Chapter 6

Lagrangian Dynamics

In this chapter we will:

- develop the basic formalism of classical mechanics starting from a variational principle
- introduce and explore the concepts of generalized coordinates and generalized momenta
- study how conservation laws in physics are connected to symmetries of the Universe

Following the remarkable impact of Newton's Principia in the late 17th century, classical mechanics was reinvented in the 18th century by Leonhard Euler (1707-1783) and Joseph-Louis Lagrange (1736-1813) in terms of variational principles. Their ideas were further refined in the early 19th century by Carl Gustav Jacob Jacobi (1804-1851) and William Rowan Hamilton (1805-1865). According to the variational principle approach, particles do not follow trajectories because they are pushed around by external forces, as Newton proposed. Instead, among all possible trajectories between two points, they choose the one which minimizes a particular integral of their kinetic and potential energies called the action.

The new approach to classical mechanics offered a set of mathematical tools that greatly simplified the solution of complex problems. Perhaps more importantly, it provided a recipe for attacking such problems; a recipe that worked with minor changes in most occasions. Lagrange himself boasted in his book Méchanique Analytique (1788) that "The reader will find no figures in this work. The methods which I set forth do not require either constructions or geometrical or mechanical reasoning, but only algebraic operations, subject to a regular and uniform rule of procedure" (as quoted in [1]). Following Lagrange's own book title, this new approach to classical mechanics has been called Analytical Mechanics to differentiate it from Newton's mechanics that was based on vectors and on the concept of forces.

Despite advocating a revolutionary approach to describing the world, the main goal of the proponents of Analytical Mechanics was modest. They did not aim to invent new physics but simply to reproduce exactly Newton's equations of motion

and predictions. This led to a widespread belief best summarized by Ernst Mach in his book The Science of Mechanics (1893): "No fundamental light can be expected from this branch of mechanics. On the contrary, the discovery of matters of principle must be substantially completed before we can think of framing analytical mechanics, the sole aim of which is perfect practical mastery of problems" (also as quoted in [1]). The 20th century could not have proven Mach's prophecy any more false.

In this chapter, we will develop the formalism of analytical mechanics starting from the point of view that led to the major developments in Physics in the 20th century. We will assert that mechanical systems are described by equations that arise from a variational principle. We will then obtain the resulting equations by requiring them to obey particular symmetries, such as the fact that the equations of mechanics should be invariant to Galilean transformations and to agree with qualitative aspects of experiments, like the attractive nature of the gravitational force. The history of the development of these ideas, which is not necessarily in the order that they will be discussed in this chapter, can be found in several excellent books[1,2].

6.1 The Lagrangian Action for Particles in a Conservative Force Field

Our goal in this section is to derive the equations of motion for a system of particles in the presence of a potential force field as a solution to a variational problem. The independent variable in the problem will clearly be time and the dependent functions will be the three dimensional positions of each particle. We will aim to find a function L such that the paths of the particles between times t_1 and t_2 extremize the integral

$$S = \int_{t_1}^{t_2} L(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3, ...; t)dt \ .$$

We will call the integral S the action of the system and the function L its Lagrangian. Hereafter, we will use overdots to denote a time derivative of a function, i.e., $\dot{x}_1 \equiv dx_1/dt$.

It is important to emphasize first that the solution to our problem will not be unique because many different Lagrangian functions will give rise to the same equation of motion and, therefore, to the same predictions that can be compared to observations. Consider for example two Lagrangians L and L' that differ only by a full time derivative of some function g that contains derivatives of one order less than the Lagrangian, i.e.,

$$L'(x_i, \dot{x}_i, ..., x_i^n) = L(x_i, \dot{x}_i, ..., x_i^n) + \frac{d}{dt} g(x_i, \dot{x}_i, ..., x_i^{n-1}) \ .$$

The actions for the two Lagrangians will also be related since

$$\begin{aligned} S' &= \int_{t_1}^{t_2} L'(x_i, \dot{x}_i, ..., x_i^n)dt = \int_{t_1}^{t_2} L(x_i, \dot{x}_i, ..., x_i^n)dt + \int_{t_1}^{t_2} \frac{dg}{dt}dt \\ &= S + \left\{ g\left[x_i(t_2), \dot{x}_i(t_2), ..., x_i^{(n-1)}(t_2)\right] - g\left[x_i(t_2), \dot{x}_i(t_2), ..., x_i^{(n-1)}(t_1)\right] \right\} \ . \end{aligned}$$

The difference between the two terms in the curly brackets will be independent of the path, because it is determined by the values of the first $n-1$ derivatives of the paths evaluated at the boundaries, which are kept fixed. As a result, the path that extremizes the action S will also extremize the action S'.

With this consideration in mind, we are now in the position to search for a Lagrangian that we will use to describe the motion of particles in a potential force field. In our quest, we will be guided by general assumptions and symmetries in the problem, which we will now discuss in some detail.

The trajectory of a particle is uniquely specified by its initial position and velocity— We will first confine our attention to the case of a single particle in a potential force field. In general, the Lagrangian L may depend on time derivatives of the coordinates of up to order n. The resulting Euler-Lagrange equation for each coordinate will be of order $2n$ and it will require $2n$ initial conditions. We know empirically, as testified by the success of the Newtonian theory, that we can predict the trajectory of an object uniquely given its initial position and velocity, i.e., using only two initial conditions. As a result, the Euler-Lagrange equations will need to be of order 2 and the Lagrangian of order one. In other words, the expectation that the equation of motion will be of second order implies that the Lagrangian may depend only on position and velocity but not on acceleration or higher derivatives of the position with respect to time.

The equation of motion of a free particle is invariant to translations and rotations— We will now consider the equation of motion of a particle in the absence of a force field. The equation of motion, and hence the Lagrangian, in general, cannot depend on the position vector of the particle with respect to any coordinate system, because we normally assume that space is uniform. Moreover, the Lagrangian cannot depend on the orientation of the velocity vector because we also normally assume that space is isotropic. As a result, invariance of the equation of motion to translations and rotations implies that the Lagrangian of a free particle is a function of the square of its velocity vector $v^2 \equiv \dot{\vec{x}}^2$, i.e., $L = L(v^2)$.

The equation of motion of a free particle is invariant to Galilean transformations— We will now consider the motion of a free particle in two inertial frames related by a Galilean transformation: the first (unprimed) frame is at rest with respect to an inertial observer and the second (primed) frame is moving with a uniform, infinitesimal velocity $\delta\vec{u}$ with respect to the first frame. The velocity vectors of the particle in the two frames will be related by the Galilean transformation

$$\vec{v}' = \vec{v} + \delta\vec{u} \, .$$

Similarly, the Lagrangians in the two frames will be related by

$$\begin{aligned} L'(v'^2) &= L\left[(\vec{v}+\delta\vec{u})^2\right] \\ &= L\left(v^2 + 2\vec{v}\cdot\delta\vec{u} + \delta u^2\right) \\ &\simeq L\left(v^2\right) + 2\left(\frac{dL}{dv^2}\right)\vec{v}\cdot\delta\vec{u} \, , \end{aligned}$$

where, in the last step, we performed a Taylor expansion of $L(v^2)$ and kept terms of up to first order in the infinitesimal velocity $\delta\vec{u}$. In order for the equation of motion

of the particle to be invariant under Galilean transformations, the two Lagrangian may differ only by a total time derivative of the coordinates. However, the term

$$2\left(\frac{dL}{dv^2}\right)\vec{v}\cdot\delta\vec{u} = \left[2\left(\frac{dL}{dv^2}\right)\delta\vec{u}\right]\cdot\frac{d\vec{x}}{dt}$$

is already proportional to the first derivative of the position vector $\vec{x}$ with respect to time. In order for the whole term to remain equal to a total time derivative, the product in the square brackets should not depend on the velocity of the particle, or equivalently,

$$\frac{dL}{dv^2} = \text{constant} \Rightarrow L = \alpha v^2 + \beta \ ,$$

where α and β are two constants that we have not yet specified.

The Euler-Lagrange equation that arises from this general Lagrangian for a free particle is

$$\begin{aligned} &\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0 \\ \Rightarrow \quad &\frac{d^2x_i}{dt^2} = 0 \Rightarrow u_i = \frac{dx_i}{dt} = \text{constant} \ , \end{aligned}$$

independent of the values of the parameters α and β. This is simply Newton's first law of motion, which states that the velocity of an object in a inertial frame remains constant, when no force is acting on it. In the variational approach we are following here, Newton's first law of motion is indeed a consequence of our assumption that the laws of mechanics are invariant to Galilean transformations.

The potential force field is described by a single scalar function of the coordinates— The presence of a force fields breaks the translation invariance of space, because the particle experiences, in general, forces of different magnitude and orientation at different locations in space. For the potential force fields that we considered so far, the interaction between the field and the particle depends only on its coordinates and not on its velocity. In this case, the Lagrangian of a particle in a potential field, which itself is a scalar quantity, will depend on the coordinates of the particle in ways that can be described by a scalar function, which we will denote for now by $F(\vec{x},t)$, i.e.,

$$L = \alpha v^2 + \beta + F(\vec{x},t) = \alpha\sum_{i=1}^{3}\dot{x}_i^2 + \beta + F(x_i,t) \ .$$

This is the most general form of a Lagrangian that is consistent with the symmetries we have imposed so far. We have also written it in coordinate form, to facilitate the derivation of the following equations.

The Euler-Lagrange equation that correspond to this Lagrangian is

$$\begin{aligned} &\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0 \\ \Rightarrow \quad &\frac{\partial F}{\partial x_i} - \alpha\frac{d^2x_i}{dt^2} = 0 \ . \end{aligned}$$

In order to render this equation equivalent to Newton's second law, we simply need to set $\alpha = m/2$, where m is the mass of the particle, and identify the scalar function

F with minus the potential energy V of the particle in the field. The constant β does not enter in the equation of motion and we, therefore, set it to zero.

What we have just proven is that Newton's second law for a particle in a conservative field can be derived as the Euler-Lagrange equation for a Lagrangian

$$L = T - V = \frac{1}{2}mu^2 - V(\vec{x}, t) \ .$$

We can also easily generalize this proof to a case of N particles interacting via a conservative force and show that the equation of motion for each particle can be derived as the Euler-Lagrange equation that corresponds to the Lagrangian

$$L = T - V = \sum_{j=1}^{N} \frac{1}{2} m_j u_j^2 - V(\vec{x}_j, t) \ , \tag{6.1}$$

Lagrangian for a system of particles

where u_j is the velocity of each particle, $\vec{x}_j$ is its coordinate vector, m_j is its mass, and V is the potential energy of interaction between the particles. In other words, we have proven that the trajectories of N particles interacting via a conservative force will be such that the action

$$S = \int_{t_1}^{t_2} (T - V) dt \tag{6.2}$$

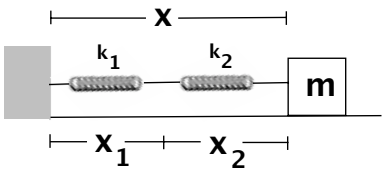
Action for a System of Interacting Particles

will be extremized, where T and V are the total kinetic and potential energy for the system. We call this integral the action of the dynamical system and we call this statement Hamilton's Principle, to honor the Irish mathematician William Rowan Hamilton, who first expressed it in this form.. However, in order to differentiate the formulation of mechanics in terms of Euler-Lagrange equations from Hamilton's own approach, which we will explore in the following chapter, we will call the formalism we follow here Lagrange mechanics, or Lagrange dynamics.

The most remarkable aspect of our ability to derive Newton's second law from a variational principle is captured in the form of the Lagrangian (6.1): the dynamics of an ensemble of N particles that are interacting with each other via conservative forces can be completely described in terms of the single, scalar function L. It remains true that, in order to calculate the positions of the individual particles as a function of time, we still need to solve a vector differential equation for each particle. However, we can now derive each and every one of these differential equations from the scalar function L.

The variational approach we have followed so far, allowed us to reach Newton's first and second laws starting from a discussion of symmetries in space and the Galilean invariance of the theory. In Section 6.4, we will see how symmetries also lead to conservation laws for the linear and angular momenta of a system and, therefore, to Newton's third law of action and reaction. Before embarking into the proof of conservation theorems, however, we will first explore some examples and discuss another aspect of Lagrangian mechanics that makes it extremely useful as a tool for solving complex problems.

Example 6.1: Springs in Series



In this example, we will use Hamilton's principle to calculate the effective constant of a compound spring we construct by attaching in series two massless springs.

We will consider two springs with constants k_1 and k_2 and with unextended lengths l_1 and l_2. We attach the springs in series, making a compound spring of length $l_0 = l_1 + l_2$, place it on a horizontal surface, and connect one of its ends to a vertical wall and the other end to an object of mass m. We will denote by x the distance of the edge of the massive object from the wall, by x_1 the length of the first spring, and by x_2 the length of the second spring. These three lengths satisfy the relation

$$x = x_1 + x_2 \ .$$

When the object is in motion, the total kinetic energy of the system is simply

$$T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m \left(\dot{x}_1 + \dot{x}_2 \right)^2 \ ,$$

since the springs are assumed to be massless. On the other hand, the potential energy in the system is equal to the sum of the potential energies of the two springs, i.e.,

$$V = \frac{1}{2} k_1 (x_1 - l_1)^2 + \frac{1}{2} k_2 (x_2 - l_2)^2 \ .$$

The Lagrangian of the system is

$$L(x_1, x_2) = T - V = \frac{1}{2} m \left(\dot{x}_1 + \dot{x}_2 \right)^2 - \frac{1}{2} k_1 (x_1 - l_1)^2 + \frac{1}{2} k_2 (x_2 - l_2)^2 \ .$$

Note that the Lagrangian cannot be expressed in terms of the total extension x but it depends separately on the individual lengths x_1 and x_2. As a result, we will need to write two equations, one for each length.

The Euler-Lagrange equation for length x_1 is

$$\frac{\partial L}{\partial x_1} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) = 0 \Rightarrow -k_1 (x_1 - l_1) - m \left(\ddot{x}_1 + \ddot{x}_2 \right) = 0 \ . \tag{6.3}$$

Similarly, the Euler-Lagrange equation for length x_2 is

$$\frac{\partial L}{\partial x_2} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_2} \right) = 0 \Rightarrow -k_2 (x_2 - l_2) - m \left(\ddot{x}_1 + \ddot{x}_2 \right) = 0 \ .$$

Combining these two equations, we obtain the algebraic relation

$$k_1 (x_1 - l_1) = k_2 (x_2 - l_2) \ . \tag{6.4}$$

Even though we never introduced the concept of a force in our discussion of Lagrange mechanics, we can appreciate the fact that this algebraic equation expresses the balance of forces at the point where the two springs are attached.

Our goal is to write a single equation for the displacement of the massive object, as

$$m \ddot{x} = -k (x - l_0) \ ,$$

which will allow us to read off the effective constant of the compound spring. With this in mind, we use the rational identity

$$\frac{A}{B} = \frac{C}{D} \Rightarrow \frac{A+B}{B} = \frac{C+D}{D}$$

to write equation (6.4) as

$$\frac{k_1}{k_2} = \frac{x_2 - l_2}{x_1 - l_1} \Rightarrow \frac{k_1 + k_2}{k_2} = \frac{x_1 + x_2 - (l_1 + l_2)}{x_1 - l_1} \ .$$

We now solve the last equation for $(x_1 - l_1)$, multiply the result by k_1, and find

$$k_1(x_1 - l_1) = \frac{k_1 k_2}{k_1 + k_2}(x - l_0) \ .$$

Finally, we insert this expression into the differential equation (6.3) to obtain

$$m\ddot{x} = -\frac{k_1 k_2}{k_1 + k_2}(x - l_0) \ .$$

The compound spring, therefore, behaves as a single spring with a constant

$$k = \frac{k_1 k_2}{k_1 + k_2} \Rightarrow \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \ .$$

Our result is easy to understand, since the least stiff spring is the one that extends the most and, therefore, dominates the dynamics of the system.

6.2 Lagrangian Dynamics with Generalized Coordinates

In Section 6.1, we showed that the paths of an ensemble of particles that are interacting with each other via conservative forces are such that the integral (6.2) is extremized. We used symmetries to obtain the form of the Lagrangian function and then proved the equivalence of the resulting Euler-Lagrange equations to Newton's second law.

In our proof, we wrote explicitly the potential and kinetic energy of the system in Cartesian coordinates. The form of the coordinates, however, did not enter Hamilton's principle, or the definition of the integral (6.2). Indeed, the path that extremizes the action will be the same, no matter what coordinates we used to describe it. For example, if we have a single particle in a potential, choose a set of non-Cartesian coordinates, q_i, $i = 1, 2, 3$, and express the kinetic and potential energy in terms of these coordinates, i.e., $T(q_i, t)$ and $V(q_i, t)$, then the differential equations that will allow us to solve for the time evolution of each coordinate are simply the Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \ . \tag{6.5}$$

Euler-Lagrange equations for generalized coordinates

The main difference with the case of Cartesian coordinates is that the expression for the kinetic energy will not, in general, be equal to $m\dot{q}_i^2/2$.

We use the term generalized coordinates to describe any set of coordinates that uniquely describe the dynamical state of a system. In the same spirit, we use the term generalized velocities to describe the time derivatives of generalized coordinates. It is important to emphasize here that the generalized coordinates and velocities need not have units of length and speed. Even though we often use as generalized coordinates geometric quantities, such as angles and arc lengths, any set of quantities that have a one-to-one correspondence with the coordinates of the particles are equally suitable in Lagrangian dynamics.

Example 6.2: Equations of Motion in a Central Potential using Spherical-Polar Coordinates

In this example, we will revisit the derivation of the equations of motion of a particle in curvilinear coordinate system, which we originally performed in Section 2.4 using vector calculus. In order to simplify the problem, we will consider the motion of a single particle in a central potential.

Because the potential is spherically symmetric, it seems reasonable to derive the equations of motion in a spherical-polar coordinate system (r, θ, ϕ) with its origin placed at the center of the potential. The radius r and the angles θ and ϕ will be the three generalized coordinates that we will use to describe the location of the particle.

The corresponding generalized velocities are $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$, which are *not* the same as the components of the velocity vector in the spherical-polar coordinate system. Indeed, as we proved in Section 2.4, the velocity vector is equal to

$$\vec{u} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + (r\sin\theta)\dot{\phi}\hat{e}_\phi \ ,$$

where $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_\phi$ are the basis vectors.

The kinetic energy of a particle of mass m is

$$T = \frac{1}{2}m\left(\vec{u}\cdot\vec{u}\right) = \frac{1}{2}\left[\dot{r}^2 + r^2\dot{\theta}^2 + (r^2\sin^2\theta)\dot{\phi}^2\right].$$

In the same coordinate system, the potential energy of the particle is only a function of the radius r, i.e., $V = V(r)$. As a result, the Lagrangian for the particle is

$$L = T - V = \frac{1}{2}m\left(\vec{u}\cdot\vec{u}\right) = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2 + (r^2\sin^2\theta)\dot{\phi}^2\right] - V(r)$$

and the equations of motion are the corresponding Euler-Lagrange equations.

In the radial direction, the Euler-Lagrange equation is

$$\begin{aligned} & \frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = 0 \\ \Rightarrow \quad & m\left[r\dot{\theta}^2 + (r\sin^2\theta)\dot{\phi}^2\right] - \frac{\partial V}{\partial r} - m\ddot{r} = 0 \\ \Rightarrow \quad & \ddot{r} - r\dot{\theta}^2 - (r\sin^2\theta)\dot{\phi}^2 = -\frac{1}{m}\left(\frac{\partial V}{\partial r}\right) \ . \end{aligned} \tag{6.6}$$

Similarly, the Euler-Lagrange equation in the polar direction is

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\begin{aligned}
&\Rightarrow \quad m\left(r^2\sin\theta\cos\theta\right)\dot{\phi}^2 - m\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \\
&\Rightarrow \quad \left(r^2\sin\theta\cos\theta\right)\dot{\phi}^2 - 2r\dot{r}\dot{\theta} - r^2\ddot{\theta} = 0 \\
&\Rightarrow \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} - (r\sin\theta\cos\theta)\dot{\phi}^2 = 0 \qquad (6.7)
\end{aligned}$$

and in the azimuthal direction is

$$\begin{aligned}
&\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = 0 \\
\Rightarrow \quad &-m\frac{d}{dt}\left(r^2\sin^2\theta\dot{\phi}\right) = 0 \\
\Rightarrow \quad &r^2(\sin^2\theta)\ddot{\phi} + 2r\dot{r}\dot{\phi}\sin^2\theta + 2r^2\dot{\phi}\sin\theta\cos\theta\dot{\theta} = 0 \\
\Rightarrow \quad &r(\sin\theta)\ddot{\phi} + 2\dot{r}\dot{\phi}\sin\theta + 2r(\cos\theta)\dot{\theta}\dot{\phi} = 0\,. \qquad (6.8)
\end{aligned}$$

Note that, in the derivation of the last two equations we assumed $r \neq 0$ and $\sin\theta \neq 0$. This assumption does not reduce the generality of the expressions, since the spherical-polar coordinate system is not regular at the origin ($r = 0$) and along the pole ($\theta = 0$). If the form of the potential takes the object through one of these irregularities, then the spherical-polar coordinate system will not be appropriate to describe the motion.

As expected, equations (6.6)-(6.8) are identical to those we derived in Section 2.4 using vector calculus and Newton's second law. The elegance of the derivation with variational methods, however, lies on the fact that it required neither "constructions (n)or geometrical or mechanical reasoning, but only algebraic operations, subject to a regular and uniform rule of procedure" in Lagrange's own words.

6.3 Generalized Momenta and Forces

In the previous section, we discussed how it is straightforward to see from the mathematical expression of Hamilton's principle that the Euler-Lagrange equations are the equations of motion of a system in any set of generalized coordinates. We will now prove this statement explicitly in order to introduce the concepts of generalized momenta and forces. These concepts are not important in Lagrangian dynamics, which deals only with generalized coordinates and velocities. However, they will allow us to incorporate in Lagrangian dynamics the effects of non-conservative forces, such as friction and aerodynamic drag, as well as to derive conservation laws. Moreover, generalized momenta will play a dominant role in our discussion of Hamiltonian dynamics in the following chapter.

For simplicity, we will consider a single particle in a conservative force field. The kinetic and potential energy of the particle are most easily expressed in a Cartesian coordinate system x_i, $i = 1, 2, 3$, since

$$T = \frac{1}{2}m\sum_{i=1}^{3}\dot{x}_i^2$$

and the potential energy in a conservative field may not depend explicitly on velocity, i.e., $V = V(x_i, t)$. We will also consider a set of generalized coordinates

q_j, $j = 1, 2, 3$, and a set of transformation rules between the Cartesian and the generalized coordinates

$$x_i = \mathcal{X}_i(q_j, t) \ , \tag{6.9}$$

which, in principle, may depend on time.

The expression for the kinetic energy in terms of the generalized coordinates and their derivatives will differ from the one in Cartesian coordinates in one important aspect: it will not have a simple quadratic dependence on the generalized velocities. We can prove this last statement explicitly by taking the time derivative of both sides of the transformation law (6.9) using the chain rule,

$$\dot{x}_i = \sum_{k=1}^{3} \frac{\partial \mathcal{X}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathcal{X}_i}{\partial t} \ , \tag{6.10}$$

and inserting this equation to the expression for the kinetic energy in Cartesian coordinates to obtain

$$T = \frac{1}{2} m \sum_{i=1}^{3} \left(\sum_{k=1}^{3} \frac{\partial \mathcal{X}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathcal{X}_i}{\partial t} \right)^2 \ . \tag{6.11}$$

It will be useful to note for our discussion later in this chapter that, when the transformation law between the generalized and Cartesian coordinates is independent of time, then the kinetic energy can be written as a general quadratic function of the generalized velocities, i.e.,

$$\begin{aligned} T &= \frac{1}{2} m \sum_{i=1}^{3} \left(\sum_{k=1}^{3} \frac{\partial \mathcal{X}_i}{\partial q_k} \dot{q}_k \right) \left(\sum_{l=1}^{3} \frac{\partial \mathcal{X}_i}{\partial q_l} \dot{q}_l \right) \\ &= \frac{1}{2} \sum_{k=1}^{3} \sum_{l=1}^{3} m_{kl} \dot{q}_k \dot{q}_l \ , \end{aligned} \tag{6.12}$$

where the terms

$$m_{kl} \equiv m \sum_{i=1}^{3} \frac{\partial \mathcal{X}_i}{\partial q_k} \frac{\partial \mathcal{X}_i}{\partial q_l} \tag{6.13}$$

depend only on the transformation laws. Note that $m_{kl} = m_{lk}$, i.e., the array $[m_{kl}]$ is symmetric.

With these considerations in mind, we are now ready to prove that the Euler-Lagrange equations for the generalized coordinates

$$\begin{aligned} & \frac{\partial L}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) = 0 \\ \Rightarrow \quad & \frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) = 0 \end{aligned} \tag{6.14}$$

are equivalent to Newton's second law. In Cartesian coordinates, Newton's second law takes the form

$$m \ddot{x}_i = - \frac{\partial V}{\partial x_i} \ ,$$

which we can write in terms of the components of the linear momentum $p_i = m \dot{x}_i$ as

$$- \frac{\partial V}{\partial x_i} - \frac{d}{dt} \left(\dot{p}_i \right) = 0 \ , \tag{6.15}$$

Comparison of equation (6.14) to equation (6.15) suggests that we identify the generalized momentum of the particle as the vector with coordinates

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \ . \tag{6.16}$$

Generalized Momentum

The generalized momentum p_j is also called the conjugate momentum to coordinate q_j. It is tempting to define a generalized force that is equal to $\partial(T-V)/\partial q_j$ so that the equation of motion in generalized coordinates retains the form of Newton's second law. As we will see below, however, such a definition mixes together effects that are caused by real forces with those that are solely introduced by the non-Cartesian and/or non-inertial character of the coordinates. Instead, we define the generalized force as

$$Q_j = -\frac{\partial V}{\partial q_j} \tag{6.17}$$

Generalized Force

so that the work done by the force as an object moves from point A to point B is independent of the path but depends only on the values of the potential energy, V_A and V_B, at these points, i.e.,

$$W_{A\to B} = \int_A^B \sum_{j=1}^{3} Q_j dq_j = V_B - V_A \ .$$

We will now use equation (6.10) to evaluate the two terms of the Euler-Lagrange equation (6.14) that involve derivatives of the kinetic energy, i.e.,

$$\begin{aligned}
\frac{\partial T}{\partial q_j} &= \frac{1}{2}m\sum_{i=1}^{3}\frac{\partial}{\partial q_j}\dot{x}_i^2 = m\sum_{i=1}^{3}\dot{x}_i\frac{\partial \dot{x}_i}{\partial q_j} \\
&= m\sum_{i=1}^{3}\dot{x}_1\frac{\partial}{\partial q_j}\left(\sum_{k=1}^{3}\frac{\partial \mathcal{X}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathcal{X}_i}{\partial t}\right) \\
&= m\sum_{i=1}^{3}\dot{x}_i\left(\sum_{k=1}^{3}\frac{\partial^2 \mathcal{X}_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 \mathcal{X}_i}{\partial q_j \partial t}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial T}{\partial \dot{q}_j} &= \frac{1}{2}m\sum_{i=1}^{3}\frac{\partial}{\partial \dot{q}_j}\dot{x}_i^2 = m\sum_{i=1}^{3}\dot{x}_i\frac{\partial \dot{x}_i}{\partial \dot{q}_j} \\
&= m\sum_{i=1}^{3}\dot{x}_1\frac{\partial}{\partial \dot{q}_j}\left(\sum_{k=1}^{3}\frac{\partial \mathcal{X}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathcal{X}_i}{\partial t}\right) \\
&= m\sum_{i=1}^{3}\dot{x}_i\frac{\partial \mathcal{X}_i}{\partial q_j} \ .
\end{aligned}$$

The time derivative of the last term is

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial q_j}\right) &= m\sum_{i=1}^{3}\frac{d}{dt}\left(\dot{x}_i\frac{\partial \mathcal{X}_i}{\partial q_j}\right) \\
&= m\sum_{i=1}^{3}\ddot{x}_i\frac{\partial \mathcal{X}_i}{\partial q_j} + m\sum_{i=1}^{3}\dot{x}_i\left(\sum_{k=1}^{3}\frac{\partial^2 \mathcal{X}_i}{\partial q_k \partial q_j} + \frac{\partial^2 \mathcal{X}_i}{\partial t \partial q_j}\right) \ .
\end{aligned}$$

Inserting these expressions in the Euler-Lagrange equation, we find

$$\begin{aligned} & \frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = 0 \\ \Rightarrow \quad & -m\sum_{i=1}^{3} \ddot{x}_i \frac{\partial \mathcal{X}_i}{\partial q_j} - \frac{\partial V}{\partial q_j} = 0 \\ \Rightarrow \quad & -m\sum_{i=1}^{3} \ddot{x}_i \frac{\partial \mathcal{X}_i}{\partial q_j} - \sum_{i=1}^{3} \frac{\partial V}{\partial x_i}\frac{\partial \mathcal{X}_i}{\partial q_j} = 0 \\ \Rightarrow \quad & \sum_{i=1}^{3} \left(m\ddot{x}_i + \frac{\partial V}{\partial x_i}\right)\frac{\partial \mathcal{X}_i}{\partial q_j} = 0 \; . \end{aligned} \tag{6.18}$$

In order for this last expression to be true for all points of space, the term within the parentheses has to vanish, i.e.,

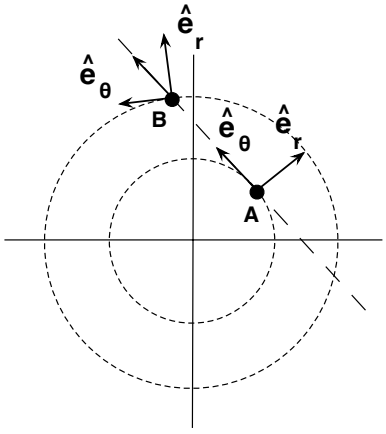
$$m\ddot{x}_i + \frac{\partial V}{\partial x_i} = 0 \; ,$$

which is nothing but Newton's second law of motion. Indeed, as expected, the Euler-Lagrange equation in generalized coordinates is equivalent to the Newtonian equation of motion in Cartesian coordinates.

Using our definition of generalized momenta and forces, we can now write the Euler-Lagrange equation in the form

$$\dot{p}_j = Q_j + \frac{\partial T}{\partial q_j} \; ,$$

which shows that the generalized momentum of an object may change even in the absence of external forces. Although this might appear at first to violate Newton's first law of motion, the following example illustrates the origin of this effect.



Consider the motion of a free object on a plane, which we describe in a spherical polar coordinate system, as shown in the margin figure. Because no force is acting on the object, its trajectory will be a straight line. At some time t_1, the object is at point A with its momentum vector perpendicular to its coordinate vector $\vec{r}$, i.e., the radial component of its momentum is zero. At same later time t_2, the object has moved to point B. At that time, its momentum is no longer perpendicular to its coordinate vector but has non-zero radial and azimuthal component. In other words, both the radial and azimuthal components of its momentum have changed, even though no force is acting on the object. This is clearly a result of the change in orientation of the basis vectors in the curvilinear coordinate system from point A to point B and not of a violation of Newton's first law of motion. It is also important to keep in mind here that the generalized coordinates need not be defined in an inertial frame, in which case Newton's laws of motion are not valid.

A final byproduct of the derivation we followed in this section is a relation between a Newtonian force in Cartesian coordinates and its corresponding generalized force in generalized coordinates. According to Newton's second law, the force acting on an object is equal to its mass times the acceleration, i.e., $F_i = m\ddot{x}_i$. Using equation (6.18) and the definition (6.17) of a generalized force, we obtain

$$Q_j = \sum_{i=1}^{3} F_i \frac{\partial \mathcal{X}_i}{\partial q_j} \; . \tag{6.19}$$

Even though the concept of a force is not necessary in predicting the motion of an object in Lagrangian mechanics, we will use this relation in Section 6.5 below in order to incorporate in this framework the effect of non-conservative forces.

Example 6.3: The angular momentum of an object in cylindrical coordinates

In this example, we will show that the angular momentum of an object with respect to the polar axis in a cylindrical coordinate system is the conjugate momentum to the azimuthal angle ϕ.

The conjugate momentum to the ϕ coordinate of an object is related to its kinetic energy T by

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} \, .$$

In order to evaluate this derivative we need to express the kinetic energy in terms of the time derivatives of the cylindrical coordinates of the object. The easiest way to achieve this is starting from the expression of the kinetic energy in a Cartesian coordinate system, which is always equal to

$$T = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 \right) \, ,$$

and then applying the transformation law to cylindrical coordinates.

The transformations between the Cartesian and the cylindrical coordinates are (see Section 1.6.2)

$$\begin{aligned} x_1 &= \rho \cos\phi \\ x_2 &= \rho \sin\phi \\ x_3 &= z \, . \end{aligned} \tag{6.20}$$

Using the chain rule to calculate the time derivatives of both sides in each equation we obtain

$$\dot{x}_1 = \frac{\partial x_1}{\partial r}\dot{r} + \frac{\partial x_1}{\partial \phi}\dot{\phi} + \frac{\partial x_1}{\partial z}\dot{z} = \cos\phi\dot{\rho} - \rho(\sin\phi)\dot{\phi} \, ,$$

$$\dot{x}_2 = \frac{\partial x_2}{\partial r}\dot{r} + \frac{\partial x_2}{\partial \phi}\dot{\phi} + \frac{\partial x_2}{\partial z}\dot{z} = \sin\phi\dot{\rho} + \rho(\cos\phi)\dot{\phi} \, ,$$

and

$$\dot{x}_3 = \frac{\partial x_3}{\partial r}\dot{r} + \frac{\partial x_3}{\partial \phi}\dot{\phi} + \frac{\partial x_3}{\partial z}\dot{z} = \dot{z} \, .$$

Inserting these relations into the expression for the kinetic energy we find, after some algebra,

$$\begin{aligned} T &= \frac{1}{2} m \left[\left(\cos\phi\dot{\rho} - \rho\sin\phi\dot{\phi} \right)^2 + \left(\sin\phi\dot{\rho} + \rho\cos\phi\dot{\phi} \right)^2 + \dot{z}^2 \right] \\ &= \frac{1}{2} m \left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2 \right) \, . \end{aligned} \tag{6.21}$$

The conjugate momentum to the azimuthal coordinate ϕ is, therefore,

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi} \, ,$$

The angular momentum of the object, on the other hand, is defined as

$$\vec{L} = m\vec{r} \times \vec{u} \, ,$$

where

$$\vec{r} = \rho \hat{e}_\rho + z\hat{e}_z$$

is the position vector of the particle and

$$\vec{u} = \dot{\rho}\hat{e}_\rho + \rho\dot{\phi}\hat{e}_\phi + \dot{z}$$

is its velocity in terms of the basis vectors of the cylindrical coordinate system (see section 2.4). We, therefore, obtain

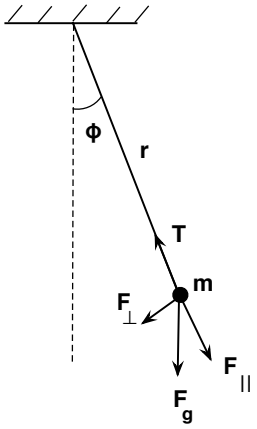
$$\begin{aligned} \vec{L} &= m\left(\rho\hat{e}_\rho + z\hat{e}_z\right) \times \left(\dot{\rho}\hat{e}_\rho + \rho\dot{\phi}\hat{e}_\phi + \dot{z}\hat{e}_z\right) \\ &= m\left[\rho^2\dot{\phi}\left(\hat{e}_\rho \times \hat{e}_\phi\right) + \rho\dot{z}\left(\hat{e}_\rho \times \hat{e}_z\right) + z\dot{\rho}\left(\hat{e}_z \times \hat{e}_\rho\right) + z\rho\dot{\phi}\left(\hat{e}_z \times \hat{e}_\phi\right)\right] \\ &= -mz\rho\dot{\phi}\hat{e}_\rho + m\left(z\dot{\rho} - \rho\dot{z}\right)\hat{e}_\phi + m\rho^2\dot{\phi}\hat{e}_z \, . \end{aligned}$$

The component of the angular momentum along the polar axis is then simply

$$L_z = m\rho^2\dot{\phi}\hat{e}_z = p_\phi \, ,$$

i.e., equal to the conjugate momentum to the azimuthal coordinate.

6.4 Motion with Constraints



The motion of objects is often constrained by forces that are very difficult to predict, either from first principles or using empirical laws. One of the simplest examples is the pendulum shown in the margin figure, which is constructed by hanging a small object of mass m from a very light, non-extendable thread of length l. Clearly, the thread exerts a force on the mass in order to constrain its motion along a circular arc. The direction of the force is parallel to the thread, but the magnitude is unknown.

In Newtonian mechanics we often address situations with forces of unknown magnitude. In the example of the pendulum, the tension of the thread is along the direction of the thread. As a result, we may decompose the gravitational force into two components, one parallel and one perpendicular to the thread, and write two equations of motion, one along each direction. These two equations allow us to calculate the displacement θ of the pendulum from the vertical direction as a function of time, as well as the magnitude T of the tension from the thread that is required to keep the pendulum intact. In this approach, the magnitude of the force of the constraint is part of the solution. Alternatively, we may write an equation for the rate of change of the angular momentum $\vec{L} = m\vec{r} \times \vec{u}$ of the mass with respect to the pivot point of the pendulum as

$$\frac{d\vec{L}}{dt} = \vec{N} \Rightarrow mr^2\frac{d\phi}{dt} = F_{\rm g}\sin\phi \, ,$$

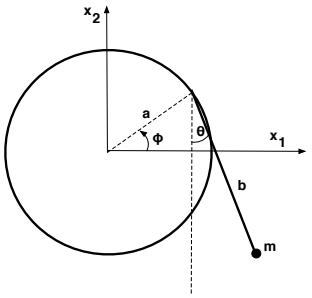
which does not depend on the tension, since the latter does not exert any torque. In this second approach, we calculate the displacement of the pendulum from the vertical direction as a function of time, without ever dealing with the unknown tension from the thread.

In Lagrangian mechanics, we similarly also use a number of different approaches to deal with constraint motions of objects, depending on the character of the constraints. If they can be expressed as algebraic relations between the generalized coordinates, i.e., if the constraints are holonomic, then we may define an appropriate set of generalized coordinates such that the constraint equations are trivially satisfied. In this case, the forces of the constraints do not enter the setup or the solution of the problem. If, on the other hand, we wish to calculate the forces of constraints or we are dealing with non-holonomic constraints, then we may use the method of Lagrange multipliers. The functional forms of the unspecified multipliers will emerge as part of the solution and will be, as we will see later in this section, related to the force of the constraints. The superiority of using Lagrangian mechanics to solve problems with constraint dynamical systems lies on the fact that we may address such problem with knowing a priori neither the magnitude nor the direction of the forces of constraints.

In the following two examples, we will explore the dynamics of two different constraint systems, following the two approaches we outlined above.

Example 6.4: A Pendulum Hanging from a Spinning Ring

In this example we will derive the equation of motion for a pendulum of length b that is attached on the rim of a vertical ring of radius a, which we spin at a constant angular frequency ω. For simplicity, we will assume that the pendulum consists of a massless rod at the end of which we have attached an object of mass m.



The orientation of the ring is specified by the angle ϕ defined between the spoke of the attachment point of the pendulum and the horizontal plane. On the other hand, the orientation of the pendulum is uniquely specified by the the Cartesian coordinates x_1 and x_2 of its massive end. As a result, the dynamical state of the system is specified by three coordinates: ϕ, x_1 and x_2. These three coordinates, however, have to satisfy two constraints. First, the ring is forced to spin at a constant angular frequency, i.e.,

$$\phi = \omega t \ ,$$

with an appropriate choice of the zero point in time. Second, the length of the pendulum is equal to b, i.e.,

$$\Big[(x_1 - a\cos\phi)^2 + (x_2 - a\sin\phi)^2 \Big]^{1/2} = b \ .$$

Because we have three coordinates and two constraints, the system has only one degree of freedom. As a result, we will only need to derive one Euler-Lagrange equation for an appropriately chosen generalized coordinate. We choose this coordinate to be the angle θ that the pendulum makes with the vertical direction. The

transformation relations between the Cartesian coordinates of the mass m attached to the end of the pendulum and the angle θ are

$$x_1 = a\cos\phi + b\sin\theta$$

and

$$x_2 = a\sin\phi - b\cos\theta \ .$$

Similarly, the relations between the Cartesian components of its velocity and the time derivative of the angle θ are

$$\dot{x}_1 = \frac{\partial x_1}{\partial\phi}\dot{\phi} + \frac{\partial x_1}{\partial\theta}\dot{\theta} = -a\omega\sin\phi - b\cos\theta\dot{\theta}$$

and

$$\dot{x}_2 = \frac{\partial x_2}{\partial\phi}\dot{\phi} + \frac{\partial x_2}{\partial\theta}\dot{\theta} = a\omega\cos\phi + b\sin\theta\dot{\theta} \ ,$$

where we have used the constraint on the angle ϕ to write $\dot{\phi} = \omega$.

In order to write the kinetic and potential energy of the mass m in terms of the generalized coordinate θ, we start from the known expressions in terms of its Cartesian coordinates and then apply the above transformations. We find, after a small amount of algebra,

$$\begin{aligned} T &= \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) \\ &= \frac{1}{2}m\left[a^2\omega^2 + b^2\dot{\theta}^2 - 2ab\omega\dot{\theta}\sin(\omega t)\cos\theta + 2ab\omega\dot{\theta}\cos(\omega t)\sin\theta\right] \\ &= \frac{1}{2}m\left[a^2\omega^2 + b^2\dot{\theta}^2 + 2ab\omega\dot{\theta}\sin(\theta - \omega t)\right] \end{aligned}$$

and

$$V = mgx_2 = mg\left[a\sin(\omega t) - b\cos\theta\right] \ .$$

The Lagrangian of the system is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}m\left[a^2\omega^2 + b^2\dot{\theta}^2 + 2ab\omega\dot{\theta}\sin(\theta - \omega t)\right] - mg\left[a\sin(\omega t) - b\cos\theta\right] \end{aligned}$$

and the Euler-Lagrange equation becomes

$$\begin{aligned} &\frac{\partial L}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = 0 \\ \Rightarrow \quad &-g\sin\theta - b\ddot{\theta} + a\omega^2\cos(\theta - \omega t) = 0 \ . \end{aligned}$$

It is reasonable to assume at this point that $b \neq 0$, since $b = 0$ corresponds to the trivial situation of a mass m attached on a vertical ring that we spin at a constant rate. We can, therefore, divide both sides of the above equation by b and rearrange its terms to write

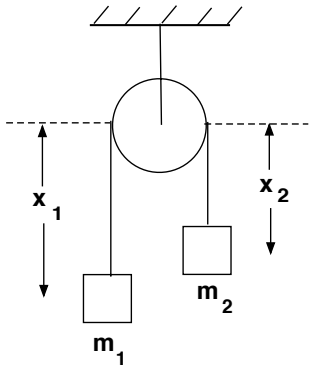
$$\ddot{\theta} = -\frac{g}{b}\sin\theta + \frac{a\omega^2}{b}\cos\left(\theta - \omega t\right) \ .$$

This equation describes the motion of an oscillator, driven periodically by external forcing. When $\omega = 0$, i.e., when we do not spin the vertical ring but keep it in place, the equation of motion becomes

$$\ddot{\theta} = -\frac{g}{b}\sin\theta \simeq -\frac{g}{b}\theta$$

where in the last equality we also made the assumption that $\theta \ll 1$, to show explicitly that it describes the motion of a simple harmonic oscillator. We will study in detail the dynamics of free and driven oscillators in Chapter 11.

Example 6.5: Atwood's machine



The second dynamical system with constraints that is commonly explored in an introduction to Lagrangian mechanics is Atwood's machine. As shown in the margin figure, the machine consists of two masses m_1 and m_2 attached by a massless, non-extendable rope of length l, which is placed around a massless and frictionless pulley.

The two masses m_1 and m_2 are constraints to move along the vertical direction. As a result, we can uniquely specify their location by choosing as generalized coordinates the vertical distances x_1 and x_2 of each mass from the location of the pulley, which we assume to have a negligible size. The two distances, however, are constraints by the length of the rope such that

$$x_1 + x_2 = l \ .$$

This constraint also translates into a constraint on their velocities as

$$\dot{x}_1 = -\dot{x}_2 \ .$$

The kinetic and potential energies of the system of two masses are

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 = \frac{1}{2} \left(m_1 + m_2 \right) \dot{x}_1^2$$

and

$$V = -m_1 g x_1 - m_2 g x_2 = \left(m_2 - m_1 \right) g x_1 - m_2 g l \ ,$$

respectively, where we have set the zero point of the potential energy at the height of the pulley. The Lagrangian of the system becomes

$$L = T - V = \frac{1}{2} \left(m_1 + m_2 \right) \dot{x}_1^2 - \left(m_2 - m_1 \right) g x_1 - m_2 g l$$

and the Euler-Lagrange equation for the coordinate x_1 is

$$\begin{aligned} & \frac{\partial L}{\partial x_1} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_1} \right) = 0 \\ \Rightarrow \quad & \left(m_1 - m_2 \right) g - \left(m_1 + m_2 \right) \ddot{x}_1 = 0 \\ \Rightarrow \quad & \ddot{x}_1 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) g \ . \end{aligned}$$

This equation is consistent with our expectation that, if $m_1 > m_2$, the first object will accelerate downwards, pulling the second, lighter object with it. If $m_1 = m_2$, then the two objects balance each other and neither object feels any acceleration.

The Euler-Lagrange equation describes motion with constant acceleration and can be integrated simply to give

$$x_1 = x_1^0 + \frac{1}{2} \left(\frac{m_1 - m_2}{m_1 + m_2} \right) g t^2 \ ,$$

where x_1^0 is the position of mass m_1 at time $t = 0$ and we assumed that the initial velocity is equal to zero. At this point, we can also use the constraint relation to write the solution for the motion of mass m_2 as

$$\begin{aligned} x_2 &= L - x_1 = L - x_1^0 - \frac{1}{2}\left(\frac{m_1 - m_2}{m_1 + m_2}\right) g t^2 \\ &= x_2^0 - \frac{1}{2}\left(\frac{m_1 - m_2}{m_1 + m_2}\right) g t^2 \ , \end{aligned}$$

where we defined x_2^0 as the initial position of mass m_2.

In the two examples we just explored, the constraints in the motion of the objects were imposed as holonomic, algebraic equations between their coordinates. Moreover, in neither of the two examples were we interested in the forces of the constraints. For these two reasons, we chose appropriate generalized coordinates to describe the motions of the systems such that the constraints were trivially satisfied. The net result, in each case, was an Euler-Lagrange equation that without any complications imposed by the constraints or their forces. As we discussed earlier, however, such an approach is not possible for all dynamical systems and, in some cases, might not even be desirable. The method of Lagrange multipliers allows us, in general, to address constrained dynamical problems, whether the constraints are holonomic or not and whether we are interested in calculating the forces on the constraints or not.

In order to see the connection between the Lagrange multipliers and the forces on constraints, we will study a dynamical system that is uniquely described by a set of N generalized coordinates q_i, $i = 1, ..., N$ and a Lagrangian function $L(q_i, \dot{q}_i)$. We will also assume that the evolution of this system is subject to K constraints, each of which can be expressed in terms of a relation of the form

$$g_j(q_1, ..., q_N; \dot{q}_1, ..., \dot{q}_N; t) = 0, \qquad j = 1, ..., K \ .$$

As we discussed in Section 5.5, in order to solve such a problem we introduce K Lagrangian multipliers $\lambda_j(t)$, $j = 1, ..., K$, and construct a new Lagrangian function

$$\mathcal{L} \equiv L(q_i, \dot{q}_i) + \sum_{j=1}^{K} \lambda_j(t) q_j(q_i, \dot{q}_i) \ .$$

The equations of motion of the constrained system are then the Euler-Lagrange equations for the new Lagrangian function,

$$\begin{aligned} & \frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = 0 \\ \Rightarrow \quad & \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \sum_{j=1}^{K} \lambda_j(t) \left[\frac{\partial g_j}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial g_j}{\partial \dot{q}_i}\right)\right] = 0 \ . \end{aligned} \tag{6.22}$$

The set of N equations (6.22), augmented by the K constraints (6.4), will allow us to solve for the N generalized coordinates $q_i(t)$ and the K Lagrange multipliers $\lambda_j(t)$, all of which depend, in general, on time.

What if, on the other hand, we could model the microphysical interaction between the object we are studying and each of the constraints and had devised a relation for the force of the constraint? For example, in Atwood's machine discussed in Example 6.5, this would amount to modeling the infinitesimal stretching of the rope that connects the two masses and devising an empirical relation to calculate the instantaneous tension on the rope. In that case, we wouldn't have to model the motion of the constrained dynamical system using the method of Lagrange multipliers. Instead, we would use the Cartesian components $F_{j,k}$ $(k = 1, ..., 3)$ of the Newtonian force of the $j-$th constraint to obtain the components of the corresponding generalized force using equation (6.19), i.e.,

$$Q_{j,i} = \sum_{i=1}^{3} F_{j,k} \frac{\partial \mathcal{X}_k}{\partial q_i} \ . \tag{6.23}$$

We would then add this force to the Euler-Lagrange equation of the unconstrained system and write

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \sum_{j=1}^{K} Q_{j,i} = 0 \ . \tag{6.24}$$

The set of N differential equations (6.24) provide a description of the evolution of the constrained dynamical system that is completely equivalent to the one provided by equations (6.22). Indeed, if we have modeled the forces of the constraints correctly, then the same path, which we will denote by $q_i^0(t)$, $i = 1, ..., N$, should satisfy both sets of equations. Comparison of equations (6.22) and (6.24) shows that this would be true, if the forces of the constraints are related to the Lagrange multipliers by

$$\boxed{Q_{j,i} = \lambda_j(t)\left[\frac{\partial g_j}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial g_j}{\partial \dot{q}_i}\right)\right] \ ,} \tag{6.25}$$

Forces of Constraints and Lagrange Multipliers

where it is understood that all derivatives in the right-hand side of this relation are evaluated along the path $q_i^0(t)$ that satisfies the constrained Euler-Lagrange equations.

Equation (6.25) illustrates the connection between the Lagrange multipliers and the forces of constraints that we alluded to in Chapter 5. It also demonstrates the utility of Lagrangian dynamics in calculating forces of constraints without any prior knowledge of the magnitude or direction of the forces. Note, however, that the generalized forces calculated with this equation need not be forces in the traditional, Newtonian sense of the term. However, given the components of the generalized forces of the constraints, we can always calculate the corresponding Newtonian forces by inverting the system of equations (6.23) that connects them.

In the following example, we will revisit Atwood's machine and solve it using the method of Lagrange multipliers in order to calculate the forces of the constraints, i.e., the tension on the rope.

Example 6.6: Revisiting Atwood's Machine with Lagrange Multipliers

Our goal in this example is to write the equations of motion for each of the two masses in Atwood's machine (see Example 6.5), while imposing the constraint in their motion as a Lagrange multiplier.

Using the same coordinate system as in Example 6.5, we write the kinetic and potential energies of the system of two masses as

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$$

and

$$V = -m_1 g x_1 - m_2 g x_2 \ ,$$

respectively. The Lagrangian of the system, therefore, becomes

$$L = T - V = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + m_1 g x_1 + m_2 g x_2 \ .$$

Both the Lagrangian as well as the constraint

$$g(x_1, x_2) = x_1 + x_2 - l = 0$$

are independent of time. As a result, when we construct the combined Lagrangian

$$\mathcal{L} = L + \lambda g(x_1, x_2) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + m_1 g x_1 + m_2 g x_2 + \lambda(x_1 + x_2 - l) \ ,$$

the Lagrange multiplier λ will also be independent of time.

The two Euler-Lagrange equations that correspond to the coordinates x_1 and x_2 are

$$\begin{aligned} & \frac{\partial L}{\partial x_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = 0 \\ \Rightarrow \quad & m_1 g - m_1\ddot{x}_1 - \lambda = 0 \end{aligned} \tag{6.26}$$

and

$$\begin{aligned} & \frac{\partial L}{\partial x_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = 0 \\ \Rightarrow \quad & m_2 g - m_2\ddot{x}_2 - \lambda = 0 \ , \end{aligned} \tag{6.27}$$

respectively. These are two differential equations for three unknowns: x_1, x_2, and λ. The third equation needed to close the system of equations is provided by the constraint.

Subtracting equation (6.27) from equation (6.26) allows us to eliminate the Lagrange multiplier as

$$(m_1 - m_2)\, g - (m_1\ddot{x}_1 - m_2\ddot{x}_2) = 0 \ .$$

Taking the second time derivative of the constraint provides a relation between the two accelerations as

$$\frac{d^2}{dt^2}g(x_1, x_2) = 0 \Rightarrow \ddot{x}_1 + \ddot{x}_2 = 0 \ .$$

Combining the last two equations finally leads to two uncoupled differential equations for the two coordinates, i.e.,

$$\begin{aligned} \ddot{x}_1 &= \frac{m_1 - m_2}{m_1 + m_2}g \\ \ddot{x}_2 &= \frac{m_2 - m_1}{m_1 + m_2}g \ . \end{aligned} \tag{6.28}$$

As expected, the two equations of motion are the same as the ones we derived in Example 6.4, where we imposed the constraint on the motion of the two masses by substitution and not with the method of Lagrange multipliers. The method we are following here, however, allows us to calculate the force of the constraint, i.e., the tension on the rope. Adding equations (6.27) and (6.26) leads to

$$(m_1 - m_2)\, g - (m_1 \ddot{x}_1 + m_2 \ddot{x}_2) - 2\lambda = 0 \; .$$

Inserting then into this equation the relations (6.28) for the two accelerations provides an expression for the constant Lagrange multiplier

$$\lambda = 2g \frac{m_1 m_2}{m_1 + m_2} \; .$$

The generalized forces on the two masses are related to the Lagrange multiplier and the constraint by equation (6.25), i.e.,

$$\begin{aligned} Q_1 &= \lambda \frac{\partial g}{\partial x_1} = \frac{2 m_1 m_2}{m_1 + m_2} g \\ Q_2 &= \lambda \frac{\partial g}{\partial x_2} = \frac{2 m_1 m_2}{m_1 + m_2} g = Q_1 \; . \end{aligned}$$

Because we have used Cartesian coordinates as the generalized coordinates in this problem, the Newtonian forces on the masses m_1 and m_2 are equal to the generalized forces. In other words, the tension on the rope is equal to

$$T = \frac{2 m_1 m_2}{m_1 + m_2} g \; .$$

The maximum tension the rope needs to withstand is equal to the weight of the heaviest object. Indeed, setting without loss of generality $m_2 \leq m_1$ and expressing the tension in terms of the mass ratio $R \equiv m_1/m_2 \geq 1$ we find

$$T = \frac{2}{R+1} m_1 g \leq m_1 g \; .$$

The equality in the above relation occurs when $R = 1$, i.e., when the two objects have the same mass.

6.5 Motion Under the Influence of Non-Conservative Forces

Hamilton's principle, which was introduced in the beginning of this chapter as the foundation of Lagrangian mechanics, is based on the concept of conservative forces, for which a potential energy of interaction can be uniquely defined. In several situations in everyday life, however, we deal with non-conservative forces, such as friction and aerodynamic drag. In these section we will discuss how to incorporate the effects of non-conservative forces in the Euler-Lagrange equations that describe the evolution of a dynamical system in terms of generalized coordinates.

Before embarking into the study of non-conservative forces in Lagrangian mechanics, however, it is important to emphasize that the requirement that forces are

conservative is not a serious limitation of Hamilton's principle as a fundamental description of the physical world. Indeed, as we discussed in Chapters 2 and 3, all fundamental macroscopic forces in the Universe are believed to be conservative in nature. Non-conservative forces appear only when we try to model macroscopically the complex interactions between systems using empirical models. For example, when we model friction between two surfaces, we consider only the macroscopic translation of the two surfaces and not the microscopic motions of their atoms. In Lagrangian language, we do not take into account the internal degrees of freedom that describe the rapid oscillations of the atoms of one surface and are excited by interactions with the atoms of the other surface. The excitation of such oscillations, which we describe thermodynamically as an increase in the temperature of each surface, requires mechanical energy that is taken from the kinetic energy of the macroscopic motion of the surfaces. Moreover, in a macroscopic model of the force of friction we do not consider the fact that these oscillating atoms will emit photons and radiate away the energies of their random motions. In other words, neglecting the internal degrees of freedom as well as the fact that the system of the two surfaces is not thermodynamically closed makes our empirical model of the friction force to be non-conservative.

Empirical models of non-conservative forces are very useful in reducing the complexity of problems in the physical world. For this reason, we need to find a way to incorporate them in the framework of Lagrangian mechanics. Our discussion of generalized forces and of the equivalence between Newtonian and Lagrangian mechanics offers such a way.

We consider first the evolution of a dynamical system in the absence of non-conservative forces. Assuming that there are N degrees of freedom in this problem, we can identify an appropriate set of N generalized coordinates q_j, $j = 1, ..., N$ that describe uniquely the dynamical state of the system. We denote by $x_i = \mathcal{X}_i(q_j, t)$ the transformation laws between a set of Cartesian coordinates x_i, $i = 1, ..., N$ and the generalized coordinates and express the Lagrangian of the system $L(q_j, \dot{q}_j)$ in terms of the latter coordinates and their time derivatives. The Euler-Lagrange equations for this system, in the absence of non-conservative forces are simply

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \, .$$

We now consider the effect on this dynamical system of a non-conservative interaction that we describe empirically by a Newtonian force with components F_i, $i = 1, ..., N$ in a Cartesian coordinate system. The equivalence between Newtonian and Lagrangian mechanics we proved in Section 6.3 implies that the equations of motion in the two frameworks will be equivalent if we augment the Euler-Lagrange equations by a generalized force Q_j, $j = 1, ..., N$ as

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = -Q_j \, .$$

where the components of the generalized force are related to the Cartesian components of the non-conservative force and the transformation between Cartesian and

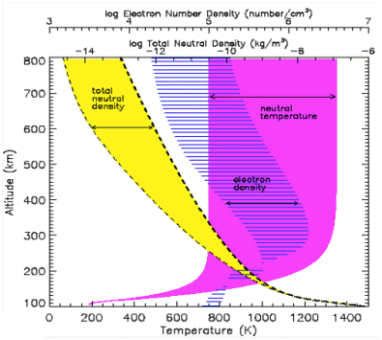
generalized coordinates by equation (6.19), i.e.,

$$Q_j = \sum_{i=1}^{3} F_i \frac{\partial \mathcal{X}_i}{\partial q_j} \ .$$

In the following application, we will use the methods of Lagrangian mechanics to investigate the influence of atmospheric drag on the motion of satellites in a low Earth orbits.

6.6 Application: The Reentry of Low Earth Orbit Satellites

The vast majority of artificial satellites are placed in orbits with relatively low distances from the Earth's surface. Traditionally, objects with distances smaller than about 2000 km from the Earth's surface are characterized as low Earth orbit satellites. At distances as small as $100 - 300$ km, the density in the Earth's atmosphere is low when compared to that on the surface but high enough to slow down the motion of the satellites and cause them to loose altitude and eventually burn on their way to the surface. The density and temperature in these layers of the atmosphere (called the thermosphere) depend strongly on the Solar activity. The margin figure shows the range of temperatures (rightmost shaded region) and densities (leftmost shaded region) in the thermosphere for different levels of solar activity. At an altitude of 100 miles (or 160 km), the density is as low as $\simeq 10^{-9}$ kg m^{-3} and drops to $\simeq 10^{-14} - 10^{-12}$ kg m^{-3} at 300 miles (or 480 km).



Our goal in this section is to write the equations of motion for a satellite in low earth orbit, taking into account the atmospheric drag. We will use the empirical model for the drag force that we developed in Chapter 2, i.e.,

$$\vec{F}_{\rm d} = -\frac{1}{2}\rho C_{\rm a} A |\vec{u}| \vec{u} \ ,$$

where ρ is the density of the atmosphere, $C_{\rm a}$ is the coefficient of aerodynamic drag for the satellite, which depends on its exact shape, A is the projected surface area of the satellite perpendicular to its direction of motion, and $\vec{u}$ is its velocity vector.

The velocity vector of the satellite, the drag force, and the Earth's gravitational acceleration $\vec{g}$ lie on a single plane that passes through the Earth's center. This will be the plane of the satellite's orbit, since no interaction exist in the system that will force it to move away from that plane. We will, therefore, consider a set of Cartesian coordinates x_1 and x_2 on the orbital plane, with their origin at the center of the Earth, as well as a set of polar coordinates r and ϕ. The transformations between the Cartesian and polar coordinates are

$$\begin{aligned} x_1 &= r\cos\phi \\ x_2 &= r\sin\phi \end{aligned}$$

and between the corresponding velocities are

$$\dot{x}_1 = \frac{\partial x_1}{\partial r}\dot{r} + \frac{\partial x_1}{\partial \phi}\dot{\phi} = -r\dot{\phi}\sin\phi + \dot{r}\cos\phi$$

$$\dot{x}_2 = \frac{\partial x_2}{\partial r}\dot{r} + \frac{\partial x_2}{\partial \phi}\dot{\phi} = r\dot{\phi}\cos\phi + \dot{r}\sin\phi \; .$$

We will now use these transformations to write the kinetic energy of the satellite in terms of its generalized velocities as

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) = \frac{1}{2}m\left(r^2\dot{\phi}^2 + \dot{r}^2\right) \; ,$$

where m is the mass of the satellite. We will also use the general expression for the potential energy in the Earth's gravitational field,

$$V = -\frac{GM_\oplus m}{r} \; ,$$

where $M_\oplus$ is the mass of the Earth, to write the Lagrangian of the system as

$$L = T - V = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) = \frac{1}{2}m\left(r^2\dot{\phi}^2 + \dot{r}^2\right) + \frac{GM_\oplus m}{r} \; .$$

We will finally apply the transformations between the Cartesian and generalized coordinates in order to derive the components of the generalized aerodynamic drag forces starting from the Cartesian components of the Newtonian force

$$\begin{aligned} F_{d,1} &= -\frac{1}{2}\rho C_{\rm a} A\left(\dot{x}_1^2 + \dot{x}_2^2\right)^{1/2}\dot{x}_1 \\ F_{d,2} &= -\frac{1}{2}\rho C_{\rm a} A\left(\dot{x}_1^2 + \dot{x}_2^2\right)^{1/2}\dot{x}_2 \; . \end{aligned}$$

After a small amount of algebra, we find

$$\begin{aligned} Q_r &= F_1\frac{\partial x_1}{\partial r} + F_2\frac{\partial x_2}{\partial r} \\ &= -\frac{1}{2}\rho C_{\rm a} A\left(\dot{r}^2 + r^2\dot{\phi}^2\right)^{1/2}\dot{r} \end{aligned}$$

and

$$\begin{aligned} Q_\phi &= F_1\frac{\partial x_1}{\partial \phi} + F_2\frac{\partial x_2}{\partial \phi} \\ &= -\frac{1}{2}\rho C_{\rm a} A\left(\dot{r}^2 + r^2\dot{\phi}^2\right)^{1/2} r^2\dot{\phi} \; . \end{aligned}$$

Inserting these expression into the Euler-Lagrange equations we obtain

$$\begin{aligned} &\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = -Q_r \\ \Rightarrow \quad & mr\dot{\phi}^2 - m\ddot{r} - \frac{GM_\oplus m}{r^2} = \frac{1}{2}\rho C_{\rm a} A\left(\dot{r}^2 + r^2\dot{\phi}^2\right)^{1/2}\dot{r} \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial L}{\partial \phi} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = -Q_\phi \\ \Rightarrow \quad & -mr^2\ddot{\phi} - 2mr\dot{r}\dot{\phi} = \frac{1}{2}\rho C_{\rm a} A\left(\dot{r}^2 + r^2\dot{\phi}^2\right)^{1/2} r^2\dot{\phi} \; . \end{aligned}$$

Rearranging some terms and dividing both sides of these equations by the mass m of the satellite we obtain

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM_\oplus}{r^2} - \frac{1}{2}\left(\frac{C_{\rm a}A}{m}\right)\rho\left(\dot{r}^2 + r^2\dot{\phi}^2\right)^{1/2} \tag{6.29}$$

$$r\ddot{\phi} + 2\dot{r}\dot{\phi} = \frac{1}{2}\left(\frac{C_{\rm a}A}{m}\right)\rho\left(\dot{r}^2 + r^2\dot{\phi}^2\right)^{1/2} r\dot{\phi} \; . \tag{6.30}$$

These are two, coupled second-order differential equations in time for the two generalized coordinates r and ϕ. They can be integrated easily on a computer, with the two components of the position and the velocity of the satellite as initial conditions.

For the purposes of our discussion here, we can estimate the characteristic timescale at which the satellite reenters the Earth's atmosphere by making a number of simplifying assumptions. We will consider a satellite in a circular orbit, at a high enough altitude that the aerodynamic drag force is negligible compared to the force of gravity. In this case, the satellite mostly stays on the circular orbit, while slowly drifting towards the Earth's surface. If we denote by $P = 2\pi/\dot{\phi}$ the orbital period of the satellite at any given altitude and by

$$\tau \equiv \left| \frac{r}{\dot{r}} \right|$$

the characteristic timescale for reentry, the above assumptions imply that $\tau >> P$.

In the radial equation of motion, i.e., equation (6.29), the drag force is negligible compared to the gravitational force, by assumption. Moreover, the ratio of the two terms on the left-hand side of the same equation scales as

$$\frac{\ddot{r}}{r\dot{\phi}^2} = \frac{\frac{d^2r}{dt^2}}{r\left(\frac{d\phi}{dt}\right)^2} \simeq \frac{\frac{r}{\tau^2}}{\frac{r}{P^2}} = \left(\frac{P}{\tau}\right)^2 \ll 1$$

and, therefore, the radial acceleration term $\ddot{r}$ is negligible. Equation (6.29) becomes simply

$$\dot{\phi}^2 \simeq \frac{GM_\oplus}{r^3} \ ,$$

which is the usual expression for the angular velocity of a satellite in orbit.

In the azimuthal equation of motion, i.e., equation (6.30), we can also compare the ratio of the first two terms by noting that the azimuthal acceleration is expected to scale as

$$\ddot{\phi} = \frac{d\dot{\phi}}{dt} \simeq \frac{d}{dt}\left(\frac{GM_\oplus}{r^3}\right)^{1/2} = -\frac{3}{2}\left(\frac{GM_\oplus}{r^3}\right)^{1/2}\frac{\dot{r}}{r} \simeq -\frac{3}{2}\left(\frac{\dot{\phi}}{\tau}\right) = -3\pi\left(\frac{1}{P\tau}\right) \ .$$

As a result, the ratio of the two first terms in equation (6.30) is

$$\frac{r\ddot{\phi}}{2\dot{r}\dot{\phi}} \simeq \frac{1}{2}\left(\frac{r}{\dot{r}}\right)\left(\frac{\ddot{\phi}}{\dot{\phi}}\right) \simeq -\frac{3}{4} \ .$$

The two terms, therefore, are of comparable magnitude and, therefore, each one should be comparable to the term in the right-hand side of equation (6.30). However, we can also compare the two terms in parenthesis in the right-hand side of the same equation and find that

$$\frac{r^2\dot{\phi}^2}{\dot{r}^2} \simeq 4\pi^2\left(\frac{\tau}{P}\right)^2 \gg 1$$

i.e.,

$$\dot{r}^2 + r^2\dot{\phi}^2 \simeq r^2\dot{\phi}^2 \ .$$

With all these approximations in mind, we can finally write

$$2\dot{r}\dot{\phi} \simeq \frac{1}{2}\left(\frac{C_{\rm a}A}{m}\right)\rho r^2\dot{\phi}^2$$

from which we can estimate the characteristic timescale for reentry as

$$\tau = |\frac{r}{\dot{r}}| = 4\left(\frac{m}{C_{\rm a}A}\right)\frac{1}{\rho r\dot{\phi}} = \frac{4}{\sqrt{GM_\oplus}}\left(\frac{m}{C_{\rm a}A}\right) r^{1/2}\rho^{-1} \; .$$

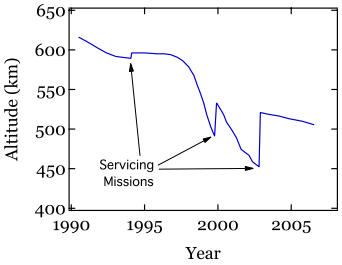
As we might have expected, the timescale for reentry is inversely proportional to the projected surface area per unit mass of the satellite, because satellites with large such ratios experience significant aerodynamic drag. Moreover, the timescale increase with increasing distance from the center of the Earth as well as with decreasing density of the atmosphere, for the same reason.

As an example, we can estimate the reentry timescale for the Hubble Space Telescope, which is orbiting the Earth at an altitude of $\simeq 550$ km. The mass of the satellite is about 11,000 kg and its projected area is $\simeq 60$ m^2. Setting the mass and radius of the Earth at $M_\oplus = 5.97\times10^{24}$ kg and $R_\oplus = 6.38\times10^8$ cm, respectively, as well as an average value for the density of the atmosphere at an altitude of 550 km (see the discussion in the beginning of this section) we find that the characteristic reentry timescale becomes

$$\tau \simeq \frac{1.5}{C_{\rm a}}\left(\frac{M_\oplus}{5.97\times10^{24}\ {\rm kg}}\right)^{-1/2}\left(\frac{m}{11,000\ {\rm kg}}\right)\left(\frac{A}{60\ {\rm m}^2}\right)^{-1} \left(\frac{r}{6380+480\ {\rm km}}\right)^{1/2}\left(\frac{\rho}{5\times10^{-14}\ {\rm g\ m^{-3}}}\right)^{-1}\ {\rm yr} \; ,$$

i.e., of the order of a couple of years. The Hubble Space Telescope, however, has been in orbit since April 1990. Unless special consideration had been given in advance of placing the Space Telescope in orbit, it would have reentered long ago.

Several satellites carry with them propellant and small engines, which are fired at regular intervals and replace the satellite to its original orbit. The Hubble Space Telescope, on the other hand, did not rely on such a process to avoid reentry. Instead, every few years, the Space Telescope was captured by the Space Shuttle, for repairs and to update the scientific instruments on board (see the margin figure). At the end of each servicing mission, the Space Telescope was replaced in a higher orbit, extending this way its lifetime.



The margin figure shows the evolution of the altitude of the Space Telescope with time. The slow decay of the orbit at timescales of ∼years is evident. Each sharp increase of the altitude marks one of the servicing missions, which captured the Space Telescope, repaired it, and reset its orbit. The missions shown took place in December 1993, in December 1999, and in March 2002. Between the servicing missions, the rate of orbital decay shows large variations, as it can been seen, for example, before and after the March 2002 orbital boost. This was caused by a large variation in the solar activity, which peaked at around 2002 and affected the density and temperature in the thermosphere.

6.7 Time and Energy in Lagrangian Dynamics

In the beginning of this chapter we introduced the foundations of Lagrangian dynamics based on the concepts of kinetic and potential energies. However, the total

mechanical energy of the system, i.e., the sum of its kinetic and potential energies, never entered our discussion. In fact, Hamilton's principle states that the evolution of the dynamical system is such that the *difference* between the kinetic and potential energies, $L = T - V$, is extremized and not their sum, $T + V$. Tracing the origin of the minus sign in front of the potential energy requires a discussion of the fundamental difference between time and space in the theories of special and general relativity. We will return to this issue in Chapters 20-22. However, in this section, we will discuss the role of the mechanical energy in Lagrangian dynamics and its intimate relation to the concept of time. In fact, we will prove that, under some general assumptions, the mechanical energy of a system is the conjugate momentum to time.

In Section 6.2, we defined the conjugate momentum p_j to a generalized coordinate q_j as

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j} \ ,$$

where T is the kinetic energy of the system and $\dot{q}_j$ is the time derivative of the coordinate. This definition shows that, in order to reach our goal, we will need to elevate time from being the independent variable to becoming a generalized coordinate in the evolution of a dynamical system.

Although in the development of Lagrangian dynamics so far we have used time as the independent variable, nothing limits us to use instead a different variable to trace uniquely the evolution of a dynamical system. As an illustrative example, suppose that we want to describe the motion of a single object in a potential field. Our choice so far has been to use an outside observer, who we will call Alice, to measure time with her clock and mark the generalized coordinates of the object as a function of that time, e.g., $q_1(t)$, $q_2(t)$, and $q_3(t)$. Suppose now that we will also consider a different observer, Bob, who moves along with the particle and uses a different clock to measure time. The catch is that Bob's clock is broken and measures "time" at a different rate, which he denotes by τ. Throughout the motion of the object, Bob records its coordinates as a function of his measurement of time, i.e., $q_1(\tau)$, $q_2(\tau)$, and $q_3(\tau)$. Moreover, at each instant, Bob asks Alice for her time and records it as $t(\tau)$. Bob's description of the motion of the object is completely equivalent to that obtained by Alice. In fact, with a proper translation of their measurements, they will both agree on the motion of the particle. The only difference is that Alice used (the correct) time t as the independent variable, whereas Bob used this other variable τ as the independent variable. For Bob, time is a generalized coordinate.

The velocity along each of the original generalized coordinates that Alice measures will be related to the velocity measured by Bob by

$$\dot{q}_i = \frac{dq_i}{dt} = \frac{\frac{dq_i}{d\tau}}{\frac{dt}{d\tau}} = \frac{q_i'}{t'} \ . \tag{6.31}$$

In this equation, we have used overdots to denote the velocities measured by Alice and primes for the velocities measured by Bob. Because, in Bob's description, the time t is a generalized coordinate, he can also define a corresponding "velocity" as

$t' = dt/d\tau$.

According to Hamilton's principle, the evolution of the system will be such that, in Alice's description, its path will extremize the integral

$$S = \int_A^B L(q_1, ..., q_N, \dot{q}_1, ..., \dot{q}_N; t)dt \ ,$$

where we have generalized the Lagrangian L to depend on N coordinates, the corresponding velocities, and time. Since Alice and Bob agree on the observed dynamical evolution of the system, the same integral has to be extremized, even when expressed in terms of Bob's measurements. Using equation (6.31) and the fact that $dt = t'd\tau$, we can rewrite the above integral as

$$S = \int_A^B L\left(q_1, ..., q_N, t, \frac{q'_1}{t'}, ..., \frac{q'_N}{t'}\right) t' d\tau \ .$$

Note that we have displaced the location of the variable t in the arguments of the Lagrangian to show explicitly that it is now a generalized coordinate.

Equation (6.7) shows that Bob may also use a principle similar to Hamilton's to calculate the evolution of the system. The only difference with Alice is that his Lagrangian action needs to be

$$L'(q_1, ..., q_N, t, q'_1, ..., q'_N, t') = L\left(q_1, ..., q_N, t, \frac{q'_1}{t'}, ..., \frac{q'_N}{t'}\right) t' \ .$$

In this case, the evolution of the system will be such that the integral

$$S = \int_A^B L'(q_1, ..., q_N, t, q'_1, ..., q'_N, t')d\tau \ ,$$

is extremized.

Having completed the transformation of time from an independent variable to a generalized coordinate, we can now calculate the conjugate momentum to time according to Bob. We have

$$\begin{aligned} p_t &\equiv \frac{d}{dt'} L'(q_1, ..., q_N, t, q'_1, ..., q'_N, t') \\ &= \frac{d}{dt'}\left[t' L\left(q_1, ..., q_N, t, \frac{q'_1}{t'}, ..., \frac{q'_N}{t'}\right)\right] \\ &= L + t' \sum_{i=1}^{N} \left(\frac{\partial L}{\partial \dot{q}_i}\right)\left(\frac{d\dot{q}_i}{dt'}\right) \ , \end{aligned}$$

where we have used equation (6.31) to convert Bob's velocities, q'_i, back to Alice's velocities, $\dot{q}_i$. Using the same relation between the two sets of velocities we can also write

$$\frac{d\dot{q}_i}{dt'} = \frac{d}{dt'}\left(\frac{q'_i}{t'}\right) = -\frac{q'_i}{t'^2} = -\frac{\dot{q}_i}{t'} \ .$$

Inserting this equation into the expression for the conjugate momentum to time we finally obtain

$$p_t = L - \sum_{i=1}^{N} \dot{q}_i \left(\frac{\partial L}{\partial \dot{q}_i}\right) = L - \sum_{i=1}^{N} \dot{q}_i p_i \ .$$

For reasons that will become obvious very shortly, we define the Hamiltonian of they dynamical system as

$$H = \sum_{i=1}^{N} \dot{q}_i p_i - L \ . \tag{6.32}$$

Hamiltonian of a Dynamical System

We will now show that, under two general but important assumptions, the Hamiltonian of a dynamical system is equal to its mechanical energy, i.e., $H = T + V$.

We will consider a dynamical system in which *(i)* the potential energy does not depend on any generalized velocity, i.e.,

$$\frac{\partial V}{\partial \dot{q}_j} = 0 \ , \qquad j = 1, ..., N$$

and *(ii)* the transformation between a set of Cartesian coordinates x_i, $i = 1, .., N$ and the generalized coordinates is independent of time, i.e.,

$$x_i = \mathcal{X}_i(q_j) \ .$$

The first of these requirements allows us to write the generalized momenta as

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} \ .$$

The second requirement, on the other hand, helps us write the kinetic energy as the double sum (see equation [6.12])

$$T = \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{N} m_{kl} \dot{q}_k \dot{q}_l \ ,$$

with the coefficients m_{kl} depending, in principle, only on the generalized coordinates and time. Using this expression for the kinetic energy, we find

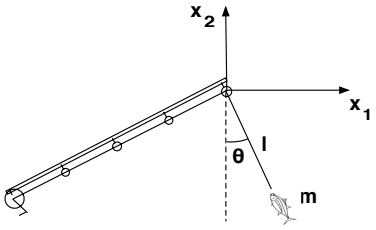
$$\begin{aligned} p_j &= \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{3} m_{kl} \frac{\partial \dot{q}_k}{\partial \dot{q}_j} \dot{q}_l + \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{3} m_{kl} \dot{q}_k \frac{\partial \dot{q}_l}{\partial \dot{q}_j} \\ &= \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{3} m_{kl} \delta_{kj} \dot{q}_l + \frac{1}{2} \sum_{k=1}^{N} \sum_{l=1}^{3} m_{kl} \dot{q}_k \delta_{lj} \\ &= \frac{1}{2} \sum_{l=1}^{N} m_{jl} \dot{q}_l + \frac{1}{2} \sum_{k=1}^{N} m_{kj} \dot{q}_k \\ &= \sum_{l=1}^{N} m_{jl} \dot{q}_l \ , \end{aligned}$$

where, in the last equation, we took into consideration the fact that the matrix with coefficients m_{kj} is symmetric, i.e., $m_{kj} = m_{jk}$. Inserting this expression for the generalized momenta into the definition of the Hamiltonian (6.32), we find

$$\begin{aligned} H &= \sum_{j=1}^{N} \dot{q}_j \left(\sum_{l=1}^{N} m_{jl} \dot{q}_l \right) - (T - V) \\ &= \sum_{j=1}^{N} \sum_{l=1}^{N} m_{jl} \dot{q}_j \dot{q}_l - T + V \\ &= 2T - T + V \\ &= T + V \ . \end{aligned}$$

This final result completes our proof that, under the two assumptions discussed earlier, the Hamiltonian of a dynamical system, which is the conjugate momentum to time, is equal to the mechanical energy in the system.

Example 6.7: Reeling in a Fish caught with a Fishing Rod



In this example, we will study a situation in which the Hamiltonian of a system is not equal to its mechanical energy. In particular, we will write the equations of motion for a fish caught by a line and is being reeled in by a fisherman. We will assume that the fishing rod is not flexible, that the line is massless, and that the fisherman reels it in at a constant rate a. We will also neglect air resistance and consider for simplicity only the motion of the fish on a vertical plane.

We start by setting up a Cartesian coordinate system as shown in the figure. The position of the fish is uniquely described by its two Cartesian coordinates x_1 and x_2. However, its motion is constrained in such a way that the length of the line from the top of the fishing rod is l, which is reduced from an initial length l_0 at a constant rate a, i.e.,

$$l = l_0 - at \; .$$

The presence of one constraint leads to the conclusion that this dynamical system has only one degree of freedom, which we choose to be the angle θ between the fishing line and the vertical direction.

The transformations between the generalized coordinate and the Cartesian coordinates are

$$\begin{aligned} x_1 &= l\sin\theta = (l_0 - at)\sin\theta \\ x_2 &= -l\cos\theta = (l_0 - at)\cos\theta \end{aligned} \tag{6.33}$$

so that the Cartesian components of the velocity are

$$\begin{aligned} \dot{x}_1 &= a\sin\theta + l\dot{\theta}\cos\theta \\ \dot{x}_2 &= -a\cos\theta + l\dot{\theta}\sin\theta \; . \end{aligned}$$

We use now these transformation to write the kinetic energy of the fish, to which we assign a mass m, in terms of its generalized coordinate and velocity as

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right) = \frac{1}{2}m\left(a^2 + l^2\theta^2\right)$$

and the potential energy as

$$V = -mgl\cos\theta \; .$$

The Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left(a^2 + l^2\theta^2\right) + mgl\cos\theta$$

and the Euler-Lagrange equation for the coordinate θ becomes

$$\begin{aligned} & \frac{\partial L}{\partial\theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = 0 \\ \Rightarrow \quad & -mgl\sin\theta - \frac{d}{dt}\left(ml^2\dot{\theta}\right) = 0 \\ \Rightarrow \quad & \ddot{\theta} = -\frac{g}{l}\sin\theta + 2\frac{a}{l}\dot{\theta} \; . \end{aligned}$$

When the length of the line remains fixed, i.e., when $a = 0$, the last equation describes a simple pendulum, as expected.

The Hamiltonian of the system is

$$\begin{aligned} H &= \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L \\ &= \frac{1}{2}ml^2\dot{\theta}^2 - \frac{1}{2}m\alpha^2 - mgl\cos\theta \end{aligned} \tag{6.34}$$

and is not equal to the mechanical energy, E, since

$$E = T + V = \frac{1}{2}m\left(a^2 + l^2\theta^2\right) - mgl\cos\theta \neq H \ . \tag{6.35}$$

This is not surprising, since the transformations (6.33) between the Cartesian coordinates, x_1 and x_2, and the generalized coordinate θ, depend on the length l, which is an explicit function of time.

6.8 Symmetries and Conservation Laws

The concepts of generalized coordinates and momenta in a dynamical system leads to one of the most striking applications of Lagrangian dynamics to the understanding of the physical world: the fact that every symmetry in a dynamical system is closely related to a conservation law. In this section, we will discuss how the conservation of linear momentum, angular momentum, and energy in a system are related to the invariance of the corresponding Lagrangian to translations, to rotations, and to displacements in time, respectively.

We will first consider a dynamical system described by a Lagrangian that is independent of a particular general coordinate, q_i, i.e.,

$$\frac{\partial L}{\partial q_i} = 0 \ .$$

We will call such a coordinate *cyclic* or *ignorable*. The Euler-Lagrange equation for this coordinate then takes the form

$$\begin{aligned} & \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \\ \Rightarrow \quad & 0 - \frac{dp_i}{dt} = 0 \\ \Rightarrow \quad & p_i = \text{constant} \ . \end{aligned}$$

This last relation shows that the generalized momentum that is conjugate to a cyclic coordinate remains constant throughout the evolution of the dynamical system. We will now explore the implications of this result in three different cases.

6.8.1 Linear Momentum Conservation

We will consider first a Lagrangian of a dynamical system expressed in terms of a set of Cartesian coordinates. If the system consists of M particles, each with mass

m_k, $k = 1, ..., M$ and each particle is allowed to move in a three-dimensional space with coordinates $x_{i,k}$, $i = 1, ..., 3$, then the Lagrangian of the system is simply

$$L = \frac{1}{2} \sum_{k=1}^{M} m_k \sum_{i=1}^{3} \dot{x}_{i,k}^2 - V(x_{i,k}) \ .$$

We will then assume that the dynamical system is invariant to translations along one of the coordinates, say x_j. In other words, we will assume that, if we were to add any constant ϵ to the $x_{j,k}$ coordinate of each of the constituents particles

$$x_{j,k} \rightarrow x_{j,k} + \epsilon, \qquad k = 1, ..., M$$

then the Lagrangian will remain unchanged. This implies that the Lagrangian density does not depend explicitly on the $x_{j,k}$ coordinate of any of the constituents particles of the system and, therefore, the conjugate momentum to this Cartesian coordinate is constant throughout the evolution of the dynamical system.

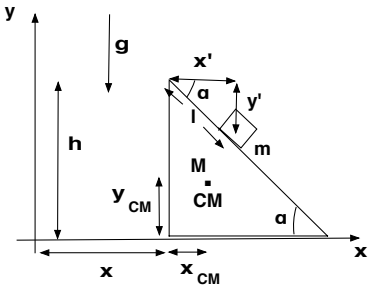
The conjugate momentum to the Cartesian coordinate x_j is

$$p_j \equiv \frac{\partial L}{\partial \dot{x}_j} = \sum_{k=1}^{M} m_k \dot{x}_{j,k} \ ,$$

which is the j-th coordinate of the total linear momentum of the system. This demonstrates that invariance of a Lagrangian to Cartesian translations along any direction always implies conservation of the component of linear momentum along the same direction.

Example 6.8: Motion on a movable inclined plane

When we study the motion of an object sliding down an inclined plane, we usually consider the plane to be immovable. In this example, we will relax this assumption and study the motion of a small object of mass m sliding down a wedge of mass M that is allowed to move on a horizontal plane. We will denote by α the opening angle of the wedge and by h its height and assume that all surfaces are frictionless. We begin by setting a two-dimensional Cartesian coordinate system (x, y), as shown in the margin figure. In order to describe fully the motion of the system that consists of the wedge and the small object of mass m, we need in principle four coordinates: the coordinates $x_{\rm M}$ and $y_{\rm M}$ of the center of mass for the wedge and the coordinates $x_{\rm m}$ and $y_{\rm m}$ of the center of mass of the small object. However, two geometric constraints are imposed on the motion of the two masses: the wedge is allowed to move only along the horizontal plane and the small object is allowed to move only on the surface of the wedge. For these reasons, the system has two degrees of freedom and requires two generalized coordinates for describing uniquely its dynamical state.



The only external force that affects the two masses is gravity, which acts along the vertical direction. Moreover, the only external constraint, i.e., the horizontal plane, also exerts a force along the vertical direction. As a result, we expect the system to be invariant to translations along the horizontal direction. It is, therefore,

to our advantage to choose at least one of the generalized coordinates to be a Cartesian coordinate along the $x-$axis, anticipating that its conjugate momentum will be conserved throughout the evolution of the system.

We choose as generalized coordinates the distance x of the vertical side of the wedge from our coordinate system and the distance l of the center of mass of the small object from the top of the wedge. If we denote by $x_{\rm CM}$ and $y_{\rm CM}$ the distances along the two axis of the center of mass of the wedge from its bottom-left corner, then

$$\begin{aligned} x_{\rm M} &= x + x_{\rm CM} \\ y_{\rm M} &= y_{\rm cm} \end{aligned}$$

from which we can calculate the corresponding velocities as

$$\begin{aligned} \dot{x}_{\rm M} &= \dot{x} \\ \dot{y}_{\rm M} &= 0 \, . \end{aligned}$$

The coordinates of the center of mass of the small object are

$$\begin{aligned} x_{\rm m} &= x + l\cos\alpha \\ y_{\rm m} &= h - l\sin\alpha \end{aligned}$$

from which we can also calculate the corresponding velocities as

$$\begin{aligned} \dot{x}_{\rm m} &= \dot{x} + \dot{l}\cos\alpha \\ \dot{y}_{\rm m} &= \dot{l}\sin\alpha \, . \end{aligned}$$

The kinetic energy of the system is

$$\begin{aligned} T &= \frac{1}{2}M\left(\dot{x}_{\rm M}^2 + \dot{y}_{\rm M}^2\right) + \frac{1}{2}m\left(\dot{x}_{\rm m}^2 + \dot{y}_{\rm m}^2\right) \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{l}^2\cos^2\alpha + 2\dot{x}\dot{l}\cos\alpha + \dot{l}^2\sin^2\alpha\right) \\ &= \frac{1}{2}\left(M+m\right)\dot{x}^2 + \frac{1}{2}m\dot{l}^2 + m\dot{x}\dot{l}\cos\alpha \end{aligned}$$

and the potential energy, measured from the horizontal plane, is

$$\begin{aligned} V &= Mgy_{\rm M} + mgy_{\rm m} \\ &= Mgy_{\rm CM} + mg(h - l\sin\alpha) \, . \end{aligned}$$

As expected, the Lagrangian of the system does not depend on the x generalized coordinate, i.e.,

$$L = \frac{1}{2}\left(M+m\right)\dot{x}^2 + \frac{1}{2}m\dot{l}^2 + m\dot{x}\dot{l}\cos\alpha - Mgy_{\rm CM} - mg(h - l\sin\alpha) \, .$$

Therefore, we do not need to write the Euler-Lagrange equation for this coordinate, as its conjugate momentum

$$p \equiv \frac{\partial L}{\partial \dot{x}} = \left(M+m\right)\dot{x} + m\dot{l}\cos\alpha$$

is a constant of the motion, with a value determined by the initial conditions. If we assume that both the wedge and the small object are initially at rest, then $p = 0$ and

$$(M+m)\,\dot{x} + m\dot{l}\cos\alpha = 0 \Rightarrow \dot{x} = -\frac{m\cos\alpha}{M+m}\dot{l}\,. \tag{6.36}$$

The Euler-Lagrange equation for the second generalized coordinate is

$$\begin{aligned} &\frac{\partial L}{\partial l} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{l}}\right) = 0 \\ \Rightarrow \quad & mg\sin\alpha - m\ddot{l} - m\ddot{x}\cos\alpha = 0\,. \end{aligned} \tag{6.37}$$

Equations (6.36) and (6.37) form a set of two differential equations for the generalized coordinates. In order to combine them, we first take the time derivative of both sides of equation (6.36) and insert the expression for $\ddot{x}$ into equation (6.37) to obtain

$$\ddot{l} = \frac{\sin\alpha}{1 - \left(\frac{m}{M+m}\right)\cos^2\alpha}g$$

and

$$\ddot{x} = \frac{m\sin\alpha\cos\alpha}{M+m-m\cos^2\alpha}g\,.$$

These two equations show that, unless $\alpha = 0$, the horizontal motion of the wedge as well as the motion of the small mass on the surface of the wedge are characterized by constant accelerations. When $\alpha = 0$, the wedge is completely horizontal and neither of the two objects move. On the other hand, when the mass of the wedge is much larger the mass of the small object, i.e., when $M >> m$, the two expression reduce to

$$\ddot{l} = g\sin\alpha$$

and

$$\ddot{x} = 0\,.$$

This is the familiar result that the wedge remains fixed and the small mass is accelerated by the component of the force that is parallel to the inclined plane.

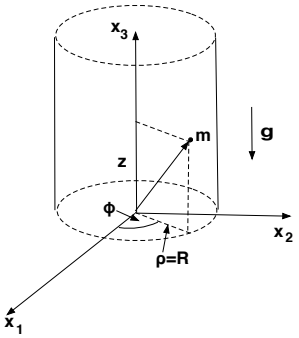
6.8.2 Angular Momentum Conservation

As a second example, we will consider a dynamical system with a Lagrangian that is invariant to rotations around a particular axis. In this case, we will set up a cylindrical coordinate system with its $z-$axis parallel to the symmetry axis of the system. Invariance to rotations along the azimuthal direction ϕ of the cylindrical coordinate system means that the conjugate momentum to the ϕ coordinate remains constant throughout the evolution of the dynamical system. However, in Example 6.3, we showed that the conjugate momentum to the ϕ coordinate in cylindrical coordinates is the angular momentum of the dynamical system along the $z-$axis. In other words, the invariance of a Lagrangian to rotations around an axis always implies conservation of the angular momentum of the system with respect to the symmetry axis.

Example 6.9: Motion on the inside of a cylinder

In this example, we will consider the motion of a small object of mass m on the inside surface of a vertical cylinder of radius R, in the presence of a vertical gravitational acceleration g. We will assume that the initial velocity of the object is large enough to keep it at all times in contact with the cylindrical surface. We will also assume that the small object is not spherical or cylindrical, so that it cannot roll in its motion. We will revisit the problem of a rolling ball inside a cylinder in Chapter 23, where we will study why golf balls bounce out of golf holes.

The motion of the small object inside the cylinder can be described in terms of three Cartesian coordinates x_1, x_2, and x_3. However, the motion is also subject to one constraint, that the object is always on the inside of the cylindrical surface. Therefore, we need two generalized coordinates to uniquely specify the position of this object.



The only external force in this system is the force of gravity, which acts along the vertical direction. The force of the constraint, on the other hand, is always along the radius of the cylindrical surface. As a result, we expect the system to be invariant to rotations around the axis of the cylindrical surface. In anticipation of a conservation law related to this symmetry, we will set up a cylindrical coordinate system as shown in the margin figure and use the cylindrical azimuth ϕ and the vertical coordinate z as the generalized coordinates for this problem.

In Example 6.3, we derived a general expression for the kinetic energy of an object in cylindrical coordinates. Setting $\rho = R$ and $\dot{\rho} = 0$, expression (6.21) becomes

$$T = \frac{1}{2}m\left(R^2\dot{\phi}^2 + \dot{z}^2\right) .$$

The potential energy, measured from the $z = 0$ plane, is

$$V = mgz$$

so that the Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\left(R^2\dot{\phi}^2 + \dot{z}^2\right) - mgz .$$

As expected, the Lagrangian does not depend on the azimuth ϕ and, therefore, the corresponding momentum, which is the angular momentum along the $z-$axis,

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi}$$

remains constant throughout the motion of the object. Conservation of angular momentum in this case implies that the ϕ-velocity of the object, i.e., its angular velocity, remains constant, with a value determined by the initial conditions.

The Euler-Lagrange equation along the vertical direction is

$$\begin{aligned} & \frac{\partial L}{\partial z} - \frac{\partial L}{\partial t}\left(\frac{\partial L}{\partial \dot{z}}\right) = 0 \\ \Rightarrow \quad & -mgz - m\ddot{z} = 0 \\ \Rightarrow \quad & \ddot{z} = -g . \end{aligned} \tag{6.38}$$

The object, therefore, follows a helix, keeping a constant angular velocity along the azimuthal direction and exhibiting free fall along the vertical direction.

6.8.3 Energy Conservation

In this final example, we will consider a dynamical system with a Lagrangian that is invariant to translations in time. In Hamilton's principle, time is the independent variable. However, we can always use the construction introduced in Section 6.7 to elevate the role of time from being an independent variable to becoming a generalized coordinate. In this case, the fact that the Lagrangian is independent of time implies that the Hamiltonian, which is the conjugate momentum to time, remains constant throughout the evolution of the dynamical system.

It is important to emphasize here, however, that conservation of the Hamiltonian of a dynamical system does not necessarily imply conservation of energy. Indeed, as we showed in Section 6.7, the Hamiltonian of a system is equal to its mechanical energy only if two conditions are satisfied: *(i)* the potential energy does not depend explicitly on the generalized velocity, and *(ii)* the transformation between a set of Cartesian coordinates and the generalized coordinates does not depend on time. If these two conditions are satisfied and the Lagrangian of a dynamical system does not depend explicitly on time, then the mechanical energy of the system remains constant throughout its evolution. If, on the other hand, either of the two conditions is not satisfied and the Lagrangian does not depend on time, then the Hamiltonian of the system will remain constant, but not necessarily its mechanical energy.

It might appear surprising, at first, that the mechanical energy of a dynamical system that is described by a Lagrangian may not be conserved. In fact, in the beginning of this chapter, we derived the form of the Lagrangian of an object under the assumption that the object is under the influence of only potential forces. There are, however, two distinct ways in which the mechanical energy of a dynamical system described by potential forces may not be conserved.

First, the potential field may not necessarily describe the interaction between constituents of the system under study, but rather the potential of an externally imposed interaction. For example, the potential energy in the above example, describes the interaction between the mass m and the Earth, even though the latter is external to the system under study. If such an externally imposed potential is time dependent, then the mechanical energy of the system is not conserved. In this case, the source or sink of energy is whatever mechanism causes the external potential to change in time.

Second, a time dependence in a Lagrangian may be introduced by a time dependent constraint on the generalized coordinates. Such a constraint typically reflects the effects of an external force on the constituents of the system, which can inject or remove mechanical energy from the system. Example 6.7 treated such a situation, in which mechanical energy was injected into the system via the reeling of the fishing rod.

Further Reading

1. *Variation Principles of Mechanics*, by C. Lanczos, 4th edition (Dover)
2. *A History of Mechanics*, by René Dugas (Dover)

Suggested Problems

1. Consider two Lagrangians L and L' that differ by a term that can be written as a time derivative of a function $g(q_i, t)$ of the generalized coordinates and time, i.e.,
$$L'(q_i, \dot{q}_i, t) = L(q_i, t) + \frac{d}{dt} g(q_i, t) \ .$$
Write the Euler-Lagrange equations that correspond to the Lagrangian L' and show by explicit calculation that these equations are identical to those arising from the Lagrangian L.

2. A vertical, extended bungee cord exerts a force on the bungee jumper that is described by Hooke's law,
$$F = -k(l - l_0) \ ,$$
where k is the force constant, l is the instantaneous length of the cord, and l_0 is its non-extended length. Calculate the acceleration on a bungee jumper of mass M, taking into account the fact that the cord is not massless but has a linear mass density λ. You may assume that the gravitational acceleration g is vertical and uniform.

3. Write the equations of motion for the pendulum discussed in Example 6.4, when the vertical ring is allowed to rotate freely.

4. A pendulum consists of a rod of mass $m_{\rm r}$ and a ball of mass $m_{\rm b}$ attached to the rod's end. Find the frequency of small oscillations of the pendulum, when it is placed in a uniform vertical gravitational field $\vec{g}$.

5. Consider a simple plane pendulum consisting of a mass m attached to a string of length l. After the pendulum is set into motion, the length of the string is shortened at a constant rate
$$\frac{dl}{dt} = -a = constant \ . \tag{6.39}$$
The suspension point remains fixed. Compute the Lagrangian and Hamiltonian functions. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.