Solution to "Hilbert Spaces of Entire Functions"

Kevin Linghu
dlinghu@caltech.edu

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Abstract

From the past two years in mathematical research I gradually realized I kept forgetting stuff and therefore, I decide to write down what I did before I forget it. I hope this would save my time and memory. All numbered problems are from Dr. de Branges' book "Hilbert Spaces of Entire Functions".

Problem 1

Let $f(z)$ be a function which is analytic and has a nonnegative real part in the upper half-plane. Assume that $\Re f(z)$ has a continuous extension to the closed half-plane and that $h(z)$ is a bounded, continuous function of real x such that $0 \leqslant h(z) \leqslant \Re f(x)$ for all real x. Show that

$$\Re f(x+iy) \geqslant \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{h(t)dt}{(t-x)^2+y^2}$$

for $y > 0$.

Remark. *This problem, together with problem 2, 3, 4 and some approximation argument yields Herglotz's representation theorem: f analytic function with nonnegative real parts on the upper half-plane iff f is a Poisson transform of a positive regular measure μ, i.e $\mu \in \mathcal{M}_{\Pi}^{+}(\mathbb{T})$. Also note that I didn't assume $h(x)$ is bounded—I didn't note this condition until I finished the proof. But never mind, we'll use this result for unbounded $h(x)$ in problem 2 anyway.*

Proof. Let $h_n(x) := \max(h(x), n)$ and $g_n(x) := e^{h_n(x)}$, then g_n satisfies condition of Theorem 2, which gives $F_n(z)$ s.t.

$$\log|F_n(z)| = \frac{y}{\pi} \int_{\mathbb{R}} \frac{h_n(t)dt}{(t-x)^2+y^2}$$
$$|F_n(x)| = g_n(x)$$

Now let $\phi(z) := \frac{F_n(z)}{e^{f(z)}}$. $\phi(z)$ is analytic in the upper half-plane, and $|\phi(z)| = \frac{|F_n(z)|}{e^{\Re f(z)}}$ has a continuous extension to $\mathbb{C}_+ \cup \mathbb{R}$. To use Phragmén-Lindelöf principle (Theorem 1), if suffices to check

$$\liminf_{a\to\infty} \frac{1}{a} \int_0^{\pi} \log^+ |\phi(ae^{i\theta})| \sin\theta d\theta = 0$$

Since $\Re f(z) \geqslant 0$ on $\mathbb{C}_+$, we have

$$\begin{aligned}\log^+ \left|\frac{F_n(z)}{e^{f(z)}}\right| &= (\log|F_n(z)| - \Re f(z))^+ \\ &\leqslant \log|F_n(z)| \\ &= \frac{y}{\pi} \int_{\mathbb{R}} \frac{h_n(t)dt}{(t-x)^2+y^2}\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{a}\int_0^\pi \log^+|\phi(ae^{i\theta})|\sin\theta d\theta &\leqslant \frac{1}{a}\int_0^\pi \frac{y}{\pi}\int_{\mathbb{R}}\frac{h_n(t)dt}{(t-x)^2+y^2}\sin\theta d\theta \\
&\leqslant \frac{n}{\pi}\int_0^\pi \sin\theta\int_{\mathbb{R}}\frac{dt}{(t-a\cos\theta)^2+a^2\sin^2\theta}d\theta \\
&= \frac{n}{\pi}\int_0^\pi \sin\theta\int_{\mathbb{R}}\frac{dt}{t^2+a^2\sin^2\theta}d\theta \quad (\text{let } t=(a\sin\theta)s) \\
&= \frac{n}{a\pi}\int_0^\pi\int_{\mathbb{R}}\frac{ds}{s^2+1}d\theta \\
&= \frac{n\pi}{a} \\
\liminf_{a\to\infty}\frac{1}{a}\int_0^\pi \log^+|\phi(ae^{i\theta})|\sin\theta d\theta &\leqslant \liminf_{a\to\infty}\frac{n\pi}{a} \\
&= 0
\end{aligned}$$

Now by Theorem 1 we have

$$\begin{aligned}
|\phi(z)| &= |\frac{F_n(z)}{e^{f(z)}}| \leqslant 1 \\
\Re f(z) &\geqslant \frac{y}{\pi}\int_{\mathbb{R}}\frac{h_n(t)dt}{(t-x)^2+y^2}
\end{aligned}$$

Now use monotone convergence theorem we get:

$$\Re f(z) \geqslant \frac{y}{\pi}\int_{\mathbb{R}}\frac{h(t)dt}{(t-x)^2+y^2}$$

□

Problem 2

Let $f(z)$ be a function which is analytic and has a nonnegative real part in the upper half-plane. If $\Re f(z)$ has a continuous extension to the closed half-plane, show that there exists a function $g(z)$, which is analytic and has a nonnegative real part in the upper half-plane, such that

$$\Re f(x+iy) = \Re g(x+iy) + \frac{y}{\pi}\int_{\mathbb{R}}\frac{\Re f(t)dt}{(t-x)^2+y^2}$$

for $y>0$. Show that $\Re g(z)$ is continuous in the closed half-plane and that $\Re g(x)=0$ for all real x.

Proof. From problem 1 (for unbounded $h(x)$), we know

$$\frac{1}{\pi}\int_{\mathbb{R}}\frac{\Re f(t)dt}{t^2+1} \leqslant \Re f(i)$$

then by Theorem 2, there exists $\phi(z)$, s.t.

$$\begin{aligned}
\Re\phi(z) &= \frac{y}{\pi}\int_{\mathbb{R}}\frac{\Re f(t)dt}{(t-x)^2+y^2} \\
\Re\phi(x) &= \Re f(x)
\end{aligned}$$

Let $g(z) := f(z)-\phi(z)$, then

$$\Re g(z) = \Re f(z) - \Re\phi(z) = \Re f(z) - \frac{y}{\pi}\int_{\mathbb{R}}\frac{\Re f(t)dt}{(t-x)^2+y^2} \geqslant 0$$

and since both $\Re\phi(z)$ and $\Re f(z)$ have a continuous extension to closed half-plane, so is $\Re g(z)$, and $\Re g(x) = \Re f(x) - \Re\phi(x) = 0$ on $\mathbb{R}$. □

Problem 3

Let $g(z)$ be a function which is analytic and has a nonnegative real part in the upper half-plane. Assume that $\Re g(z)$ is continuous in the closed half-plane, and that $\Re g(x) = 0$ for all real x. Show that $\Re g(x+iy) = py$ where p is a constant.

Proof. By reflection principle we can extend domain of $g(z)$ to $\mathbb{C}$ by $g(z) := -g^{\#}(z)$ for $z \in \mathbb{C}_-$. In particular, $\Re g(z) = -\Re g(\bar{z})$. I'll first show

$$\Re g(z) = \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4ay\Re g(ae^{i\theta}) \sin\theta d\theta}{|ae^{i\theta} - z|^2 |a^{-i\theta} - a|^2} \tag{1}$$

for $a > 0$, $|z| < a$ and $y > 0$. Since $\Re g(z)$ is harmonic, using Poisson's formula we have

$$\begin{aligned}
\Re g(z) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 - |z|^2}{|ae^{i\theta} - z|^2} \Re g(ae^{i\theta}) d\theta \\
&= \frac{1}{2\pi} \int_0^{\pi} \frac{a^2 - |z|^2}{|ae^{i\theta} - z|^2} \Re g(ae^{i\theta}) d\theta \\
&+ \frac{1}{2\pi} \int_\pi^{2\pi} \frac{a^2 - |z|^2}{|ae^{i\theta} - z|^2} \Re g(ae^{i\theta}) d\theta \\
&= I + II
\end{aligned}$$

Since $\Re g(z) = -\Re g(\bar{z})$, we have

$$\begin{aligned}
II &= \frac{1}{2\pi} \int_\pi^{2\pi} \frac{a^2 - |z|^2}{|ae^{i\theta} - z|^2} \Re g(ae^{i\theta}) d\theta \quad \text{(let } \theta = \theta - 2\pi \text{, abuse of notation)} \\
&= \frac{1}{2\pi} \int_{-\pi}^{0} \frac{a^2 - |z|^2}{|ae^{i\theta} - z|^2} \Re g(ae^{i\theta}) d\theta \quad \text{(let } \theta = -\theta\text{)} \\
&= \frac{1}{2\pi} \int_0^{\pi} \frac{a^2 - |z|^2}{|ae^{-i\theta} - z|^2} \Re g(ae^{-i\theta}) d\theta \\
&= -\frac{1}{2\pi} \int_0^{\pi} \frac{a^2 - |z|^2}{|ae^{-i\theta} - z|^2} \Re g(ae^{i\theta}) d\theta
\end{aligned}$$

Also note that

$$\begin{aligned}
\frac{1}{|ae^{i\theta} - z|^2} - \frac{1}{|ae^{-i\theta} - a|^2} &= \frac{2\Re(zae^{-i\theta} - zae^{-i\theta})}{|ae^{i\theta} - z|^2 |ae^{-i\theta} - a|^2} \\
&= \frac{4ay \sin\theta}{|ae^{i\theta} - z|^2 |ae^{-i\theta} - a|^2}
\end{aligned}$$

and we get (1). Fix $a > 0$, let $z \to 0$ from $\mathbb{C}_+$, we have

$$\lim_{z \to 0} \frac{\Re g(z)}{y} = \frac{1}{2\pi a} \int_0^\pi 4y \Re g(ae^{i\theta}) \sin\theta d\theta$$

exists. Hence RHS is independent of a. Now we have

$$\begin{aligned}
\frac{\Re g(z)}{y} &= \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4a\Re g(ae^{i\theta}) \sin\theta d\theta}{|ae^{i\theta} - z|^2 |a^{-i\theta} - a|^2} \\
&= \lim_{a \to +\infty} \frac{a^2 - |z|^2}{2\pi} \int_0^\pi \frac{4a\Re g(ae^{i\theta}) \sin\theta d\theta}{|ae^{i\theta} - z|^2 |a^{-i\theta} - a|^2} \\
&= \lim_{a \to +\infty} \frac{1}{2\pi a} \int_0^\pi 4y \Re g(ae^{i\theta}) \sin\theta d\theta
\end{aligned}$$

is a constant, and we're done. □

Problem 4

In problem 2 and 3, show that

$$p = \lim_{y \to \infty} \frac{\Re f(iy)}{y}$$

Proof. According to problem 2 and 3, it suffices to show

$$\lim_{y\to\infty}\int_{\mathbb{R}}\frac{\Re f(t)dt}{t^2+y^2}=0$$

$\forall\epsilon>0$, choose N large enough s.t.

$$\int_{-\infty}^{-N}\frac{\Re f(t)dt}{t^2+1}\leqslant\frac{\epsilon}{3}$$
$$\int_{N}^{\infty}\frac{\Re f(t)dt}{t^2+1}\leqslant\frac{\epsilon}{3}$$

For $y>N^2$ and $t\in[-N,N]$,

$$\frac{t^2+1}{t^2+y}\leqslant\frac{N^2+1}{N^2+y}\leqslant\frac{y+1}{y^2+y}=\frac{1}{y}$$

Hence

$$\int_{-N}^{N}\frac{\Re f(t)dt}{t^2+y}\leqslant\frac{1}{y}\int_{-N}^{N}\frac{\Re f(t)dt}{t^2+1}$$

choose even larger y we get

$$\int_{\mathbb{R}}\frac{\Re f(t)dt}{t^2+y^2}\leqslant\epsilon$$

and since ϵ is arbitrary we're done. □

Problem 5

$\varphi^{\#}(z)=-\varphi(z)$ and

$$\Re\varphi(x+iy)=py+\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{(t-x)^2+y^2}$$

when $y>0$. Show that

$$\frac{\varphi(z)-\varphi(\bar{w})}{\pi i(\bar{w}-z)}=\frac{p}{\pi}+\frac{1}{\pi^2}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{(t-z)(t-\bar{w})}$$

when z and w are not real.

Proof. There exists some constant c, such that

$$\varphi(z)=c+\frac{pz}{i}+\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{1+z^2}{1+t^2}\frac{d\mu(t)}{t-z}+\frac{z}{\pi}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{1+t^2}$$

WLOG we assume both z and w are on the upper half-plane, and by direct calculation we can get what we want. Similarly for other z and w. □

Remark. *One important thing to note is, μ is regular. Any μ given by Poisson representation is regular, just like any dB function given by Nevanlinna matrix is regular.*

Problem 6

Show that a polynomial is of Pólya class if it has no zeros in the upper half-plane.

Proof. By definition of Pólya class, we only need to deal with polynomial $E=z-\lambda$ with degree one, $\Im\lambda\leqslant 0$. And then it's trivial. □

Problem 7

Show that

$$\left|(1-z)\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^r}{r}\right)-1\right|\leqslant\exp(|z|^{r+1})-1$$

for all complex z, $r=1,2,3,\cdots$.

Proof. Let $F(z) := (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^r}{r}\right) - 1$, then $F(0) = 0$, and

$$\begin{aligned}F'(z) &= (-1 + (1-z)(1 + z + \cdots + z^{r-1}))\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^r}{r}\right)\\ &= -z^r \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^r}{r}\right)\end{aligned}$$

I'll prove the following inequality first:

$$x + \frac{x^2}{2} + \cdots + \frac{x^r}{r} \leqslant x^{r+1} + \log(1+r) \tag{2}$$

For $x \leqslant 1$, the inequality is well known:

$$LHS \leqslant 1 + \sum_{k=2}^{r} \int_{k-1}^{k} \frac{1}{k} ds < 1 + \sum_{k=2}^{r} \int_{k-1}^{k} \frac{1}{s} ds = 1 + \int_1^r \frac{1}{s} ds = 1 + \log(r) < 1 + \log(1+r)$$

For $x > 1$,

$$LHS - LHS \text{ at } 1 = \int_1^x (1 + s + \cdots + s^{r-1}) ds < \int_1^x r s^r ds < \int_1^x (r+1) s^r ds = RHS - RHS \text{ at } 1$$

Now

$$\begin{aligned}\Big|F(z) - F(0)\Big| &\leqslant \int_0^{|z|} \left|F'\left(t\frac{z}{|z|}\right)\right| dt\\ &\leqslant \int_0^{|z|} t^r \exp\left(t + \cdots + \frac{t^r}{r}\right) dt\\ &\leqslant \int_0^{|z|} (1+r) t^r \exp(t^{r+1}) dt\\ &= \exp(|z|^{r+1}) - 1\end{aligned}$$

□

Problem 8

Show that

$$1 + |ab - 1| \leqslant (1 + |a-1|)(1 + |b-1|)$$

for all complex numbers a and b.

Proof. Let $a_0 := a - 1, b_0 := b - 1$, and then the inequality becomes

$$|a_0 b_0 + a_0 + b_0| \leqslant |a_0 b_0| + |a_0| + |b_0|$$

□

Problem 9

Let (z_n) be a sequence of numbers such that $y_n \geqslant 0$ for every n and

$$\sum_1^{+\infty} \frac{1 + y_n}{x_n^2 + y_n^2} < \infty$$

Show that the product

$$E(z) = \prod_1^{\infty} \left(1 - \frac{z}{\bar{z}_n}\right) e^{h_n z}$$

converges uniformly on bounded sets if

$$h_n = \frac{x_n}{x_n^2 + y_n^2}$$

Show that the limit is an entire function of Pólya class.

Proof. Note that if $f_1, \cdots, f_n$ are of Pólya class, so is $\prod_1^n f_i$. According to problem 6, $1 - \frac{z}{\bar{z}_n}$ is of Pólya class. $e^{h_n z}$ is of Pólya class as well since $h_n \in \mathbb{R}$. Let

$$P_r(z) = \prod_{n=1}^{r} \left(1 - \frac{z}{\bar{z}_n}\right) e^{h_n z}$$

then P_r is of Pólya class, $\forall r \in \mathbb{N}^+$. We'll show P_r is uniformly convergent on bounded set so

$$E(z) = \exp\left(i(\sum_{n=1}^{+\infty} \frac{y_n}{x_n^2 + y_n^2})\right) \lim_{r\to+\infty} P_r(z)$$

is well-defined and entire. Now I'll show

$$|P_s(z) - P_r(z)| \leqslant \exp\{\sum_{n=1}^{s} |z/\bar{z}_n|^2\} - \exp\{\sum_{n=1}^{r} |z/\bar{z}_n|^2\}$$

when $r < s$. First, use problem 7 and 8, let $a_n(z) := \left(1 - \frac{z}{\bar{z}_n}\right) e^{h_n z}$

$$\begin{aligned}
|P_s/P_r - 1| &= \left|\prod_{n=r+1}^{s} a_n - 1\right| \\
&\leqslant \prod_{n=r+1}^{s} (1 + |a_n - 1|) - 1 \\
&\leqslant \exp(\sum_{n=r+1}^{s} |z/\bar{z}_n|^2) - 1
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|a_n| &\leqslant |a_n - 1| + 1 \leqslant \exp(|z/\bar{z}_n|^2) \\
P_r &\leqslant |a_n - 1| + 1 \leqslant \exp(\sum_{n=1}^{r} |z/\bar{z}_n|^2)
\end{aligned}$$

Hence $|P_s(z) - P_r(z)| \leqslant \exp\{\sum_{n=1}^{s} |z/\bar{z}_n|^2\} - \exp\{\sum_{n=1}^{r} |z/\bar{z}_n|^2\}$, use condition $\sum_{n=1}^{+\infty} \frac{1}{x_n^2+y_n^2} < +\infty$ we can see $P_r(z)$ is uniformly convergent on bounded sets, hence $E(z)$ is uniformly convergent on bounded sets. And obviously Pólya class is closed when taking uniform limit, so $E(z)$ is of Pólya class. □

Problem 10

Let $E(z)$ be a polynomial of Pólya class such that $E(0) = 1$, and let $E(z) = A(z) - iB(z)$ where $A(z)$ and $B(z)$ are polynomials which are real for real z. Show that

$$\log|E(z)| \leqslant xA'(0) + yB'(0) + \frac{1}{2}[A'(0)^2 - A''(0) + B'(0)]|z|^2$$

for all complex z.

Proof. Since E is of Pólya class and $E(0) = 1$, we have $E(z) = \prod_{i=1}^{n} \left(1 - \frac{z}{\bar{z}_i}\right)$, where $z_i = x_i + iy_i \in \mathbb{C}_+$. First let's calculate $A'(0), A''(0)$ and $B'(0)$. Obviously $E'(0) = -\sum_{i=1}^{n} \frac{1}{\bar{z}_i}$, $(E^{\#})'(0) = -\sum_{i=1}^{n} \frac{1}{z_i}$, hence

$$\begin{aligned}
A'(0) &= \frac{1}{2}(E'(0) + (E^{\#})'(0)) = -\sum_{i=1}^{n} \frac{x_i}{|z_i|^2} \\
B'(0) &= \frac{i}{2}(E'(0) + (E^{\#})'(0)) = \sum_{i=1}^{n} \frac{y_i}{|z_i|^2} \\
E''(0) &= \sum_{i\neq j} \frac{1}{\bar{z}_i\bar{z}_j} \\
(E^{\#})''(0) &= \sum_{i\neq j} \frac{1}{z_i z_j} \\
A''(0) &= \sum_{i\neq j} \frac{\Re(z_i z_j)}{|z_i z_j|^2} = \sum_{i\neq j} \frac{x_i x_j - y_i y_j}{|z_i z_j|^2}
\end{aligned}$$

Hence

$$A'(0)^2 - A''(0) + B'(0)^2 = \sum_{i=1}^{n} \frac{1}{|z_i|^2} + 2\sum_{i\neq j} \frac{y_i y_j}{|z_i z_j|^2} \geqslant \sum_{i=1}^{n} \frac{1}{|z_i|^2}$$

Then it suffices to show

$$\log|1 - z/\bar{z}_i| \leqslant -\frac{x x_i}{|z_i|^2} + \frac{y y_i}{|z_i|^2} + \frac{|z|^2}{2|z_i|^2}$$

i.e.

$$\begin{aligned}
\log|1-z| &\leqslant -\Re z + \frac{|z|^2}{2} \\
(1-x)^2 + y^2 &\leqslant e^{x^2 - 2x + y^2} \\
(1-x)^2 + y^2 &\leqslant e^{(x-1)^2 + y^2 - 1}
\end{aligned}$$

Now use $t \leqslant e^{t-1}$ for $t \geqslant 0$ we are done. □

Problem 11

If $a > 0$ is given, find a sequence $\{P_n(z)\}$ of polynomials, which have only real zeros, such that $e^{-az^2} = \lim P_n(z)$ uniformly on bounded sets.

Proof. Let $P_n(z) := \left(1 - \frac{az^2}{n}\right)^n$. Obviously $P_n(z)$ goes to $f(z) := e^{-az^2}$ pointwisely, and only have real zeros. For the uniform convergence, note that

$$\begin{aligned}
|P_n(z) - f(z)| &\leqslant \sum_{k=2}^{n} \left(1 - (1 - \frac{k-1}{n}) \cdots (1 - \frac{1}{n})\right) \frac{|z|^k}{k!} + \sum_{k=n+1}^{+\infty} \frac{|z|^k}{k!} \\
&\leqslant \sum_{k=2}^{n} \left(1 - (1 - \frac{k-1}{n})^{k-1}\right) \frac{|z|^k}{k!} + \sum_{k=n+1}^{+\infty} \frac{|z|^k}{k!} \\
&\leqslant \sum_{k=2}^{n} \frac{(k-1)^2}{n} \frac{|z|^k}{k!} + \sum_{k=n+1}^{+\infty} \frac{|z|^k}{k!} \\
&= I + II
\end{aligned}$$

Now for any $C > 0$ and bounded set $\{z : |z| \leqslant C\}$, choose n big enough s.t. $n! > C^n$ and $n > C$.

$$\begin{aligned}
I &\leqslant \frac{|z|^2}{n} \left(\frac{1}{2}1 + |z| + \frac{|z|^2}{2} + \cdots + \frac{|z|^{n-2}}{(n-2)!}\right) \\
&\leqslant \frac{C^2(e^C + \frac{1}{2})}{n} \\
II &\leqslant \frac{|z|}{n+1} + \frac{|z|^2}{(n+1)^2} + \cdots \\
&\leqslant \frac{C}{n+1-C}
\end{aligned}$$

and now we get uniform convergence. □

Remark. *Your first thought might be using finite Maclaurin series to approximate e^{-az^2}, but it's easy to see they have either 1 or 0 zero on the real line. Actually, if we don't use $\lim(1+z/n)^n = e^z$, the classical approach would be using associated Jensen polynomials. That is, for $f = \sum a_k \frac{z^k}{k!}$, define*

$$g_n(z) = \sum_{k=0}^{n} \binom{n}{k} a_k z^k$$

Jensen proved the following statement:
f is Laguerre-Pólya iff g_n only has real zeros and converges to f locally uniformly.
Laguerre-Pólya is a special case of Pólya, with $|E| \geqslant |E^\#|$ on $\mathbb{C}_+$ substituted by $|E| = |E^\#|$ on $\mathbb{C}_+$.

Problem 12

If b is a given number, $\Re b > 0$, find a sequence $\{P_n(z)\}$ of polynomials of Pólya class such that $e^{-ibz} = \lim P_n(z)$ uniformly on bounded sets.

Proof. Let $P_n(z) := \left(1 - \frac{ibz}{n}\right)^n$, and the only root is $z_0 = -i\frac{n}{b}$.

$$\Im z_0 = -\Re\frac{n}{b} = -\frac{n}{|b|^2}\Re b \leqslant 0$$

the rest follows from the same argument as problem 11. □

Problem 13

If $E(z)$ is a given entire function of Pólya class, show that there exists a sequence $\{P_n(z)\}$ of polynomials of Pólya class such that $E(z) = \lim P_n(z)$ uniformly on bounded sets.

Remark. *Now we have: f Pólya iff f can be approximated locally uniformly by Pólya polynomials.*

Proof. Use Theorem 7 to get factorization of $E(z)$, and use results from problem 9, 11, 12 we're done. □

Problem 14

Let $E(z)$ be an entire function which has no zeros for $y > 0$, such that $|E(x - iy)| \leqslant |E(x + iy)|$ for $y > 0$. Show that

$$|E(x - iy)| < |E(x + iy)|$$

for $y > 0$ unless $E(z)$ and $\bar{E}(\bar{z})$ are linearly dependent.

Proof. Apply maximum principle to $\frac{E^\#}{E}$ on $\mathbb{C}_+$. □

Remark. *This theorem says E is degenerate iff E and $E^\#$ are linearly dependent.*

Problem 15

If $E(z)$ is an entire function of Pólya class, show that $|E(x+iy)|$ is an increasing function of $y > 0$ for each fixed x unless $E(z) = E(0)e^{hz}$ for some real number h.

Proof. Proof by contradiction. By Theorem 7 we have

$$E(z) = A(z)B(z)C(z)D(z)$$

where $A(z) = E^{(r)}(0)\frac{z^r}{r!}$, $B(z) = e^{-az^2}$, $C(z) = e^{-ibz}$ and $D(z) = \prod(1 - \frac{z}{\bar{z}_n})e^{h_n z}$. For each fixed x, $A(z)$ is not strictly increasing only if $A(z) \equiv E(0)$, $B(z)$: $a = 0$, $C(z)$: $\Re b = 0$, $D(z) : D(z) \equiv 1$. Hence E is not strictly increasing unless

$$E(z) = E(0)e^{ibz} = E(0)e^{hz}$$

for some real h since b is purely imaginary. □

Problem 16

Let $E(z)$ be an entire function of Pólya class such that $|E(x - iy)| < |E(x + iy)|$ for $y > 0$. Show that $E(z) = A(z) - iB(z)$ where $A(z)$ and $B(z)$ are entire functions of Pólya class which are real for real z.

Proof. Let $A = \frac{E+E^\#}{2}$, $B = \frac{E^\# - E}{2i}$. Obviously A is real entire. Since $|E(z)| > |E^\#(z)|$ on $\mathbb{C}_+$, $A(z) \neq 0$ on $\mathbb{C}_+$. It suffices to show for fixed x, $|A(x + iy)|$ is nondecreasing for $y > 0$. From problem 13 we know there exists Pólya polynomials $\{P_n(z)\}$ s.t. P_n goes to E locally uniformly. Let $P_n = A_n - iB_n$ s.t. A_n and B_n are real entire, then $A_n = \frac{P_n + P_n^\#}{2}$ is a polynomial as well. If $A_n(z) = 0$ for some $z \in \mathbb{C}_+$, then $|P_n(z)| = |P_n^\#(z)|$, all roots of P_n lie on $\mathbb{R}$. Since $|E(z)| > |E^\#(z)|$, $\forall z \in \mathbb{C}_+$, we can choose a subsequence of $\{P_n\}$ s.t. P_n has at least one non-real root. Now A_n has no zeros in the upper half-plane, by problem 6 A_n is of Pólya class. Since A_n goes to A locally uniformly, A is of Pólya class. The same argument applies to B. □

Remark. *Actually, the converse is true: E is of Pólya class iff A, B are of Laguerre-Pólya class.*

Problem 17

Let $E(z)$ be an entire function of Pólya class which is not a constant. Show that $E'(z)$ is of Pólya class.

Proof. Let nontrivial Pólya polynomials $\{P_n\}$ go to E locally uniformly, then $P_n'(z)$ goes to $E'(z)$ locally uniformly. It suffices to show $P_n'(z)$ is of Pólya class, i.e. $P_n'(z)$ has no zero on $\mathbb{C}_+$. This is obvious because

$$\Re\left(i\frac{P_n'(z)}{P_n(z)}\right) = \frac{\partial}{\partial y}\log|P_n(z)| > 0$$

□

Problem 18

Show that $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ is of Pólya class. Determine the factorization given by Theorem 7. By computing the second derivative of $\cos z$ at the origin, show that

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

Proof. Since e^{-iz} is of Pólya class, by problem 16, $\cos z$ is of Pólya class. By Theorem 7,

$$\cos z = e^{-az^2} e^{-ibz} \prod_{n=-\infty}^{+\infty} \left(1 - \frac{z}{n\pi + \frac{\pi}{2}}\right) e^{\frac{z}{n\pi+\frac{\pi}{2}}}$$

Since $\cos'(0) = -\sin(0) = 0$, by simple calculation we can get $b = 0$. Since $\cos z$ is of genus 1, $a = 0$. So we get the canonical factorization:

$$\cos z = \prod_{n=-\infty}^{+\infty} \left(1 - \frac{z}{n\pi + \frac{\pi}{2}}\right) e^{\frac{z}{n\pi+\frac{\pi}{2}}}$$

Taking derivatives we get

$$\begin{aligned}
-\sin z &= \sum_{n=-\infty}^{+\infty} -\frac{z}{(n\pi+\frac{\pi}{2})^2} e^{\frac{z}{n\pi+\frac{\pi}{2}}} \prod_{m\neq n}\left(1 - \frac{z}{m\pi + \frac{\pi}{2}}\right) e^{\frac{z}{m\pi+\frac{\pi}{2}}} \\
&:= zG(z) \\
-\cos z &= G(z) + zG'(z) \\
\cos 0 = -G(0) &= \sum_{n=-\infty}^{+\infty} \frac{1}{(n\pi+\frac{\pi}{2})^2} = \sum_{n=0}^{+\infty} \frac{8}{\pi^2}\frac{1}{(2n+1)^2}
\end{aligned}$$

□

Problem 20

Show that a function $F(z)$, which is analytic in the upper half-plane, is of bounded type in the upper half-plane if its real part is nonnegative in the half-plane.

Proof. Trivial. Consider

$$\frac{F}{F+1}\bigg/\frac{1}{F+1}$$

both numerator and denominator are analytic and bounded so by definition F is of bounded type. □

Problem 21

Show that the sum and product of two functions which are of bounded type in the upper half-plane are functions of bounded type in the half-plane.

Proof. Trivial. By definition. □

Problem 22

Show that a polynomial is a function of bounded type in the upper half-plane.

Proof. Use problem 21 to reduce to $P(z) = z - w$. If $\Im w < 0$, then

$$P(z) = 1 \bigg/ \frac{1}{z-w}$$

is of bounded type in the upper half-plane. Otherwise use $P(z) = \frac{z-w}{z-\bar{w}} \bigg/ \frac{1}{z-\bar{w}}$. $\square$

Problem 23

Let $\{z_n\}$ be a sequence of numbers such that $y_n > 0$ for every n and

$$\sum \frac{y_n}{x_n^2 + y_n^2} < \infty$$

Show that the Blaschke product

$$B(z) = \prod_1^\infty \frac{1 - z/z_n}{1 - z/\bar{z}_n}$$

converges uniformly on every bounded set which lies at a positive distance from the numbers $\{\bar{z}_n\}$. Show that $B(z)$ is analytic and bounded by 1 in the upper half-plane and that $B(z)B^{\#}(z) = 1$.

Remark. *This problem gives condition of zeros of Blaschke product, and shows it's entire and bounded by 1 on $\mathbb{C}_+$. And the proof is very similar to problem 9 so I'll skip some details.*

Proof. Let $a_n(z) := \frac{1-z/z_n}{1-z/\bar{z}_n}$, then

$$\begin{aligned}
1 + |a_n - 1| &= 1 + \left| \frac{\frac{1}{\bar{z}_n} - \frac{1}{z_n}}{\frac{1}{z} - \frac{1}{\bar{z}_n}} \right| \\
&\leqslant 1 + \frac{1}{\rho(z)} \frac{2y_n}{|z_n|^2} \\
&\leqslant \exp\left(\frac{2}{\rho(z)} \frac{y_n}{x_n^2 + y_n^2} \right)
\end{aligned}$$

Now let $B_r(z) := \prod_{n=1}^r a_n(z)$

$$\begin{aligned}
|B_s(z)/B_r(z) - 1| &\leqslant \exp\left(\left(\sum_{n=r+1}^s \frac{y_n}{x_n^2 + y_n^2} \right) \frac{2}{\rho(z)} \right) - 1 \\
|B_r(z)| &\leqslant 1 + |B_r(z) - 1| \\
&\leqslant \exp\left(\left(\sum_{n=1}^r \frac{y_n}{x_n^2 + y_n^2} \right) \frac{2}{\rho(z)} \right) \\
|B_s(z) - B_r(z)| &\leqslant \exp\left(\left(\sum_{n=1}^s \frac{y_n}{x_n^2 + y_n^2} \right) \frac{2}{\rho(z)} \right) - \exp\left(\left(\sum_{n=1}^r \frac{y_n}{x_n^2 + y_n^2} \right) \frac{2}{\rho(z)} \right)
\end{aligned}$$

similar to problem 9, $B(z)$ is uniformly convergent on every bounded set where $\inf \rho(z) > 0$. $B(z)B^{\#}(z) = 1$ follows from $B_r(z)B_r^{\#}(z) = 1$. $\square$

Problem 24

Let $F(z)$ be a function which is analytic and of bounded type in the upper half-plane. Show that there exists a function $Q(z)$, which is analytic and bounded by 1 and which has no zeros in the upper half-plane, such that $P(z) = Q(z)F(z)$ is bounded by 1 in the half-plane.

Remark. *Now for any $F \in \mathcal{N}(\mathbb{C}_+)$, $F = \frac{P}{Q}$, where $|P|, |Q| \leqslant 1$ and $Q(z) \neq 0$ on $\mathbb{C}_+$.*

Proof. By Theorem 8,

$$F(z) = G(z) \prod \frac{1 - \frac{z}{z_n}}{1 - \frac{z}{\bar{z}_n}}$$

where $G(z)$ is of bounded type and has no zeros in the half-plane. By definition, $G(z) = \frac{P(z)}{Q(z)}$ where $|P|, |Q| \leqslant 1$ on $\mathbb{C}_+$. Suppose $\{w_n\}$ are zeros of Q on $\mathbb{C}_+$, note that they are also zeros of P. By similar argument of proof to Theorem 8,

$$Q(z) = \left(\prod \frac{1 - \frac{z}{w_n}}{1 - \frac{z}{\bar{w}_n}} \right) \tilde{Q}(z)$$

$$P(z) = \left(\prod \frac{1 - \frac{z}{w_n}}{1 - \frac{z}{\bar{w}_n}} \right) \tilde{P}(z)$$

and $|\tilde{P}|, |\tilde{Q}| \leqslant 1$ and $\tilde{Q}$ doesn't have any zero on $\mathbb{C}_+$. Now

$$F(z) = \frac{\tilde{P}}{\tilde{Q}} \prod \frac{1 - \frac{z}{z_n}}{1 - \frac{z}{\bar{z}_n}}$$

$$\tilde{Q}(z) F(z) = \tilde{P}(z) \prod \frac{1 - \frac{z}{z_n}}{1 - \frac{z}{\bar{z}_n}}$$

and since $|\tilde{Q}(z)| \leqslant 1, |\tilde{P}(z)| \leqslant 1, \left| \prod \frac{1 - \frac{z}{z_n}}{1 - \frac{z}{\bar{z}_n}} \right| \leqslant 1$ we're done. □

Problem 25

Show that

$$y = \frac{1 - |z|^2}{2\pi} \int_0^{\pi} \frac{4a \sin^2 \theta d\theta}{|e^{i\theta} - z|^2 |e^{-i\theta} - z|^2}$$

for $|z| < 1$ and $y > 0$.

Proof. See the proof to problem 3, with $g(z) = -iz$, $a = 1$. □

Problem 28

Let $F(z)$ be a function which is analytic and of bounded type in the upper half-plane. Show that the mean type of $F(z - a)$ is equal to the mean type of $F(z)$ for every real number a.

Proof. By Theorem 9 (Nevanlinna's factorization),

$$F(z) = B(z) e^{-ihz} e^{G(z)}$$

where

$$\Re G(x + iy) = \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - x)^2 + y^2}$$

This factorization is unique (See Theorem 5.5, Garnett's book). Substitute z with $z - a$ we get

$$F(z - a) = B(z - a) e^{-ihz} e^{G(z) + iha}$$

By definition of Blaschke product $B(z) = \prod_{n=1}^{+\infty} \frac{1 - \frac{z}{z_n}}{1 - \frac{z}{\bar{z}_n}}$ we can see $B(z - a)$ is a Blaschke product as well. $\tilde{G} := G + iha$ satisfy the same condition as G so $F(z)$ and $F(z - a)$ have same mean types. □

Problem 29

Let $F(z)$ and $G(z)$ be functions which are analytic and of bounded type in the upper half-plane. Show that the mean type of $F(z) + G(z)$ does not exceed the maximum of the mean types of $F(z)$ and $G(z)$. Show that the mean type of $F(z)G(z)$ is the sum of the mean types of $F(z)$ and $G(z)$.

Proof. The second part is trivial by Theorem 9 (Nevanlinna's factorization). The first part is easy using the fact $|F(iy)| \leqslant e^{y(p_1 + \epsilon)}$, $|G(iy)| \leqslant e^{y(p_2 + \epsilon)}$ for $\epsilon > 0$ and large y. So $|F(iy) + G(iy)| \leqslant 2e^{y(\max(p_1, p_2) + \epsilon)}$, and we're done. □

Problem 30

Show that a function which is analytic and has a non-negative real part in the upper half-plane has zero mean type in the half-plane if it does not vanish identically.

Proof. Let $g := \sqrt{f}$, then $\Re g \geqslant 0$ and $\Re g \leqslant |g| \leqslant \sqrt{2}\Re g$. By Theorem 4 (Poisson's representation) we know:

$$\Re g(x+iy) = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t-x)^2+y^2}$$
$$g(x+iy) = -ipz + \mathcal{S}\mu + ib$$

where $p, b \geqslant 0$, and

$$\mathcal{S}\mu = \frac{1}{\pi i}\int_{-\infty}^{+\infty}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)d\mu(t)$$

is the Schwarz-Herglotz transform of $\mu \in \mathcal{M}^+_{\Pi}(\mathbb{R})$. Actually, if we let p be the point mass at infinity, then the Schwarz-Herglotz transform of $\hat{\mu} := \mu \oplus p$ is:

$$\mathcal{S}\hat{\mu} = -ipz + \mathcal{S}\mu$$

Now use Theorem 10, i.e. mean type $h = \limsup_{y\to+\infty} \frac{\log|g(iy)|}{y}$. WLOG we can assume $p = b = 0$, $\int_{-\infty}^{+\infty} \frac{d\mu(t)}{1+t^2} = 1$, and by Jensen's inequality we have:

$$\exp\left(\int_{-\infty}^{+\infty} \log\left(\frac{1+t^2}{y^2+t^2}\right)\frac{d\mu(t)}{1+t^2}\right) \leqslant \int_{-\infty}^{+\infty}\frac{1+t^2}{y^2+t^2}\frac{d\mu(t)}{1+t^2}$$
$$\int_{-\infty}^{+\infty} \log\left(\frac{1+t^2}{y^2+t^2}\right)\frac{d\mu(t)}{1+t^2} \leqslant \log\left(\int_{-\infty}^{+\infty}\frac{1+t^2}{y^2+t^2}\frac{d\mu(t)}{1+t^2}\right) = \log\left(\int_{-\infty}^{+\infty}\frac{d\mu(t)}{y^2+t^2}\right)$$

Now

$$\begin{aligned}
\frac{\log|g(iy)|}{y} &\geqslant \frac{\log \Re g}{y} \\
&= \frac{\log y - \log\pi}{y} + \frac{1}{y}\log\left(\int_{-\infty}^{+\infty}\frac{d\mu(t)}{y^2+t^2}\right) \\
&\geqslant \frac{\log y - \log\pi}{y} + \int_{-\infty}^{+\infty}\frac{1}{y}\log\left(\frac{1+t^2}{y^2+t^2}\right)\frac{d\mu(t)}{1+t^2} \\
&\geqslant \frac{\log y - \log\pi}{y} - \int_{-\infty}^{+\infty}\frac{\log y^2}{y}\frac{d\mu(t)}{1+t^2} \\
&\to 0 \\
\frac{\log|g(iy)|}{y} &\leqslant \frac{\log 2}{2y} + \frac{\log\Re g}{y} \\
&\leqslant \frac{\log 2 + 2\log y - 2\log\pi}{2y} + \frac{1}{y}\log\left(\int_{-\infty}^{+\infty}\frac{d\mu(t)}{y^2+t^2}\right) \\
&\leqslant 0
\end{aligned}$$

Hence g has mean type 0. Note that $\log|f(iy)| = 2\log|g(iy)|$, hence f has mean type 0 as well, and $f \in \mathcal{N}^+$. ◻

Remark. *Another elegant approach for $\mathbb{D}$, but I'm not sure if it will work for $\mathbb{C}_+$ since I don't know the connection between $H^p(\mathbb{C}_+)$ and $H^p(\mathbb{D})$ when $p < 1$. Anyway, if we only deal with $\mathbb{D}$, $\Re f \geqslant 0$ implies f is outer and $f \in H^p$, $\forall 0 < p < 1$. This is proved by V.Smirnov in 1928 and the proof is quite straightforward, using mean value property of harmonic function $\Re f^p$ and $|f|^p \leqslant c_p \Re f^p$. For more details see Nikolski's book, Theorem 4.2.2. Now use the fact $H^p = L^p \cap \mathcal{N}^+$ we know $f \in \mathcal{N}^+$, and $\mathcal{N}^+ = \{f \in \mathcal{N}, \text{mean type} \leqslant 0\}$. Apply the same argument to $1/f$ and we're done.*

Problem 31

Show that a nonzero polynomial has zero mean type in the upper half-plane.

Proof. Let $P(z) = (z - z_1)\cdots(z - z_n)$, then use formula $h = \limsup_{y\to+\infty} \frac{\log|P(iy)|}{y}$. ◻

Problem 33

If a function $F(z)$ is analytic and of bounded type in the upper half-plane, if it has no zeros in the half-plane, and if $\log F(z)$ is defined continuously in the half-plane, show that

$$\frac{|z-\bar{z}||\log F(z)|}{|z+i|^2}$$

is bounded in the half-plane.

Proof. Since F has no zeros in $\mathbb{C}_+$, by Theorem 9,

$$F(z) = e^{-ihz} e^{U(z)-V(z)}$$

where

$$\Re U(x+iy) = \frac{y}{\pi}\int_{\mathbb{R}} \frac{d\sigma(t)}{(t-x)^2+y^2}$$
$$\Re V(x+iy) = \frac{y}{\pi}\int_{\mathbb{R}} \frac{d\nu(t)}{(t-x)^2+y^2}$$

and $\sigma, \nu \in \mathcal{M}_{\Pi}^{+}(\mathbb{R})$. Obviously, it suffices to prove

$$\frac{|z-\bar{z}||U(z)|}{|z+i|^2}$$

is bounded in the half-plane.

$$\begin{aligned} |U(z)| &= \left|ib + \frac{1}{\pi i}\int_{\mathbb{R}} \frac{1+tz}{t-z}\frac{d\sigma}{1+t^2}\right| \\ &\leqslant |b| + \frac{1}{\pi}\int_{\mathbb{R}} \frac{|1+tz|}{|t-z|}\frac{d\sigma}{1+t^2} \end{aligned}$$

and we only need to deal with the second term. I'll use the following inequality:

$$\left|\frac{t-i}{t-z}\right| \leqslant \frac{|z-i|+|z+i|}{|z-\bar{z}|}$$

and the proof is quite straightforward:

$$\begin{aligned} |t-i||z-\bar{z}| &= |tz - t\bar{z} - iz + i\bar{z}| \\ &\leqslant |(tz + i\bar{z} - |z|^2 - it) - (t\bar{z} + iz - |z|^2 - it)| \\ &\leqslant |(t-\bar{z})(z-i) - (t-z)(\bar{z}-i)| \\ &= |t-z||z-i| + |t-z||z+i| \\ &= |t-z|(|z-i|+|z+i|) \end{aligned}$$

Now

$$\begin{aligned} \frac{|z-\bar{z}|}{|z+i|^2}\int_{\mathbb{R}} \frac{|1+tz|}{|t-z|}\frac{d\sigma}{1+t^2} &\leqslant \int_{\mathbb{R}} \frac{|1+tz|}{|t-i||z+i|}\frac{|z-i|+|z+i|}{|z+i|}\frac{d\sigma}{1+t^2} \\ &\leqslant \int_{\mathbb{R}} \frac{|1+tz|}{|t-i||z+i|}\frac{2d\sigma}{1+t^2} \end{aligned}$$

It suffices to show $\frac{|1+tz|}{|t-i||z+i|}$ is bounded:

$$\begin{aligned} \frac{|1+tz|}{|t-i||z+i|} &\leqslant \frac{|(t-i)(z+i) - it + iz|}{|t-i||z+i|} \\ &\leqslant 1 + \frac{|t|}{|t-i||z+i|} + \frac{|z|}{|t-i||z+i|} \\ &\leqslant 3 \end{aligned}$$

$\square$

Remark. *The proof could be more straightforward. de Branges gave the inequality*

$$\left|\frac{t-i}{t-z}\right| \leqslant \frac{|z-i|+|z+i|}{|z-\bar{z}|}$$

in hint and that's how the proof was made—I'll also use this to prove Kreĭn's theorem (problem 37). The inequality $\frac{|z-\bar{z}||\log F(z)|}{|z+i|^2}$ *gives a nice growth estimate for* $F \in \mathcal{N}(\mathbb{C}_+)$.

Problem 34

Let $E(z)$ be an entire function which has no zeros in the upper half-plane and which satisfies the inequality $|E(x-iy)| \leqslant |E(x+iy)|$ for $y>0$. Show that $E(z)$ is of Pólya class if there exists an entire function $F(z)$ of Pólya class such that $\frac{E(z)}{F(z)}$ is of bounded type in the upper half-plane.

Remark. *Let $F \equiv 1$ we get an important corollary: If $E \in \mathcal{N}(\mathbb{C}_+)$ is zero free on $\mathbb{C}_+$ and is dB, then E is of Pólya class. And it seems like I didn't follow de Branges' idea—I used Kreĭn's theorem (problem 37), which was proved later. But the proof is independent of this result. This original proof requires F to be zero free, which is a strong assumption. For complete proof please refer to Michael Kaltenböck and Harald Woracek's paper "Pólya class theory for Hermite-Biehler functions of finite order", which is more general and applies to generalized Pólya class with k-th order. The main technique is Phragmén-Lindelöf Principle and Theorem 9, 10, 14.*

Proof. One direction is obvious: Let $F=E$. Now suppose we have F of Pólya class s.t. $\frac{E}{F} \in \mathcal{N}(\mathbb{C}_+)$. First I'll show $\frac{E}{F}$ is of exponential type. By Kreĭn's theorem, it suffices to show

$$\frac{E}{F}, \frac{E^\#}{F^\#} \in \mathcal{N}(\mathbb{C}_+)$$

The first one is given, and

$$\frac{E^\#}{F^\#} = \frac{E^\#}{E}\frac{E}{F} \Big/ \frac{F^\#}{F} \in \mathcal{N}(\mathbb{C}_+)$$

so $\frac{E}{F}$ is of exponential type, hence it has genus 0 or 1. Let's suppose genus is 1 first. By canonical factorization we have

$$\frac{E}{F} = e^{az+b} z^r \prod \left(1 - \frac{z}{\bar{z}_n}\right) e^{\frac{z}{\bar{z}_n}}$$

and $\sum \frac{1}{|\bar{z}_n|^2} < \infty$ since it's of genus 1, $\sum \frac{y_n}{x_n^2+y_n^2} < \infty$ since it's of bounded type. By Theorem 7,

$$F(z) = F^{(s)}(0)\frac{z^s}{s!} e^{-a_1 z^2} e^{-ib_1 z} \prod \left(1 - \frac{z}{\bar{w}_n}\right) e^{h_n z}$$

where $h_n = \frac{\Im w_n}{|w_n|^2}$. Put the two formulas together and take $(r+s)$-th derivative we get

$$\frac{F^{(s)}(0)e^b}{s!}(r+s)! = E^{(r+s)}(0)$$

and if we combine zeros and label them $\{\bar{z}_n\}$

$$E(z) = E^{(r+s)}(0)\frac{z^{r+s}}{(r+s)!} e^{-a_1 z^2} e^{-ib_1 z} e^{az} \prod \left(1 - \frac{z}{\bar{z}_n}\right) e^{h_n z}$$

$$\frac{|E^\#(z)|}{|E(z)|} = e^{2i(\Re b_1 - \Im a)z} \prod \frac{1-\frac{z}{z_n}}{1-\frac{z}{\bar{z}_n}}$$

where $h_n = \frac{y_n}{x_n^2+y_n^2}$. Since $|E^\#(iy)| \leqslant |E(iy)|$, we get

$$e^{-2(\Re b_1 - \Im a)y} \prod \left| \frac{1-\frac{iy}{z_n}}{1-\frac{iy}{\bar{z}_n}} \right| \leqslant 1$$

$$-2(\Re b_1 - \Im a)y + \log \prod \left| \frac{1-\frac{iy}{z_n}}{1-\frac{iy}{\bar{z}_n}} \right| \leqslant 0$$

From proof to Theorem 10 we get

$$\lim_{y\to\infty} \frac{\log \prod \left| \frac{1-\frac{iy}{z_n}}{1-\frac{iy}{\bar{z}_n}} \right|}{y} = 0$$

so we can conclude $\Im a \leqslant \Re b_1$. Now the representation for $E(z)$ is of the form of Theorem 7, hence it's of Pólya class. The case genus is 0 can be treated similarly. $\square$

Problem 35

If $E(z)$ is an entire function of Pólya class such that $E^{\#}(z) = E(-z)$, show that

$$|E(z)| \leqslant |E(i|z|)|$$

for all complex z.

Proof. By Theorem 7 we have

$$E(z) = E^{(r)}(0)\frac{z^r}{r!}e^{-az^2}e^{-ibz}\prod\left(1-\frac{z}{\bar{z}_n}\right)e^{h_n z}$$

Since $E^{\#}(z) = E(-z)$, $E(\bar{z}_n) = 0$ implies $E(-\bar{z}_n)$, hence all roots come in pair, with only possible exception on the imaginary axis. Let $G(z) := \prod\left(1-\frac{z}{\bar{z}_n}\right)e^{h_n z}$, then

$$\begin{aligned} G(z) &= \prod\left(1+\frac{z}{iy_n}\right)\prod\left(1-\frac{z}{\bar{z}_n}\right)\left(1+\frac{z}{z_n}\right) \\ &= \prod\left(1+\frac{z}{iy_n}\right)\prod\frac{x_n^2+(y_n-iz)^2}{|z_n|^2} \\ &\leqslant G(i|z|) \end{aligned}$$

And note that $G^{\#}(z) = G(-z)$, so is e^{-az^2}, let $H(z) = e^{-ibz}$, then

$$\begin{aligned} H^{\#}(z) &= H(-z) \\ e^{i\bar{b}z} &= e^{ibz} \\ \bar{b} &= b \end{aligned}$$

Hence b is real and $|H(z)| = e^{by} \leqslant e^{b|z|} = |H(i|z|)|$. Finally,

$$|e^{-az^2}| = e^{a(y^2-x^2)} \leqslant e^{a|z|^2}$$

we now conclude $|E(z)| \leqslant |E(i|z|)|$. □

Problem 37

Prove Kreĭn's theorem that an entire function $F(z)$ is of exponential type if it is of bounded type in the upper half-plane and if $F^{\#}(z)$ is of bounded type in the upper half-plane. Showt that the exponential type of $F(z)$ is the maximum of the mean types of $F(z)$ and $F^{\#}(z)$ in the upper half-plane.

Remark. *Kreĭn's theorem states*

$$Cart = \mathcal{N}(\mathbb{C}_+)\cap\mathcal{N}(\mathbb{C}_-) = Exp\cap\{f : \log^+|f| \in L^1_\Pi\}$$

Proof. Fix any z_0, since $\log|F|$ is subharmonic, we have

$$\begin{aligned} \log|F(z_0)| &\lesssim \iint\limits_{B(z_0,1)} \log^+|F|dxdy \\ &\lesssim (1+|z_0|^2)\iint\limits_{B(z_0,1)}\frac{\log^+|F|}{1+x^2}dxdy \\ &\lesssim C(1+|z_0|^2) \end{aligned}$$

so $|F| \lesssim e^{A|z|^2}$ for A, B. Now I'll show $|F(z)| \lesssim e^{C|z|}$ in any fixed Stolz angle, i.e. $\{z = x+iy : |z| > 1, y > 0, \frac{|x|}{y} \leqslant a\}$ for some $a > 0$. Use Nevanlinna's factorization (Theorem 9) we can get

$$|F(z)| \leqslant e^{hy}e^{\frac{y}{\pi}\int_{\mathbb{R}}\frac{d|\mu|}{(t-x)^2+y^2}}$$

By the inequality we proved in problem 33,

$$\begin{aligned}\int_{\mathbb{R}} \frac{d|\mu|}{(t-x)^2+y^2} &\leqslant \int_{\mathbb{R}} \frac{(|z-i|+|z+i|)^2}{(t^2+1)4y^2} d|\mu| \\ &\leqslant \int_{\mathbb{R}} \frac{|z-i|^2+|z+i|^2}{(t^2+1)2y^2} d|\mu| \\ &\leqslant \int_{\mathbb{R}} \frac{x^2+y^2+1}{(t^2+1)y^2} d|\mu| \\ &\leqslant \int_{\mathbb{R}} (1+a^2+\frac{1}{y})\frac{d|\mu|}{t^2+1} \\ &\leqslant \int_{\mathbb{R}} (1+a^2+\frac{1}{\sqrt{1+a^2}})\frac{d|\mu|}{t^2+1}\end{aligned}$$

Hence $|F(z)| \lesssim e^{C|z|}$ on any Stolz angle and it's reflection over $\mathbb{R}$. We can conclude that $|F(z)| \leqslant e^{C_1|z|}$ on $\{z = x+iy : y > \frac{1}{\sqrt{1+a^2}}, x = ay\}$ and $|F(z)| \leqslant e^{C_2|z|}$ on $\{z = x+iy : y < -\frac{1}{\sqrt{1+a^2}}, x = ay\}$. Let $C = \max(C_1, C_2)\sqrt{\frac{a^2}{1+a^2}}$, then $|F(z)| \lesssim e^{C|x|}$ on $\{z : |y| > \frac{1}{\sqrt{1+a^2}}, x = a|y|\}$. Since F is bounded in the unit disk, we can choose C even larger s.t. $|F| \lesssim e^{C|x|}$ on $\{z : x = a|y|\}$. Now we want to show $|F(z)| \lesssim e^{C|z|}$ in sector $\{z : x \leqslant a|y|\}$. Let $a = \frac{1}{\tan\frac{\pi}{6}}$ and $G(z) = F(z^{\frac{1}{3}})e^{-Cz^{\frac{1}{3}}}$, then G is bounded on the imaginary axis, and for z in $\{z : \Re z \geqslant 0\}$,

$$|G(z)| \lesssim e^{C|z|^{\frac{2}{3}}}$$

By Phragmén-Lindelöf principle $|G|$ is bounded in $\{z : \Re z \geqslant 0\}$, and hence $|F|$ is bounded in the sector. Apply similar argument to the other sector we're done. Still need to show max of mean types is the exponential type. □

Problem 38

Show that an entire function of zero exponential type is bounded in the complex plane, and hence is a constant, if it is bounded on the real axis,

Remark. *Similarly, we can change the condition into "bounded on imaginary axis".*

Proof. Trivial. By Phragmen-Lindelof principle (version of Theorem 1). □

Problem 39

Show that an entire function $F(z)$ is a constant if $F(z)$ and $F^{\#}(z)$ are of bounded type in the upper half-plane and if $F(z)$ is bounded on the imaginary axis.

Proof. By Kreĭn's theorem, F is of exponential type. Moreover, since F is bounded on the imaginary axis, the mean type for F and $F^{\#}$ on the upper half-plane is 0, and then F is of zero exponential type. By problem 38 F is a constant. □

Problem 40

If $\mathcal{B}(E)$ is a given space, show that

$$K(w,w) = \frac{B(w)\bar{A}(w) - A(w)\bar{B}(w)}{\pi(w-\bar{w})}$$

is a continuous function of w.

Proof. Trivial for non-real w. For real w, taking the limit and since A,B,$A^{\#}$,$B^{\#}$ are differentiable we're done. □

Problem 44

If $\mathcal{B}(E)$ is a given space, show that $E(z) = S(z)E_0(z)$ where $\mathcal{B}(E_0)$ exists, $E_0(z)$ has no real zeros, and $S(z)$ is an entire function which is real for real z. Show that $F(z) \to S(z)F(z)$ is an isometric transformation of $\mathcal{B}(E_0)$ onto $\mathcal{B}(E)$.

Proof. It suffices to prove two spaces are equal as sets, and the isometric part follows trivially. Use the following definition of dB space:

$$\mathcal{B}(E_0) = \{\frac{F}{E_0} \in H^2(\mathbb{C}_+), \frac{F^\#}{E_0} \in H^2(\mathbb{C}_+)\}$$

we can see $F(z) \to S(z)F(z)$ maps $\mathcal{B}(E_0)$ into $\mathcal{B}(E)$. For the onto part, take $F \in \mathcal{B}(E)$, only need to show $\frac{F}{S}$ is entire. If $S(z) = 0$ for some complex z, then $E(z) = 0$, $\frac{F}{E}, \frac{F^\#}{E} \in H^2(\mathbb{C}_+)$ implies F has a zero with at least same order as S. Hence $\frac{F}{S}$ is entire, and the map is onto. □

Remark. *This can be used to get rid of real zeros of dB function.*

Problem 45

Let $\mathcal{B}(E)$ be a given space. If w is a nonreal number, show that $\frac{F(z)}{z-w}$ belongs to $\mathcal{B}(E)$ whenever F belongs to $\mathcal{B}(E)$ and vanishes at w. Show that the same conclusion holds for a real number w if and only if $E(w) \neq 0$.

Remark. *This implies for any nonreal w, there exists $F \in \mathcal{B}(E)$ s.t. $F(w) \neq 0$, otherwise $\mathcal{B}(E)$ doesn't contain any nonzero element, i.e. $\mathcal{B}(E) = \{0\}$.*

Proof. If $w \notin \mathbb{R}$, $\frac{F}{z-w}$ is entire and check the definition of dB space we know $\frac{F}{z-w} \in \mathcal{B}(E)$.
Now assume $E(w) \neq 0$ for some real w, obviously $\tilde{F} := \frac{F}{z-w}$ is still entire. And it's easy to check $\frac{\tilde{F}}{E}, \frac{\tilde{F}^\#}{E} \in \mathcal{N}^+$, so it suffices to show $\frac{\tilde{F}}{E} \in L^2(\mathbb{R})$. Now choose $\epsilon > 0$ s.t. $|E| \geqslant \delta > 0$ on $(w-\epsilon, w+\epsilon)$, then

$$\begin{aligned}\int_{\mathbb{R}} \frac{|F|^2}{|t-w|^2|E|^2}dt &= \int_{-\infty}^{w-\epsilon} + \int_{w-\epsilon}^{w+\epsilon} + \int_{w+\epsilon}^{+\infty} \frac{|F|^2}{|t-w|^2|E|^2}dt \\ &\leqslant \frac{2}{\epsilon^2}\|F\|_E + \frac{C}{\delta^2} \\ &< \infty\end{aligned}$$

On the other hand, let $E(w) = 0$. Use result from problem 44, WLOG we assume $E'(w) \neq 0$. Suppose the statement is true. WLOG we can assume $F'(w) \neq 0$, otherwise we can keep taking derivative until $F^{(n)}(w) \neq 0$. Anyway, we have

$$\begin{aligned}+\infty &> \int_{w-\epsilon}^{w+\epsilon} \frac{|F(t)|^2}{|t-w|^2} \\ &\sim \int_{w-\epsilon}^{w+\epsilon} \frac{1}{|t-w|^2}dt \\ &= +\infty\end{aligned}$$

A contradiction. □

Problem 46

If $\mathcal{B}(E)$ is a given space, show that there is at most one real number α, module π, such that $e^{i\alpha}E(z) - e^{-i\alpha}E^\#(z)$ belongs to $\mathcal{B}(E)$.

Remark. *$e^{i\alpha}E(z) - e^{-i\alpha}E^\#(z) = uA(z) + vB(z)$, where $\bar{u}v = u\bar{v}$.*

Proof. Trivial since $E \notin \mathcal{B}(E)$. □

Problem 47

Let $f(z)$ be a function which is analytic in the complex plane except for isolated singularities at points (t_n) on the real axis. Suppose that $f^{\#}(z) = f(z)$ and that $\Re(-if(z)) > 0$ for $f > 0$. Show that there exist positive numbers p_n and a nonnegative number p such that

$$\frac{f(z) - \bar{f}(w)}{z - \bar{w}} = p + \sum \frac{p_n}{(t_n - z)(t_n - \bar{w})}$$

when z and w are not real. Show that

$$p_n = \lim_{z \to t_n} (t_n - z) f(z)$$

for every n.

Proof. I tried residue theorem first, but then I realized the most difficult part is to show the sum of partial fractions converges. My second approach is to use result from problem 5 directly.
Let $\varphi(z) = -if(z)$, then $\varphi^{\#}(z) = -\varphi(z)$ and $\varphi(z)$ has nonnegative real parts on the upper half-plane, hence by problem 5 we can get

$$\frac{f(z) - f(\bar{w})}{z - \bar{w}} = p + \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t - z)(t - \bar{w})}$$

by construction, $\mu(x) = \lim_{\epsilon \to 0} \mu_\epsilon(x)$, $\forall x$. Moreover,

$$\mu_\epsilon(b) - \mu_\epsilon(a) = \lim_{y \searrow 0} \int_a^b \Re\varphi(x + i(y + \epsilon))dx$$

Now for any interval $[a, b]$ which doesn't contain singularities of f, we can see that $\mu(b) = \mu(a)$, hence μ only has point masses. Suppose $\mu(t_n) = p_n$, then

$$\frac{f(z) - f(\bar{w})}{z - \bar{w}} = p + \sum \frac{p_n}{(t_n - z)(t_n - \bar{w})}$$

multiplying both sides by $(z - t_n)$ and let z go to t_n we can see

$$p_n = \lim_{z \to t_n} (t_n - z) f(z), \ \forall n$$

□

Problem 48

If $E(z)$ is a given entire function which satisfies the inequality $|E(x - iy)| < |E(x + iy)|$ for $y > 0$, show that there exists a continuous function $\varphi(x)$ of real x such that $E(x)e^{i\varphi(x)}$ is real for all values of x. If $\varphi(x)$ is any such function, show that

$$\varphi'(x) = \pi \frac{K(x, x)}{|E(x)|^2} > 0$$

for all real x. Such a function is said to be a phase function associated with $E(z)$.

Proof. We know between any two consecutive zeros of A, $\varphi - \arctan \frac{B}{A} \equiv C$ for some $C \in \mathbb{R}$. By taking derivative we know

$$\varphi' = \frac{B' - BA'()}{A^2 + B^2} = \frac{B'A - BA'}{|E|^2}$$

On the other hand,

$$K_x(x) = \lim_{z \to x} \frac{A(x)B(z) - A(z)B(x)}{\pi(z - x)} = \frac{B'(x)A(x) - B(x)A'(x)}{\pi}$$

hence $\varphi'(x) = \frac{\pi K_x(x)}{|E(x)|^2} > 0$ for $x \in \mathbb{R} \backslash Z(A)$. At zeros of A, since the left derivative is equal to the right derivative, both of which are positive, we know φ is differentiable at x and $\varphi'(x) = \frac{\pi K_x(x)}{|E(x)|^2} > 0$. □

Remark. *In the proof we show, if E is non-degenerate dB, then $\frac{K(x,x)}{|E(x)|^2}$ is strictly positive. This proof can be simplified if we use the fact $K(w, z)$ is entire, and $K(w, w) = 0$ for some $w \in \mathbb{C}$ implies $K(w, z)$ has norm 0 in $\mathcal{B}(E)$, hence it's 0 almost everywhere on $\mathbb{R}$, hence constantly 0 on $\mathbb{C}$, and $A(z)$ and $B(z)$ must be linearly dependent, and then E is degenerate, a contradiction.*

Problem 55

Let $F(z)$ and $G(z)$ be polynomials which are real for real z and have only real simple zeros. Assume that there exists a continuous increasing function $\psi(x)$ of real x such that the zeros of $G(z)$ are the points x where $\psi(x) \equiv 0$ modulo π and the zeros of $F(z)$ are the points x where $\psi(x) \equiv \frac{1}{2}\pi$ modulo π. Show that $G'(x)F(x) - F'(x)G(x)$ is of constant sign on the set where $\psi(x) \equiv 0$ modulo $\frac{1}{2}\pi$.

Proof. Let a, b be two consecutive points where $\psi(a) \equiv \psi(b) \equiv 0$ modulo $\frac{1}{2}\pi$. WLOG assume $\psi(a) \equiv 0$ modulo π and $\psi(b) \equiv \frac{\pi}{2}$ modulo π, and $a < b$. Then $G(a) = 0$ and $F(b) = 0$. Note that $G(b) \neq 0$ and $\frac{G'(a)}{G(b)} > 0$, $F(a) \neq 0$ and $\frac{F'(a)}{F(a)} < 0$. Multiplying by $F(a)G(b)$ we get both $G'(a)F(a)$ and $-F'(a)G(b)$ are positive or negative. □

Problem 56

If the degree of $G(z)$ does not exceed the degree of $F(z)$ in Problem 55, show there exists a real number h such that

$$\frac{G(z)}{F(z)} = h + \sum_{F(t)=0} \frac{G(t)}{F'(t)(z-t)}$$

Proof. Let $f(z) := \frac{G(z)}{F(z)} - \sum_{F(t)=0} \frac{G(t)}{F'(t)(z-t)}$, then clearly f is an entire function. Let $|z|$ go to $+\infty$, if $\deg G < \deg F$, then obviously $f(z) \to 0$. If $\deg G = \deg F = n$, let a_n, b_n be the leading coefficients, resp. Then $\frac{G}{F}$ goes to $\frac{a_n}{b_n}$. For each t s.t. $F(t) = 0$, $\frac{G(t)}{F'(t)(z-t)} \sim \frac{a_n}{nb_n}$. Since F has n zeros (on the real line, by assumption), then we're done. □

Problem 57

If $F(z)$ and $G(z)$ are not both constants in Problem 55, show that either $F(z) - iG(z)$ or $F(z) + iG(z)$ satisfies the inequality $|E(x - iy)| < |E(x + iy)|$ for $y > 0$.

Proof. WLOG we assume the degree of $G(z)$ does not exceed the degree of $F(z)$ in Problem 55. By Problem 56, we have

$$\frac{G(z)}{F(z)} = h + \sum_{F(t)=0} \frac{G(t)}{F'(t)(z-t)}$$

where h is a real number. Since $\frac{G(t)}{F'(t)}$ has constant signs on the set $\{t : F(t) = 0\}$, then $\Im \frac{G(z)}{F(z)} > 0$ or < 0 for $z \in \mathbb{C}_+$, hence $F(z) - iG(z)$ or $F(z) + iG(z)$ satisfies the inequality $|E(x - iy)| < |E(x + iy)|$ for $y > 0$. □

Problem 58

If s and t are positive numbers such that $s < t$, show that

$$\left| \frac{1 - \frac{z}{t}}{1 - \frac{z}{s}} - 1 \right| \leqslant \exp\left[\frac{\frac{1}{s} - \frac{1}{t}}{\left|\frac{1}{s} - \frac{1}{z}\right|} \right] - 1$$

Proof.

$$LHS = \left| \frac{\frac{z}{s} - \frac{z}{t}}{1 - \frac{z}{s}} \right| = \left| \frac{\frac{1}{s} - \frac{1}{t}}{\frac{1}{z} - \frac{1}{s}} \right|$$

then use the inequality $x \leq e^x - 1$ for $x \geqslant 0$. □

Problem 59

Let (s_n) and (t_n) be unbounded, increasing sequences of positive numbers such that

$$s_n < t_n < s_{n+1} < t_{n+1}$$

for every n. Show that

$$\prod_{n=1}^{\infty} \frac{1 - \frac{z}{t_n}}{1 - \frac{z}{s_n}}$$

converges if $z \neq s_n$ for every n. If $\rho(z) = \min \left| \frac{1-\frac{z}{t_n}}{1-\frac{z}{s_n}} \right|$, show that the convergence is uniform in any set on which $\rho(z)$ is bounded away from zero.

Proof. This follows directly from Problem 58 and Leibniz Test for alternating series. $\square$

Problem 60

Let $\psi(x)$ be a continuous, increasing function of real x which has 0 as a value. Show that there exists a function $f(z)$ with these properties:

(1) The function is analytic in the complex plane except for isolated singularities on the real axis, $f^{\#}(z) = f(z)$, and $\Re - if(z) > 0$ for $y > 0$.

(2) The zeros of $f(z)$ are real and simple and are the points x where $\psi(x) \equiv 0$ modulo π.

(3) The zeros of $1/f(z)$ are real and simple and are the points x where $\psi(x) \equiv \frac{\pi}{2}$ modulo π.

Proof. Let $\{s_n\}_{n=-\infty}^{+\infty}$, $\{t_n\}_{n=-\infty}^{+\infty}$ be the points where $\psi(x) \equiv 0$ modulo π, $\psi(x) \equiv \frac{\pi}{2}$ modulo π resp and satisfies $s_n < t_n < s_{n+1} < t_{n+1}$ $\forall n$. First we assume there're infinitely many such points. Let

$$f_n := \prod_{k=-n}^{n} \frac{z - t_k}{z - s_k}$$

Since $s_k < t_k$, by plotting $z - t_k$, $z - s_k$ as vectors it's easy to see the argument of f_n is less than π, and is increasing w.r.t. n. Hence $\Im f_n(z) > 0$ for $z \in \mathbb{C}_+$, and the convergence is guaranteed by Problem 59 (with similar result for negative s_k and t_k), then we're done. The case where ψ has a largest/smallest s_k/t_k can be treated similarly. $\square$

Remark. *This problem, together with Problem 59, gives one direction of the famous Hermite-Biehler criterion (See "Distribution of Zeros of Entire Functions" by B.Ja.Levin, page 308, Theorem 1) for real entire function $F(z)$ s.t. $\Im F(z) > 0$ for $z \in \mathbb{C}_+$. The other direction can be proved by modifying the proof to Problem 62.*

Problem 61

In Problem 60 show that there exists an entire function $A(z)$, which is real for real z and which has only real simple zeros, and whose zeros are the points x where $\psi(x) \equiv \frac{\pi}{2}$ modulo π. Show that $B(z) = A(z)f(z)$ is an entire function which is real for real z. Show that $E(z) = A(z) - iB(z)$ is an entire function which has no real zeros and which satisfies the inequality $|E(x - iy)| < E(x + iy)$ for $y > 0$. Show that there exists a phase function $\phi(x)$ associated with $E(z)$ such that $\phi(x) = \psi(x)$ whenever $\phi(x) \equiv 0$ modulo $\frac{\pi}{2}$ or $\psi(x) \equiv 0$ modulo $\frac{\pi}{2}$.

Proof. Using Weierstrass canonical product we can get such $A(z)$ (we'll refine this in Problem 64), and $B(z) := A(z)f(z)$ is a well-defined entire function since all f's poles are simple. Since $\Im \frac{B}{A} = \Im f(z) > 0$ on $\mathbb{C}_+$, we can conclude $E := A - iB$ is a dB function. And the rest is just trivial: $\phi(x) = 0$ iff E is real iff $B = 0$ iff $f = 0$ iff $\phi = 0$ (modulo π). $\square$

Problem 62

Let $F(z)$ and $G(z)$ be entire functions which are real for real z and which have only real simple zeros. Assume that there exists a continuous, increasing function $\psi(x)$ of real x such that the zeros of $G(z)$ are the points x where $\psi(x) \equiv 0$ modulo π and the zeros of $F(z)$ are the points x where $\psi(x) \equiv \frac{\pi}{2}$ modulo π. If $F(z)$ or $G(z)$ has a zero and if $G(z)/F(z)$ is of bounded type for $y > 0$, show that either $F(z) - iG(z)$ or $F(z) + iG(z)$ satisfies the inequality $|E(x - iy)| < |E(x + iy)|$ for $y > 0$.

Proof. By Problem 60 we know there exists a real meromorphic f s.t. f has the same zeros and poles with $\frac{G}{F}$, and since all zeros/poles are simple, it's easy to see $\Phi := \frac{1}{f}\frac{G}{F}$ is real entire, and is of bounded type in $\mathbb{C}_+$. By Problem 34, Φ is of Pólya class. And since it's zero-free, by factorization of function of Pólya class (Lemma 2 in Section 7), we have $\Phi(z) = ce^{-az^2}e^{-ibz}$, where c is a constant, $a \geqslant 0$, $\Re b \geqslant 0$. Since Φ is of bounded type (i.e. the mean type is finite), then $a = 0$. Since $\Phi^{\#} = \Phi$, c is real and b is purely imaginary. And since functions in $\mathcal{N}(\mathbb{C}_+)$ satisfy $\log^+ |\Phi| \in L^1_\Pi$ (Kreĭ's Theorem), $b = 0$. Hence $\Im \frac{G}{F} > 0$ or < 0 on $\mathbb{C}_+$, depending on the sign of c. Hence $F - iG$ or $F + iG$ is dB. In particular it's non-degenerate. $\square$

Remark. *Basically this problem says given real entire F and G with interlacing zeros, if $\frac{G}{F} \in \mathcal{N}(\mathbb{C}_+)$, then one of $F \pm iG$ is non-degenerate dB.*

Problem 63

Let $E(z)$ be an entire function of Pólya class which has no real zeros and which satisfies the inequality $|E(x-iy)| < |E(x+iy)|$ for $y > 0$. Let $\phi(x)$ be a phase function associated with $E(z)$. Show that there exists a number $p \geqslant 0$ such that

$$\frac{\partial}{\partial y} \log|E(x+iy)| = py + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\phi(t)}{(t-x)^2 + y^2}$$

for $y > 0$.

Proof. By definition of Pólya class, $\frac{\partial}{\partial y} \log|E(x+iy)| > 0$ on $\mathbb{C}_+$. Here we get strict inequality by Problem 14, since we're given $|E(x-iy)| < |E(x+iy)|$ on $\mathbb{C}_+$. By Cauchy-Riemann equation, $\frac{du}{dy} = -\frac{dv}{dx} = \Re(if)'$, hence $i(\log E)'$ is analytic and has positive real part on $\mathbb{C}_+$, by Theorem 4 (Poisson representation) we have

$$\Re \log E(x+iy) = py + \frac{y}{\pi} \int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t-x)^2 + y^2}$$

In particular, $\mu(b) - \mu(a) = \lim_{y \to 0+} \int_a^b \Re\left(i(\log E(x+iy))'\right) dx$ at points of continuity of μ, then μ and ϕ differ by a constant, and the proof is complete. □

Problem 64

Show that $E(z)$ in Problem 61 can be chosen of Pólya class if

$$\int_{-\infty}^{+\infty} \frac{d\psi(t)}{1+t^2} < \infty$$

Proof. Basically we want to get a bound on $\sum_n s_n^2$ and use Problem 9 to get $A(z)$. Suppose s_n is defined s.t. $\psi(s_n) = n\pi + \frac{\pi}{2}$. First we have

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{d\psi(t)}{1+t^2} &\geqslant \int_0^{+\infty} \frac{d\psi(t)}{1+t^2} \\
&\geqslant \sum_{n=0}^{+\infty} \int_{s_n}^{s_{n+1}} \frac{d\psi(t)}{1+t^2} \\
&\geqslant \sum_{n=0}^{+\infty} \int_{s_n}^{s_{n+1}} \frac{d\psi(t)}{1+s_n^2} \\
&= \sum_{n=0}^{+\infty} \frac{\pi}{1+s_n^2}
\end{aligned}$$

from which we can see s_n goes to $+\infty$ pretty fast and $\sum_{n=0}^{+\infty} \frac{1}{s_n^2}$ is convergent. Similar for negative n. Hence by Problem 9 we can choose A to be in Pólya class, and so is B. □

Remark. *Problem 58-64 gives a full picture on relation between spectral phase function and dB function. Actually, given any two interlacing sequences, we can construct a dB function $E = A - iB$ s.t. one sequence is the zeros of A while another one is the zeros of B. In particular, the two sequences can determine a phase function ψ, and if ψ is regular, i.e. $\int_{\mathbb{R}} \frac{d\psi}{1+t^2} < \infty$, we can choose E to be in Pólya class. Moreover, Problems 313 describes the relation between A's zeros and operator norm. Basically Pólya class is related to Hilbert-Schmidt class operators.*

Problem 65

Let $f(z)$ be a function which has an absolutely convergent representation

$$f(z) = \int_{-\infty}^{+\infty} \frac{h(t)d\mu(t)}{t-z}$$

for $y > 0$, where $h(x)$ is a Borel measurable function of real x and $\mu(x)$ is a nondecreasing function of real x. Show that $f(z)$ is analytic and of bounded type in the upper half-plane and that it has nonpositive mean type.

Remark. *Basically this problem says Cauchy transfrom of nice function is of bounded type on the half-plane.*

Proof. WLOG we can assume h is real since linear combination of functions of bounded type is still of bounded type. Let $\mu = \mu_1 + \mu_2$, where μ_1 remains constant on every interval h is negative and μ_2 remains constant on $\cdots$ positive. Now

$$f = i\int_{-\infty}^{+\infty} \frac{-ih(t)d\mu_1(t)}{t-z} + i\int_{-\infty}^{+\infty} \frac{-ih(t)d\mu_2(t)}{t-z}$$

We have

$$\Re\int_{-\infty}^{+\infty} \frac{-ih(t)d\mu_1(t)}{t-z} = \int_{-\infty}^{+\infty} \frac{yh(t)d\mu_1(t)}{(t-x)^2+y^2}$$

and notice that $d\mu_1$ is 0 for h negative hence the real part is nonnegative. By Problem 20 we 're done. $\square$

Problem 67

Let $\mathcal{B}(E)$ be a given space and let $S(z) = A(z)u + B(z)v$ where u and v are numbers, not both zero, such that $\bar{u}v = u\bar{v}$. Show that $S \in Ass(\mathcal{B}(E))$ and that the identity

$$\begin{aligned}0 = &\langle F(t)S(\alpha), \frac{G(t)S(\beta) - S(t)G(\beta)}{t-\beta}\rangle - \langle \frac{F(t)S(\alpha) - S(t)F(\alpha)}{t-\alpha}, G(t)S(\beta)\rangle \\ &+ (\alpha - \bar{\beta})\langle \frac{F(t)S(\alpha) - S(t)F(\alpha)}{t-\alpha}, \frac{G(t)S(\beta) - S(t)G(\beta)}{t-\beta}\rangle\end{aligned}$$

holds for all elements F and G of $\mathcal{B}(E)$ and all complex numbers α and β.

Proof. WLOG we assume $S(z) = B(z)$. If $B(z) \notin \mathcal{B}(E)$, then by Theorem 22 $\mathcal{B}(E)$ sits isometrically in $L^2(d\mu_0)$, where $\mu_0(t_n) = \frac{\pi}{\phi'(t_n)}\frac{1}{|E(t_n)|^2}$, where $\{t_n\}$ are zeros of $B(z)$. Then

$$\begin{aligned}&\langle F(t)S(\alpha), \frac{G(t)S(\beta) - S(t)G(\beta)}{t-\beta}\rangle - \langle \frac{F(t)S(\alpha) - S(t)F(\alpha)}{t-\alpha}, G(t)S(\beta)\rangle \\ &+ (\alpha - \bar{\beta})\langle \frac{F(t)S(\alpha) - S(t)F(\alpha)}{t-\alpha}, \frac{G(t)S(\beta) - S(t)G(\beta)}{t-\beta}\rangle \\ =&\sum_n \frac{F(t_n)S(\alpha)\overline{G(t_n)S(\beta)}}{t_n - \bar{\beta}}\mu_0(t_n) - \sum_n \frac{F(t_n)S(\alpha)\overline{G(t_n)S(\beta)}}{t_n - \alpha}\mu_0(t_n) \\ &+ (\alpha - \bar{\beta})\sum_n \frac{F(t_n)S(\alpha)\overline{G(t_n)S(\beta)}}{(t_n-\alpha)(t_n-\bar{\beta})}\mu_0(t_n) \\ =&0\end{aligned}$$

Now suppose $B \in \mathcal{B}(E)$, define Bezoutian operator $\mathbb{B}_{S,\alpha}$ by

$$(\mathbb{B}_{S,\alpha}F)(z) = \frac{F(z)S(\alpha) - S(z)F(\alpha)}{z-\alpha}$$

where $S \in Ass(\mathcal{B}(E))$. Now $S = B$, and by proof to Theorem 22 we know $\mathcal{B}(E)$ is spanned by $\{K(t_n, z)\}$ and $B(z)$, all of which are orthogonal to each other. I'll show $\mathbb{B}_{B,\alpha}$ maps $Span\{K(t_n, z)\}$ to itself and B to 0:

$$\begin{aligned}(\mathbb{B}_{B,\alpha}K(t_n,\cdot))(z) &= \frac{K(t_n,z)B(\alpha) - B(z)K(t_n,\alpha)}{z-\alpha} \\ &= \frac{B(\alpha)}{t_n-\alpha}\frac{B(z)\overline{A(t_n)}}{\pi(z-t_n)} \\ &= \frac{B(\alpha)}{t_n-\alpha}K(t_n,z) \\ (\mathbb{B}_{B,\alpha}B)(z) &= 0\end{aligned}$$

Now for $F, G \in \mathcal{B}(E)$, we decompose them into $F_1 + F_2$, $G_1 + G_2$, where $F_1, G_1 \in Span\{K(t_n, z)\}$ and $F_2, G_2 \in Span\{B\}$ (scalar multiplication of B), then

$$\begin{aligned}
&\langle F(t)S(\alpha), (\mathbb{B}_{S,\beta}\, G)(t)\rangle - \langle(\mathbb{B}_{S,\alpha}\, F)(t), G(t)S(\beta)\rangle \\
&+ (\alpha - \bar{\beta})\langle(\mathbb{B}_{S,\alpha})(t), (\mathbb{B}_{S,\beta}\, G)(t)\rangle \\
=&\langle F_1(t)S(\alpha) + F_2(t)S(\alpha), (\mathbb{B}_{S,\alpha G_1})(t)\rangle + \langle(\mathbb{B}_{S,\alpha}\, F_1)(t), G_1(t)S(\beta) + G_2(t)S(\beta)\rangle \\
&+ (\alpha - \bar{\beta})\langle(\mathbb{B}_{S,\alpha}\, F_1)(t), (\mathbb{B}_{S,\beta}\, G)(t)\rangle \\
=&\langle F_1(t)S(\alpha), (\mathbb{B}_{S,\alpha G_1})(t)\rangle + \langle(\mathbb{B}_{S,\alpha}\, F_1)(t), G_1(t)S(\beta)\rangle \\
&+ (\alpha - \bar{\beta})\langle(\mathbb{B}_{S,\alpha}\, F_1)(t), (\mathbb{B}_{S,\beta}\, G_1)(t)\rangle
\end{aligned}$$

Now since $F_1, G_1 \in Span\{K(t_n, z)\}$, we can calculate the inner products in subspace $Span\{K(t_n, z)\}$ which is also a Hilbert space, and the equality is proved. □

Remark. *This problem, together with Problem 68, explains why associate function $S(z) = uA(z) + vB(z)$ are special. For such associate function S, $\mathbb{B}_{S,\alpha}$ is self-adjoint when $\alpha \in \mathbb{R}$.*

Problem 68

Let $\mathcal{B}(E)$ be a given space and let $S \in Ass(\mathcal{B}(E))$. Assume that the identity of Problem 67 holds for all elements F and G of $\mathcal{B}(E)$ and all complex numbers α and β. Show that $S(z) = A(z)u + B(z)v$ for some numbers u and v such that $\bar{u}v = u\bar{v}$.

Proof. Suppose S satisfies the identity of Problem 67, let $t_1, t_2, x \in \mathbb{R}$, $F = K_{t_1}$ and $G = K_{t_2}$, and $\alpha = \beta = x$, then the identity becomes

$$\left(K_{t_1}(t)S(x), \frac{K_{t_2}(t)S(x) - S(t)K_{t_2}(x)}{t - x}\right) = \left(\frac{K_{t_1}(t)S(x) - S(t)K_{t_1}(x)}{t - x}, K_{t_2}(t)S(x)\right)$$
$$S(x)\frac{K_{t_2}(t_1)\overline{S(x)} - \overline{S(t_1)}K_{t_2}(x)}{t_1 - x} = \overline{S(x)}\frac{K_{t_1}(t_2)S(x) - S(t_2)K_{t_1}(x)}{t_2 - x}$$

Let $t_1 = t_2$ where $S(t_1) \neq 0$, then it becomes $S(x)\overline{S(t_1)} = \overline{S(x)}S(t_1)$, which further implies $\frac{S(z)}{S(t_1)}$ is real entire. WLOG we can assume $S(z)$ is real entire, now let $t_1, t_2 \in \mathbb{R}$ be any two real numbers, note that $K_{t_2}(t_1) = K_{t_1}(t_2)$, then the equation above becomes

$$S(x)^2 K_{t_2}(t_1)\frac{t_2 - t_1}{(t_1 - x)(t_2 - x)} - S(x)S(t_1)\frac{K_{t_2}(x)}{t_1 - x} + S(x)S(t_2)\frac{K_{t_1}(x)}{t_2 - x} = 0$$
$$S(x)K_{t_2}(t_1)\frac{t_2 - t_1}{(t_1 - x)(t_2 - x)} - S(t_1)\frac{K_{t_2}(x)}{t_1 - x} + S(t_2)\frac{K_{t_1}(x)}{t_2 - x} = 0$$

Hence

$$S(x) = \frac{1}{\pi K_{t_2}(t_1)(t_2 - t_1)}\left(S(t_1)\left(B(x)A(t_2) - A(x)B(t_2)\right) - S(t_2)\left(B(x)A(t_1) - A(x)B(t_1)\right)\right)$$

where RHS is a linear combination of $B(x)$ and $A(x)$. Since S, A, B are entire, the equality holds for any complex number z. Therefore $S(z) = A(z)u + B(z)v$ for real u and v. In general case u, v may not be real, but $\bar{u}v = \bar{v}u$. □

Problem 69

Let $\mathcal{B}(E)$ be a given space and let $S(z)$ be a nonzero entire function such that $\frac{F(z)S(w) - S(z)F(w)}{z - w}$ belongs to $\mathcal{B}(E)$ whenever $F(z)$ belongs to $\mathcal{B}(E)$. Assume that there exists a nondecreasing function $\mu(x)$ of real x such that $S(x)$ is μ-equivalent to zero and such that $\mathcal{B}(E)$ is contained isometrically in $L^2(\mu)$. Show that $S(z) = A(z)u + B(z)v$ for some numbers u and v such that $\bar{u}v = \bar{v}u$. Show that $\mu(x)$ is a step function whose points of increase are zeros of $S(z)$ and that $\mu(t+) - \mu(t-) = \frac{1}{K(t,t)}$ at each such zero. Show that $\mathcal{B}(E)$ fills $L^2(\mu)$.

Proof. $S = Au + Bv$ follows from Problem 68 directly, as it's easy to show the identity in Problem 67 holds using inner product in $L^2(\mu)$. We know S is μ-equivalent to 0, then μ must be a step functions whose support is contained in the zeros of S. Using Theorem 22, we know $\mathcal{B}(E)$ is spanned by K_{t_n} where t_n's are zeros of S, and possibly S itself. Since $K_{t_n}(t_n) \neq 0$ and $K_{t_n}(t_m) = 0$ for $m \neq n$, hence $\text{supp}(\mu) = Z(S)$, and it's the Herglotz measure of S (as given by Theorem 22), and consequently $\mu(t+) - \mu(t-) = \frac{1}{K_t(t)}$. Since S is μ-equivalent to 0, $S \notin \mathcal{B}(E)$, and $\mathcal{B}(E)$ fills $L^2(\mu)$ since now it's just spanned by K_{t_n}'s. □

Problem 70

Let $\mathcal{B}(E)$ be a given space and let $S(z)$ be an entire function such that $\frac{S(z)}{E(z)}$ and $\frac{S^{\#}(z)}{E(z)}$ are of bounded type in the upper half-plane. Assume that $E(z)$ has no real zeros and that $\mu(x)$ is a given non-decreasing function of real x such that $\mathcal{B}(E)$ is contained isometrically in $L^2(\mu)$. Assume that there exists a nonzero entire function $Q(z)$ which is μ-equivalent to zero such that $\frac{F(z)Q(w)-Q(z)F(w)}{z-w}$ belongs to $\mathcal{B}(E)$ whenever $F(z)$ belongs to $\mathcal{B}(E)$. If

$$\int_{-\infty}^{+\infty} \frac{|S(t)|^2 d\mu(t)}{1+t^2} < \infty$$

if

$$\limsup_{y\to+\infty} \left|\frac{S(iy)}{Q(iy)}\right| < \infty$$

and if

$$\limsup_{y\to+\infty} \left|\frac{S(-iy)}{Q(iy)}\right| < \infty$$

show that $\frac{F(z)S(w)-S(z)F(w)}{z-w}$ belongs to $\mathcal{B}(E)$ whenever $F(z)$ belongs to $\mathcal{B}(E)$.

Remark. *The conditions are actually necessary as well.*

Proof. See my thesis. □

Problem 71

Let $\mathcal{B}(E)$ be a given space such that $\mathcal{B}(E)$ has no real zeros, and let $S(z)$ be an entire function which is real for real z and has no zeros, such that $\frac{S(z)}{E(z)}$ is of bounded type in the upper half-plane. Let $\mu(x)$ be a nondecreasing function of real x such that $\mathcal{B}(E)$ is contained isometrically in $L^2(\mu)$. If

$$\int_{-\infty}^{+\infty} \frac{|S(t)|^2}{1+t^2} d\mu(t) < \infty$$

show that $\frac{E(z)}{S(z)}$ is of Pólya class and that $\frac{F(z)S(w)-S(z)F(w)}{z-w}$ belongs to $\mathcal{B}(E)$ whenever $F(z)$ belongs to $\mathcal{B}(E)$.

Proof. Let $G(z) := \frac{E(z)}{S(z)}$, then G is of bounded type in $\mathbb{C}_+$ since it's given that $\frac{S(z)}{E(z)}$ is of bounded type in $\mathbb{C}_+$ and S is zero-free on $\mathbb{C}_+$. G is dB since S is real entire and E is dB, then by Problem 34 G is of Pólya class. This part is wrong.
First we assume there's no $Q \in Ass(\mathcal{B}(E))$ s.t. $Q = 0$ μ-a.e.. By Theorem 26, in order to prove $S \in Ass(\mathcal{B}(E))$ it suffices to show

$$\limsup_{y\to+\infty} \left|\frac{S(iy)}{E(iy)}\right| < \infty$$
$$\limsup_{y\to+\infty} \left|\frac{S(-iy)}{E(iy)}\right| < \infty$$

Since G is nonzero and of Pólya class, $|G(iy)|$ is nondecreasing for $y \geqslant 0$, and this proves the first inequality. Note that

$$\left|\frac{S(-iy)}{E(iy)}\right| = \left|\frac{S(-iy)}{S(iy)}\frac{S(iy)}{E(iy)}\right| = \left|\frac{S(iy)}{E(iy)}\right|$$

since S is real entire, hence $S \in Ass(\mathcal{B}(E))$.
Now assume there's $Q \in Ass(\mathcal{B}(E))$ which is equal to 0 μ-a.e.. By Problem 69, $Q = uA + vB$ where $\bar{u}v$ is real. WLOG we can assume $Q = e^{i\alpha}E - e^{-i\alpha}E^{\#}$. Then

$$\left|\frac{S(iy)}{Q(iy)}\right| = \left|\frac{1}{G(iy)}\right| \left|\frac{1}{e^{i\alpha} - e^{-i\alpha}\frac{G^{\#}(iy)}{G(iy)}}\right|$$

where $G = \frac{E}{S}$ is of Pólya class. Since $G(iy)$ is nondecreasing for $y \geqslant 0$, the first factor is bounded above. As for the second factor, by factorization of functions of Pólya class, we have

$$\frac{G^{\#}(iy)}{G(iy)} = e^{-2by} \prod \frac{z_n - iy}{\bar{z}_n - iy}$$

which has module strictly less than 1 for $y > 0$, hence the second factor is bounded above as well. By Problem 70 we can conclude $S \in Ass(\mathcal{B}(E))$. □

Problem 72

Let $\mathcal{B}(E_a)$ and $\mathcal{B}(E_b)$ be given spaces such that $\mathcal{B}(E_a)$ is contained isometrically in $\mathcal{B}(E_b)$ and E_a has no real zeros. Let $S(z)$ be an entire function which has no zeros. If $\frac{F(z)S(w)-S(z)F(w)}{z-w}$ belongs to $\mathcal{B}(E_b)$ whenever $F(z)$ belongs to $\mathcal{B}(E_b)$, show that it belongs to $\mathcal{B}(E_a)$ whenever $F(z)$ belongs to $\mathcal{B}(E_a)$.

Proof. Since $\mathbb{B}_{S,w}$ is linear in S, we can decompose $S = S_1 - iS_2$ where S_1, S_2 are real entire, hence WLOG we can just assume S itself is real entire. Since $S \in Ass(\mathcal{B}(E_b))$, $\frac{S}{E_b}$ is of bounded type on $\mathbb{C}_+$. Pick $F \in \mathcal{B}(E_a)$, then $\frac{F}{E_a}$ is of bounded type on $\mathbb{C}_+$. Since F is in $\mathcal{B}(E_b)$ as well, $\frac{F}{E_b}$ is of bounded type on $\mathbb{C}_+$ too, hence $\frac{E_b}{E_a} \in \mathcal{N}(\mathbb{C}_+)$, and moreover $\frac{S}{E_a} \in \mathcal{N}(\mathbb{C}_+)$. By Theorem 25, $\int_{\mathbb{R}} \left|\frac{S(t)}{E_b(t)}\right|^2 \frac{dt}{1+t^2} < \infty$. Since $\mathcal{B}(E_a) \sqsubseteq \mathcal{B}(E_b) \sqsubseteq L^2\left(\frac{dt}{|E_b(t)|^2}\right)$, by Problem 71 $S \in Ass(\mathcal{B}(E_a))$. □

Remark. *This problem says, if $\mathcal{B}(E_a) \sqsubseteq \mathcal{B}(E_b)$ and E_a is strict, then for zero-free $S \in Ass(\mathcal{B}(E_b))$, we have $S \in Ass(\mathcal{B}(E_a))$.*

Problem 74

Let $\mathcal{B}(E)$ be a given space and let α be a real number such that $S(z) = e^{i\alpha}E(z) - e^{-i\alpha}E^{\#}(z)$ does not belong to $\mathcal{B}(E)$. Show that $F(z) \mapsto \frac{F(z)S(w)-S(z)F(w)}{z-w}$ is an everywhere defined and bounded transformation in $\mathcal{B}(E)$ for every complex number w. For each fixed $F(z)$ in $\mathcal{B}(E)$, show that $\frac{F(z)S(w)-S(z)F(w)}{z-w}$ depends continuously on w in the metric of $\mathcal{B}(E)$.

Proof. By Theorem 22,

$$\|F\|_E^2 = \sum_{\phi(t_n)\equiv\alpha} \frac{\pi}{\phi'(t_n)} \left|\frac{F(t_n)}{E(t_n)}\right|^2$$

Note that if $\phi(t_n) \equiv \alpha \mod \pi$, then $S(t_n) = 0$. Then we have

$$\left\|\frac{F(z)S(w)-S(z)F(w)}{z-w}\right\|_E^2 = \sum_{\phi(t_n)\equiv\alpha} \frac{\pi}{\phi'(t_n)} \left|\frac{F(t_n)}{E(t_n)}\right|^2 \left|\frac{S(w)}{t_n-w}\right|^2$$

The last factor $\left|\frac{S(w)}{t_n-w}\right|^2$ is bounded on the set $\{t \big| \phi(t) \equiv \alpha \mod \pi\}$, hence the operator is bounded. Let's denote the operator by R_w, then for fixed $F \in \mathcal{B}(E)$,

$$\|R_{w_1}F - R_{w_2}F\|_E^2 = \sum_{\phi(t_n)\equiv\alpha} \frac{\pi}{\phi'(t_n)} \left|\frac{F(t_n)}{E(t_n)}\right|^2 \left|\frac{S(w_1)}{t_n-w_1} - \frac{S(w_2)}{t_n-w_2}\right|^2$$

For fixed w_1, the last factor goes to 0 as $w_2 \to w_1$ uniformly. □

Remark. *Such associated function S plays an important role in dB theory. Check Theorem 22, 29, and Problem 67, 68, 79, 87 for more information. And this problem says for such $S = e^{i\alpha}E - e^{-i\alpha}E^{\#}$, $\mathbb{B}_{S,w}$ is well-defined and bounded as an operator on $\mathcal{B}(E)$. Moreover, for fixed $F \in \mathcal{B}(E)$, $\mathbb{B}_{S,w}F$ depends continuously on w in $\mathcal{B}(E)$.*

Problem 76

In Problem 75 show that

$$\begin{aligned}
0 = &\langle F(t)S(\alpha), \frac{G(t)S(\beta)-S(t)G(\beta)}{t-\beta}\rangle - \langle\frac{F(t)S(\alpha)-S(t)F(\alpha)}{t-\alpha}, G(t)S(\beta)\rangle \\
&+ (\alpha-\bar{\beta})\langle\frac{F(t)S(\alpha)-S(t)F(\alpha)}{t-\alpha}, \frac{G(t)S(\beta)-S(t)G(\beta)}{t-\beta}\rangle
\end{aligned}$$

whenever $F(z)$ belongs to $\mathcal{B}(E)$ and vanishes at α, and $G(z)$ belongs to $\mathcal{B}(E)$ and vanishes at β.

Proof. For F s.t. $F(\alpha) = 0$, and G s.t. $G(\beta) = 0$, the RHS becomes

$$\begin{aligned}
&\langle F(t)S(\alpha), \frac{G(t)S(\beta)}{t-\beta}\rangle - \langle\frac{F(t)S(\alpha)}{t-\alpha}, G(t)S(\beta)\rangle + (\alpha-\bar{\beta})\langle\frac{F(t)S(\alpha)}{t-\alpha}, \frac{G(t)S(\beta)}{t-\beta}\rangle \\
=&\int_{\mathbb{R}} \left(\frac{F(t)S(\alpha)\overline{G(t)S(\beta)}}{t-\bar{\beta}} - \frac{F(t)S(\alpha)\overline{G(t)S(\beta)}}{t-\alpha} + \frac{(\alpha-\bar{\beta})F(t)S(\alpha)\overline{G(t)S(\beta)}}{(t-\alpha)(t-\bar{\beta})}\right)\frac{dt}{|E(t)|^2} \\
=&0
\end{aligned}$$

$\square$

Remark. *This problem says*

$$\langle F(t)S(\alpha), (\mathbb{B}_{S,\beta}\, G)(t)\rangle - \langle(\mathbb{B}_{S,\alpha}\, F)(t), G(t)S(\beta)\rangle + (\alpha - \bar{\beta})\langle(\mathbb{B}_{S,\alpha}\, F)(t), (\mathbb{B}_{S,\beta}\, G)(t)\rangle$$

for $F, G \in \mathcal{B}(E)$ s.t. $F(\alpha) = G(\beta) = 0$.

Problem 79

If $C(z)$ and $D(z)$ are linearly dependent in Theorem 27, show that $S(z) = A(z)u + B(z)v$ for some numbers u and v such that $\bar{u}v = u\bar{v}$.

Remark. *Basically Theorem 27 says $S \in Ass(\mathcal{B}(E))$ iff $\exists$ real entire C and D, such that $M_S = \frac{1}{S}\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a dB matrix. Note that in standard notation we should take transpose of M_S.*

Proof. We will use two useful results (easy to prove). First, Lemma 4.2 in Misha's notes II, for dB matrix $M(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & \delta(z) \end{pmatrix}$, γ and δ (or α and β) are linearly dependent, i.e. $\tilde{E}$ degenerate (or E degenerate) iff

$$M(z) = \begin{pmatrix} 1 & az \\ 0 & 1 \end{pmatrix} U$$

where $a \geqslant 0$ and U is a constant J-unitary matrix. The second result, Cor 3.2 in Misha's notes II, says U is J-unitary iff $U = \lambda U_1$, where $|\lambda| = 1$ and $U_1 \in SL_2(\mathbb{R})$.
Now suppose C and D are linearly dependent, then so is $\gamma(z)$ and $\delta(z)$. This implies

$$\frac{1}{S}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = M_S = \lambda \begin{pmatrix} 1 & az \\ 0 & 1 \end{pmatrix} U_1$$

Let $U_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, we have

$$\begin{aligned} \frac{A}{S} &= \lambda(a_1 + c_1 az) \\ \frac{B}{S} &= \lambda(b_1 + d_1 az) \\ d_1\frac{A}{S} - c_1\frac{B}{S} &= \lambda(a_1 d_1 - b_1 c_1) = \lambda \\ S &= \frac{d_1 A}{\lambda} - \frac{c_1 B}{\lambda} \end{aligned}$$

Let $u = \frac{d_1}{\lambda}$, $v = -\frac{b_1}{\lambda}$, then $u\bar{v} = -b_1 d_1$ is real, and we're done. $\square$

Problem 83

If $\mathcal{B}(E)$ is a given space, show that the hypotheses of Theorem 28 are satisfied with $S(z) = E(z)$, $C(z) = -B(z)$ and $D(z) = A(z)$. Show that the transformation $F(z) \to \frac{1}{\sqrt{2}}\begin{pmatrix} F(z) \\ iF(z) \end{pmatrix}$ is an isometry of $\mathcal{B}(E)$ onto $\mathcal{B}_S(M)$.

Proof. Conditions of Theorem 28 are satisfied by problem 81 and the fact $A^{\#} = A, B^{\#} = B$. The isometry part is by construction. $\square$

Problem 85

Show that an element $S(z)$ of a space $\mathcal{B}(E)$ is of the form $S(z) = A(z)u + B(z)v$ for some numbers u and v if and only if

$$\frac{K(w,z)S(w) - K(w,w)S(z)}{z - w} = \frac{K(\bar{w},z)S(\bar{w}) - K(\bar{w},\bar{w})S(z)}{z - \bar{w}} \tag{3}$$

for all complex z and w. If $S(z)$ is of this form, show that $\bar{u}v = u\bar{v}$.

Proof. First we assume the equality holds. Rearranging the equality and using the fact (Problem 40) $K(\bar{w},\bar{w}) = K(w,w)$ we get:

$$\frac{K(w,z)S(w)}{z-w} - \frac{K(\bar{w},z)S(\bar{w})}{z-\bar{w}} = K(w,w)S(z)\left(\frac{1}{z-w} - \frac{1}{z-\bar{w}}\right) \tag{4}$$

Multiply both sides by $(z-w)(z-\bar{w})$ and divide by $(w-\bar{w})$ we can get

$$\left(\frac{\overline{A(w)}S(w) - A(w)S(\bar{w})}{K(w,w)(w-\bar{w})}\right)B(z) - \left(\frac{\overline{B(w)}S(w) - B(w)S(\bar{w})}{K(w,w)(w-\bar{w})}\right)A(z) = S(z)$$

$$\frac{(\mathbb{B}_{S,w}\,B)(\bar{w})}{K(w,w)}A(z) - \frac{(\mathbb{B}_{S,w}\,A)(\bar{w})}{K(w,w)}B(z) = S(z)$$

Hence $S(z)$ is a linear combination of A and B. Let $S = uA + vB$. Now by the third axiom of dB space: $S \in \mathcal{B}(E) \Rightarrow S^{\#} \in \mathcal{B}(E)$, hence $S^{\#} = A\bar{u} + B\bar{v} \in \mathcal{B}(E)$. But $E \notin \mathcal{B}(E)$, A and B can't be in $\mathcal{B}(E)$ at the same time, which implies

$$\det\begin{pmatrix} u & v \\ \bar{u} & \bar{v}\end{pmatrix} = 0$$

that is, $\bar{u}v = u\bar{v}$.

Now assume $S = uA + vB$. Note that both sides of (4) are linear in S, so WLOG we assume $u = 1$, $v = 0$, i.e. $S(z) = A(z)$, then LHS of (4) becomes

$$\begin{aligned}
&\frac{(B(z)|A(w)|^2 - \overline{B(w)}A(w)A(z)) - (B(z)|A(w)|^2 - B(w)\overline{A(w)}A(z))}{\pi(z-\bar{w})(z-w)} \\
&= \frac{B(w)\overline{A(w)} - A(w)\overline{B(w)}}{\pi(z-\bar{w})(z-w)}A(z) \\
&= \frac{(w-\bar{w})K(w,w)}{(z-\bar{w})(z-w)}A(z) \\
&= \left(\frac{1}{z-w} - \frac{1}{z-\bar{w}}\right)K(w,w)A(z)
\end{aligned}$$

which is equal to the RHS of (4). □

Remark. *Let $\mathbb{B}_{S,w}$ denote Bezoutian operator, then this problem says*

$$\mathbb{B}_{S,w}\,K_w = \mathbb{B}_{S,\bar{w}}\,K_{\bar{w}}$$

if and only if $S = uA + vB$ where $\bar{u}v = \bar{v}u$.

Problem 86

Show that a space $\mathcal{B}(E_b)$ has dimension 1 if and only if

$$(A_b(z), B_b(z)) = (A_a(z), B_a(z))\begin{pmatrix} 1-\beta z & \alpha z \\ -\gamma z & 1+\beta z\end{pmatrix}$$

where A_a and B_a are linearly dependent entire functions which are real for real z, and where α, β, γ are real numbers, not all zero, such that $\alpha \geqslant 0$, $\gamma \geqslant 0$, and $\alpha\gamma = \beta^2$. Show that

$$\alpha = \pi u\bar{u}, \beta = \pi u\bar{v} = \pi v\bar{u}, \gamma = \pi v\bar{v}$$

for some numbers u and v such that

$$S(z) = A_a(z)u + B_a(z)v = A_b(z)u + B_b(z)v$$

is an element of norm 1 in $\mathcal{B}(E_b)$.

Remark. *This problem, together with problem 87, 88 give a full description of finite dimensional dB spaces.*

Proof. "⇒" Since $\dim\mathcal{B}(E_b) = 1$, $dom(z)$ must be empty, and by Theorem 29 we know $\mathcal{B}(E_b)$ is generated by $S(z) = uA_b + vB_b$. Then by problem 85, $\bar{u}v = u\bar{v}$. Obviously we can choose u, v real and $S = uA_b + vB_b$ has norm 1 in $\mathcal{B}(E_b)$. Now choose real c, d s.t. $V := \begin{pmatrix} u & c \\ v & d\end{pmatrix} \in SL_2(\mathbb{R})$. Let

$$(\tilde{A}_b, \tilde{B}_b) = (A_b, B_b)\begin{pmatrix} u & c \\ v & d\end{pmatrix}$$

Then $S = \tilde{A}_b \in \mathcal{B}(\tilde{E}_b)$ with norm 1. To save time I'll just write $E = \tilde{E}_b, A = \tilde{A}_b, B = \tilde{B}_b$ for now. Note that $\mathcal{B}(E) = \mathcal{B}(E_b)$ since $V \in SL_2(\mathbb{R})$. For some real t, $K(t,z) \in \mathcal{B}(E)$, hence $\exists\ \lambda$ s.t.

$$K(t,z) = \frac{B(z)A(t) - A(z)B(t)}{\pi(z-t)} = \lambda A(z)$$

Since both sides are real on $\mathbb{R}$, λ is real as well. And also we can see that $\frac{B}{A} = c_1 z + c_2$, and moreover $c_1 > 0, c_2 \in \mathbb{R}$. Now use the fact A has norm 1 in $\mathcal{B}(E)$:

$$\int_{\mathbb{R}} \frac{A^2}{A^2+B^2} dt = 1$$

using translation we can get $c_1 = \pi$. Now let $\tilde{A}_a = \tilde{A}_b$, and $\tilde{B}_a = c_2 \tilde{A}_b$, then

$$(\tilde{A}_b, \tilde{B}_b) = (\tilde{A}_a, \tilde{B}_a)\begin{pmatrix} 1 & \pi z \\ 0 & 1 \end{pmatrix}$$

Now let $(A_a, B_a) = (\tilde{A}_a, \tilde{B}_a)V^{-1}$, then

$$\begin{aligned}(A_b, B_b) &= (\tilde{A}_b, \tilde{B}_b)V^{-1} \\ &= (A_a, B_a)V\begin{pmatrix} 1 & \pi z \\ 0 & 1 \end{pmatrix}V^{-1} \\ &= (A_a, B_a)\begin{pmatrix} 1-\beta z & \alpha z \\ -\gamma z & 1+\beta z \end{pmatrix}\end{aligned}$$

and obviously A_a and B_a are linearly dependent (by definition $\tilde{B}_a = c_2 \tilde{A}_a$), real entire. And it's trivial to check

$$\alpha = \pi u \bar{u}, \beta = \pi u \bar{v} = \pi v \bar{u}, \gamma = \pi v \bar{v}$$

$S = uA_a + vB_a$ follows by direct calculation using the matrix relation above.

"$\Leftarrow$" Define $\tilde{A}_b, \tilde{B}_b, \tilde{A}_a, \tilde{B}_a$ as above. WLOG assume $\tilde{A}_a \neq 0$, then $\tilde{B}_a = \lambda \tilde{A}_a$ for some $\lambda \in \mathbb{R}$, and

$$(\tilde{A}_b, \tilde{B}_b) = (\tilde{A}_a, \tilde{B}_a)\begin{pmatrix} 1 & \pi z \\ 0 & 1 \end{pmatrix}$$

Then

$$\begin{aligned}\frac{\tilde{E}_b}{\tilde{A}_a} &= \frac{\tilde{A}_b}{\tilde{A}_a} - i\frac{\tilde{B}_b}{\tilde{A}_a} \\ &= 1 - i(\lambda + \pi z)\end{aligned}$$

Let $E_0 := 1 - i(\lambda + \pi z)$, I'm gonna show it's dB, i.e. $\lambda \geqslant 0$. $\Im \frac{\tilde{B}_b}{\tilde{A}_b} > 0$ on $\mathbb{C}_+$ implies $\Im(\pi z + \lambda) > 0$ on $\mathbb{C}_+$, hence $\lambda \geqslant 0$. Now use the result of problem 44, we get

$$\dim(\mathcal{B}(E_b)) = \dim(\mathcal{B}(\tilde{E}_b)) = \dim(\mathcal{B}(E_0)) = \dim \mathbb{C} = 1$$

□

Problem 87

Let $\mathcal{B}(E_b)$ be a given space which has dimension greater than 1 and in which multiplication by z is not densely defined. Show that

$$(A_b(z), B_b(z)) = (A_a(z), B_a(z))\begin{pmatrix} 1-\beta z & \alpha z \\ -\gamma z & 1+\beta z \end{pmatrix}$$

for some space $\mathcal{B}(E_a)$ which is contained isometrically in $\mathcal{B}(E_b)$ and for some numbers α, β, γ, not all zero such that $\alpha \geqslant 0$, $\gamma \geqslant 0$, and $\alpha\gamma = \beta^2$. Show that

$$\alpha = \pi u \bar{u}, \beta = \pi u \bar{v} = \pi v \bar{u}, \gamma = \pi v \bar{v}$$

for some numbers u and v such that

$$S(z) = uA_a(z) + vB_a(z) = uA_b(z) + vB_b(z)$$

is an element of norm 1 in $\mathcal{B}(E_b)$ which spans the orthogonal complement of $\mathcal{B}(E_a)$.

Proof. Similar to problem 86, let's assume $u = 0, v = 1$ first. That is, we assume $B_b \in \mathcal{B}(E_b)$ and has norm 1. The reason we choose B_b rather than A_b is, in Theorem 22, de Branges chooses B_b so we can use some results directly. According to the proof of Theorem 22, we have (follows from problem 5, 47 in case you don't have the book):

$$K_b(w,z) = \frac{p_b}{\pi} B_b(z)\bar{B}_b(w) + \sum_{B_b(t_n)=0} \frac{A_b(t_n)}{\pi B_b'(t_n)} \frac{B_b(z)}{z-t_n} \frac{\bar{B}_b(w)}{\bar{w}-t_n} \tag{5}$$

where p_b comes from the Poisson representation of $\frac{A_b}{B_b}$:

$$-\Im \frac{A_b}{B_b} = p_b y + \frac{y}{\pi} \int_{\mathbb{R}} \frac{d\mu}{(x-t)^2 + y^2} \tag{6}$$

By (5), and note that $\frac{B_b(z)}{z-t_n} = \pi \frac{K_b(t_n,z)}{\bar{E}_b(t_n)}$ we have:

$$\begin{aligned} B_b(w) &= \langle B_b(t), K_b(w,t) \rangle \\ &= B_b(w) \langle B_b(t), \frac{p_b}{\pi} B(t) \rangle \end{aligned}$$

and since we assume B_b has norm 1, $p_b = \pi$. Now define A_a, B_a as

$$(A_a, B_a) = \begin{pmatrix} 1 & 0 \\ \pi z & 1 \end{pmatrix} (A_b, B_b)$$

Then

$$-\Im \frac{A_a}{B_a} = -\Im \frac{A_b}{B_b} - \pi y = \frac{y}{\pi} \int_{\mathbb{R}} \frac{d\mu}{(x-t)^2 + y^2} > 0$$

hence $E_a = A_a - iB_a$ is dB and $p_a = 0$ (defined as in (6)), and since $B_a = B_b$, we have

$$K_a(w,z) = \sum_{B_b(t_n)=0} \frac{A_b(t_n)}{\pi B_b'(t_n)} \frac{B_b(z)}{z-t_n} \frac{\bar{B}_b(w)}{\bar{w}-t_n}$$

belongs to $\mathcal{B}(E_a)$. In particular, when $w = t_n$, $K_b(t_n, z) = K_a(t_n, z) \in \mathcal{B}(E_a)$. Now I'll show $B_b \notin \mathcal{B}(E_a)$. Suppose $B_b \in \mathcal{B}(E_a)$, then

$$\begin{aligned} B_b(w) &= \langle B_b(t), K_a(w,t) \rangle \\ &= B_b(w) \langle B_b(t), \frac{p_a}{\pi} B(t) \rangle \\ &= 0 \end{aligned}$$

a contradiction. The "isometrically" part now comes from Theorem 22 itself directly:

$$\begin{aligned} E_a(t_n) &= A_a(t_n) = A_b(t_n) + \pi t_n B_b(t_n) = A_b(t_n) = E_b(t_n) \\ \phi_a'(t_n) &= \pi \frac{K_a(t_n,t_n)}{|E_a(t_n)|^2} = \pi \frac{K_b(t_n,t_n)}{|E_b(t_n)|^2} = \phi_b'(t_n) \end{aligned}$$

where the formula for $\phi'(x)$ comes from problem 48. For the general case, similar to 86, we still assume u, v are real. Let

$$(\tilde{A}_b, \tilde{B}_b) = (A_b, B_b) \begin{pmatrix} c & u \\ d & v \end{pmatrix}$$

where $\begin{pmatrix} c & u \\ d & v \end{pmatrix} \in SL_2(\mathbb{R})$, hence $\mathcal{B}(E_b) = \mathcal{B}(\tilde{E}_b)$ and

$$\begin{aligned} (A_b(z), B_b(z)) &= (\tilde{A}_b, \tilde{B}_b) \begin{pmatrix} v & -u \\ -d & c \end{pmatrix} \\ &= (\tilde{A}_a(z), \tilde{B}_a(z)) \begin{pmatrix} 1 & 0 \\ -\pi z & 1 \end{pmatrix} \begin{pmatrix} v & -u \\ -d & c \end{pmatrix} \\ &= (\tilde{A}_a(z), \tilde{B}_a(z)) \begin{pmatrix} v & -u \\ -d & c \end{pmatrix} \begin{pmatrix} c & u \\ d & v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\pi z & 1 \end{pmatrix} \begin{pmatrix} v & -u \\ -d & c \end{pmatrix} \\ &= (A_a, B_a) \begin{pmatrix} 1 - \beta z & \alpha z \\ -\gamma z & 1 + \beta z \end{pmatrix} \end{aligned}$$

□

Problem 88

Show that multiplication by z is not densely defined in a space $\mathcal{B}(E)$ if the space has finite dimension. Show that a space $\mathcal{B}(E)$ has finite dimension r if, and only if, $E(z) = S(z)E_0(z)$ where $S(z)$ is an entire function which is real for real z and $E_0(z)$ is a polynomial of degree r which has no real zeros.

Proof. This follows from problem 86, 87 and 44. I'll use induction but the direct proof wouldn't be more difficult. For the base case, by problem 86 we have

$$(A, B) = (A_0(z), B_0(z)) \begin{pmatrix} 1-\beta z & \alpha z \\ -\gamma z & 1+\beta z \end{pmatrix}$$

where A_0 and B_0 are linearly dependent, $\alpha = \pi u^2, \beta = \pi uv, \gamma = \pi v^2$ as usual, and $uA_0 + vB_0 \in \mathcal{B}(E)$ with norm 1. WLOG assume $A_0 \neq 0$ and $B_0 = cA_0$, $c \in \mathbb{R}$. Let $S = A_0$, then S is real entire. Note that

$$E = S(1, c) \begin{pmatrix} 1-\beta z & \alpha z \\ -\gamma z & 1+\beta z \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

and it suffices to show

$$(1, c) \begin{pmatrix} 1-\beta z & \alpha z \\ -\gamma z & 1+\beta z \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = (P_0, Q_0) \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

is a polynomial of degree 1 without real zeros. Suppose it has real zero x_0, then $P_0(x_0) = Q_0(x_0) = 0$ and $(1, c) \begin{pmatrix} 1-\beta x_0 & \alpha x_0 \\ -\gamma x_0 & 1+\beta x_0 \end{pmatrix} = 0$ Since the matrix is invertible (determinant is 1), we get a contradiction. Now suppose it has degree 0, then by direct calculation and looking at the coefficient of z we get $vc+u = 0$, then $v \neq 0$ and $c = -\frac{u}{v}$. By assumption, $cA_0 - B_0 \equiv 0$, hence $uA_0 + vB_0 = 0$, a contradiction to the fact it has norm 1.

For the induction step, by problem 87 we have

$$(A, B) = (\tilde{A}(z), \tilde{B}(z)) \begin{pmatrix} 1-\beta z & \alpha z \\ -\gamma z & 1+\beta z \end{pmatrix}$$

and by induction assumption $\tilde{E} = S\tilde{E}_0$, where S is real entire and $\tilde{E}_0$ is a polynomial of degree $r-1$, without real zeros. See the remark for the reason why $\tilde{E}_0$ is dB. Let $E_0 = P_0 - iQ_0$ where P_0 and Q_0 are real entire polynomials, then

$$E = S(\tilde{P}_0(z), \tilde{Q}_0(z)) \begin{pmatrix} 1-\beta z & \alpha z \\ -\gamma z & 1+\beta z \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

It suffices to show

$$(\tilde{P}_0(z), \tilde{Q}_0(z)) \begin{pmatrix} 1-\beta z & \alpha z \\ -\gamma z & 1+\beta z \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = (P_0, Q_0) \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

is a polynomial of degree r without real zero. By the same argument as base case we can show it doesn't have any real zero. Now assume the coefficient of z^{r-1} for $\tilde{P}_0, \tilde{Q}_0$ are a, b resp. If $P_0 - iQ_0$ has degree less than r, then $\deg P_0 < r, \deg Q_0 < r$ and by direct calculation we get:

$$au + bv = 0$$

Then $u\tilde{P}_0 + v\tilde{Q}_0$ has degree at most $r-2$, which must belong to $\mathcal{B}(\tilde{P}_0 - i\tilde{Q}_0)$, and $u\tilde{A} + v\tilde{B} = S(u\tilde{P}_0 + v\tilde{Q}_0) \in \mathcal{B}(\tilde{E})$ by problem 44, a contradiction to the result from problem 87:

$$u\tilde{A} + v\tilde{B} = uA + vB \in \mathcal{B}(E), \notin \mathcal{B}(\tilde{E})$$

□

Remark. *Since $\dim(\mathcal{B}(E)) > 0$, E is non-degenerate, hence it's zero free on $\mathbb{C}_+$. If $E = SE_0$, where S is real entire and E_0 is a polynomial without real zeros, then since $S \neq 0$ on $\mathbb{C}_+$, S is a degenerate dB function. E_0 has no zero on $\mathbb{C}_+$, hence it's a dB polynomial. To summarize, we get the following result: If E is dB and $\dim(\mathcal{B}(E)) = r < \infty$, then $E = SE_0$, where S is a real entire dB function (means no zero on $\mathbb{C}_+$), and E_0 is a dB polynomial of degree r. Moreover, $\mathcal{B}(E_0)$ is regular since by Theorem 25, $1 \in Ass(\mathcal{B}(E_0))$ and $S \in Ass(\mathcal{B}(E))$.*

Problem 89

Let $\mathcal{B}(E)$ be a given space and let $\varphi(x)$ be a choice of phase function associated with $E(z)$. Show that there exists a number $p = p(\alpha) \geqslant 0$ for every real number α such that

$$\Re \frac{e^{i\alpha}E(z) + e^{-i\alpha}E^{\#}(z)}{e^{i\alpha}E(z) - e^{-i\alpha}E^{\#}(z)} = py + \sum \frac{1}{\varphi'(t)} \frac{y}{(t-x)^2 + y^2}$$

for $y > 0$, where summation is over all real numbers t such that $\varphi(t) \equiv \alpha$ module π. Show that $p > 0$ if, and only if, $e^{i\alpha}E(z) - e^{-i\alpha}E^{\#}(z)$ belongs to $\mathcal{B}(E)$.

Proof. First let's deal with the case $\alpha = 0$, then

$$\frac{e^{i\alpha}E(z) + e^{-i\alpha}E^{\#}(z)}{e^{i\alpha}E(z) - e^{-i\alpha}E^{\#}(z)} = \frac{2A(z)}{-2iB(z)} = i\frac{A}{B}$$

Let $f(z) = -\frac{A(z)}{B(z)}$, then f is analytic in the complex plane except for isolated singularities at points (t_n), where $B(t_n) = 0$, i.e. $\varphi(t_n) = 0$. And since $f = f^{\#}$ and $\Re - if = -\Im f < 0$ as E is non-degenerate dB, by problem 47 we have

$$\frac{f(z) - \bar{f}(w)}{z - \bar{w}} = p + \sum \frac{p_n}{(t_n - z)(t_n - \bar{w})}$$

for non-real z and w. Let $w = z$, then it becomes

$$\frac{\Im f}{y} = p + \sum \frac{p_n}{|t_n - z|^2}$$

then

$$\Re i\frac{A}{B} = -\Im\frac{A}{B} = \Im f = py + \sum \frac{p_n}{|t_n - z|^2}$$

note that $p_n = \lim_{z \to t_n}(t_n - z)f(z) = \frac{A(t_n)}{B'(t_n)} = \frac{1}{\varphi'(t_n)}$ by problem 48. $p > 0$ iff $B \in \mathcal{B}(E)$, and this can be seen from proof to Theorem 22, the decomposition of $K(w, z)$.
For general α, define

$$(\tilde{A}, \tilde{B}) = (A, B)\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

now

$$\Re \frac{e^{i\alpha}E(z) + e^{-i\alpha}E^{\#}(z)}{e^{i\alpha}E(z) - e^{-i\alpha}E^{\#}(z)} = \Re \frac{1}{i} \frac{\cos\alpha A + \sin\alpha B}{\sin\alpha A - \cos\alpha B} = \Re i \frac{\tilde{A}}{\tilde{B}}$$

which reduces to the special case $\alpha = 0$ for $\tilde{E} = \tilde{A} - i\tilde{B} =$. Since dB space is invariant under $SL_2(\mathbb{R})$ transform of (A, B), it suffices to show $\varphi'(t_n) = \tilde{\varphi}'(t_n)$, where $\varphi(t_n) = \alpha$, i.e. $\tilde{\varphi}(t_n) = 0$.

$$\tilde{\varphi}'(t_n) = \frac{\tilde{B}'(t_n)}{\tilde{A}(t_n)} = \frac{-\sin\alpha A'(t_n) + \cos\alpha B'(t_n)}{\cos\alpha A + \sin\alpha B}$$

note that at t_n, $Ee^{i\alpha}$ is real, that is, $A\sin\alpha = B\cos\alpha$. Since $\alpha \in (0, \pi)$ mod π, $\sin\alpha \neq 0$, and $B \neq 0$:

$$\begin{aligned}
\tilde{\varphi}'(t_n) &= \frac{-\sin\alpha BA'(t_n) + \cos\alpha BB'(t_n)}{\cos\alpha AB + \sin\alpha B^2} \\
&= \frac{-\sin\alpha BA'(t_n) + \sin\alpha AB'(t_n)}{\sin\alpha A^2 + \sin\alpha B^2} \\
&= \frac{-BA'(t_n) + AB'(t_n)}{A^2 + B^2} \\
&= \frac{\pi K(x, x)}{|E|^2}
\end{aligned}$$

by problem 48, and we're done. □

Remark. *This is a special case of Theorem 32, where $W = e^{-2i\alpha}$ is a constant Schur function.*

Remark. *The second part can be proved by the discussion in the proof to Theorem 27, Page 75.*

Problem 90

Let $\mathcal{B}(E)$ be a given space and let $\mu(x)$ be a nondecreasing function of real x such that

$$\int_{-\infty}^{+\infty} \left|\frac{F(t)}{E(t)}\right|^2 dt = \int_{-\infty}^{+\infty} \left|\frac{F(t)}{E(t)}\right|^2 d\mu(t)$$

for every $F(z)$ in $\mathcal{B}(E)$. Show that there exists a function $W(z)$, analytic and bounded by 1 in the upper half-plane, such that

$$\Re\frac{E(z)+E^{\#}(z)W(z)}{E(z)-E^{\#}(z)W(z)} = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t-x)^2+y^2} \tag{7}$$

for $y>0$. If $\mu(x)$ is constant in an interval (a,b), show that $W(z)$ is analytic across (a,b) if defined in the lower half-plane by $W^{\#}(z)W(z)=1$.

Remark. *This is Alexander-Sarason Theorem, which is a special case of Nevanlinna's characterization of spectral measure for general associated function S. The proof would be quite easy using model space K_Θ. This is a partial converse of Theorem 32, and it seems like Theorem 30, 31 are presented because of this problem. Note that we have the following corollary: $\mathcal{B}(E) \sqsubseteq L^2\left(\frac{\mu}{|E|^2}\right)$ implies μ is regular.*

Proof. The first equation follows directly from Theorem 30 and 31, with $S=E, C=-B, D=A$. Now suppose $\mu(x)$ is a constant on (a,b), which means $d\mu=0$. Taking limit $x\in(a,b),\ y\to 0$ then we know W can be continuously extended to (a,b). Taking the same limit in (7) gives us $|W|^2=1$ on (a,b). Then it's locally nonzero and we can define, for some neighbourhood of (a,b) in $\mathbb{C}^-$, $W(z)=\frac{1}{W^{\#}(z)}$. □

Remark. 1. *I'm not sure if W can be extended to the whole $\mathbb{C}^-$, because W may have zeros on the upper plane. Unless we only require W to be meromorphic?*

2. *Similarly, if y_0 is in the essential support of $d\mu$, every neighborhood of y_0 has an arbitrarily large number of points where E is real.*

Problem 91

Let $W(z)$ be a function which is analytic and bounded by 1 in the upper half-plane and which is analytic across an interval (a,b) of the real axis when defined in the lower half-plane by $W^{\#}(z)W(z)=1$. Show that $W(z)=\exp(2i\psi(x))$ for $a<x<b$ where $\psi(x)$ is a nondecreasing, differentiable function of x.

Proof. By the construction of W on $\mathbb{C}^-$, and the assumption W is bounded by 1 in the upper half-plane, we know

$$|W(\bar{z})| \geqslant 1 \geqslant |W(z)|$$

for $z\in\mathbb{C}^+$. Then $\frac{d\log|W(z)|}{dy}\leqslant 0$ for $z\in(a,b)$. Let $f=\log W$, by Cauchy-Riemann equation, $f'=u_x+iv_x=u_x-iu_y$. But here we have $u_x=0$ since $\log|W|\equiv 0$ and $u_y\leqslant 0$. If we denote $W(z)=\exp(2i\psi(x))$, then $\frac{f'}{2i}=\psi'(x)\geqslant 0$. Hence $\psi(x)$ is a nondecreasing, differentiable function of x. □

Problem 92

In Problem 90, let $\phi(x)$ be a phase function associated with $E(z)$. Show that $\phi(b)-\phi(a)\leqslant\pi$ and that the inequality is strict unless $W(z)$ is a constant of absolute value 1.

Proof. Now assume μ doesn't support (a,b), we want to show

$$\phi(b)-\phi(a)\leqslant\pi$$

Suppose not, then the increment of the argument of the function $f=\frac{E^{\#}}{E}W$ on (a,b) is bigger than 2π. This is because $\frac{E^{\#}}{E}=\exp(2i\phi)$ and in last theorem we proved the argument of W is nondecreasing. $|f|=1$ on $\mathbb{R}$ so $\exists\xi\in(a,b)$ s.t. $f(\xi)=1$. Go back to (7) we will have

$$\infty = \frac{y}{\pi}\int \frac{d\mu(t)}{(t-x)^2+y^2}\bigg|_{z=\xi} = 0$$

A contradiction.
And the inequality is strict, suppose not then by the same argument as above, W's argument must be a constant otherwise there exists $\xi\in(a,b,)$ s.t. $f(\xi)=1$. In this case μ is the Clark measure of $\Theta=\frac{E^{\#}}{E}$, and we know

$$K_\Theta = L^2(d\mu)$$

Hence

$$\mathcal{B}(E) = EK_\Theta = L^2(\frac{d\mu}{|E|^2})$$

See Problem 89 for more details. $\square$

Problem 93

Here I'll just prove the special case when $\phi(b,t) - \phi(b,s) = \pi$. Choose $\alpha = \exp(2\pi\phi(b,s))$. Then $\Theta_b(s) = \Theta_b(t) = \alpha$. Now use sampling formula

$$\|F\|^2_{\mathcal{B}(E_b)} = 2\pi \sum_{\Theta_b(\xi)=\alpha} \left|\frac{F(\xi)}{E(\xi)}\right|^2 \frac{1}{|\Theta_b'|} \tag{8}$$

which doesn't use the value of F between s and t, hence μ_α doesn't support (s,t).
On the other hand, let μ_α be the Clark measure associated with (α, Θ_b), then

$$K_{\Theta_b} = L^2(\mu_\alpha)$$

We have

$$\begin{aligned}\mathcal{B}(E_a) &\sqsubset \mathcal{B}(E_b)\\ K_{\Theta_a} &\sqsubset K_{\Theta_b} = L^2(\mu_\alpha)\\ \mathcal{B}(E_a) &\sqsubset L^2(\frac{\mu_\alpha}{|E|^2})\end{aligned}$$

By the result of Problem 92 we get $\phi(a,t) - \phi(a,s) \leqslant \pi = \phi(b,t) - \phi(b,s)$.

Problem 94

Let $f(z)$ be a function which is analytic and has a nonnegative real part in the upper half-plane. Assume that

$$\Re f(x+iy) = py + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{d\mu(t)}{(t-x)^2+y^2}$$

for $y > 0$ where $p \geqslant 0$ and $\mu(x)$ is a nondecreasing function of real x which is constant in an interval (a,b). Let $z = x + iy$ where $y > 0$ and $a < x < b$. Show that

$$\Re f(x+iy) \leqslant \frac{(c-x)^2+h^2}{(c-x)^2+y^2}\frac{y}{h}\Re f(c+ih)$$

for $0 < y < h$, where $c = a$ if $x \leqslant \frac{a+b}{2}$ and $c = b$ if $x \geqslant \frac{a+b}{2}$.

Proof. This statement is FALSE without additional condition. For example, let μ be purely point and has point mass ϵ at $a = -1$, and 1 at $b = 1$. Let $p = 0$, and by Theorem 3 (Stieljes Inversion Formula) we know there exists f analytic on $\mathbb{C}_+$ and

$$\Re f = \frac{y}{\pi}\left(\frac{\epsilon}{(-1-x)^2+y^2} + \frac{1}{(1-x)^2+y^2}\right)$$

Now let $x = 0$ and $y = 1$, then $|c - x| = 1$ and the statement becomes

$$\left(\frac{\epsilon}{(-1-x)^2+y^2} + \frac{1}{(1-x)^2+y^2}\right)((c-x)^2+y^2) \leqslant \left(\frac{\epsilon}{(-1-a)^2+h^2} + \frac{1}{(1-a)^2+y^2}\right)((c-x)^2+h^2)$$
$$\left(\frac{\epsilon}{2}+\frac{1}{2}\right)2 \leqslant \left(\frac{\epsilon}{h^2}+\frac{1}{5}\right)(1+h^2)$$

Pick h close to 1 we'll get a contradiction. $\square$

Problem 100

Let $\mathcal{B}(E(a))$ and $\mathcal{B}(E(b))$ be given spaces such that $\mathcal{B}(E(a))$ is contained isometrically in $\mathcal{B}(E(b))$ and $E(a,z)/E(b,z)$ has no real zeros. Let $\mathcal{B}(M_1(a,b))$ and $\mathcal{B}(M_2(a,b))$ be spaces such that

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M_k(a,b,z)$$

for $k = 1, 2$. Show that $M_1(a,b,z) = M_2(a,b,z)$.

Remark. *This problem, together with Theorem 33, claims that given E_b and E_a, then the transition Nevanlinna matrix exists if $\mathcal{B}(E_a) \sqsubseteq \mathcal{B}(E_b)$ and $\frac{E_a}{E_b}$ has no real zeros, and it is unique. And also note that, if we take $E_a \equiv 1$, then $(A_a, B_a) = (1, 0)$, and the result is clearly false by construction of dB space, and we know the Nevanlinna matrix is unique up to a constant multiple of E_b. So it's necessary to assume $\mathcal{B}(E_a) \neq 0$, i.e. E_a is not degenerate. This can also be seen from the proof: we need to pick nonzero $F \in \mathcal{B}(E_a)$.*

Remark. *This part is taken from Misha's note III.*

Proof. Let $M(a,z) = \begin{pmatrix} A_a & B_a \\ -B_a & A_a \end{pmatrix}$, we know it's dB associated with $S_a = E_a$. Let

$$M_k(b,z) = M(a,z)M_k(a,b,z)$$

and $M_k(b,z) = \begin{pmatrix} A_b & B_b \\ C_k & D_k \end{pmatrix}$. Since $M_k(b,z)$ is dB, $\frac{D_k + iC_k}{A_b - iB_b}$ has nonnegative real part on $\mathbb{C}_+$ (see proof to Theorem 28) and is equal to

$$\Re\frac{D_k + iC_k}{A_b - iB_b} = p_k y + \frac{y}{\pi}\int_{\mathbb{R}} \frac{|S/E_b|^2 dt}{(t-x)^2 + y^2}$$

where $S = E_a$ by definition of $M_k(b,z)$. Let $M_1(a,b,z)$ be the one given by Theorem 33, i.e. $p_1 = 0$. Hence $E_{b,2}$ and $E_{b,1}$ differs by $-ipz + iq$, and

$$M_2(b,z) = \begin{pmatrix} 1 & 0 \\ -pz + q & 1 \end{pmatrix} M_1(b,z)$$

where $p \geqslant 0, q \in \mathbb{R}$. Note that $F \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} F \\ iF \end{pmatrix}$ maps $\mathcal{B}(E_a)$ isometrically onto $\mathcal{B}_S(M(a,z))$, and $\mathcal{B}(E_a)$ is contained isometrically into $\mathcal{B}(E_b)$, which can be mapped into $\mathcal{B}_S(M_k(b,z))$, and onto for $\mathcal{B}_S(M_1(b,z))$ but not necessarily onto for $\mathcal{B}(M_2(b,z))$. Now for $\begin{pmatrix} F \\ iF \end{pmatrix} \in \mathcal{B}_S(M(a,z))$, by construction of $\mathcal{B}_S(M_2(a,b))$, $\exists G \in \mathcal{B}(E_b)$ and $\lambda \in \mathbb{C}$, s.t.

$$\begin{pmatrix} F \\ iF \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -pz + q & 1 \end{pmatrix}\begin{pmatrix} G \\ \tilde{G} \end{pmatrix} + S(z)\begin{pmatrix} 0 \\ \lambda \end{pmatrix}$$

The first row reduces to $G = F$, and calculating norm of LHS in $\mathcal{B}_S(M_1(a,b))$, and norm of RHS in $\mathcal{B}_S(M_2(a,b))$ we get $\lambda = 0$, now the second row gives $p = q = 0$, hence $M(a,b)$ is unique. $\square$

Problem 101

If $\mathcal{B}(M(a,c))$ is a given space and if there exists a constant $\begin{pmatrix} u \\ v \end{pmatrix}$ of norm 1 in $\mathcal{B}(M(a,c))$, show that $\bar{u}v = u\bar{v}$ and that a space $\mathcal{B}(M(a,b))$ exists,

$$M(a,b,z) = \begin{pmatrix} 1 - 2\pi u\bar{v}z & 2\pi u\bar{u}z \\ -2\pi v\bar{v}z & 1 + 2\pi u\bar{v}z \end{pmatrix}$$

Show that $\mathcal{B}(M(a,b))$ is contained isometrically in $\mathcal{B}(M(a,c))$, that $M(a,c,z) = M(a,b,z)M(b,c,z)$ for some space $\mathcal{B}(M(b,c))$, and that $\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \to M(a,b,z)\begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$ is an isometric transformation of $\mathcal{B}(M(b,c))$ onto the orthogonal complement of $\mathcal{B}(M(a,b))$ in $\mathcal{B}(M(a,c))$.

Proof. Obviously we can assume u, v are real. For the special case $u = 0$, by construction we know,

$$\begin{aligned} M(a,c,z) &= \begin{pmatrix} 1 & 0 \\ -pz & 1 \end{pmatrix} M_1(a,c,z) \\ &:= M(a,b,z)M(b,c,z) \\ M_1(a,c,z) &= \begin{pmatrix} 1 & 0 \\ pz & 1 \end{pmatrix} M(a,c,z) \end{aligned}$$

We can see $M(a,b,z)$ and $M^{-1}(a,b,z)$ is Nevanlinna (you may refer to my notes on Nevanlinna matrix for definition and basic properties) via direct calculation. Since product of Nevanlinna matrices is still Nevanlinna, $\mathcal{B}(M(b,c))$ exists. By construction, $\mathcal{B}(M(a,c))$ is the set of

$$\lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -pz & 1 \end{pmatrix} \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix}$$

with norm $\frac{2\pi|\lambda|^2}{p} + \left\| \begin{pmatrix} F_+(z) \\ F_-(z) \end{pmatrix} \right\|_{\mathcal{B}(M(b,c))}$ Note that $\begin{pmatrix} 0 \\ v \end{pmatrix}$ has norm 1, $p = 2\pi v^2$, and we're done for the special case. For general case, let $U = \frac{1}{u^2+v^2} \begin{pmatrix} v & u \\ -u & v \end{pmatrix}$ and $\tilde{M} = U^* M U$, then $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{B}(\tilde{M}(a,c))$, and the rest follows. □

Remark. *Check problem 87 for (formally) similar results as well.*

Problem 102

Let $\mathcal{B}(E(a))$, $\mathcal{B}(E(c))$, and $\mathcal{B}(M(a,c))$ be given spaces such that

$$(A(c,z), B(c,z)) = (A(a,z), B(a,z))M(a,c,z)$$

and $\mathcal{B}(E(a))$ is not contained isometrically in $\mathcal{B}(E(c))$. If $M(a,c,z) = M(a,b,z)M(b,c,z)$ as in problem 101, show that there exists a space $\mathcal{B}(E(b))$ such that

$$(A(b,z), B(b,z)) = (A(a,z), B(a,z))M(a,b,z)$$

and $\mathcal{B}(E(b))$ is contained isometrically in $\mathcal{B}(E(c))$.

Proof. $\mathcal{B}(E(a))$ is not contained isometrically in $\mathcal{B}(E(c))$, and by Theorem 34 we know there exists $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{B}(M(a,c))$ s.t. $uA_a + vB_a \in \mathcal{B}(E_a)$. Like we did for problem 86, 87, we assume u, v are real. Choose u, v s.t. $\begin{pmatrix} u \\ v \end{pmatrix}$ has norm 1 in $\mathcal{B}(M(a,c))$, then by problem 101 we get factorization

$$M(a,c,z) = M(a,b,z)M((b,c,z)$$

where $M(a,b,z) = \begin{pmatrix} 1 - 2\pi u\bar{v}z & 2\pi u\bar{u}z \\ -2\pi v\bar{v}z & 1 + 2\pi u\bar{v}z \end{pmatrix}$. It's easy to check

$$M(a,b,z) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

then

$$uA_a + vB_a = (A_a, B_a) \begin{pmatrix} u \\ v \end{pmatrix} = (A_a, B_a)M(a,b,z) \begin{pmatrix} u \\ v \end{pmatrix} = (A_b, B_b) \begin{pmatrix} u \\ v \end{pmatrix} = uA_b + vB_b$$

By Theorem 34, $\mathcal{B}(E_a) \subseteq \mathcal{B}(E_b)$, then $uA_b + vB_b \in \mathcal{B}(E_b)$. To show $\mathcal{B}(E_b) \sqsubseteq \mathcal{B}(E_c)$, by Theorem 34, it suffices to show there is no $\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \in \mathcal{B}(M(b,c))$, s.t. $u_2 A_b + v_2 B_b \in \mathcal{B}(E_b)$. Suppose it exists. Since we already know $uA_b + vB_b \in \mathcal{B}(E_b)$, if there exists such $\begin{pmatrix} u_2 \\ v_2 \end{pmatrix}$, it must be equal to (up to multiplication by $e^{i\theta}$) $\begin{pmatrix} u \\ v \end{pmatrix}$ By problem 101, $M(a,b,z) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$ is in the orthogonal complement of $\mathcal{B}(M(a,b))$ in $\mathcal{B}(M(a,c))$. I'll show

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{B}(M(a,b))$$

as well and then we'll get a contradiction. To simplify notation, let $M = M(a,b) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, $E = A - iB$, $\tilde{E} = C - iD$. It's easy to see $\frac{1}{y} \Re \frac{i\tilde{E(iy)}}{E(iy)}$ goes to 0 when y goes to $+\infty$. By construction, $\mathcal{B}(M)$ is just the map of Hilbert transform. Obviously constant function $f \equiv u \in \mathcal{B}(E)$, and it suffices to show the Hilbert transform of f is v. By Theorem 27,

$$\tilde{f}(w) = \langle f, \frac{1 - A(t)\bar{D}(w) + B(t)\bar{C}(w)}{\pi(t - \bar{w})} \rangle_{\mathcal{B}(E)}$$

it's easy to get $\frac{1-A(t)\bar{D}(w)+B(t)\bar{C}(w)}{\pi(t-\bar{w})} = 2uv$, and

$$\begin{aligned}\|1\|^2_{\mathcal{B}(E)} &= \int_{\mathbb{R}} \frac{1}{(1-2\pi uvx)^2 + (2\pi ux)^2} \\ &= \int_{\mathbb{R}} \frac{1}{4\pi^2u^2(u^2+v^2)\left(x - \frac{v}{2\pi u(u^2+v^2)}\right) + \frac{u^2}{u^2+v^2}} \\ &= \frac{1}{2u^2}\end{aligned}$$

since $\int_{\mathbb{R}} \frac{dx}{ax^2+b} = \frac{\pi}{\sqrt{ab}}$ for nonnegative a, b. And we're done. $\square$

Remark. *The proof also yields an important property of $\mathcal{B}_S(M)$: if $M = \begin{pmatrix} 1-2\pi u\bar{v}z & 2\pi|u|^2z \\ -2\pi|v|^2z & 1+2\pi u\bar{v}z \end{pmatrix}$ for some complex numbers u, v, then $\mathcal{B}_S(M)$ is one dimensional, generated by vector $\binom{u}{v}$, whose norm in $\mathcal{B}_S(M)$ is exactly* 1.

Problem 110

If $\mathcal{B}(M)$ is a given space which has finite dimension r, show that

$$M(z) = \begin{pmatrix} 1-\beta_1 z & \alpha_1 z \\ -\gamma_1 z & 1+\beta_1 z \end{pmatrix} \cdots \begin{pmatrix} 1-\beta_r z & \alpha_r z \\ -\gamma_r z & 1+\beta_r z \end{pmatrix} M(0)$$

where (α_k), (β_k), (γ_k) are real numbers such that $\alpha_k \geqslant 0$, $\gamma_k \geqslant 0$, and $\alpha_k\gamma_k = \beta_k^2$ for $k = 1, \cdots, r$.

Proof. The proof consists of two parts. First, I'll show each component of M is a polynomial; Secondly, a polynomial Nevanlinna matrix has to be of this product form.
First, M has to be a polynomial matrix, i.e. all entries are polynomials. Use real representation of dB matrix, we have

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

By construction, let $E = A - iB$, then $\mathcal{B}(E)$ has finite dimension as well. By problem 88, $E = SE_0$, where S is real entire and E_0 is a strict dB polynomial. From the remark to problem 88 we know $S \in \mathcal{B}(E_0) + z\,\mathcal{B}(E_0)$, so it's a polynomial as well. The same argument applies to C, D as well.
Now let's show a polynomial Nevanlinna matrix has to be of this product form. For some reason in the proof I took conjugation of everything, for more details please refer to my note "Existence Theorem")
$M = \begin{pmatrix} P & R \\ Q & S \end{pmatrix}$ The difference between degree of $P - iQ$, $R - iS$ is at most 1 because Nevanlinna matrix is unique up to $pz + q$. Let's conjugate M by $U = \begin{pmatrix} v & u \\ -u & v \end{pmatrix}$ ($u\bar{v} = \bar{u}v$, $|u|^2 + |v|^2 = 1$) to get $\widetilde{M} = U^\# M U$, and the second dB function of $\widetilde{M}$ would be one degree higher than the first dB function. To be more precise, suppose $\widetilde{M} = \begin{pmatrix} \widetilde{P} & \widetilde{R} \\ \widetilde{Q} & \widetilde{S} \end{pmatrix}$, then $\deg(\widetilde{R} - i\widetilde{S}) > \deg(\widetilde{P} - i\widetilde{Q})$. Actually $\deg(\widetilde{R} - i\widetilde{S}) = \deg(P - iQ) + 1$ since the degree difference can not be greater than 1.
Thus we have $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{B}(\widetilde{M})$ by construction of $\mathcal{B}(\widetilde{M})$, also by construction we know:

$$\widetilde{M} = M_1 \begin{pmatrix} 1 & -\gamma z \\ 0 & 1 \end{pmatrix}$$

where $\gamma > 0$. Choose the maximal positive $\gamma > 0$, then $\deg(R_1 - iS_1) = \deg(P_1 - iQ_1)$, and $\deg(M_1) = \deg(M) - 1$.

$$\begin{aligned} M &= UM_1 \begin{pmatrix} 1 & -\gamma z \\ 0 & 1 \end{pmatrix} U^\# = UM_1U^\# \begin{pmatrix} 1-\beta_r z & -\gamma_r z \\ \alpha_r z & 1+\beta_r z \end{pmatrix} \\ &\deg(UM_1U^\#) = r - 1 \end{aligned}$$

Then use induction w.r.t. r. $\square$

Problem 111

Let $\mathcal{B}(M_a), \mathcal{B}(M_b), \mathcal{B}(M_c)$ be spaces such that

$$M_c = M_a M(a,c) \text{ and } M_c = M_b M(b,c)$$

for some spaces $\mathcal{B}(M(a,c))$ and $\mathcal{B}(M(b,c))$. If $\mathcal{B}(M_c)$ has dimension 0 or 1, show that either

$$M_b = M_a M(a,b)$$

for some space $\mathcal{B}(M(a,b))$ or

$$M_a = M_b M(b,a)$$

for some space $\mathcal{B}(M(b,a))$.

Proof. The proof seems long. Save for later. □

Problem 115

Let $\mathcal{B}(M)$ be a finite dimensional space such that $M(0) = 1$. Show that

$$iM'(0)J = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \geqslant 0$$

and that

$$M(z) = \sum_{n=0}^{\infty} M^{(n)}(0)\frac{z^n}{n!}$$

where $\sigma(M^{(n)}(0)) \leqslant (\alpha+\gamma)^n$ for every $n = 1, 2, 3, \cdots$.

Proof. By problem 110 we can factorize M into elementary factors:

$$M(z) = M_1(z)\cdots M_r(z), M_j(z) = \begin{pmatrix} 1-\beta_j z & -\gamma_j z \\ \alpha_j z & 1+\beta_j z \end{pmatrix},\ \alpha_j, \gamma_j \geqslant 0,\ \beta_j^2 = \alpha_j\gamma_j$$

Let $t(M) := Tr(M'(0)I)$, where $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then

$$t_j = t(M_j) = \alpha_j + \gamma_j,\ t(M) = \sum_{j=1}^{r} t_j$$

Then

$$M^{(k)}(0) = k! \sum_{1\leqslant j_1<j_2<\cdots<j_k\leqslant r} M'_{j_1}(0)\cdots M'_{j_k}(0)$$

and use $\|A_1\cdots A_n\|_2 \leqslant \|A_1\|_2\cdots\|A_n\|_2$, we can get

$$\begin{aligned}
\sigma(M^{(k)}(0)) &\leqslant k! \sum_{1\leqslant j_1<j_2<\cdots<j_k\leqslant r} \sigma(M'_{j_1}(0)\cdots M'_{j_k}(0)) \\
&\leqslant k! \sum_{1\leqslant j_1<j_2<\cdots<j_k\leqslant r} t_{j_1}\cdots t_{j_k} \\
&\leqslant \left(\sum_{j=1}^{r} t_j\right)^k \\
&= t(M)^k
\end{aligned}$$

□

Problem 116

If $\mathcal{B}(E_b)$ is a finite dimensional space and if $h \geqslant 0$, show that there exists a space $\mathcal{B}(M(a,b))$ such that $M(a,b,0) = 1$, $B'(a,b,0) - C'(a,b,0) = h$, and

$$(A_b, B_b) = (A_a, B_a)M(a,b)$$

for some entire functions A_a and B_a, which are real for real z, such that

$$\frac{B_a \bar{A}_a - A_a \bar{B}_a}{z - \bar{z}} \geqslant 0$$

for all complex z.

Proof. First we assume E_b is regular. By Theorem 27 we have C_b, D_b s.t. Let $M_b = \begin{pmatrix} A_b & B_b \\ C_b & D_b \end{pmatrix}$ is Nevanlinna, and $\mathcal{B}(E_b)$ can be mapped onto $\mathcal{B}(M_b)$ isometrically (i.e. $\lim_{y\to} \Re \frac{1}{y} \frac{D_b(iy)+iC_b(iy)}{E_b(iy)} = 0$). Now that $\mathcal{B}(M_b)$ has finite dimension. By problem 110,

$$M_b(z) = \prod_k \begin{pmatrix} 1 - \beta_k z & \alpha_k z \\ -\gamma_k z & 1 + \beta_k z \end{pmatrix} M_b(0)$$

First let's assume $M_b(0) = I$, the identity matrix, i.e. M_b is normalized. Let $t(M_b) = \operatorname{Tr}(M_b'(0)I)$, if $t(M_b) < h$, then let $\tilde{M}_b = \begin{pmatrix} 1 & 0 \\ -(h - t(M_b))z & 1 \end{pmatrix} M_b$, and $t(\tilde{M}_b) = h$. Let $(A_a, B_a) = (1, 0)$, then

$$(A_b, B_b) = (\tilde{A}_b, \tilde{B}_b) = (1,0)\tilde{M}_b = (A_a, B_a)\tilde{M}_b$$

and we're done. If $t(M_b) \geqslant h$, split the factorization into two parts,

$$M_b(z) = M_a(z) M(a,b,z)$$

s.t. $t(M(a,b)) = h$, and since M_a is Nevanlinna, $A_a - iB_a$ is dB, might be degenerate though. And obviously A_a, B_a are real for real z. For the general case, let $U = M_b(0)$, and components of

$$U^{-1} \begin{pmatrix} 1 - \beta_k z & \alpha_k z \\ -\gamma_k z & \beta_k z \end{pmatrix} U = U^{-1} V U$$

are still polynomials of degree 1. By direct calculation we know it has the form

$$\tilde{V} = \begin{pmatrix} 1 - az & bz \\ cz & 1 + dz \end{pmatrix}$$

with $b \geqslant 0, c \leqslant 0$. Note that $\operatorname{Tr} \tilde{V} = \operatorname{Tr} V = 2$, hence $a = d$. $\det \tilde{V} = \det V = 1$ implies $bc = a^2$, so $\tilde{V}$ has the same form as V and is Nevanlinna, hence

$$\begin{aligned}
(A_b, B_b) &= (\tilde{A}_b, \tilde{B}_b)U \\
&= (\tilde{A}_a, \tilde{A}_a) \prod_k \begin{pmatrix} 1 - \beta_k z & \alpha_k z \\ -\gamma_k z & \beta_k z \end{pmatrix} U \\
&= (\tilde{A}_a, \tilde{A}_a) U \prod_k U^{-1} \begin{pmatrix} 1 - \beta_k z & \alpha_k z \\ -\gamma_k z & \beta_k z \end{pmatrix} U \\
&= (\tilde{A}_a, \tilde{A}_a) U \prod_k \begin{pmatrix} 1 - \tilde{\beta}_k z & \tilde{\alpha}_k z \\ -\tilde{\gamma}_k z & \tilde{\beta}_k z \end{pmatrix} \\
&= (A_a, B_a) M(a,b)
\end{aligned}$$

obviously $t(M(a,b)) = h$ and $A_a - iB_a$ is dB, and the rest follows.
Now for general E, by problem 88 $E = SE_0$, where S, E_0 entire and $S = S^{\#}$ (S real entire), E_0 is a dB polynomial without real zeros. By Theorem 25, E_0 is regular, then by previous result we have

$$(A_{0,b}(z), B_{0,b}(z)) = (A_{0,a}(z), B_{0,a}(z)) M(a,b,z)$$

Multiply everything by $S(z)$, we have

$$(A_b(z), B_b(z)) = (A_a(z), B_a(z)) M(a,b,z)$$

where $A_a = SA_{0,a}$ and $B_a = SB_{0,a}$. Moreover,

$$\frac{B_a(z)\bar{A}_a(z) - A_a(z)\bar{B}_a(z)}{z - \bar{z}} = |S(z)|^2 \frac{B_{0,a}(z)\bar{A}_{0,a}(z) - A_{0,a}(z)\bar{B}_{0,a}(z)}{z - \bar{z}} \geqslant 0$$

□

Remark. *Note that M can be chose as a polynomial dB matrix. This can be generalized to infinite dimensional space, see Theorem 36. h may be related to the existence of $\mathcal{B}(E_a)$, i.e. A_a, B_a linearly dependent or not.*

Problem 117

In Theorem 36 show that a space $\mathcal{B}(E_a)$ exists, $E_a(z) = A_a(z) - iB_a(z)$, if $A_a(z)$ and $B_a(z)$ are linearly independent.

Proof. Since

$$\frac{B_a(z)\bar{A}_a(z) - A_a(z)\bar{B}_a(z)}{z - \bar{z}} \geqslant 0$$

for $z \in \mathbb{C}_+$, we have $\Im \frac{A}{B} \leqslant 0$, i.e. E is a dB function, which might be degenerate though. If A_a and B_a are linearly independent, then E is not degenerate, hence a space $\mathcal{B}(E_a)$ exists. □

Problem 118

If $A_a(z)$ and $B_a(z)$ are linearly dependent in Theorem 36, show that $E_a(z) = A_a(z) - iB_a(z)$ has only real zeros and that $E_b(z)/E_a(z)$ is an entire function. Show that

$$\frac{F(z)E(a,w) - E(a,z)F(w)}{z - w}$$

belongs to $\mathcal{B}(E_b)$ whenever $F(z)$ belongs to $\mathcal{B}(E_b)$.

Remark. *Usually if $\mathcal{B}(E_a) \sqsubseteq \mathcal{B}(E_b)$, then $E_a \in Ass(\mathcal{B}(E_a)) \subseteq Ass(\mathcal{B}(E_b))$. But here E_a is degenerate, and $\mathcal{B}(E_a)$ doesn't exist, i.e. is equal to $\{0\}$.*

Proof. We have

$$(A_b, B_b) = (A_a, B_a)M$$

Now let C_b, D_b be real entire s.t.

$$M_b := \begin{pmatrix} A_b & B_b \\ C_b & D_b \end{pmatrix} = \begin{pmatrix} A_a & B_a \\ -B_a & A_a \end{pmatrix} M := M_a M$$

It's easy to check $M_a I \bar{M}_a = E_a I \bar{E}_a$, and since M is Nevanlinna, we have

$$\begin{aligned} \frac{M_b I \bar{M}_b - E_a I \bar{E}_a}{z - \bar{z}} &= \frac{M_a M I \bar{M} \bar{M}_a - E_a I \bar{E}_a}{z - \bar{z}} \\ &\geqslant \frac{M_a I \bar{M}_a - E_a I \bar{E}_a}{z - \bar{z}} \\ &= 0 \end{aligned}$$

hence $\frac{1}{E_a} M_b$ is a dB matrix, and by Theorem 27 we know $E_a \in Ass(\mathcal{B}(E_b))$. □

Problem 122

Show that $A(a,z) = \lim A_n(a,z)$, $B(a,z) = \lim B_n(a,z)$, and $M(a,b,z) = \lim M_n(a,b,z)$ as $n \to \infty$ in the proof of Theorem 36.

Proof. See my notes on existence theorems. □

Problem 123

If $\mathcal{B}(M)$ is a given space and if $M(0) = 1$, show that there exists a sequance $\{\mathcal{B}(M_n)\}$ of finite dimensional spaces such that $M_n(0) = 1$ and $B'_n(0) - C'_n(0) = B'(0) - C'(0)$ for every n, and such that $M(z) = \lim M_n(z)$ for all complex z.

Problem 126

If $\mathcal{B}(E_0)$ is a given space and if $t \leqslant 0$, let $\mathcal{B}(E_0)$ be the unique space such that $M(t,0,0) = 1$,

$$-M'(t,0,0)I = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix} = m(t)$$

where $\alpha(t) + \gamma(t) = t$, and

$$(A_0(z), B_0(z)) = (A_t(z), B_t(z))M(t,0,z)$$

for entire functions $A_t(z)$ and $B_t(z)$, which are real for real z, such that

$$\frac{B_t(z)\bar{A}_t(z) - A_t(z)\bar{B}_t(z)}{z - \bar{z}} \geqslant 0$$

for all complex z. Show that $m(t)$ is a nondecreasing function of t and that its entries are continuous, real valued functions of t. Show that $A_t(w)$ and $B_t(w)$ are continuous functions of t for every w and that

$$(A_b(w), B_b(w))I - (A_a(w), B_a(w))I = w \int_a^b (A_t(w), B_t(w))dm(t)$$

whenever $-\infty < a < b \leqslant 0$. Show that $A_a(z)$ and $B_a(z)$ are linearly dependent if $a < b$ and if $A_b(z)$ and $B_b(z)$ are linearly dependent. If there exists a value of t such that $A_t(z)$ and $B_t(z)$ are linearly dependent, show that there exists a largest value of t with this property, say $t = s_-$. Otherwise define $s_- = -\infty$. Show that a space $\mathcal{B}(E_t)$ exists when $t > s_-$.

Remark. *This problem says, given any dB space, we can find a chain, which may start from empty space but end with the given dB space.*

Proof. The existence is given by Theorem 36: choose $h = -t$ for $t < 0$, and we have the inequality $K_t(z,z) \geqslant 0$. For $a < b < 0$, by problem 112 and the fact $t(M(a,0)) = -a > -b = t(M(b,0))$, we have

$$(A_b, B_b) = (A_a, B_a)M(a,b)$$
$$M(a,0) = M(a,b)M(b,0)$$

The following is similar to the proof to Theorem 37. We know $m(b) - m(a) = M'(a,0)I - M'(b,0)I = M'(a,b)I \geqslant 0$. Since $\alpha(t) + \gamma(t) = t$ and $M'(a,b) \geqslant 0$, α, γ are nondecreasing and continuous. Since $(\beta(b) - \beta(a))^2 \leqslant (\alpha(b) - \alpha(a))^2(\gamma(b) - \gamma(a))^2$, $\beta(t)$ is continuous as well. Similar to Theorem 37, we get continuity of A_t and B_t. As for the integral equation, since

$$(A_b, B_b) = (A_a, B_a)M(a,b)$$

for $a < b \leqslant 0$, it suffices to show

$$M(a,b)I - I = w \int_a^b M(a,t)dm(t)$$

and the proof is the same as the one to Theorem 37. The existence of s_- means the supreme can be reached, and the remained part is trivial. □

Problem 128

If $\mathcal{B}(M)$ is a given space, show that the functions $A(z) - iB(z)$ and $D(z) + iC(z)$ are of bounded type in the upper half-plane and have qual mean types in the half-plane. Show that each of the functions $A(z), B(z), C(z), D(z)$ is of bounded type in the upper half-plane and that it has the same mean type in the half-plane as $A(z) - iB(z)$ and $D(z) + iC(z)$ unless it vanishes identically. The common mean type of these functions is taken as the definition of the mean type of $M(z)$.

Proof. By Theorem 28 and 27 we know $1 \in Ass(\mathcal{B}(E))$, where $E = A - iB$. By Theorem 25 we know $\frac{1}{E} \in \mathcal{N}(\mathbb{C}_+)$, and hence $E \in \mathcal{N}(\mathbb{C}_+)$. From proof to Theorem 28 we have (let $\tilde{E} = C - iD$):

$$\Re i\frac{\tilde{E}}{E} \geqslant 0$$

hence $i\frac{\tilde{E}}{E} \in \mathcal{N}(\mathbb{C}_+)$ and

$$\frac{i\tilde{E}}{E} = -ipz + ib + \frac{1}{\pi i}\int \frac{1+tz}{(t-z)(1+t^2)}d\mu$$

Since $\tilde{E}$ doesn't vanish constantly, RHS is not zero function neither. Use the inequality in problem 33 we have (for $|z| > 1$),

$$\begin{aligned}\left|\frac{1+tz}{t-z}\right| &\leqslant \frac{|1+tz|(|z-i|+|z+i|)}{|t-i||z-\bar{z}|} \\ &\leqslant \frac{|z|(1+|t|)(|z-i|+|z+i|)}{\sqrt{1+t^2}|z-\bar{z}|} \\ &\leqslant \sqrt{2}\frac{|z|(|z-i|+|z+i|)}{|z-\bar{z}|}\end{aligned}$$

(I must have done this for some other problem) we can see the mean type of RHS is 0, hence $D+iC$ has the same mean type as $A-iB$. Since $A=\frac{E+E^{\#}}{2}$, $|A| \leqslant |E|$, $h_A \leqslant h_E$, where h_A, h_E denote mean types of A, E resp. Also $h_B \leqslant h_E$. On the other hand, by problem 29, $h_E \leqslant \max\{h_A, h_B\}$. Suppose $h_A \geqslant h_B$, then A doesn't vanish constantly, and $h_A \neq \infty$, $h_E = h_A$. If B doesn't vanish constantly, then $\Im\frac{A}{B} \leqslant 0$ on $\mathbb{C}_+$, B has the same mean type as A as we proved earlier. Same argument applies to C and D. ☐

Remark. *By problem 34, a dB function of bounded type is of Pólya class. Using Theorem 27 and 28 we conclude, if $1 \in Ass(\mathcal{B}(E))$, i.e. E is regular, then E is of Pólya class.*

Problem 129

If $\mathcal{B}(M)$ is a given space, show that the mean type of $M(z)$ is nonnegative and that it is zero if $A(z)$ and $B(z)$ are linearly dependent.

Proof. By remark to last problem we know E is of Pólya class, and since $\det M \equiv 1$, E is zero free. We have the following lemma in the proof to Theorem 7: if E is of Pólya class and has no zeros, then $E(z) = E(0)e^{-az^2}e^{-ibz}$, where $a \geqslant 0$ and $\Re b \geqslant 0$. By problem 128 and the fact $A = A^{\#}, B = B^{\#}$, E is of bounded type in $\mathbb{C}_+$ and $\mathbb{C}_-$, by Kreĭn's Theorem, E is of exponential type, and this implies $a = 0$. Now $\Re b$ is the mean type of E, and it's nonnegative. If A and B are linearly dependent, $|E| = |E^{\#}|$. In particular, take $z = iy$, then $|E(iy)| = |E^{\#}(iy)|$ becomes $e^{y\Re b} = e^{-y\Re b}$, hence mean type $\Re b$ is 0. ☐

Problem 130

Let $\mu(x)$ be a nondecreasing function of real x which has $r+1$ points of increase, $r = 0, 1, 2, \cdots$. Show that the polynomials of degree at most r are a Hilbert space which satisfies the axioms (H1), (H2) and (H3) in the metric of $L^2(\mu)$. Show that the space is a space $\mathcal{B}(E)$ for some polynomial $E(z)$ of degree $r+1$ which has no real zeros. Show that there exist entire functions $C(z)$ and $D(z)$, which are real for real z, such that

$$\begin{aligned}A(z)D(z) - B(z)C(z) &= 1 \\ \Re(A(z)\bar{D}(z) - B(z)\bar{C}(z)) &\geqslant 1\end{aligned}$$

for all complex z, $\frac{D(z)+iC(z)}{E(z)}$ has no real singularities and

$$\lim_{y\to+\infty} \frac{1}{y}\frac{D(iy)+iC(iy)}{E(iy)} = 0$$

Show that the corresponding space $\mathcal{B}(M)$ has dimension $r+1$ and that $D(z)+iC(z)$ is a polynomial of degree $r+1$. Show that there exists a number W of absolute value 1 such that

$$\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{d\mu(t)}{(t-x)^2+y^2} = \Re\frac{(D(z)+iC(z))+(D(z)-iC(z))W}{(A(z)-iB(z))-(A(z)+iB(z))W}$$

for $y > 0$.

Remark. *This problem says, given any such μ, we can find dB $\mathcal{B}(E)$ which sits inside $L^2(\mu)$. The equality can be used to generalize the result to any regular measure. See problem 137 for more details.*

Proof. I'll prove this statement in a constructive way, using orthogonal polynomials. We know normalized orthogonal polynomials $\{p_n\}_{n=0}^r$ is a base for $L^2(\mu)$, and the reproducing kernel is given by

$$K(w,z) = \sum_{n=0}^{r} p_n(z)\bar{p_n}(w)$$

Now suppose μ has jumps at $x_0, \cdots, x_r$, let $p_{r+1} = a\prod_{k=0}^{r}(x - x_k)$, where a is the leading coefficient of p_r, then

$$xp_r(x) = p_{r+1}(x) + \sum_{k=0}^{r} c_k p_k(x)$$

Taking inner products with p_j, $j \leqslant n-2$ for both sides we know $c_j = 0$ for $j \leqslant n-2$, and we can write

$$xp_r(x) = p_{r+1}(x) + c_r p_r(x) + c_{r-1}p_{r-1}(x)$$

$c_{r-1} = a_n$ because $(xp_r, p_{r-1}) = a_n$. For more details see Barry Simon's book "Orthogonal Polynomials on the Unit Circle", part I, Page 12-13. Let $b_{r+1} = c_r$. The three term relation for orthogonal polynomials are given by

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x)$$
$$yp_n(y) = a_{n+1}p_{n+1}(y) + b_{n+1}p_n(y) + a_n p_{n-1}(y)$$

multiply first row by $p_n(y)$, second row by $p_n(x)$ and subtract second row from the first row, summing up from $n = 0$ to r, we get

$$(x - y)\sum_{n=0}^{r} p_n(x)p_n(y) = p_{r+1}(x)p_r(y) - p_{r+1}(y)p_r(x)$$

and let $x = z, y = \bar{w}$ we get

$$K(w, z) = \frac{p_{r+1}(z)p_r(\bar{w}) - p_{r+1}(\bar{w})p_n(z)}{z - \bar{w}}$$

which is exactly the reproducing kernel of $\mathcal{B}(E)$ for $E = \sqrt{\pi}(p_r - ip_{r+1})$, hence μ is the sampling measure of $\mathcal{B}(E)$ on zero set of B. Here E is a dB function because be definition of p_{r+1}, p_{r+1} and p_r have interlacing zeros and p_{r+1} is one degree higher. Suppose at x_k, μ has point mass m_k, then $\pi A(x_n)B'(x_n) = m_k$. Let $A = \sqrt{\pi}p_r$, $B = \sqrt{\pi}p_{r+1}$, use Theorem 27 to find suitable C and D. Since $\mathcal{B}(M)$ is isometric to $\mathcal{B}(E)$, it has dimension $r + 1$ and by problem 110 C and D are real polynomials of degree at most $r + 1$. Since $\tilde{E} = C - iD$ is unique up to a real multiple of E, we can let $\tilde{E}$ be a polynomial of degree $r + 1$. And now let's consider function $i\frac{D}{B}$. Since $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is Nevanlinna, $B + iD$ is dB as well, then $\Re i\frac{D}{B} \geqslant 0$ on $\mathbb{C}_+$, there exists ν s.t.

$$\Re i\frac{D(z)}{B(z)} = py + \frac{y}{\pi}\int_{\mathbb{R}} \frac{d\nu(t)}{(t - x)^2 + y^2}$$

$p = 0$ since

$$\lim_{y\to+\infty} \frac{1}{y}\frac{D(iy) + iC(iy)}{E(iy)} = 0$$

Let y go to zero we can see ν is purely point, and $supp(\nu) = Z(B)$. The point mass is $\pi\frac{D(x_n)}{B'(x_n)}$, and the rest follows because $AD - BC = 1$ becomes $AD = 1$ on $Z(B)$. Note that this corresponds to the special case $W \equiv 1$. $\square$

Problem 135

Let $\mathcal{B}(E(a))$ be a given space and let $W(a, z)$ be a function which is analytic and bounded by 1 for $y > 0$. Assume that $W(a, z)$ is not identically 1 and that

$$\frac{1 + W(a, z)}{1 - W(a, z)} = \frac{[D(a, b, z) + iC(a, b, z)] + [D(a, b, z) - iC(a, b, z)]W(b, z)}{[A(a, b, z) - iB(a, b, z)] - [A(a, b, z) + iB(a, b, z)]W(b, z)} \tag{9}$$

where $\mathcal{B}(M(a, b))$ exists and $W(b, z)$ is analytic and bounded by 1 for $y > 0$. If $C(a, z) = -B(a, z)$, $D(a, z) = A(a, z)$ and

$$M(b, z) = M(a, z)M(a, b, z)$$

show that

$$\frac{E(a, z) + E^{\#}(a, z)W(a, z)}{E(a, z) - E^{\#}(a, z)W(a, z)} = \frac{[D(b, z) + iC(b, z)] + [D(b, z) - iC(b, z)]W(b, z)}{[A(b, z) - iB(b, z)] - [A(b, z) + iB(b, z)]W(b, z)}$$

for $y > 0$.

Proof. For some reason (see Problem 158 for more details) I prefer to assume W_a is not identically -1, and prove the reciprocal of LHS is equal to the reciprocal of RHS. The RHS can be rewritten as

$$\frac{[D_b(z)+iC_b(z)]+[D_b(z)-iC_b(z)]W_b(z)}{[A_b(z)-iB_b(z)]-[A_b(z)+iB_b(z)]W_b(z)} = i\frac{\tilde{E}_b-\tilde{E}_b^{\#}W_b}{E_b-E_b^{\#}W_b} = i\frac{C_b\left(i\frac{1-W_b}{1+W_b}\right)+D_b}{A_b\left(i\frac{1-W_b}{1+W_b}\right)+B_b} \tag{10}$$

Similarly, (9) can be rewritten as

$$i\frac{1-W_a}{1+W_a} = \frac{E_{\mathrm{a}\to\mathrm{b}}-E_{\mathrm{a}\to\mathrm{b}}^{\#}W_b}{\tilde{E}_{\mathrm{a}\to\mathrm{b}}-\tilde{E}_{\mathrm{a}\to\mathrm{b}}^{\#}W_b} = \frac{A_{\mathrm{a}\to\mathrm{b}}\left(i\frac{1-W_b}{1+W_b}\right)+B_{\mathrm{a}\to\mathrm{b}}}{C_{\mathrm{a}\to\mathrm{b}}\left(i\frac{1-W_b}{1+W_b}\right)+D_{\mathrm{a}\to\mathrm{b}}} \tag{11}$$

Since $M_b = M_a M_{a\to b}$, we have

$$\begin{aligned}
\begin{pmatrix} A_b\left(i\frac{1-W_b}{1+W_b}\right)+B_b \\ C_b\left(i\frac{1-W_b}{1+W_b}\right)+D_b \end{pmatrix} &= \begin{pmatrix} A_b & B_b \\ C_b & D_b \end{pmatrix}\begin{pmatrix} i\frac{1-W_b}{1+W_b} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} A_a & B_a \\ C_a & D_a \end{pmatrix}\begin{pmatrix} A_{a\to b} & B_{a\to b} \\ C_{a\to b} & D_{a\to b} \end{pmatrix}\begin{pmatrix} i\frac{1-W_b}{1+W_b} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} A_a & B_a \\ C_a & D_a \end{pmatrix}\begin{pmatrix} A_{\mathrm{a}\to\mathrm{b}}\left(i\frac{1-W_b}{1+W_b}\right)+B_{\mathrm{a}\to\mathrm{b}} \\ C_{\mathrm{a}\to\mathrm{b}}\left(i\frac{1-W_b}{1+W_b}\right)+D_{\mathrm{a}\to\mathrm{b}} \end{pmatrix} \\
&= \begin{pmatrix} A_a\left\{A_{\mathrm{a}\to\mathrm{b}}\left(i\frac{1-W_b}{1+W_b}\right)+B_{\mathrm{a}\to\mathrm{b}}\right\}+B_a\left\{C_{\mathrm{a}\to\mathrm{b}}\left(i\frac{1-W_b}{1+W_b}\right)+D_{\mathrm{a}\to\mathrm{b}}\right\} \\ C_a\left\{A_{\mathrm{a}\to\mathrm{b}}\left(i\frac{1-W_b}{1+W_b}\right)+B_{\mathrm{a}\to\mathrm{b}}\right\}+D_a\left\{C_{\mathrm{a}\to\mathrm{b}}\left(i\frac{1-W_b}{1+W_b}\right)+D_{\mathrm{a}\to\mathrm{b}}\right\} \end{pmatrix}
\end{aligned} \tag{12}$$

where $C_a = -B_a, D_a = A_a$. Take the ratio of two rows and apply (11), we have

$$\begin{aligned}
&\frac{[D_b(z)+iC_b(z)]+[D_b(z)-iC_b(z)]W_b(z)}{[A_b(z)-iB_b(z)]-[A_b(z)+iB_b(z)]W_b(z)} \\
&= i\frac{C_b\left(i\frac{1-W_b}{1+W_b}\right)+D_b}{A_b\left(i\frac{1-W_b}{1+W_b}\right)+B_b} \quad \text{(by (10))} \\
&= i\frac{C_a\left\{A_{\mathrm{a}\to\mathrm{b}}\left(i\frac{1-W_b}{1+W_b}\right)+B_{\mathrm{a}\to\mathrm{b}}\right\}+D_a\left\{C_{\mathrm{a}\to\mathrm{b}}\left(i\frac{1-W_b}{1+W_b}\right)+D_{\mathrm{a}\to\mathrm{b}}\right\}}{A_a\left\{A_{\mathrm{a}\to\mathrm{b}}\left(i\frac{1-W_b}{1+W_b}\right)+B_{\mathrm{a}\to\mathrm{b}}\right\}+B_a\left\{C_{\mathrm{a}\to\mathrm{b}}\left(i\frac{1-W_b}{1+W_b}\right)+D_{\mathrm{a}\to\mathrm{b}}\right\}} \quad \text{(by (12))} \\
&= i\frac{C_a\left(i\frac{1-W_a}{1+W_a}\right)+D_a}{A_a\left(i\frac{1-W_a}{1+W_a}\right)+B_a} \quad \text{(by (11))} \\
&= i\frac{C_a(1-W_a)-iD_a(1+W_a)}{A_a(1-W_a)-iB_a(1+W_a)} \\
&= i\frac{\tilde{E}_a-\tilde{E}_a^{\#}W_a}{E_a-E_a^{\#}W_a} \\
&= \frac{E_a+E_a^{\#}W_a}{E_a-E_a^{\#}W_a}
\end{aligned}$$

where $\tilde{E}_a = C_a - iD_a = -B_a - iA_a = (-i)(A_a - iB_a) = (-i)E_a, \tilde{E}_a^{\#} = iE_a^{\#}$. □

Problem 137

Let $\mu(x)$ be a nondecreasing function of real x, which is not a constant, such that $\int_{-\infty}^{+\infty}\frac{d\mu(t)}{1+t^2} < \infty$. Show that there exists a space $\mathcal{B}(E)$ contained isometrically in $L^2(\mu)$ such that $E(z)$ is of bounded type in the upper half-plane and has no real zeros.

Remark. *In other words, given any regular measure, there exists a strict dB function E of Pólya class, such that $\mathcal{B}(E) \sqsubseteq L^2(\mu)$.*

Proof. See my notes on existence theorems. □

Problem 138

Let $\mathcal{B}(E_0)$ be a given space such that $E_0(z)$ has no real zeros and let $\mu(x)$ be a nondecreasing function of real x such that $\mathcal{B}(E_0)$ is contained isometrically in $L^2(\mu)$. For each number $b \geqslant 0$ show that there exists a unique space $\mathcal{B}(E_b)$ such that

$$(A_b(z), B_b(z)) = (A_0(z), B_0(z))M_{0\to b}(z)$$

for a space $\mathcal{B}(M_{0\to b})$ with $M_{0\to b}(0) = 1$,

$$M'_{0\to b}(0)I = \begin{pmatrix}\alpha(b) & \beta(b)\\ \beta(b) & \gamma(b)\end{pmatrix} = m(b)$$

and $\alpha(b) + \gamma(b) = b$, and such that there exists a function $W_b(z)$, analytic and bounded by 1 for $y > 0$, and a number $p(b) \geqslant 0$ such that

$$\Re\frac{E_b(z) + E_b^{\#}(z)W_b(z)}{E_b(z) - E_b^{\#}(z)W_b(z)} = p(b)y + \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{|E_b(t)|^2 d\mu(t)}{(t-x)^2 + y^2}$$

for $y > 0$. Show that $m(t)$ is a nondecreasing function of f and that its entries are real valued, continuous functions of t. Show that $E_t(w)$ is a continuous function of $t \geqslant 0$ for every w and that

$$(A_b(w), B_b(w))I - (A_a(w), B_a(w))I = w\int_a^b (A_t(w), B_t(w))dm(t)$$

for $0 \leqslant a < b < \infty$.

Proof. See my notes "Forward Extension of dB Chain". □

Problem 139

Let $\{\mathcal{B}(E_t)\}$ be a family of spaces and let

$$m(t) = \begin{pmatrix}\alpha(t) & \beta(t)\\ \beta(t) & \gamma(t)\end{pmatrix}$$

be a nondecreasing, matrix valued function of t, both defined in an interval $s_- < t < s_+$. Assume that the entries of $m(t)$ are continuous, real valued function of t, that $E_t(w)$ is a continuous function of t for every w, and that

$$(A_b(w), B_b(w))I - (A_a(w), B_a(w))I = w\int_a^b (A_t(w), B_t(w))dm(t)$$

whenever $s_- < a < b < s_+$. Show that

$$\begin{aligned}&\left[B_b(z)\overline{A_b(w)} - A_b(z)\overline{B_b(w)}\right] - \left[B_a(z)\overline{A_a(w)} - A_a(z)\overline{B_a(w)}\right]\\ =&(z-\bar{w})\int_a^b (A_t(z), B_t(z))dm(t)\begin{pmatrix}\overline{A_t(w)}\\ \overline{B_t(w)}\end{pmatrix}\end{aligned}$$

for all complex z and w.

Proof. Note that

$$B_b(z)\overline{A_b(w)} - A_b(z)\overline{B_b(w)} = (A_b(z), B_b(z))I\begin{pmatrix}\overline{A_b(w)}\\ \overline{B_b(w)}\end{pmatrix} = (A_b(z), B_b(z))I\begin{pmatrix}A_b(\bar{w})\\ B_b(\bar{w})\end{pmatrix}$$

Then

$$\begin{aligned}\frac{d}{dt}\left[B_t(z)\overline{A_t(w)} - A_t(z)\overline{B_t(w)}\right] &= (A_t(z), B_t(z))^{\cdot} I\begin{pmatrix}A_t(\bar{w})\\ B_t(\bar{w})\end{pmatrix} + (A_t(z), B_t(z))I\begin{pmatrix}A_t(\bar{w})\\ B_t(\bar{w})\end{pmatrix}^{\cdot}\\ &= z(A_t(z), B_t(z))\,\mathcal{H}(t)\begin{pmatrix}A_t(\bar{w})\\ B_t(\bar{w})\end{pmatrix} - \bar{w}(A_t(z), B_t(z))\,\mathcal{H}(t)\begin{pmatrix}A_t(\bar{w})\\ B_t(\bar{w})\end{pmatrix}\\ &= (z-\bar{w})(A_t(z), B_t(z))\,\mathcal{H}(t)\begin{pmatrix}A_t(\bar{w})\\ B_t(\bar{w})\end{pmatrix}\end{aligned}$$

whose integral form is what we need. □

Problem 140

Let $\{B(E_{+,t})\}$ and $\{B(E_{-,t})\}$ be families of spaces and let $m(t) = \begin{pmatrix}\alpha(t) & \beta(t)\\ \beta(t) & \gamma(t)\end{pmatrix}$ be a nondecreasing, matrix valued function of t, both defined in an interval $s_- < t < s_+$. Assume that the entries of $m(t)$ are continuous, real valued function of t, that $E_{+,t}(w)$ and $E_{-,t}(w)$ are continuous functions of t for every w, and that

$$(A_{+,b}(w), B_{+,b}(w))I - (A_{+,a}(w), B_{+,a}(w))I = w\int_a^b (A_{+,t}(w), B_{+,t}(w))dm(t)$$
$$(A_{-,b}(w), B_{-,b}(w))I - (A_{-,a}(w), B_{-,a}(w))I = w\int_a^b (A_{-,t}(w), B_{-,t}(w))dm(t)$$

whenever $s_- < t < s_+$. Show that

$$\begin{aligned}&\left[B_{+,b}(z)\overline{A_{-,b}(w)} - A_{+,b}(z)\overline{B_{-,b}(w)}\right] - \left[B_{+,a}(z)\overline{A_{-,a}(w)} - A_{+,a}(z)\overline{B_{-,a}(w)}\right]\\ =&(z-\bar{w})\int_a^b (A_{+,t}(z), B_{+,t}(z))dm(t)\begin{pmatrix}\overline{A_{-,t}(w)}\\ \overline{B_{-,t}(w)}\end{pmatrix}\end{aligned} \tag{13}$$

for all complex z and w. If there is some choice of a such that $E_{+,a}(z) = E_{-,a}(z)$ for all complex z, show that $E_{+,z} = E_{-,z}$ for all t, $s_- < t < s_+$, and for all complex z.

Proof. eq. (13) can be proved in the same way as Problem 139. Now let

$$\hat{A}_t(z) := A_{+,t}(z) - A_{-,t}$$
$$\hat{B}_t(z) := B_{+,t}(z) - B_{-,t}$$

Then $\hat{A}_t(z), \hat{B}_t(z)$ are continuous in t for fixed z, and $\hat{A}_a(z) = \hat{B}_a(z) = 0, \forall z \in \mathbb{C}$ by assumption, and taking the difference of the two given integral equations we can get

$$(\hat{A}_b(z), \hat{B}_b(z))I = z\int_a^b (\hat{A}_t(z), \hat{B}_t(z))dm(t)$$

for all $b \in (s_-, s_+)$.

To show $E_+(z) = E_-(z)$, it suffices to show $\hat{A}_b(z) = \hat{B}_b(z) = 0$, $\forall b \in (s_-, s_+)$. First WLOG we fix $z \in \mathbb{C}$ and $b > a$. Since $\hat{A}_t(z), \hat{B}_t(z)$ are continuous in t, let C be a uniform upper bound of $\hat{A}_t(z), \hat{B}_t(z)$ for $t \in [a, b]$. Since $m(t)$ is continuous, we can choose a partition of $[a, b]$, say $\{t_0, \cdots, t_n\}$ s.t.

$$\alpha(t_k) - \alpha(t_{k-1}) < \frac{\epsilon}{2|z|}$$
$$\beta(t_k) - \beta(t_{k-1}) < \frac{\epsilon}{2|z|}$$
$$\gamma(t_k) - \gamma(t_{k-1}) < \frac{\epsilon}{2|z|}$$

Then for any $s \in [a, t_1]$,

$$\begin{aligned}|\hat{B}_s(z)| &= \left| z\int_a^s (\hat{A}_t(z), \hat{B}_t(z))\begin{pmatrix}d\alpha(t)\\ d\beta(t)\end{pmatrix}\right|\\ &\leqslant |z|\int_a^s C(d\alpha(t) + d\beta(t))\\ &\leqslant |z|\int_a^{t_1} C(d\alpha(t) + d\beta(t))\\ &\leqslant C\epsilon\end{aligned}$$

which is a smaller upper bound for $\hat{B}_s(z)$ for $s \in [a, t_1]$. The same estimate holds for $\hat{A}_s(z)$, and then doing this recursively we can get upper bound $C\epsilon^k$ for any $k \in \mathbb{N}$, then $\hat{A}_s(z) = \hat{B}_s(z) = 0$ for $s \in [a, t_1]$. Doing this inductively we can conclude $\hat{A}_s(z) = \hat{B}_s(z) = 0$ for $s \in [a, b]$. The proof for $b < a$ is similar. $\square$

Remark. *Basically this problem says if we have two families of dB functions, which are associated with the same Hamiltonian. Suppose they're equal for some index $a \in (s_-, s_+)$, then they are equal $\forall t \in (s_-, s_+)$.*

Problem 141

In Problem 139 let $M(a,t,w)$ be the unique, continuous, matrix valued function of t, $s_- < a \leqslant t < s_+$, such that

$$M(a,b,w)I - I = w\int_a^b M(a,b,w)dm(t)$$

for $a \leqslant b < s_+$. By Theorem 38 the entries of $M(a,b,z)$ are entire functions of z for any fixed a and b, and a space $\mathcal{B}(M(a,b))$ exists. Show that

$$(A_b(z), B_b(z)) = (A_a(z), B_a(z))M(a,b,z)$$

whenever $s_- < a \leqslant b < s_+$.

Proof. This follows directly from Problem 140. ☐

Problem 148

In Theorem 40 show that $\mathcal{B}(E_a)$ is not contained isometrically in $L^2(\mu)$ when the index a is singular with respect to $m(t)$.

Proof. Suppose a is singular w.r.t. $m(t)$ and $\mathcal{B}(E_a) \subseteq L^2(\mu)$. Let b be the right endpoint of the singular interval containing a, then b is regular, and by Theorem 40 $\mathcal{B}(E_b) \subseteq L^2(\mu)$, so $\mathcal{B}(E_a) \subseteq \mathcal{B}(E_b)$. It's easy to calculate $M_{a\to b}$ given the fact (a,b) is singular:

$$M_{a\to b} = \begin{pmatrix} 1 - luvz & lu^2z \\ -lv^2z & 1 + luvz \end{pmatrix}$$

where $l = Tr(m(b)) - Tr(m(a))$ and $u,\, v \in \mathbb{R}$ and $u^2 + v^2 = 1$. By the remark to Problem 102 we know $\binom{u}{v} \in \mathcal{B}(M_{a\to b})$, then by Theorem 33, $uA_a + vB_a$ lies in the orthogonal complement of $\mathcal{B}(E_a)$ in $\mathcal{B}(E_b)$. On the other hand, let c be the left endpoint of the singular interval, then

$$(A_a, B_a) = (A_c, B_c)M_{c\to a}$$

and by Theorem 34, $uA_c + vB_c \in \mathcal{B}(E_a)$. By the proof to Problem 102, $uA_c + vB_c = uA_a + vB_a$, and then we have a contradiction. Hence $\mathcal{B}(E_a)$ can't sit in $L^2(\mu)$ isometrically if a is singular w.r.t. $m(t)$. ☐

Problem 149

In Theorem 40 let b be a regular point which is not the left end point of an interval of singular points. Show that $\mathcal{B}(E_b)$ is the intersection of the spaces $\mathcal{B}(E_c)$ such that c is regular and $b < c$.

Proof. It's easy to see the intersection $\mathcal{M} := \cap_{c>b \text{ regular}} \mathcal{B}(E_c)$ satisfies $(H1), (H2)$ and $(H3)$ so it's a dB space, and moreover we have $\mathcal{B}(E_b) \sqsubseteq \mathcal{M} \sqsubseteq \mathcal{B}(E_b)$, $\forall c > b$ regular. Let $F \in \mathcal{M}$ be orthogonal to $\mathcal{B}(E_b)$, then

$$(F, K_{b,w}) = 0, \quad \forall w \in \mathbb{C}$$

By continuity of E_t,

$$\begin{aligned} F(w) &= (F, K_{c,w}), \quad \forall c > b \text{ regular} \\ F(w) &= \lim_{c\searrow b}(F, K_{c,w}) \\ &= (F, K_{b,w}) \\ &= 0 \end{aligned}$$

then $F \equiv 0$, hence $\mathcal{B}(E_b) = \mathcal{M}$. ☐

Problem 150

In Theorem 40 let b be a regular point which is not the right end point of an interval of singular points. Show that $\mathcal{B}(E_b)$ is the closed span of the spaces $\mathcal{B}(E_a)$ such that a is regular and $a < b$.

Proof. Similar to the proof to Problem 149. ☐

Problem 151

If the regular points have an upper bound in Theorem 40, show that there is a largest regular point b and that $\mathcal{B}(E_b)$ fills $L^2(\mu)$.

Proof. For any $c \in (b, s_+)$, since $\mathcal{B}(E_c)$ is not contained in $L^2(\mu)$ isometrically, by Problem 138 and Theorem 32, multiplication by z is not densely defined in $\mathcal{B}(E_c)$. By Theorem 87, there exists $a < c$ s.t.

$$(A_c, B_c) = (A_a, B_a) \begin{pmatrix} 1 - \pi u_c v_c & \pi u_c^2 z \\ -\pi v_c^2 z & 1 + \pi u_c v_c z \end{pmatrix}$$

and by the second half of Problem 87 we know $\mathcal{B}(E_a) \sqsubseteq L^2(\mu)$,and by Problem 148 a must be a regular point, then $a \leqslant b$ (by Problem 126 we know $\mathcal{B}(E_a)$ is in the dB chain). If $a < b$, then it's easy to show b is contained in the singular interval, hence $a = b$. Since we know two adjacent singular intervals must have the same ratio $\frac{u}{v}$, and $\alpha(c) + \gamma(c) = c$, then it's easy to show

$$M_{b \to c}(z) = \begin{pmatrix} 1 - uv(c-b)z & u^2 z \\ -\pi v^2 z & 1 + \pi uvz \end{pmatrix}$$

where $u, v \in \mathbb{R}$ and $u^2 + v^2 = 1$. The rest depends on the construction of forward extension. By construction we can show W_b is a constant, and it's easy to see μ is a sampling measure of $\mathcal{B}(E_b)$, so $L^2(\mu)$ is filled by $\mathcal{B}(E_b)$. □

Problem 155

Let $\mathcal{B}(M_a)$, $\mathcal{B}(M_{a\to b})$, and $\mathcal{B}(M_b)$ be spaces such that $M_b(z) = M_a(z) M_{a\to b}(z)$ and such that $A_a(z)$ and $B_a(z)$ are linearly independent. Show that when z is in the upper half-plane,

$$w \to \frac{[D_c(z) + iC_c(z)] + [D_c(z) - iC_c(z)]w}{[A_c(z) - iB_c(z)] - [A_c(z) + iB_c(z)]w} \tag{14}$$

is a mapping of the unit disk $|w| < 1$ onto the disk $\mathcal{D}_c(z)$ of center

$$\frac{D_c(z)\bar{A}_c(z) - C_c(z)\bar{B}_c(z)}{iA_c(z)\bar{B}_c(z) - iB_c(z)\bar{A}_c(z)}$$

and radius

$$\frac{1}{iA_c(z)\bar{B}_c(z) - iB_c(z)\bar{A}_c(z)}$$

for $c = a$ and $c = b$, and show that $\mathcal{D}_a(z)$ contains $\mathcal{D}_b(z)$.

Proof. For simplicity let's denote A_c by A, so is B, C, D and $\mathcal{D}(z)$. Note that the maps $w \to i\frac{1-w}{1+w}$ maps the open unit disk onto upper half-plane conformally, and (14) becomes

$$w \to i\frac{1-w}{1+w} \to i\frac{C\left(i\frac{1-w}{1+w}\right) + D}{A\left(i\frac{1-w}{1+w}\right) + B}$$

so it suffices to show the map

$$u \to i\frac{Cu + D}{Au + B} = i\left(\frac{C}{A} + \left(D - \frac{BC}{A}\right)\frac{1}{Au + B}\right) \tag{15}$$

maps $\mathbb{C}_+$ to the disk $\mathcal{D}(z)$. The map is a linear fractional transformation, which maps generalized circles to generalized circles. Note that the map $v \to \frac{1}{v}$ only takes lines which pass through the origin to lines (other lines will be mapped to circles), and line $Ax + B$ for $x \in \mathbb{R}$ can't pass through the origin, so the image of $\mathbb{R}$ under map (15) must be a circle. Since linear fractional transformation maps a pair of symmetric points to a pair of symmetric points (See Ahlfors' complex analysis textbook), the center is given by

$$i\frac{C\frac{-\bar{B}}{\bar{A}} + D}{A\frac{-\bar{B}}{\bar{A}} + B} = i\frac{\bar{B}C - \bar{A}D}{A\bar{B} - \bar{A}B} = \frac{\bar{A}D - \bar{B}C}{i(A\bar{B} - \bar{A}B)}$$

Since the image of ∞, which is $i\frac{C}{A}$ is on the circle, the radius is given by

$$\begin{aligned} \left| i\frac{\bar{B}C - \bar{A}D}{A\bar{B} - \bar{A}B} - i\frac{C}{A} \right| &= \left| \frac{AD - BC}{A\bar{B} - \bar{A}B} \right| \\ &= \frac{1}{i(A\bar{B} - \bar{A}B)} \end{aligned}$$

here $AD - BC = 1$ since by default M is a Nevanlinna matrix. Now it suffices to show $\mathcal{D}_b \subseteq \mathcal{D}_a$. WLOG we can apply map $w \to \frac{i}{w}$ after map (15), and denote the image of $\mathcal{D}$ by $\tilde{\mathcal{D}}$. Then

$$\tilde{\mathcal{D}}_b = \tau_{M_b}(\mathbb{C}_+) = \tau_{M_a}\tau_{M_{a\to b}}(\mathbb{C}_+) \subseteq \tau_{M_a}(\mathbb{C}_+) = \tilde{\mathcal{D}}_a$$

hence $\mathcal{D}_b \subseteq \mathcal{D}_a$. □

Remark. *This problem describes the chain of Weyl disks (for regular dB chain).*

Problem 158

Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing, matrix valued function whose entries are continuous, real valued function of t in some interval (s_-, s_+). Assume that there exists a family $\{E(t,z)\}$ of entire functions, which have no real zeros, such that $E(t,w)$ is a continuous function of t for every w and

$$(A(b,w), B(b,w))I - (A(a,w), B(a,w))I = w\int_a^b (A(t,w), B(t,w))dm(t)$$

whenever $s_- < a < b < s_+$. If a space $\mathcal{B}(E(a))$ exists for every a, $s_- < a < s_+$, show that there exists a family $\{W(a,z)\}$ of functions, analytic and bounded by 1 for $y > 0$, such that

$$\frac{1 + W(a,z)}{1 - W(a,z)} = \frac{[D(a,b,z) + iC(a,b,z)] + [D(a,b,z) - iC(a,b,z)]W(b,z)}{[A(a,b,z) - iB(a,b,z)] - [A(a,b,z) + iB(a,b,z)]W(b,z)}$$

when $s_- < a < b < s_+$. (If $W(a,z)$ is identically 1, the formula is meaningless as written but has an obvious interpretation on solving for $W(a,z)$.) Show that there exists a nondecreasing function $\mu(x)$ of real x such that

$$\Re\frac{E(a,z) + E^{\#}(a,z)W(a,z)}{E(a,z) - E^{\#}(a,z)W(a,z)} = p(a)y + \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{|E(a,t)|^2 d\mu(t)}{(t-x)^2 + y^2}$$

for $y > 0$ and all indices a, where $p(a)$ is a nonnegative constant which depends only on a.

Proof. First we show the existence of W_a, $\forall a \in (s_-, s_+)$. Since $M_{a\to b}$ is J-expansive, $\Im\frac{A_{\mathrm{a}\to\mathrm{b}}(z)i + B_{\mathrm{a}\to\mathrm{b}}(z)}{C_{\mathrm{a}\to\mathrm{b}}(z)i + D_{\mathrm{a}\to\mathrm{b}}(z)} \geqslant 0$ for $z \in \mathbb{C}_+$, i.e. $\Im\frac{E_{\mathrm{a}\to\mathrm{b}}}{\tilde{E}_{\mathrm{a}\to\mathrm{b}}} \geqslant 0$ for $z \in \mathbb{C}_+$. Let's define

$$w_{a,b} = \frac{E_{\mathrm{a}\to\mathrm{b}}}{\tilde{E}_{\mathrm{a}\to\mathrm{b}}} \quad \left(= i\frac{1 - W_{a,b}}{1 + W_{a,b}}\right)$$

$$W_{a,b} = \frac{i - w_{a,b}}{i + w_{a,b}} = \frac{i - \frac{E_{\mathrm{a}\to\mathrm{b}}}{\tilde{E}_{\mathrm{a}\to\mathrm{b}}}}{i + \frac{E_{\mathrm{a}\to\mathrm{b}}}{\tilde{E}_{\mathrm{a}\to\mathrm{b}}}}$$

then $W_{a,b}$ is analytic and bounded by 1 on $\mathbb{C}_+$. Since $M_{\mathrm{a}\to\mathrm{c}} = M_{\mathrm{a}\to\mathrm{b}}M_{\mathrm{b}\to\mathrm{c}}$, we have

$$\begin{pmatrix} A_{\mathrm{a}\to\mathrm{c}} & B_{\mathrm{a}\to\mathrm{c}} \\ C_{\mathrm{a}\to\mathrm{c}} & D_{\mathrm{a}\to\mathrm{c}} \end{pmatrix} = \begin{pmatrix} A_{\mathrm{a}\to\mathrm{b}} & B_{\mathrm{a}\to\mathrm{b}} \\ C_{\mathrm{a}\to\mathrm{b}} & D_{\mathrm{a}\to\mathrm{b}} \end{pmatrix}\begin{pmatrix} A_{\mathrm{b}\to\mathrm{c}} & B_{\mathrm{b}\to\mathrm{c}} \\ C_{\mathrm{b}\to\mathrm{c}} & D_{\mathrm{b}\to\mathrm{c}} \end{pmatrix}$$

$$\begin{pmatrix} E_{\mathrm{a}\to\mathrm{c}} \\ \tilde{E}_{\mathrm{a}\to\mathrm{c}} \end{pmatrix} = \begin{pmatrix} A_{\mathrm{a}\to\mathrm{b}} & B_{\mathrm{a}\to\mathrm{b}} \\ C_{\mathrm{a}\to\mathrm{b}} & D_{\mathrm{a}\to\mathrm{b}} \end{pmatrix}\begin{pmatrix} E_{\mathrm{b}\to\mathrm{c}} \\ \tilde{E}_{\mathrm{b}\to\mathrm{c}} \end{pmatrix}$$

$$\begin{aligned} w_{a,c} &= \frac{A_{\mathrm{a}\to\mathrm{b}}w_{b,c} + B_{\mathrm{a}\to\mathrm{b}}}{C_{\mathrm{a}\to\mathrm{b}}w_{b,c} + D_{\mathrm{a}\to\mathrm{b}}} \\ &= \frac{\frac{E_{\mathrm{a}\to\mathrm{b}} + E^{\#}_{\mathrm{a}\to\mathrm{b}}}{2}w_{b,c} + i\frac{E_{\mathrm{a}\to\mathrm{b}} - E^{\#}_{\mathrm{a}\to\mathrm{b}}}{2}}{\frac{\tilde{E}_{\mathrm{a}\to\mathrm{b}} + \tilde{E}^{\#}_{\mathrm{a}\to\mathrm{b}}}{2}w_{b,c} + i\frac{\tilde{E}_{\mathrm{a}\to\mathrm{b}} - \tilde{E}^{\#}_{\mathrm{a}\to\mathrm{b}}}{2}} \\ &= \frac{E_{\mathrm{a}\to\mathrm{b}}(i + w_{b,c}) - E^{\#}_{\mathrm{a}\to\mathrm{b}}(i - w_{b,c})}{\tilde{E}_{\mathrm{a}\to\mathrm{b}}(i + w_{b,c}) - \tilde{E}^{\#}_{\mathrm{a}\to\mathrm{b}}(i - w_{b,c})} \\ &= \frac{E_{\mathrm{a}\to\mathrm{b}} - E^{\#}_{\mathrm{a}\to\mathrm{b}}W_{b,c}}{\tilde{E}_{\mathrm{a}\to\mathrm{b}} - \tilde{E}^{\#}_{\mathrm{a}\to\mathrm{b}}W_{b,c}} \end{aligned}$$

The last one can be rewritten as

$$i\frac{1 - W_{a,c}}{1 + W_{a,c}} = \frac{E_{\mathrm{a}\to\mathrm{b}} - E^{\#}_{\mathrm{a}\to\mathrm{b}}W_{b,c}}{\tilde{E}_{\mathrm{a}\to\mathrm{b}} - \tilde{E}^{\#}_{\mathrm{a}\to\mathrm{b}}W_{b,c}} \tag{16}$$

Since $W_{a,b}$ are bounded by 1, we can choose a sequence b_n s.t. W_{a,b_n} goes to W_a locally uniformly on $\mathbb{C}_+$. Taking limit in (16) we can get

$$i\frac{1-W_a}{1+W_a} = \frac{E_{\mathrm{a}\to\mathrm{b}} - E^{\#}_{\mathrm{a}\to\mathrm{b}}W_b}{\tilde{E}_{\mathrm{a}\to\mathrm{b}} - \tilde{E}^{\#}_{\mathrm{a}\to\mathrm{b}}W_b}$$

As for the second part, by Theorem 32, for each $b \in (s_-, s_+)$, there exists a μ_b s.t.

$$\Re\frac{E_b + E_b^{\#}W_b}{E_b - E_b^{\#}W_b} = p_b y + \frac{y}{\pi}\int_{\mathbb{R}} \frac{d\mu_b(t)}{(t-x)^2+y^2} \tag{17}$$

for $y > 0$. Now we'll show $\left|\frac{E_a}{E_b}\right|^2 d\mu_b = d\mu_a$. Let's define

$$M_a(z) := \begin{pmatrix} A_a(z) & B_a(z) \\ C_a(z) & D_a(z) \end{pmatrix}$$

$$M_b(z) := M_a(z)M_{\mathrm{a}\to\mathrm{b}}(z) := \begin{pmatrix} A_b(z) & B_b(z) \\ C_b(z) & D_b(z) \end{pmatrix}$$

where $C_a = -B_a, D_a = A_a$, then it's easy to check both M_a, M_b are dB matrices with associated function $S = E_a$. By Problem 135 we know

$$\frac{E_a + E_a^{\#}W_a}{E_a - E_a^{\#}W_a} = i\frac{\tilde{E}_b - \tilde{E}_b^{\#}W_b}{E_b - E_b^{\#}W_b} \tag{18}$$

We know for given dB function E_b and associated function S, the dB pair function is unique up to a E_b multiplied by a linear function in z. In particular, by Theorem 27 we can choose $\hat{E}$ s.t. $\lim_{y\to+\infty}\frac{\hat{E}_b(iy)}{iyE_b(iy)} = 0$. Note that such $\hat{E}$ is unique up to a real multiple of E. Now by Theorem 32, we have

$$\Re i\frac{\hat{E}_b - \hat{E}_b^{\#}W_b}{E_b - E_b^{\#}W_b} = p(E_a, E_a)y + \frac{y}{\pi}\int_{\mathbb{R}} \frac{1}{(t-x)^2+y^2}\left|\frac{E_a}{E_b}\right|^2 d\mu_b(t) \tag{19}$$

Let $p = -\lim_{y\to+\infty}\frac{\tilde{E}_b(iy)}{iyE_b(iy)} \geqslant 0$, since both $-\frac{\hat{E}_b}{E_b}$ and $-\frac{\tilde{E}_b + pzE_b}{E_b}$ has nonnegative imaginary part on $\mathbb{C}_+$, same limit on the positive imaginary axis, can be continuously extended to $\mathbb{R}$ and have the same imaginary parts on $\mathbb{R}$, they differ by a real number. Now we can choose $\hat{E}_b$ s.t. the real number is 0, then $\hat{E}_b = \tilde{E}_b + pzE_b$. Plug this in (19), we get:

$$\Re i\frac{\tilde{E}_b - \tilde{E}_b^{\#}W_b}{E_b - E_b^{\#}W_b} - py = p(E_a, E_a)y + \frac{y}{\pi}\int_{\mathbb{R}} \frac{1}{(t-x)^2+y^2}\left|\frac{E_a}{E_b}\right|^2 d\mu_b(t) \tag{20}$$

Combine this with (18) we get:

$$\Re\frac{E_a + E_a^{\#}W_a}{E_a - E_a^{\#}W_a} = (p(E_a, E_a) + p)y + \frac{y}{\pi}\int_{\mathbb{R}} \frac{1}{(t-x)^2+y^2}\left|\frac{E_a}{E_b}\right|^2 d\mu_b(t) \tag{21}$$

On the other hand, by our definition of μ_a (see (17)), we have

$$\Re\frac{E_a + E_a^{\#}W_a}{E_a - E_a^{\#}W_a} = p_a y + \frac{y}{\pi}\int_{\mathbb{R}} \frac{d\mu_a(t)}{(t-x)^2+y^2}$$

and since such representation is unique, we can conclude $p_a = p(E_a, E_a) + p$ and $d\mu_a = \left|\frac{E_a}{E_b}\right|^2 d\mu_b$. Let $d\mu = \frac{d\mu_b}{|E_b|^2}$, then (17) becomes

$$\Re\frac{E_b + E_b^{\#}W_b}{E_b - E_b^{\#}W_b} = p_b y + \frac{y}{\pi}\int_{\mathbb{R}} \frac{|E_b|^2 d\mu(t)}{(t-x)^2+y^2}$$

and then the proof is complete.

□

Remark. *This problem claims any dB chain has at least one spectral measure.*

Problem 164

Let $m(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \beta(t) & \gamma(t) \end{pmatrix}$ be a nondecreasing, matrix valued function of $t > 0$ whose entries are continuous, real valued functions of t. Assume that $\alpha(t) > 0$ for $t > 0$ and that $\lim \alpha(t) = 0$ as $t \searrow 0$. Assume that $\{E_{+,t}(z)\}$ and $\{E_{-,t}(z)\}$ are given families of entire functions, which have no real zeros and which have value 1 at the origin, such that spaces $\mathcal{B}(E_{+,t})$ and $\mathcal{B}(E_{-,t})$ exist for every $t > 0$, $E_{+,t}(w)$ and $E_{-,t}(w)$ are continuous functions of t for every w, and

$$(A_{+,b}(w), B_{+,b}(w))I - (A_{+,a}(w), B_{+,a}(w))I = w \int_a^b (A_{+,t}(w), B_{+,t}(w))dm(t)$$

$$(A_{-,b}(w), B_{-,b}(w))I - (A_{-,a}(w), B_{-,a}(w))I = w \int_a^b (A_{-,t}(w), B_{-,t}(w))dm(t)$$

for $0 < a < b < \infty$, and such that $\pi K_{+,a}(0,0) = \alpha(a) = \pi K_{-,a}(0,0)$ for $a > 0$. If

$$P_t(z) = \begin{pmatrix} A_{+,t}(z) & B_{+,t}(z) \\ A_{-,t}(z) & B_{-,t}(z) \end{pmatrix}$$

show that

$$P_b(z)IP_b^*(w) - P_a(z)IP_a^*(w) = (z - \bar{w}) \int_a^b P_t(z)dm(t)P_t^*(w) \tag{22}$$

whenever $0 < a < b < \infty$ and that

$$\lim_{t \searrow 0} P_t(z)IP_t^*(w) = T_+(z)IT_-^*(w)$$

for some entire functions $T_+(z)$ and $T_-(z)$ which are real for real z. Show that $T_+(z)T_-(z)$ vanishes at the origin, and use this fact to show that it vanishes identically. Show that there exists an entire function $S(z)$, which is real for real z and which has no zeros, such that $E_{-,t}(z) = S(z)E_{+,t}(z)$ for all $t > 0$.

Proof. See my notes "Several Remarks on Uniqueness". □

Problem 174

Let $\mathcal{B}(E_1)$ and $\mathcal{B}(E_2)$ be spaces which are isometrically equal. Show that $E_1(z) = E_2(z)$ if $E_k^*(z) = E_k(-z)$ and if $E_k(0) = 1$ for $k = 1, 2$.

Proof. Since $\mathcal{B}(E_1) = \mathcal{B}(E_2)$, there exists $P \in SL(2, \mathbb{R})$ s.t. $(A_1(z), B_1(z)) = (A_2(z), B_2(z))P$, $\forall z \in \mathbb{C}$. Plug in $z = 1$ we can see that the first row of P must be $(1, 0)$. Since $\det P = 1$, the lower right component of P must be 1. So P must be like

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$$

for some real number c, so $A_1(z) = A_2(z) + cB_2(z)$. Change z into $-z$ and use the fact that A_1, A_2 are even and B_2 is odd we can get $c = 0$. So $E_1(z) = E_2(z)$. □

Remark. *This problem says any symmetric dB space is associated with a unique normailized symmetric dB function.*

Problem 175

Let $\mathcal{B}(E)$ be a given space such that $E(z)$ has no real zeros and $\frac{E^\#(-z)}{E(z)}$ is of bounded type in the upper half-plane. Let $\mu(x)$ be a nondecreasing function of real x such that $\mathcal{B}(E)$ is contained isometrically in $L^2(\mu)$. If $\mu(x)$ is an odd function of x, show that $\mathcal{B}(E)$ is symmetric about the origin.

Proof. Let $E_2(z) := E^\#(-z)$. It's not hard to see E_2 is a strict dB function. First I'll show if $F(z) \in \mathcal{B}(E_2)$, then $F(-z) \in \mathcal{B}(E)$. Note that

$$K_2(z,z) = \frac{|E_2(z)|^2 - |E_2^\#(z)|^2}{4\pi y} = \frac{|E^\#(-z)|^2 - |E(-z)|^2}{4\pi y} = K(-z, -z)$$

So by Theorem 20 we have $F(-z) \in \mathcal{B}(E)$. Similarly we can show if $F(z) \in \mathcal{B}(E)$, then $F(-z) \in \mathcal{B}(E_2)$. Now we prove $\mathcal{B}(E_2) \sqsubseteq L^2(\mu)$. Let $F \in \mathcal{B}(E_2)$, then

$$\|F\|_{\mathcal{B}(E_2)}^2 = \int_{\mathbb{R}} \left| \frac{F(t)}{E(-t)} \right|^2 dt = \|F(-z)\|_{\mathcal{B}(E)}^2 = \|F(-z)\|_{L^2(\mu)} = \|F\|_{L^2\mu}$$

the last equality comes from the assumption that μ is odd.
Since $\frac{E_2}{E} \in \mathcal{N}(\mathbb{C}_+)$, by Theorem 35 (ordering theorem), either $\mathcal{B}(E_2) \sqsubseteq \mathcal{B}(E)$ or $\mathcal{B}(E) \sqsubseteq \mathcal{B}(E_2)$. For the first case, $\forall F \in \mathcal{B}(E)$, $F(-z) \in \mathcal{B}(E_2) \sqsubseteq \mathcal{B}(E)$. For the second case, $\forall F \in \mathcal{B}(E) \sqsubseteq \mathcal{B}(E_2)$, we already proved $F(-z) \in \mathcal{B}(E)$. So in either case, if $F(z) \in \mathcal{B}(E)$, then $F(-z) \in \mathcal{B}(E)$. So $\mathcal{B}(E)$ is symmetric. $\square$

Remark. *This problem says under some technical conditions (E strict dB, $\frac{E^{\#}(-z)}{E(z)} \in \mathcal{N}(\mathbb{C}_+)$), if $\mathcal{B}(E) \sqsubseteq L^2(\mu)$ for some odd μ, then $\mathcal{B}(E)$ itself is symmetric.*

Problem 176

Let $\mathcal{B}(E_a)$ and $\mathcal{B}(E_b)$ be given spaces such that $B(E_a)$ is contained isometrically in $B(E_b)$ and such that $E_a(z)$ and $E_b(z)$ have no zeros. Show that $\mathcal{B}(E_a)$ is symmetric about the origin if $\mathcal{B}(E_b)$ is symmetric about the origin.

Proof. Similar to problem 175 we know $E_a^{\#}(-z)$ is dB as well and using the property that $\forall F \in \mathcal{B}(E_a^{\#}(-z))$, $F(-z) \in \mathcal{B}(E_a) \sqsubseteq \mathcal{B}(E_b)$ and hence $F(z) \subset \mathcal{B}(E_b)$, we get $\mathcal{B}(E_a^{\#}(-z)) \sqsubseteq \mathcal{B}(E_b)$. Pick any nonzero $F \in \mathcal{B}(E_a) \sqsubseteq \mathcal{B}(E_b)$, since $\frac{F}{E_a}$ and $\frac{F}{E_b}$ are in $\mathcal{N}(\mathbb{C}_+)$, we have $\frac{E_a}{E_b} \in \mathcal{N}(\mathbb{C}_+)$. Similarly we get $\frac{E_a^{\#}(-z)}{E_b(z)} \in \mathcal{N}(\mathbb{C}_+)$, hence $\frac{E_a^{\#}(-z)}{E_a(z)} \in \mathcal{N}(\mathbb{C}_+)$. Now by similar argument as in proving last problem we know that $\mathcal{B}(E_a)$ is symmetric about the origin. In particular, $\mathcal{B}(E_a(z)) = \mathcal{B}(E_a^{\#}(-z))$. $\square$

Remark. *This problem says, under some technical conditions (E_a, E_b strict), a subspace of symmetric dB space is still symmetric.*

Problem 177

Let $\mathcal{B}(E_a)$ and $\mathcal{B}(E_b)$ be given spaces such that

$$(A_b(z), B_b(z)) = (A_a, B_a) M_{a\to b}(z)$$

for some space $\mathcal{B}(M_{a\to b})$ such that $M_{a\to b}(0) = 1$. Let

$$\begin{aligned} A_{1,a}(z) &= A_a(-z), & B_{1,a}(z) &= -B_a(z) \\ A_{1,a\to b}(z) &= A_{a\to b}(-z), & B_{1,a\to b}(z) &= -B_{a\to b}(-z) \\ C_{1,a\to b}(z) &= -C_{a\to b}(-z), & D_{1,a\to b}(z) &= D_{a\to b}(-z) \end{aligned}$$

If $E_b^{\#}(z) = E_b(-z)$, show that spaces $\mathcal{B}(E_{1,a})$ and $\mathcal{B}(M_{1,a\to b})$ exist and that

$$(A_b(z), B_b(z)) = (A_{1,a}(z), B_{1,a}(z)) M_{1,a\to b}(z)$$

Show that $E_{1,a}(z) = E_a^{\#}(-z) = E_a(z)$ and that $M_{1,a\to b}(z) = M_{a\to b}(z)$.

Proof. First let's prove $\mathcal{B}(E_{1,a})$ and $\mathcal{B}(M_{1,a\to b})$ exist. From proof to problem 175 we know $E_{1,a}$ is a non-degenerate strict dB function and $\forall F \in \mathcal{B}(E)$, so $\mathcal{B}(E_{1,a})$ exists. To show $\mathcal{B}(M_{1,a\to b})$ exists, it suffices to show the existence of entire function S_1 s.t. (use M_1 to denote $M_{1,a\to b}$)

$$\det M_1(z) = S_1(z) S_1^{\#}(z), \quad \frac{M_1(z) I \bar{M}_1(z) - S_1(z) I \bar{S}_1(w)}{z - \bar{z}} \geqslant 0, \ \forall z \in \mathbb{C} \tag{23}$$

Since $\mathcal{B}(M_{a\to b})$ exists, there exists entire function S s.t.

$$\det M(z) = S(z) S^{\#}(z), \quad \frac{M(z) I \bar{M}(z) - S(z) I \bar{S}(w)}{z - \bar{z}} \geqslant 0, \ \forall z \in \mathbb{C} \tag{24}$$

Note that $M_1(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M(-z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that for a 2×2 matrix N,

$$N \geqslant 0 \iff \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} N \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \geqslant 0$$

Let $S_1(z) := S(-z)$, change z into $-z$ in equation (24) and use the equation

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} I \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

we can get (23).
Since E_b is symmetric, $A_b(z) = A_b(-z)$ and $B_b(z) = -B_b(-z)$,

$$\begin{aligned}
(A_b(z), B_b(z)) &= (A_b(-z), -B_b(-z)) \\
&= (A_b(-z), B_b(-z)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= (A_a(-z), B_a(-z)) M_{a\to b}(-z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= (A_{1,a}(z), -B_{1,a}(z)) M_{a\to b}(-z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= (A_{1,a}(z), B_{1,a}(a)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M_{a\to b}(-z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= (A_{1,a}(z), B_{1,a}(z)) M_{1,a\to b}(z)
\end{aligned}$$

By problem 176, $\mathcal{B}(E_a)$ is symmetric and $\mathcal{B}(E_a) = \mathcal{B}(E_a^{\#}(-z))$, hence there exists $P \in SL(2, \mathbb{R})$ s.t.

$$(A_a(z), B_a(z)) = (A_a(-z), -B_a(-z))P$$

Plug in $z = 0$ and use the fact $(A_a(0), B_a(0)) = (A_b(0), B_b(0)) = (A_b(0), 0)$, it's easy to see P must be $\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ for some $c \in \mathbb{R}$, hence $B_a(z) = -B_a(-z)$ and $A_a(z) = A_a(-z) + cB_a(z)$.
Now it suffices to show $c = 0$. Let $z = iy$, then $cB_a(iy) = A_a(iy) - A_a(-iy) = A_a(iy) - \overline{A_a \overline{iy}} = 2\Re A(iy) \in \mathbb{R}$. On the other hand, $B(iy) = -B(-iy) = -\overline{B(\overline{-iy})} = -\overline{B(iy)}$ implies $B(iy)$ is purely imaginary, so $c = 0$. Hence $E_{1,a}(z) = E_a^{\#}(-z) = E_a(z)$. The equality $M_{1,a\to b}(z) = M_{a\to b}(z)$ follows from problem 100. □

Remark. *This problem says if $(A_b, B_b) = (A_a, B_a)M_{a\to b}$ and E_b is symmetric, $\mathcal{B}(E_a), \mathcal{B}(E_b)$ and $\mathcal{B}(M_{a\to b})$ exist, then E_a is symmetric.*

Remark. *Note that we also proved the following proposition: if $\mathcal{B}(E)$ is symmetric and $B(0) = 0$, then E is symmetric.*

Problem 178

If $E^{\#}(z) = E(-z)$ and if $\mu(x) = -\mu(-x)$ in Theorem 40, show that $\beta(t)$ is a constant and that $E_a^{\#}(z) = E_a(-z)$ for all indices a.

Proof. First we'll show $E_a^{\#}(z) = E_a(-z)$ for all $a \in (s_-, s_+)$. From the proof to Theorem 40 we can see that

$$(A_b, B_b) = (A_a, B_a)M_{a\to b} \tag{25}$$

when $s_- < a \leqslant b$. By Theorem 40, $E(z) = E_c(z)$. If $a < c$, then by problem 177 we have E_a is symmetric. Now for $a > c$, there're two cases.

Case 1. There's no largest regular point. So for any $a > c$, we can always find $b > a$ which is regular, and therefore $\mathcal{B}(E_b) \sqsubseteq L^2(\mu)$. Let $F \in \mathcal{B}(E_c)$, then $\frac{F(-z)}{F(z)} \in \mathcal{N}(\mathbb{C}_+)$. And since $\frac{F(-z)}{E_b^{\#}(-z)}, \frac{F(z)}{E_b(z)} \in \mathcal{N}(\mathbb{C}_+)$ as well, then $\frac{E_b^{\#}(-z)}{E_b(z)} \in \mathcal{N}(\mathbb{C}_+)$, and by problem 175 we know E_b is symmetric. Change a into c in equation (25), we can see that $B_b(0) = 0$, so by the proposition in the second remark to problem 177, E_b must be symmetric. And then by problem 177, E_a is symmetric.

Case 2. There's one largest regular point b. For any $c \leqslant b$, by previous argument we know E_c is symmetric. For $c > b$, from the proof to Problem 151 we can see

$$(A_c, B_c) = (A_b, B_b) \begin{pmatrix} 1 - uvz & u^2 z \\ -v^2 z & 1 + uvz \end{pmatrix}$$

for some $u, v \in \mathbb{R}$, and $uA_b + vB_b \in \mathcal{B}(E_c)$. The rest depends on the construction of forward extension. By construction, $v = 0$, so it's easy to see E_c is symmetric.

The statement $\beta(t)$ is a constant comes from differentiating $M_{a\to b}(z)$ and use the results of Problem 177. □

Remark. *This problem says if a dB chain has a symmetric(odd) spectral measure, and if one dB function is symmetric, then all dB functions in the chain are symmetric.*

Problem 305

Let ρ be a nonnegative integer. An entire function $F(z)$ is said to belong to the ρ-th Laguerre class if it is real for real z, has only real zeros, and has value one at the origin, and if

$$\Re\left(iz^{-2\rho}\frac{F'(z)}{F(z)}\right) \geqslant 0$$

for $y > 0$. Show that the function

$$(1-hz)\exp\left(hz+\frac{h^2z^2}{2}+\cdots+\frac{h^{2\rho+1}z^{2\rho+1}}{2\rho+1}\right)$$

belongs to the ρ-th Laguerre class if h is real. Show that a finite product of functions which belong to the ρ-th Laguerre class. Show that a limit of functions which belong to the ρ-th Laguerre class is a function which belongs to the ρ-th Laguerre class if convergence is uniform on bounded sets. If $\{h_n\}$ is a sequence of real numbers such that

$$\sum_{n=1}^{\infty} h_n^{2\rho+2} < \infty$$

show that the product

$$P(z) = \prod_{n=1}^{\infty}(1-h_nz)\exp\left(h_nz+\frac{h_n^2z^2}{2}+\cdots+\frac{h_n^{2\rho+1}z^{2\rho+1}}{2\rho+1}\right)$$

converges uniformly on bounded sets and represents an entire function which belongs to the ρ-th Laguerre class. Show that it satisfies the estimate

$$\log(1+|P(z)-1|) \leqslant \left(\sum_{n=1}^{\infty} h_n^{2\rho+2}\right)|z|^{2\rho+2}$$

for all complex z. Show that the function $e^{-az^{2\rho+2}}$ belongs to the ρ-th Laguerre class if $a \geqslant 0$.

Proof. Let

$$F(z) := (1-hz)\exp\left(hz+\frac{h^2z^2}{2}+\cdots+\frac{h^{2\rho+1}z^{2\rho+1}}{2\rho+1}\right)$$

then

$$\begin{aligned}
F'(z) &= (-h+(1-hz)(h+\cdots+h^{2\rho+1}z^{2\rho}))\exp\left(hz+\frac{h^2z^2}{2}+\cdots+\frac{h^{2\rho+1}z^{2\rho+1}}{2\rho+1}\right)\\
&= \left(-h+(1-hz)h\frac{1-(hz)^{2\rho+1}}{1-hz}\right)\exp\left(hz+\frac{h^2z^2}{2}+\cdots+\frac{h^{2\rho+1}z^{2\rho+1}}{2\rho+1}\right)\\
&= -h(hz)^{2\rho+1}\exp\left(hz+\frac{h^2z^2}{2}+\cdots+\frac{h^{2\rho+1}z^{2\rho+1}}{2\rho+1}\right)\\
\Re\left(i\frac{F'(z)}{z^{2\rho}F(z)}\right) &= \Re\frac{-ih^{2\rho+2}z}{1-hz}\\
&= \frac{h^{2\rho+2}}{|1-hz|^2}\Im z\\
&\geqslant 0
\end{aligned}$$

Use definition it's easy to check finite product, and uniform limit of Laguerre functions is of Laguerre class. Now let's show $P(z)$ belongs to ρ-th Laguerre class. Let

$$\begin{aligned}
a_n(z) &:= (1-h_nz)\exp\left(h_nz+\frac{h_n^2z^2}{2}+\cdots+\frac{h_n^{2\rho+1}z^{2\rho+1}}{2\rho+1}\right)\\
P_r(z) &:= \prod_{n=1}^{r} a_n(z)
\end{aligned}$$

This is analogous to Problem 9 for Pólya class. We'll show

$$|P_s(z)-P_r(z)| \leqslant \exp\left(\left(\sum_{n=1}^{s} h_n^{2\rho+2}\right)|z|^2\right) - \exp\left(\left(\sum_{n=1}^{r} h_n^{2\rho+2}\right)|z|^2\right)$$

Use problem 7 and 8,

$$\begin{aligned}\left|\frac{P_s(z)}{P_r(z)}-1\right| &= |a_{r+1}\cdots a_s - 1| \\ &\leqslant (1+|a_{r+1}-1|)\cdots(1+|a_s-1|)-1 \\ &\leqslant e^{h_{r+1}^{2\rho+2}|z|^{2\rho+2}}\cdots e^{h_s^{2\rho+2}|z|^{2\rho+2}}-1 \\ &\leqslant e^{\left(\sum_{n=r+1}^{s} h_n^{2\rho+2}\right)|z|^{2\rho+2}}-1 \\ |P_r(z)| &\leqslant 1+|P_r(z)-1| \\ &\leqslant e^{\left(\sum_{n=1}^{r} h_n^{2\rho+2}\right)|z|^{2\rho+2}} \\ |P_s(z)-P_r(z)| &\leqslant e^{\left(\sum_{n=1}^{s} h_n^{2\rho+2}\right)|z|^{2\rho+2}} - e^{\left(\sum_{n=1}^{r} h_n^{2\rho+2}\right)|z|^{2\rho+2}}\end{aligned}$$

and then use the same argument as problem 9, we get local uniform convergence.
Now let $F(z) := e^{-az^{2\rho+2}}$, $a \geqslant 0$,

$$\begin{aligned}F'(z) &= -a(2\rho+2)z^{2\rho+1}e^{-az^{2\rho+2}} \\ \Re i\frac{F'}{z^{2\rho}F} &= a(2\rho+2)\Im\frac{(1-a\bar{z})z}{|1-az|^2} \\ &\geqslant 0\end{aligned}$$

for $z \in \mathbb{C}_+$. $\square$

Remark. *We probably need to add* $\lim_{z\to 0}\frac{F'(z)}{z^{2\rho+1}}$ *exists for F to be in ρ-th Laguerre class. From now on we'll use the following definition: F belongs to the ρ-th Laguerre class if:*

(1) F is real entire, has only real zeros;

(2) $F(0) = 1$;

(3) $\Re\left(i\frac{F'(z)}{z^{2\rho}F(z)}\right) \geqslant 0$;

(4) $\lim_{z\to 0}\frac{F'(z)}{z^{2\rho+1}}$ exists.

If we don't include (4)*, then $F(z) = e^{z^{2\rho+1}}$ satisfies (1), (2) and (3):*

$$\Re\left(i\frac{F'(z)}{z^{2\rho}F(z)}\right) = \Re(i(2\rho+1)) = 0$$

and this contradicts Problem 307.

Problem 306

If an entire function $F(z)$ belongs to the ρ-th Laguerre class and has a zero w, show that

$$F(z) = G(z)(1-hz)\exp\left(hz + \frac{h^2z^2}{2} + \cdots + \frac{h^{2\rho+1}z^{2\rho+1}}{2\rho+1}\right)$$

where $G(z)$ is an entire function which belongs to the ρ-th Laguerre class and $h = 1/w$.

Proof. Let $H_w(z) := (1-hz)\exp\left(hz + \frac{h^2z^2}{2} + \cdots + \frac{h^{2\rho+1}z^{2\rho+1}}{2\rho+1}\right)$ (where $w = \frac{1}{h}$), then H belongs to ρ-th Laguerre class by Problem 305, and it's easy to check G is real entire, $G(0) = 1$ and $\lim_{z\to 0}\frac{G'(z)}{z^{2\rho+1}}$ exists since $F' = G'H_h + GH_h'$. Now it suffices to check $\Re\left(i\frac{1}{z^{2\rho}}\frac{F'}{F}\right) \geqslant 0$. Since $\Re\left(i\frac{1}{z^{2\rho}}\frac{F'}{F}\right) \geqslant 0$, by Poisson representation there exists p and μ s.t. $\Re\left(i\frac{1}{z^{2\rho}}\frac{F'}{F}\right) = py + \mathcal{P}\mu$. Let $\{t_k\}$ be zeros of F, with multiplicity n_k. Let $\tilde{F}(z) := i\frac{1}{z^{2\rho}}\frac{F'}{F}$. Since $\Re\tilde{F} = 0$ on the real line, μ is discrete (since $\mu(b)-\mu(a) = \lim_{y\to 0+}\int_a^b \Re\tilde{F}(x+iy)dx$). Now let's calculate jump of μ at each t_n. Let $F = G_0H_{t_k}^{n_k}$, then

$$\Re\tilde{F} = \Re\tilde{G} + n_k\Re\tilde{H}_{t_k}$$

where $\Re\tilde{G}$ goes to 0 as y goes to 0. It's easy to calculate $\Re\tilde{H}_{t_k}$. Let $w_k = \frac{1}{t_k}$, then

$$\begin{aligned}
\tilde{H}_{w_k} &= \frac{i}{z^{2\rho}} \frac{H'_{w_k}(z)}{H_{w_k}(z)} \\
&= -i \frac{t_k^{2\rho+2} z}{1 - t_k z} \\
\Re\tilde{H}_{w_k} &= t_k^{2\rho+2} \Im \frac{z}{1 - t_k z} \\
&= t_k^{2\rho} \frac{y}{|z - w_k|^2} \\
\mu(w_k + \epsilon) - \mu(w_k - \epsilon) &= n_k t_k^{2\rho} \lim_{y \to 0+} \int_{w_k - \epsilon}^{w_k + \epsilon} \frac{y}{(x - w_k)^2 + y^2} dy \\
&= \pi t_k^{2\rho}
\end{aligned}$$

Hence $\Re\tilde{F}(z) = py + \sum_k n_k t_k^{2\rho} \frac{y}{|z - w_k|^2}$. Moreover, find j s.t. $w = w_j$, then $h = \frac{1}{w_j}$, and

$$\begin{aligned}
\Re\tilde{G}(z) &= \Re\tilde{F}(z) - \Re\tilde{H}_{w_k} \\
&= py + \sum_{k \neq j} n_k t_k^{2\rho} \frac{y}{|z - w_k|^2} + (n_j - 1) t_j^{2\rho} \frac{y}{|z - w_j|^2} \\
&\geqslant 0
\end{aligned}$$

□

Remark. *For* 0*-th Laguerre class, please refer to Čebotarev's theorem ("Distribution of Zeros of Entire functions", page 310).*

Problem 307

If an entire function $F(z)$ belongs to the ρ-th Laguerre class and has no zeros, show that

$$F(z) = \exp(-az^{2\rho+2})$$

where $a \geqslant 0$.

Proof. Let $\tilde{F}(z) = i \frac{F'(z)}{z^{2\rho} F(z)}$. Since F is zero-free and $\lim_{z \to 0} \frac{F'(z)}{z^{2\rho}} = 0$ exists, $\tilde{F}$ is entire, and $\Re F \geqslant 0$ on $\mathbb{C}_+$, $= 0$ on $\mathbb{R}$ and $\leqslant 0$ on $\mathbb{C}_-$. By Problem 2-4 we know $\tilde{F}(z) = -ia_0 z + b$, where $a_0 \geqslant 0$. Since $\tilde{F}(0) = 0$, $b = 0$. Hence $\tilde{F}(z) = -ia_0 z$, i.e. $\log F(z) = -a_0 \frac{z^{2\rho+2}}{2\rho+2} := -az^{2\rho+2}$, $F(z) = \exp(-az^{2\rho+2})$. □

Problem 308

If an entire function $F(z)$ belongs to the ρ-th Laguerre class, show that $F(z)$ is equal to

$$\exp(-az^{2\rho+2}) \prod (1 - h_n z) \exp\left(h_n z + \frac{h_n^2 z^2}{2} + \cdots + \frac{h_n^{2\rho+1} z^{2\rho+1}}{2\rho+1}\right)$$

where $a \geqslant 0$ and $\{h_n\}$ is a sequence of real numbers such that $\sum h_n^{2\rho+2} < \infty$.

Proof. Let $\{w_k\} \subseteq \mathbb{R}$ be the zeros of F, $h_k := \frac{1}{w_k}$, then by Problem 305-307 it suffices to show $\sum_{k=1}^{+\infty} h_k^{2\rho+2} < \infty$. First we have $F(z) = G_n(z) \prod_{k=1}^n H_{w_k}(z)$, where G_n and $\prod_{k=1}^n H_{w_k}$ are of ρ-th Laguerre class. Note that

$$\begin{aligned}
\Re\tilde{F}(z) &\geqslant \sum_{k=1}^n \Re\tilde{H}_{w_k}(z) \\
&= \sum_{k=1}^n \frac{y}{|z - w_k|^2} h_k^{2\rho}
\end{aligned}$$

Since $\lim_{z\to 0}\frac{F'(z)}{z^{2\rho+1}F(z)}$ exists, let $G=\frac{F'(z)}{z^{2\rho+1}F(z)}$, then $G(z)$ is analytic in a neighborhood of 0. In particular, $\Re\tilde{F}(z)=\Re(izG)$. Let $z=iy$ for small y, then

$$\begin{aligned}\Re(-yG(iy)) &\geqslant \sum_{k=1}^{n}\frac{y}{|iy-w_k|^2}h_k^{2\rho}\\ \Re(-G(iy)) &\geqslant \sum_{k=1}^{n}\frac{1}{y^2+w_k^2}h_k^{2\rho}\\ \Re(-G(0)) &\geqslant \sum_{k=1}^{n}h_k^{2\rho+2}\end{aligned}$$

Hence $\sum_{k=1}^{+\infty}h_k^{2\rho+2}\leqslant\Re(-G(0))<\infty$, and by Problem 305-307 we're done. □

Problem 309

If an entire function $F(z)$ belongs to the ρ-th Laguerre class, show that

$$\lim_{z\to 0}\frac{F'(z)}{z^{2\rho+1}}=-(2\rho+2)\delta$$

where $\delta\geqslant 0$ and

$$\log(1+|F(z)-1|)\leqslant\delta|z|^{2\rho+2}$$

for all complex z.

Proof. This statement is wrong. A counterexample is $F(z)=(1-z)e^z$, then $F'(z)=-ze^z$ and $\lim_{z\to 0}\frac{F'(z)}{z}=-1$, hence $\delta=\frac{1}{2}$. $F(1)=0$, hence $\log(1+|F(z)-1|)=\log 2\approx 0.69$. However, RHS is 0.5 which is smaller. The statement is true if we define $\delta=-\lim_{z\to 0}\frac{F'(z)}{z^{2\rho+1}}$. □

Problem 310

Let $F(z)$ be a function which is real for real z, has only real zeros, and has value one at the origin. Define $\log F(z)$ continuously in the upper half-plane so as to have limit zero at the origin. Show that $F(z)$ belongs to the ρ-th Laguerre class if, and only if,

$$\Re\left(i\frac{\log F(z)}{z^{2\rho+1}}\right)\geqslant 0$$

for $y>0$.

Remark. *The statement and proof is analogous to Theorem 14. And we need additional assumption that* $\lim_{z\to 0}\frac{F'(z)}{z^{2\rho+1}}$ *exists, as mentioned in the remark to Problem 305.*

Proof. First we assume F belongs to the ρ-th Laguerre class. By Problem 308, there're only two cases: $F=\exp(-az^{2\rho+2})$ and $F=(1-hz)\exp(hz+\cdots+\frac{(hz)^{2\rho+1}}{2\rho+1})$. The first case is trivial. For the second one,

$$\begin{aligned}\frac{\partial}{\partial h}\Re\left(i\frac{\log F(z)}{z^{2\rho+1}}\right) &= \Re\left(\frac{i}{2\rho+1}\frac{\partial}{\partial h}\left(\log(1-hz)+hz+\cdots+\frac{(hz)^{2\rho+1}}{2\rho+1}\right)\right)\\ &= \Re\left(\frac{i}{2\rho+1}\left(\frac{-z}{1-hz}+z+\cdots+z(hz)^{2\rho}\right)\right)\\ &= \Re\left(\frac{-ih^{2\rho+1}z}{|1-hz|^2}\right)\\ &= h^{2\rho+1}\frac{y}{|1-hz|^2}\end{aligned}$$

is positive (negative) for $h>0$ ($h<0$). For $h=0$, $\Re\left(i\frac{\log F(z)}{z^{2\rho+1}}\right)=0$, hence $\Re\left(i\frac{\log F(z)}{z^{2\rho+1}}\right)\geqslant 0$ for all h, when $y>0$.
Now assumes $\Re\left(i\frac{\log F(z)}{z^{2\rho+1}}\right)\geqslant 0$. If F is zero-free, $i\frac{\log F}{z^{2\rho+1}}$ is entire by our additional assumption. Now use Problem 2-4 we can get

$$i\frac{\log F}{z^{2\rho+1}}=-iaz+b$$

where $a\geqslant 0$. Plug in $z=0$ we have $b=0$, then $F=\exp(-az^{2\rho+2})$.

Now assume F has zero w_1, let $h_1 = \frac{1}{w_1}$ and the associated canonical factor be $P_1(z)$. We want to show $F_1 = \frac{F}{P_1}$ still satisfies the same inequality. This can be proved using Poisson representation like we did for Problem 308. Define $\tilde{}$-transform as $\tilde{F}(z) = i\frac{\log F(z)}{z^{2\rho+1}}$, then what we showed is $\tilde{F}(z) = \tilde{F_1} + \tilde{P_1}$, and $\Re\tilde{F}, \Re\tilde{F_1}, \Re\tilde{P_1} \geqslant 0$ on $\mathbb{C}_+$. It's easy to prove $F_n := \frac{F}{\prod_{k=1}^n P_k}$ satisfies $\Re\tilde{F} \geqslant \Re\tilde{F}_n \geqslant 0$ on $\mathbb{C}_+$.
Now it suffices to show $\sum_{k=1}^{+\infty} h_k^{2\rho+2} < \infty$. Let $Q_n(z) := \prod_{k=1}^n P_k(z)$, since $\Re\tilde{F} - \Re\tilde{Q}_n = \Re\tilde{F}_n \geqslant 0$ on $\mathbb{C}_+$, $= 0$ on $\mathbb{R}$, $\leqslant 0$ on $\mathbb{C}_-$, we have

$$\frac{\partial}{\partial y}\left(\Re\tilde{F} - \Re\tilde{Q}_n\right)\bigg|_{z=0} \geqslant 0$$

The first term is just $-\Im\tilde{F}'(0)$, which is a fixed nonnegative number. Note that $\tilde{Q}_n = \sum_{k=1}^n \tilde{P}_k$, and by L'Hôpital's rule we have

$$\begin{aligned}
\tilde{P}_k'(0) &= i\frac{(\log P_k)' z^{2\rho+1} - (\log P_k)(2\rho+1)z^{2\rho}}{z^{4\rho+2}}\bigg|_{z=0} \\
&= i\left(\frac{P_k'}{P_k z^{2\rho+1}} - (2\rho+1)\frac{\log P_k}{z^{2\rho+2}}\right)\bigg|_{z=0} \\
&= i\left(\frac{P_k'}{P_k z^{2\rho+1}} - \frac{2\rho+1}{2\rho+2}\frac{P_k'}{P_k z^{2\rho+1}}\right)\bigg|_{z=0} \\
&= -i\frac{1}{2\rho+2}h_k^{2\rho+2}
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial}{\partial y}\Re\tilde{Q}_n\bigg|_{z=0} &= -\Im(\tilde{Q}_n)'(0) \\
&= \frac{1}{2\rho+2}\sum_{k=1}^n h_k^{2\rho+2}
\end{aligned}$$

Now we have $\sum_{k=1}^n h_k^{2\rho+2} \leqslant -\Im\tilde{F}'(0)$, $\forall n$, and we can reduce to the case where F is zero-free, and the proof is complete.

□

Problem 313

Let $\mathcal{B}(E)$ be a given space such that $E(z)$ has no real zeros and $A(0) = 1$. Show that the transformation $F(z) \mapsto \frac{F(z)-A(z)F(0)}{z}$ is self-adjoint in the space. Show that the space admits an orthogonal basis consisting of eigenfunctions of the transformation. Show that the nonzero eigenvalues of the transformations are the numbers $\{\frac{1}{t_n}\}$ where $\{t_n\}$ are the zeros of $A(z)$.

Proof. In Problem 67, let $\alpha = \beta = 0$, $u = 1$, then

$$\langle F(t), (\mathbb{B}_{A,0}\, G)(t)\rangle = \langle(\mathbb{B}_{A,0}\, F)(t), G(t)\rangle$$

hence $\mathbb{B}_{A,0}$ is self-adjoint. Furthermore, we know $\{K(t_n, z)\}$ or $\{K(t_n, z)\} \cup \{A(z)\}$ forms an orthogonal basis, depending on if $A \in \mathcal{B}(E)$ or not. From the proof to Problem 67 we can see the nonzero eigenvalues are $\frac{A(0)}{t_n - 0} = \frac{1}{t_n}$, where $\{t_n\}$ are the zeros of $A(z)$. □

Problem 314

A bounded transformation T of a Hilbert space into itself is said to be of Schmidt class if

$$\sigma(T)^2 = \sum \|Tf_n\|^2 < \infty$$

for some orthonormal basis $\{f_n\}$ of the space. Show that the sum does not depend on the choice of orthonormal basis.

Proof. The independence over choice of orthonormal basis is a well-known fact in operator theory. □

Problem 315

Let $\mathcal{B}(E)$ be a given space such that $E(z)$ has no real zeros and $A(0) = 1$. Let T be the transformation

$$F(z) \mapsto \frac{F(z) - A(z)F(0)}{z}$$

in the space. Show that $T^{1+\rho}$ is of Schmidt class if $A(z)$ belongs to the ρ-th Laguerre class. Show that $\sigma(T^{1+\rho})^2 \leqslant \delta$ where

$$-(2\rho + 2)\delta = \lim_{z\to 0} \frac{A'(z)}{z^{2\rho+1}}$$

Show that equality holds if

$$\lim_{y\to+\infty} \frac{\log|A(iy)|}{y^{2\rho+2}} = 0$$

Proof. Let $\{t_n\}$ be the zeros of A, then $T^{1+\rho}K(t_n, z) = \frac{1}{t_n^{1+\rho}}K(t_n, z)$ and $T^{1+\rho}A = 0$, then by Problem 314 we have $\sigma(T^{1+\rho})^2 = \sum_n \frac{1}{|t_n|^{2\rho+2}}$. Since A belongs to ρ-th Laguerre class, by Problem 308 $\sum_n \frac{1}{|t_n|^{2\rho+2}} < \infty$. Hence $T^{1+\rho}$ is of Schmidt class if $A(z)$ belongs to the ρ-th Laguerre class.
From the proof to Problem 310 we can see $\sum_n \frac{1}{|t_n|^{2\rho+2}} \leqslant -\Im\tilde{A}'(0)$. Similar to the calculation of $\tilde{P}_k'(0)$, we can get $\tilde{A}'(0) = i\frac{1}{2\rho+2}\lim_{z\to 0}\frac{A'}{Az^{2\rho+1}} = -i\delta$, hence

$$\sigma(T^{1+\rho})^2 = \sum_n \frac{1}{|t_n|^{2\rho+2}} \leqslant -\Im\tilde{A}'(0) = \delta$$

Now suppose $\lim_{y\to+\infty}\frac{\log|A(iy)|}{y^{2\rho+2}} = 0$. Let $A(z) = e^{-az^{2\rho+2}}\prod_{k=1}^{+\infty}P_k(z)$ be the usual factorization. Let $Q_n(z) := \prod_{k=1}^{n}P_k(z)$, it's easy to check $\frac{\log|A(iy)|}{y^{2\rho+2}}$ goes to 0 as y goes to $+\infty$, and the convergence is uniform in n. Hence the condition $\lim_{y\to+\infty}\frac{\log|A(iy)|}{y^{2\rho+2}} = 0$ implies $a = 0$, i.e. $A(z)$ is just the product of canonical factors, then by the proof to Problem 310 we can see the equality holds. □

Problem 316

Let $\mathcal{B}(E)$ be a given space such that $E(z)$ has no real zeros and $A(0) = 1$. Let T be the transformation

$$F(z) \mapsto \frac{F(z) - A(z)F(0)}{z}$$

in the space. If $T^{1+\rho}$ is of Schmidt class, show that $E(z) = S(z)E_0(z)$ where $S(z)$ is an entire function which is real for real z and has no zeros and $\mathcal{B}(E_0)$ is a space such that $A_0(z)$ belongs to the ρ-th Laguerre class. Show that $A_0(z)$ can be chosen so that

$$\lim_{y\to+\infty} \frac{\log|A_0(iy)|}{y^{2\rho+2}} = 0$$

Proof. By Problem 313 and 314, let $\{t_n\}$ be the zeros of A, then $\sum_{n=1}^{+\infty}\frac{1}{t_n^{2\rho+2}} < \infty$. Let $A_0(z) := \prod_{k=1}^{+\infty}P_k(z)$, then A_0 belongs to ρ-th Laguerre class and $\lim_{y\to+\infty}\frac{\log|A_0(iy)|}{y^{2\rho+2}} = 0$. Let $S = \frac{A}{A_0}$, it's real entire and zero-free by definition of A_0. Let $E_0 = \frac{E}{S}$, then E_0 is strict, non-degenerate dB, as E is strict and non-degenerate. □