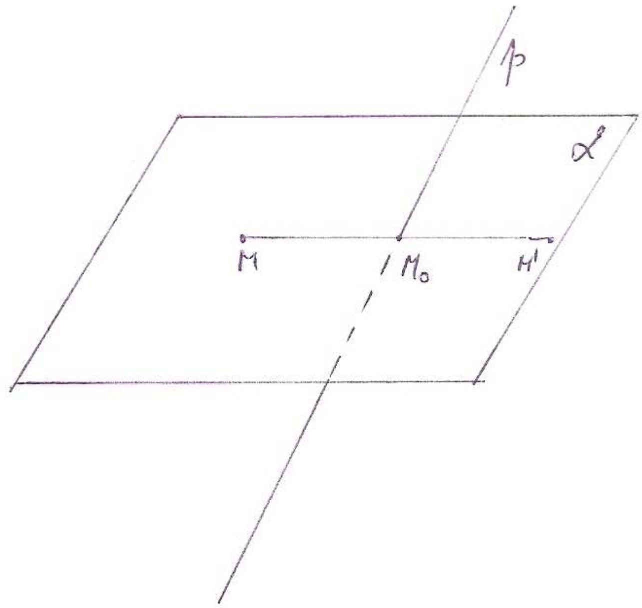
$\alpha:\; A\,(x-x_0)+B\,(y-y_0)+C\,(z-z_0)=0$

$2\,(x-4)+4\,(y-13)+5\,(z-10)=0$

$2x-4y+5z-110=0$

$\frac{x-1}{2}=\frac{y-2}{4}=\frac{z-3}{5}=t$

$$\begin{cases} x=2t+1\\ y=4t+2\\ z=5t+3\end{cases}$$



Slika 1.60:

$2\,(2t+1)+4\,(4t+2)+5\,(5t+3)-110=0$

$45t-85=0$

$t=\frac{85}{45}=\frac{17}{9}$

$x=2\cdot\frac{17}{9}+1=\frac{34}{9}+\frac{9}{9}=\frac{43}{9}$

$y=\frac{86}{9};\ z=\frac{112}{9}$

$\Rightarrow\ M_0\left(\frac{43}{9},\ \frac{86}{9},\ \frac{112}{9}\right)$

$M_0=\frac{M+M'}{2}$

$M'=2M_0-M$

$M'=\left(\frac{86}{9},\ \frac{172}{9},\ \frac{224}{9}\right)-\left(\frac{36}{9},\ \frac{117}{9},\ \frac{90}{9}\right)=\left(\frac{50}{9},\ \frac{55}{9},\ \frac{134}{9}\right)$

4. Odrediti tačku simetričnu tački $A\,(4,\ 3,\ 10)$ u odnosu na pravu $\frac{x-1}{2}=\frac{y-2}{4}=\frac{z-3}{5}$, a zatim i u odnosu na ravan α određenu tačkama $A\,(1,\ 10,\ 5)\,,\ B\,(10,\ 3,\ 4)\,,\ C\,(7,\ 2,\ 6)\,.$

$\alpha: \ 2\left(x-4\right)+4\left(y-3\right)+5\left(z-10\right)=0$

$$\begin{cases} 2x+4y+5z-70=0 \\ \frac{x-1}{2}=\frac{y-2}{4}=\frac{z-3}{5}=t \end{cases}$$

$$\begin{cases} x=2t+1 \\ y=4t+2 \\ z=5t+3 \end{cases}$$

$2\left(2t+1-4\right)+4\left(4t+2-3\right)+5\left(5t+3-10\right)=0$

$45t-45=0$

$t=1$

$$\begin{cases} x=3 \\ y=6 \\ z=8 \end{cases}$$

$A_1\left(3,\ 6,\ 8\right)$

$A_1=\frac{A+A'}{2} \ \Rightarrow \ A'=2A_1-A$

$A'=2\left(3,6,8\right)-\left(4,3,10\right)=\left(2,9,6\right)$

II deo zadatka

$A\left(1,\ 10,\ 5\right)$

$B\left(10,\ 3,\ 4\right)$

$C\left(7,\ 2,\ 6\right)$

$\overrightarrow{AB}=\left(9,\ -7,\ -1\right)$

$\overrightarrow{AC}=\left(6,\ -8,\ 1\right)$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 9 & -7 & 1 \\ 6 & -8 & 1 \end{vmatrix} = -15\vec{i} - 15\vec{j} - 30\vec{k} \parallel (1,\, 1,\, 2)$$

$$\alpha: \ (x-1) + (y-10) + 2\,(z-5) = 0$$

$$x + y + 2z - 21 = 0$$

$$p \perp \alpha \ \wedge \ A \in p$$

$$\tfrac{x-4}{1} = \tfrac{y-3}{1} = \tfrac{z-10}{2} = t$$

$$\begin{cases} x = t + 4 \\ y = t + 3 \\ z = 2t + 10 \end{cases}$$

$$(t + 4 - 1) + (t + 3 - 10) + 2\,(2t + 10 - 5) = 0$$

$$6t + 6 = 0$$

$$t = -1$$

$$\begin{cases} x = 3 \\ y = 2 \\ z = 8 \end{cases}$$

$$A_1\,(3,\, 2,\, 8)$$

$$2A' = A + A_1$$

$$A_1 = 2A' - A$$

$$A_1 = 2\,(3,\, 2,\, 8) - (4,\, 3,\, 10)$$

$$A_1 = (2,\, 1,\, 6)\,.$$

Domaći: 1. Odrediti tačku koja je simetrična tački $(2,\, 7,\, 1)$ u odnosu na ravan $x - 4y + z + 7 = 0$.

(Rešenje: $(4, -1, 3)$)

2. Naći prodornu tačku prave $\frac{x}{2} = \frac{y-1}{1} = \frac{z+1}{2}$ i ravni $x + 2y + 3z - 29 = 0$.

(Rešenje: $(6, 4, 5)$)

3. Odrediti jednačinu ravni koja sadrži pravu $\frac{x-2}{1} = \frac{y-3}{2} = \frac{z+1}{3}$ i tačku $(3, 4, 0)$.

(Rešenje: $x - 2y + z - 5 = 0$)

4. Naći jednačinu projekcije prave $\frac{x}{4} = \frac{y-4}{3} = \frac{z+1}{-2}$ na ravan $x - y + 3z + 8 = 0$.

(Rešenje: $\frac{x+9}{7} = \frac{y+1}{4} = \frac{z}{-1}$)

5. Odrediti λ tako da se prave $p: \frac{x-2}{3} = \frac{y+4}{5} = \frac{z-1}{-2}$ i $q: \frac{x-\lambda}{2} = \frac{y-3}{1} = \frac{z+5}{0}$ seku i odrediti presečnu tačku.

$\frac{x-\lambda}{2} = \frac{y-3}{1} = \frac{z+5}{0} = t$

$$\begin{cases} x = 2t + \lambda \\ y = t + 3 \\ z = -5 \end{cases}$$

$A(2, -4, 1)$

$B(\lambda, 3, -5)$

$\overrightarrow{AB} = (\lambda - 2, 7, -6)$

$$\begin{vmatrix} \lambda - 2 & 7 & -6 \\ 3 & 5 & -2 \\ 2 & 1 & 0 \end{vmatrix} = 0 \Leftrightarrow \lambda = -5$$

Kada se reši sistem $\left.\begin{array}{l}\frac{x-2}{3}=\frac{y+4}{5}=\frac{z-1}{-2}\\ \frac{x+5}{2}=\frac{y-3}{1}=\frac{z+5}{0}\end{array}\right\}$

$z=-5$

$\frac{x-2}{3}=\frac{y+4}{5}=3$

$\left\{\begin{array}{l}x=11\\ y=11\end{array}\right.$

$(11,\,11,\,-5)$

6. Odrediti rastojanje između mimoilaznih pravih:

$p:\ \frac{x-9}{4}=\frac{y+2}{-3}=\frac{z}{1}$

$q:\ \frac{x}{-2}=\frac{y+7}{9}=\frac{z-2}{2}$

$$\left|\begin{array}{ccc}4 & -3 & 1\\ -2 & 9 & 2\\ 9 & 5 & -2\end{array}\right|\neq 0$$

$$\left|\begin{array}{ccc}\vec{i} & \vec{j} & \vec{k}\\ 4 & -3 & 1\\ -2 & 9 & 2\end{array}\right|=(-15,\,-10,\,30)\parallel(-3,\,-2,\,6)$$

$-3\,(x-0)-2\,(y+7)+6\,(z-2)=0$

$-3x-2y+6z-26=0$

$d=\frac{|-3\cdot 9-2\cdot(-2)+6\cdot 0|}{\sqrt{9+4+36}}=\frac{23}{\sqrt{49}}=\frac{23}{7}$

7.Odrediti jednačinu prave koja je zajednička normala mimoilaznih pravih

$p:\ \frac{x-9}{4}=\frac{y+2}{-3}=\frac{z}{1}$

$q: \frac{x}{-2} = \frac{y+7}{9} = \frac{z-2}{2}$

Domaći.

$p: \frac{x-4}{1} = \frac{y+3}{2} = \frac{z-12}{-1}$

$q: \frac{x-3}{-7} = \frac{y-1}{2} = \frac{z-1}{3}$

Rešenje.

$p: \frac{x-7}{4} = \frac{y-3}{2} = \frac{z-9}{8}$

1.30 Sfera

$C(a, b, c)$ - centar sfere r - poluprečnik sfere

$M(x, y, z) \in S$

$S: (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ - kanonski oblik jednačine sfere

$P: Ax + By + Cz + D = 0$

$d = \frac{|A \cdot a + B \cdot b + C \cdot c + D|}{\sqrt{A^2+B^2+C^2}}$ - odstojanje centra sfere od ravni P

1. Sastaviti jednačinu sfere opisane oko tetraedra čija su temena $O(0, 0, 0), A(2, 0, 0), B(0, 5, 0), C(0, 0, 3)$.

Rešenje.

$S: (x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

$x^2 + y^2 + z^2 - 2ax - 2by - 2cz + a^2 + b^2 + c^2 - r^2 = 0$

$O \in S \Rightarrow a^2 + b^2 + c^2 - r^2 = 0$

$A \in S \Rightarrow 4 - 4a + a^2 + b^2 + c^2 - r^2 = 0$

$B \in S \Rightarrow 25 - 10b + a^2 + b^2 + c^2 - r^2 = 0$

$C \in S \ \Rightarrow \ 9 - 6c + a^2 + b^2 + c^2 - r^2 = 0$

$4 - 4a = 0 \ \Rightarrow \ a = 1$

$25 - 10b = 0 \ \Rightarrow \ b = \frac{5}{2}$

$9 - 6c = 0 \ \Rightarrow \ c = \frac{3}{2}$

$r^2 = 1 + \frac{25}{4} + \frac{9}{4} = \frac{38}{4}$

$S: \ (x-1)^2 + \left(y - \frac{5}{2}\right)^2 + \left(z - \frac{3}{2}\right)^2 = \frac{38}{4} / \cdot 4$

$S: \ 4x^2 - 8x + 4 + 4y^2 - 20y + 25 + 4z^2 - 12z + 9 = 38$

$S: \ 4x^2 + 4y^2 + 4z^2 - 8x - 20y - 12z = 0 / : 4$

$S: \ x^2 + y^2 + z^2 - 2x - 5y - 3z = 0$

2. Napisati jednačinu sferne površi čiji je centar u tački $C\,(1,\ 4,\ -7)$ i dodiruje ravan $6x + 6y - 7z + 42 = 0$.

Rešenje.

$S: \ (x-1)^2 + (y-4)^2 + (z+7)^2 = r^2$

$\alpha: \ 6x + 6y - 7z + 42 = 0$

U središtu sfere C postavimo pravu p normalnu na ravan α. Zatim odredimo prodor prave p kroz ravan α.

$p: \ \frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$

$\vec{p} = \lambda\,(6,\ 6,\ -7)$

$c \in p \ \wedge \ p \perp \alpha \ \Rightarrow \ \frac{x-1}{6} = \frac{y-4}{6} = \frac{z+7}{-7} = t \ \Rightarrow$

$$\Rightarrow \begin{cases} x = 6t + 1 \\ y = 6t + 4 \\ z = -7t - 7 \end{cases}$$

$$6\left(6t+1\right)+6\left(6t+4\right)-7\left(-7t-7\right)+42=0$$

$$36t+6+36t+24+49t+49+42=0$$

$$121t+121=0$$

$$t=-1$$

$$\begin{cases} x = -5 \\ y = -2 \\ z = 0 \end{cases}$$

$$M = p \cap \alpha$$

$$M = \left(-5,\, -2,\, 0\right)$$

$$r = \overline{CM} = \sqrt{\left(-5-1\right)^2+\left(-2-4\right)^2+\left(0+7\right)^2} = \sqrt{121} = 11$$

$$S:\ \left(x-1\right)^2+\left(y-4\right)^2+\left(z+7\right)^2=121$$

3. Sastaviti jednačinu sferne površi koja prolazi kroz kružnicu: $K:\begin{cases} \left(x-3\right)^2+\left(y-4\right)^2+z^2=36 \\ 4x+y-z-9=0 \end{cases}$ i kroz tačku $T\left(7,\, -3,\, 2\right)$.

Rešenje.

$$\left(x-3\right)^2+\left(y-4\right)^2+z^2-36+\lambda\left(4x+y-z-9\right)=0$$

$$T \in S:\ \left(7-3\right)^2+\left(-3-4\right)^2+z^2-36+\lambda\left(4\cdot 7-3-1-9\right)=0$$

$$16+49+1-36+15\lambda=0$$

$$15\lambda+30=0$$

$\lambda = -2$

$S: \; (x-3)^2 + (y-4)^2 + z^2 - 36 - 2\,(4x + y - z - 9) = 0$

$x^2 - 6x + 9 + y^2 - 8y + 16 + z^2 - 36 - 8x - 2y + 2z + 18 = 0$

$S: \; x^2 + y^2 + z^2 - 14x - 10y + 2z + 7 = 0$

4. Naći centar i poluprečnik kružnice:

$K: \begin{cases} (x-4)^2 + (y-7)^2 + (z+1)^2 = 36 \\ 3x + y - z - 9 = 0 \end{cases}$

Rešenje.

$K = S \cap \alpha$. Iz središta sfere spuštamo normalu na ravan α i odredimo njen prodor.

$n: \; \frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$

$C_S\,(4,\, 7,\, -1)$

$C_S \in n \wedge n \perp \alpha \;\Rightarrow\; n: \; \frac{x-4}{3} = \frac{y-7}{l} = \frac{z+1}{-l} = t \;\Rightarrow$

$\Rightarrow \begin{cases} x = 3t + 4 \\ y = t + 7 \\ z = -t - 1 \end{cases}$

$3\,(3t+4) + (t+7) + (t+1) - 9 = 0$

$9t + 12 + t + 7 + t + 1 - 9 = 0$

$11t + 11 = 0$

$t = -1$

$\begin{cases} x = 1 \\ y = 0 \\ z = 0 \end{cases}$

$$\overline{C_K C_S}^2 = (4-1)^2 + (7-6)^2 + (-1-0)^2$$

$$\overline{C_K C_S}^2 = 3^2 + 1^2 + 1^2 = 11$$

$$r_k^2 + \overline{C_K C_S}^2 = r_S^2$$

$$r_k^2 = r_S^2 - \overline{C_K C_S}^2$$

$$r_S^2 = 36$$

$$r_k^2 = 36 - 11$$

$$r_k^2 = 5$$

5. U tačkama prodora prave $p : \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{2}$ i sfere S : $(x-2)^2 + (y+1)^2 + (z-3)^2 = 6$ postaviti ravi koje dodiruju datu sferu.

Rešenje.

$$\tfrac{x-1}{1} = \tfrac{y}{-1} = \tfrac{z-1}{2} = t \Rightarrow \begin{cases} x = t+1 \\ y = -t \\ z = 2t+1 \end{cases}$$

$$(t+1-2)^2 + (-t+1)^2 + (2t+1-3)^2 = 6$$

$$(t-1)^2 + (-t+1)^2 + (2t-2)^2 = 6$$

$$6t^2 - 12t = 0$$

$$6t\,(t-2) = 0$$

$$t_1 = 0 \lor t_2 = 2$$

$$\begin{cases} x_1 = 1 \\ y_1 = 0 \\ z_1 = 1 \end{cases}$$

$$\begin{cases} x_2 = 3 \\ y_2 = -2 \\ z_2 = 5 \end{cases}$$

$$T_1 = (x_1,\, y_1,\, z_1) = (1,\, 0,\, 1)$$

$$T_2 = (x_2,\, y_2,\, z_2) = (3,\, -2,\, 5)$$

$$l_{t_1} :\ \tfrac{x-2}{1-2} = \tfrac{y+1}{0+1} = \tfrac{z-3}{1-3}$$

$$l_{t_1} :\ \tfrac{x-2}{-1} = \tfrac{y+1}{1} = \tfrac{z-3}{-2} / \cdot (-1)$$

$$l_{t_1} :\ \tfrac{x-2}{1} = \tfrac{y+1}{-1} = \tfrac{z-3}{2}$$

$$l_{t_2} :\ \tfrac{x-2}{3-2} = \tfrac{y+1}{-2+1} = \tfrac{z-3}{5-3}$$

$$l_{t_2} :\ \tfrac{x-2}{1} = \tfrac{y+1}{-1} = \tfrac{z-3}{2}$$

$$T_1 \in \alpha_1 \ \wedge\ \alpha_1 \perp l_{t_1} \ \Rightarrow\ \alpha_1 :\ (x-1) - (y-0) + 2\,(z-1) = 0$$

$$\alpha_1 :\ x - y + 2z - 3 = 0$$

$$T_2 \in \alpha_2 \ \wedge\ \alpha_2 \perp l_{t_2} \ \Rightarrow\ \alpha_2 :\ (x-3) - (y+2) + 2\,(z-5) = 0$$

$$\alpha_2 :\ x - y + 2z - 15 = 0$$

6. Kroz osu x postaviti tangentnu ravan sfere
$S :\ (x+5)^2 + (y-8)^2 + (z+1)^2 = 16$
Rešenje.

$$\alpha :\ Ax + By + Cz + D = 0$$

x-osa $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

$$x\, in\, \alpha \ \Rightarrow\ \begin{array}{r} A \cdot 0 + B \cdot 0 + C \cdot 0 + D = 0 \ \Rightarrow\ D = 0 \\ A \cdot 1 + B \cdot 0 + C \cdot 0 = 0 \ \Rightarrow\ A = 0 \end{array}$$

Ravan α je tangentna ravan, ako je odstojanje centra sfere od ravni jednako poluprečniku sfere

$$d = \left| \frac{B\cdot y_0 + C\cdot z_0}{\sqrt{B^2+C^2}} \right|$$

$$4 = \left| \frac{8B-C}{\sqrt{B^2+C^2}} \right| /^2$$

$$\frac{64B^2-16BC+C^2}{B^2+C^2} = 16$$

$$64B^2 - 16BC + C^2 = 16B^2 + 16C^2$$

$$48B^2 - 16BC - 15C^2 = 0$$

$$B_{1,2} = \frac{8C \pm \sqrt{64C^2+720C^2}}{48}$$

$$B_{1,2} = \frac{8C \pm 28C}{48}$$

$$B_1 = -\frac{5}{12}C,\ B_2 = \frac{3}{4}C$$

$$\alpha_1 :\ -\frac{5}{12}Cy + Cz = 0 / \cdot \left(-\frac{12}{C}\right)$$

$$\alpha_2 :\ 5y - 12z = 0$$

$$\alpha_2 :\ \frac{3}{4}Cy + Cz = 0 / \cdot \frac{12}{C}$$

$$\alpha_2 :\ 3y + 4z = 0$$

7. Napisati jednačinu sfere koja dodiruje pravu $p_1 :\ \frac{x-1}{3} = \frac{y+4}{6} = \frac{z-6}{4}$ u tački $M_1(1,\ -4,\ 6)$ i pravu $p_2 :\ \frac{x-4}{2} = \frac{y+3}{1} = \frac{z-2}{-6}$ u tački $M_2(4,\ -3,\ 2)$.

$$M_1 \in \alpha \ \wedge\ \alpha \perp p_1 \ \Rightarrow\ \alpha :\ 3(x-1) + 6(y+4) + 4(z-6) = 0$$
$$\alpha :\ 3x + 6y + 4z - 3 = 0$$

$$M_2 \in \beta \ \wedge\ \beta \perp p_2 \ \Rightarrow\ \beta :\ 2(x-4) + (y+3) - 6(z-2) = 0$$
$$\beta :\ 2x + y - 6z + 7 = 0$$

$T_0\,(x_0, y_0, z_0)$

$M_1T_0 = M_2T_0 \;\Rightarrow$

$T_0\left(\frac{5}{2},\, -\frac{7}{2},\, 4\right)$

$M_1M_2:\; \frac{x-1}{4-1} = \frac{y+4}{-3+4} = \frac{z-6}{2-6}$

$M_1M_2:\; \frac{x-1}{3} = \frac{y+4}{1} = \frac{z-6}{-4}$

$T_0 \in \gamma \,\wedge\, \gamma \perp M_1M_2 \;\Rightarrow\; \gamma:\; 3\left(x - \frac{5}{2}\right) + \left(y + \frac{7}{2}\right) - 4\,(z-4) = 0$

$\gamma:\; 3x + y - 4z - \frac{15}{2} + \frac{7}{2} + 16 = 0$

$\gamma:\; 3x + y - 4z + 12 = 0$

$C_S = \alpha \cap \beta \cap \gamma$

$$\left.\begin{array}{l} 3x + 6y + 4z - 3 = 0 \\ 2x + y - 6z + 7 = 0 \\ 3x + y - 4z + 12 = 0 \end{array}\right\}$$

$C_S\,(-5,\, 3,\, 0)$

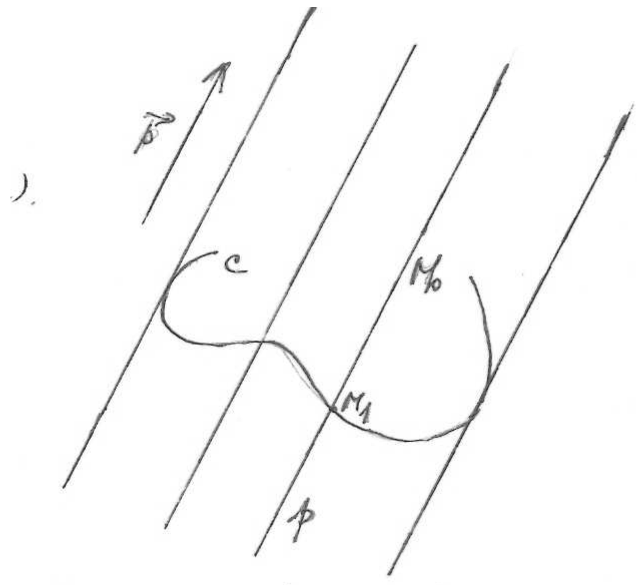
$r = CM_1 = \sqrt{(-5-1)^2 + (3+4)^2 + (0-6)^2} = \sqrt{121} = 11$

$S: (x+5)^2 + (y-3)^2 + z^2 = 121$

1.31 Cilindrične površi

Kriva c je data kao presek dve ravni

$$c:\; \left\{\begin{array}{l} S_1:\; F_1\,(x,\, y,\, z) = 0 \\ S_2:\; F_2\,(x,\, y,\, z) = 0 \end{array}\right. \quad \vec{p} = (l,\, m,\, n)$$



Slika 1.61:

Skup svih tačaka pravih paralelnih vektoru $\vec{p}$ koje prolaze kroz tačke krive c zove se cilindrična površ S. Kriva c zove se direktrisa ili vodilja cilindrične površi S, a prava p sa vektorom pravca $\vec{p}$ koja prolazi kroz tačke krive c zove se generatrisom ili izvodnicom cilindrične površi S.

$M\left(x,\,y,\,z\right)\in S$

$M\in p,\,p\parallel\vec{p}$

$M_1\left(x_1,\,y_1,\,z_1\right)\in p\cap c_1$

$M_1\in c\,\wedge\,M_1\in p$

$$\left.\begin{array}{l} F_1\left(x_1,\,y_1,\,z_1\right)=0\\ F_2\left(x_1,\,y_1,\,z_1\right)=0\\ \frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}\end{array}\right\}\Rightarrow F\left(x,\,y,\,z\right)=0,\text{ eliminacijom parametra}$$

$x_1,\,y_1,\,z_1$

$F\left(x,\,y,\,z\right)=0$ - jednačina cilindrične površi S

1. Naći jednačinu cilindrične površi koja je opisana oko sfera $x^2+y^2+z^2=16$ i $\left(x-1\right)^2+y^2+\left(z+2\right)^2=16$.

$C_1\left(0,\,0,\,0\right)$

$C_2\left(1,\,0,\,-2\right)$

$C_1C_2: \frac{x}{1} = \frac{y}{0} = \frac{z}{-2}$

$C_1 \in \alpha \wedge \alpha \perp C_1C_2 \Rightarrow \alpha: (x-0) + 0 \cdot (y-0) - 2(z+0) = 0$

$\alpha: x - 2z = 0$

$$D: \begin{cases} x^2 + y^2 + z^2 = 16 \\ x - 2z = 0 \end{cases}$$

$c_1 \overrightarrow{C_1C_2} \Rightarrow c_1: \frac{X-x}{1} = \frac{Y-y}{0} = \frac{Z-z}{-2} = t$

$$c_1: \begin{cases} X = t + x \\ Y = y \\ Z = 2t + 2 \end{cases}$$

Neka je $A(\alpha, \beta, \gamma) = D \cap G \Rightarrow A \in D \wedge A \in G$

$$\begin{cases} \alpha = t + x \\ \beta = y \\ \gamma = -2t + z \\ \alpha^2 + \beta^2 + \gamma^2 = 16 \\ \alpha - 2\gamma = 0 \end{cases}$$

$x + t + 4t - 2z = 0$

$5t + x - 2z = 0$

$t = \frac{-x+2z}{5}$

$$\begin{cases} \alpha = x + \frac{-x+2z}{5} = \frac{4x+2z}{5} \\ \beta = y \\ \gamma = -2\frac{-x+2z}{5} + z = \frac{2x+z}{5} \end{cases}$$

$\frac{(4x+2z)^2}{25} + y^2 + \frac{(2x+z)^2}{25} = 1/ \cdot 25$

$16x^2 + 16xz + 4z^2 + 25y^2 + 4x^2 + 4xz + z^2 = 400$

$20x^2+25y^2+5z^2+20xz-400=0/:5$

$4x^2+5y^2+z^2+4xz-80=0$

2. Naći jednačinu cilindrične kružnr površi, na kojoj je kružnica poluprečnika $r=9$ sa centrom $C\,(5,\,1,\,1)$ u ravni $x-2y-2z-1=0$.

Rešenje.

$$D:\left\{\begin{array}{l}(x-5)^2+(y-1)^2+(z-1)^2=9\\ x-2y-2z-1=0\end{array}\right.$$

$G\parallel\vec{a}=\{1,\,-2,\,-2\}$

$G:\ \frac{X-x}{1}=\frac{Y-y}{-2}=\frac{Z-z}{-2}=t$

$$G:\left\{\begin{array}{l}X=t+x\\ Y=-2t+y\\ Z=-2t+z\end{array}\right.$$

$A\,(\alpha,\,\beta,\,\gamma)=D\cap G\ \Rightarrow A\in D\ \wedge\ A\in G$

$$\left\{\begin{array}{l}\alpha=t+x\\ \beta=y-2t\\ \gamma=z-2t\\ (\alpha-5)^2+(\beta-1)^2+(\gamma-1)^2=9\\ \alpha-2\beta-2\gamma-1=0\end{array}\right.$$

$t+x-2y+4t-2z+4t-1=0$

$t=\frac{-x+2y+2z+1}{9}$

$$\left\{\begin{array}{l}\alpha=x+\frac{-x+2y+2z+1}{9}=\frac{8x+2y+2z+1}{9}\\ \beta=y-\frac{-x+2y+2z+1}{9}=\frac{2x+8y-4z-2}{9}\\ \gamma=z-2\frac{-x+2y+2z+1}{9}=\frac{2x-4y+5z-2}{9}\end{array}\right.$$

$\frac{8x+2y+2z+1}{81}+\frac{2x+8y-4z-2}{81}+\frac{2x-4y+5z-2}{81}=9/\cdot 81$

$64x^2+4y^2+4z^2+1936+32xy+32xz-704x+8yz-176y-176z+$
$4x^2+25y^2+16z^2+121+20xy-16xz-44x-44yz-110y+882+$
$4x^2+16y^2+25z^2+121-16xy+20xz-44x-40yz+8y-110z=729$

$$72x^2+45y^2+45z^2+36xy+36xz-72yz-792x-198y-198z+1449=0$$

3. Naći jednačinu cilindrične kružne površi kojoj je osa prava $x=3t+1$, $y=-2t-2$, $z=t+2$, a tačka $A\,(2,\,-1,\,1)$ je na površi.

$p:\ \frac{X-1}{3}=\frac{Y+2}{-2}=\frac{Z-2}{1}$

$\alpha \perp p\ \wedge\ A\in\alpha$

α je kružnica koja je presek sfere čiji je poluprečnik rastojanja tačke A od ose cilindra, a centar je u ravni α i na osi cilindra.

$\alpha:\ 3\cdot(x-2)-2\cdot(y+1)+(z-1)=0$

$\alpha:\ 3x-2y+z-9=0$

$p\cap\alpha:\ 9t+3+4t+4+t+2-9=0$

$14t=0\ \Rightarrow\ t=0\ \Rightarrow\ C\,(1,\,-2,\,2)$

$|AC|=\sqrt{1+1+1}=\sqrt{3}\ \Rightarrow\ r^2=3$

$$D:\ \begin{cases}(x-1)^2+(y+2)^2+(z-2)^2=9\\ 3x-2y+z-9=0\end{cases}$$

$\vec{G}\parallel\vec{p}=\{3,\,-2,\,1\}$

$G:\ \frac{X-x}{3}=\frac{Y-y}{-2}=\frac{Z-z}{1}=t$

$$G:\ \begin{cases}X=3t+x\\ Y=-2t+y\\ Z=2t+z\end{cases}$$

$A\left(\alpha,\,\beta,\,\gamma\right)=D\cap G\,\Rightarrow A\in D\,\wedge\,A\in G$

$$\begin{cases}\alpha=3t+x\\ \beta=2t+y\\ \gamma=t+z\\ \left(\alpha-1\right)^2+\left(\beta+2\right)^2+\left(\gamma-2\right)^2=3\\ 3\alpha-2\beta+\gamma-9=0\end{cases}$$

$9t+3x-2y+4t+z+t-9=0$

$3x-2y+z+14t=0$

$t=\frac{-3x+2y-z+9}{14}$

$\alpha=\frac{-9x+6y-3z+27}{14}+x=\frac{5x+6y-3z+27}{14}$

$\beta=\frac{6x-4y+2z-18}{14}+y=\frac{6x+10y+2z-18}{14}$

$\gamma=\frac{-3x+2y-z+9}{14}+z=\frac{-3x+2y+13z+9}{14}$

$\left(\frac{5x+6y-3z+27}{14}\right)^2+\left(\frac{6x+10y+2z-18}{14}\right)^2+\left(\frac{-3x+2y+13z+9}{14}\right)^2=3/\cdot 196$

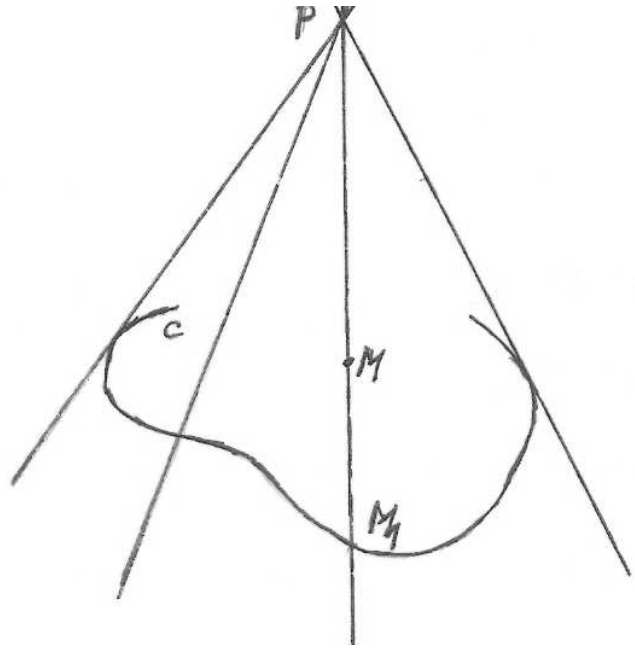
$5x^2+36y^2+9z^2+169+60xy-30xz+130x+36yz+156y-78z+36x^2+100y^2+4z^2+100+120xy+24xz+120x+40yz+200y+40z+9x^2+4y^2+169z^2+361-12xy-278xz+114x+52yz-76y-494z=588$

$70x^2+140y^2+182z^2+168xy-84xz+56yz+364x+280y-532z=-42/:14$

$5x^2+10y^2+13z^2+12xy-6xz+4yz+26x+20y-38z+3=$

1.32 Konusne površi

$$c:\begin{cases}S_1:\,F_1\left(x,\,y,\,z\right)=0\\ S_2:\,F_2\left(x,\,y,\,z\right)=0\end{cases}$$



Slika 1.62:

$P\,(x_0,\, y_0,\, z_0)$ - data tačka

Skup svih tačaka S pravih koje prolaze kroz tačku P i kroz tačke krive c zove se konusna površ, čija je direktrisa kriva c, a prave kroz P i tačke krive c su generatrise konusne površi S. Tačka P zove se vrh konusa S.

$M\,(x,\, y,\, z) \in S$

$M \in G$

$G \cap D = M_1\,(x_1,\, y_1,\, z_1)$

$\left.\begin{array}{l} F_1\,(x_1,\, y_1,\, z_1) = 0 \\ F_2\,(x_1,\, y_1,\, z_1) = 0 \\ \frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0} \end{array}\right\} \Rightarrow F\,(x,\, y,\, z) = 0$, eliminacijom parametra $x_1,\, y_1,\, z_1$

$F\,(x,\, y,\, z) = 0$ - jednačina konusne površi S.

1.Naći jednačinu konusne površi čiji je vrh u koordinatnom početku, a direktrisa je kriva

$$D : \begin{cases} x^2 - 2z + 1 = 0 \\ y - z + 1 = 0 \end{cases}$$

Rešenje.

$V\left(0,\,0,\,0\right)$

$\overrightarrow{VM} \wedge \overrightarrow{VM_1}$

$\overrightarrow{VM} = \left(x,\,y,\,z\right)$

$\overrightarrow{VM_1} = \left(\alpha,\,\beta,\,\gamma\right)$

$M_1\left(\alpha,\,\beta,\,\gamma\right) \equiv D \cap G \Rightarrow M_1 \in D \wedge M_1 \in G$

$$G:\ \frac{X}{x} = \frac{Y}{y} = \frac{Z}{z} = t \quad \Rightarrow \quad \begin{cases} X = xt \\ Y = yt \\ Z = zt \end{cases}$$

$$\begin{cases} \alpha = xt \\ \beta = yt \\ \gamma = zt \\ \alpha^2 - 2\gamma + 1 = 0 \\ \beta - \gamma + 1 = 0 \end{cases}$$

$yt - zt + 1 = 0$

$t = \frac{-1}{y-z}$

$\alpha = \frac{x}{z-y}$

$\beta = \frac{y}{z-y}$

$\gamma = \frac{z}{z-y}$

$\frac{x^2}{\left(z-y\right)^2} - \frac{2z}{z-y} + 1 = 0 / \cdot \left(z-y\right)^2$

$x^2 - 2z\left(z-y\right) + \left(z-y\right)^2 = 0$

$x^2 - 2z^2 + 2yz + z^2 - 2yz + y^2 = 0$

$x^2 + y^2 - z^2 = 0$

2. Naći jednačinu površi koja nastaje rotacijom prave $\frac{x-2}{3} = \frac{y}{-2} = \frac{z}{6}$ oko x-ose.

Rešenje. x-osa: $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

$$\begin{cases} x^2 + y^2 + z^2 = \alpha^2 \\ x = \beta \\ \frac{x-2}{3} = \frac{y}{-2} = \frac{z}{6} \end{cases}$$

$\frac{x-2}{3} = \frac{y}{-2} \Rightarrow -2x + 4 = 3y \Rightarrow y = \frac{4-2x}{3} \Rightarrow y = \frac{4-2\beta}{3}$

$\frac{y}{-2} = \frac{z}{6} \Rightarrow y = -\frac{z}{3} \Rightarrow z = -3y$

$z = 2\beta - 4$

$x^2 + y^2 + z^2 = \alpha^2 \Rightarrow \beta^2 + \left(\frac{4-2\beta}{3}\right)^2 + (2\beta - 4)^2 = \alpha^2 \Rightarrow \beta^2 + \frac{16-16\beta+4\beta^2}{9} + 4\beta^2 - 16\beta + 16 = \alpha^2 / \cdot 9 \Rightarrow$

$9\beta^2 - 16\beta + 16 + 36\beta^2 - 144\beta + 144 = 9\alpha^2$

$49\beta^2 - 9\alpha^2 - 160\beta + 160 = 0$

$49x^2 - 9\left(x^2 + y^2 + z^2\right) - 160x + 160 = 0$

$40x^2 - 9y^2 - 9z^2 - 160x + 160 = 0$

3. Naći jednačina rotacione površi koja nastaje rotacijom prave $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$ oko prave $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z+2}{3}$.

$l : \frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1}$

$p : \frac{x+1}{-1} = \frac{y-2}{2} = \frac{z+2}{3}$

$$\begin{cases} (x+1)^2 + (y-2)^2 + (z+2)^2 = \alpha^2 \\ -x + 2y + 3z = \beta \\ \frac{x+1}{2} = \frac{y-3}{3} = \frac{z+2}{-1} \end{cases}$$

$\frac{x+1}{2} = \frac{z+2}{-1} \Rightarrow -x - 1 = 2z + 4 \Rightarrow x = -2z - 5$

$\frac{y-3}{3} = \frac{z+2}{-1} \Rightarrow -y + 3 = 3z + 6 \Rightarrow y = -3z - 3$

$-x + 2y + 3z = \beta \Rightarrow 2z + 5 - 6z - 6 + 3z = \beta \Rightarrow -z - 1 = \beta \Rightarrow z = -\beta - 1$

$x = 2\beta + 2 - 5 \Rightarrow x = 2\beta - 3$

$y = 3\beta + 3 - 3 \Rightarrow y = 3\beta$

$$\begin{cases} x = 2\beta - 3 \\ y = 3\beta \\ z = -\beta - 1 \end{cases}$$

$(x+1)^2 + (y-2)^2 + (z+2)^2 = \alpha^2$

$(2\beta - 2)^2 + (3\beta - 2)^2 + (-\beta + 1)^2 = \alpha^2$

$4\beta^2 - 8\beta + 4 + 9\beta^2 - 12\beta + 4 + \beta^2 - 2\beta + 1 = \alpha^2$

$14\beta^2 - \alpha^2 - 22\beta + 9 = 0$

$14\left(-x + 2y + 3z\right)^2 - 22\left(-x + 2y + 3z\right) + 9 = (x+1)^2 + (y-2)^2 + (z+2)^2$

$14\left(x^2 + 4y^2 + 9z^2 - 4xy - 6xz + 12yz\right) + 22x - 44y - 66z + 9 = x^2 + 2x + 1 + y^2 - 4y + 4 + z^2 + 4z + 4$

$13x^2 + 55y^2 + 125z^2 - 56xy - 84xz + 168yz + 20x - 40y - 20z = 0$