# Walras' Tâtonnement in the Theory of Exchange*[1]

In Walras' theory of general equilibrium, an important role is played by the concept of *tâtonnement*. In spite of many contributions to the theory of *tâtonnement*,[2] there are still interesting problems which have not been satisfactorily solved. In the present paper, we intend to fill some of the gaps in that theory ; especially with regards to the stability problem of *tâtonnement* processes.

In [14], Walras first considers an economic system in which only exchange of commodities between the individuals takes place, and then proceeds to handle more complicated systems in which production of commodities or capital goods becomes possible. In any economic system, however, he shows by counting the numbers of the economic variables (unknowns) and the relations (equations) which prescribe those variables, that it is theoretically or mathematically possible to determine equilibrium values of the economic variables. He then shows that the problem of determination of equilibrium values of the economic variables is empirically, or in the market, solved by the *tâtonnement* process which represents the mechanism of the competitive market.

In an exchange economy, the competitive market process consists of a price adjustment by which the price of a commodity will rise or fall according to whether there is a positive excess demand or a positive excess supply of the commodity. Walras himself, however, does not make clear what is meant by his *tâtonnement* process. In particular, he has two distinct *tâtonnement* processes in mind : the one with *simultaneous* adjustment, and the other with *successive* adjustment, both with respect to prices of commodities. For example, the passages on pp. 170-2, [14], in which he attempts to prove the stability of the process, show that his process is one of successive price adjustments, while the summary on p. 172, [14], seems to suggest that his *tâtonnement* process is a simultaneous price adjustment.

Let us consider a competitive exchange economy with $n + 1$ commodities and $R$ participants. Commodities will be denoted by $i = 0, 1, \ldots, n$, while individual participants will be denoted by $r = 1, \ldots, R$. At the beginning of the market day, each individual has certain amounts of commodities, and during the market day the exchange of commodities between the individuals will take place. Let the amount of commodity $i$ initially held by individual $r$ be $y^r_i$, $i = 0, 1, \ldots, n$ ; $r = 1, \ldots, R$. Using vector notation, we may say that the vector of the initial holdings of individual $r$ is

$$y^r = (y^r_0, y^r_1, \ldots, y^r_n), \; r = 1, \ldots, R.$$

It will be assumed that each individual has a definite demand schedule when a market price vector and his income are given. Let the demand function of individual $r$ be

$$x^r(p, M^r) = [x^r_0(p, M^r), \ldots, x^r_n(p, M^r)],$$

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[1] The author is indebted to Professors K. J. Arrow, R. Dorfman, L. Hurwicz, and G. Tintner for their valuable comments and suggestions.

[2] The reader is in particular referred to an excellent note by Professor Patinkin for the most complete treatment of the theory ; [8], pp. 377-385.

where $p = (p_0, p_1, \ldots, p_n)$ is a market (accounting) price vector and $M^r$ an income of individual $r$. By a price vector $p$ is meant a non-zero vector with non-negative components.

Each demand function $x^r(p, M^r)$ is assumed to be continuous at any price vector $p$ and income $M^r$, and to satisfy the budget equation :

$$\sum_{i=0}^{n} p_i \, x_i^r(p, M^r) = M^r.$$

If a price vector $p = (p_0, p_1, \ldots, p_n)$ has been announced to prevail in the whole market, then the income $M^r(p)$ of individual $r$ is

$$M^r(p) = \sum_{i=0}^{n} p_i y_i^r, \; r = 1, \ldots, R.$$

Hence the demand function of individual $r$ becomes $x^r[p, M^r(p)]$, $r = 1, \ldots, R$. The aggregate function is now defined by

$$x_i(p) = \sum_{r=1}^{R} x_i^r[p, M^r(p)], \; i = 0, 1, \ldots, n,$$

and the excess demand function $z(p) = (z_0(p), \ldots, z_n(p))$ by

$$z_i(p) = x_i(p) - y_i$$

where

$$y_i = \sum_{r=1}^{R} y_i^r, \; i = 0, 1, \ldots, n.$$

The excess demand function $z(p)$ is *continuous* at any price vector $p$, and homogeneous of order zero.

It may also be noted that *the Walras law* holds :

$$(1) \qquad \sum_{i=0}^{n} p_i \, z_i(p) = 0, \quad \text{for any price vector } p = (p_0, p_1, \ldots, p_n).$$

A price vector $\bar{p} = (\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_n)$ is defined to be an *equilibrium price vector* if $z_i(\bar{p}) \leqq 0$, $i = 0, 1, \ldots, n$, which in view of the Walras Law (1), may be written,

$$z_i(\bar{p}) = 0, \text{ if } \bar{p}_i > 0,$$

$$z_i(\bar{p}) \leqq 0, \text{ if } \bar{p}_i = 0.$$

The main problem in the theory of exchange of commodities is now to investigate whether or not there exists an equilibrium, and if such exists, how one can determine an equilibrium. In what follows, we will rigorously show that the problem is solved by the Walras' *tâtonnement process*.[1]

[1] The existence of an equilibrium for the Walrasian model of general equilibrium was rigorously proved by, e.g., Wald [13] and Arrow and Debreu [2]. The competitive economy of exchange with which we are concerned in the present paper is a special case of the Arrow-Debreu model ; hence, their existence proof can be applied. We are, however, interested in applying the Walrasian *tâtonnement* process in order to give a proof of the existence of an equilibrium, which will be done in the Appendix.

Let us interpret the competitive exchange economy as a game which $R$ individuals and a fictitious player, say a Secretary of Market, play according to the following rules :

(i) Secretary of Market announces a price vector.

(ii) Each individual submits to Secretary of Market a "ticket" on which the quantities of demand and supply made by the individual according to the announced price vector are described.

(iii) Secretary of Market calculates the quantities of aggregate excess demands from the tickets submitted to him by the individuals.

(iv) Secretary of Market announces a new price vector such that prices of commodities which have a positive excess demand will rise, and prices of commodities which have a negative excess demand (a positive excess supply) will fall. Moves (ii) and (iii) are repeated at the new price system. The game is continued until Secretary of Market announces an equilibrium price vector.

Let $p(t) = (p_0(t), \ldots, p_n(t))$ be the price vector announced by Secretary of Market at the $t$-th stage, $t = 0, 1, 2, \ldots$, ad inf. Then the rule (iv) may be mathematically formulated as follows :[1]

$$p_i(t+1) = \max\{0, p_i(t) + f_i(t)\}, \qquad t = 0, 1, 2, \ldots; \quad i = 0, 1, \ldots, n, \tag{2}$$

where $f_i(t)$ has the same sign as $z_i[p(t)]$.

The *tâtonnement* process (2) leaves the price vector $p(t)$ invariant if and only if it is an equilibrium price vector.[2] The process (2), as will be shown in the Appendix, enables us to prove the existence of an equilibrium price vector $\bar{p} = (\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_n)$.

Now we take as a *numéraire* a commodity, say commodity 0, which has a positive price at an equilibrium, and suppose that prices are expressed in terms of the *numéraire* good, i.e., we consider only those price vectors $p = (p_0, p_1, \ldots, p_n)$ for which

$$p_0 = 1.$$

In what follows, by a price vector $p$ is meant an $n$-vector $(p_1, \ldots, p_n)$ with non-negative components.

The *simultaneous tâtonnement* process (2) accordingly may be reformulated by

$$p_i(t+1) = \max\{0, p_i(t) + f_i(t)\}, \qquad t = 0, 1, 2, \ldots; \quad i = 1, \ldots, n. \tag{3}$$

We now investigate the *stability problem*[3], i.e., whether or not the price vector $p(t)$ determined by the *tâtonnement* process (3) converges to an equilibrium price vector.

[1] It may be noted that the system of differential equations corresponding to (2) is given by

(2)′ $\dot{p}_i = \max \{0, p_i + f_i(p)\} - p_i, i = 0, 1, \ldots, n.$

The process of price adjustment represented by (2)′, is a special case of the ones discussed in Arrow, and Hurwicz [1], p. 94. The solution to the process (2)′, however, remains positive whenever the initial position is positive.

[2] See Walras [14], p. 172.

[3] For recent contributions to the stability problem, see, e.g., Arrow and Hurwicz [3] and Arrow, Block, and Hurwicz [1].

It is noted that the Lyapunov stability theorem[1] may be modified so as to be appropriately applied to our *tâtonnement* process ; namely we have the following :

STABILITY THEOREM.[2] *Let* $p(t;\, p^0)$ *be the solution to the process* (3) *with initial price vector* $p^0$. *If*

(*a*) *the solution* $p(t;\, p^0)$ *is bounded for any initial price vector* $p^0$; *and*

(*b*) *there exists a continuous function* $\Phi(p)$ *defined for all price vectors p such that* $\varphi(t) = \Phi\,[p(t;\, p^0)]$ *is strictly decreasing unless* $p(t;\, p^0)$ *is an equilibrium*;
*then the process is quasi-stable.*

Here the process (3) is called *quasi-stable* if, for any initial price vector $p^0$, every limiting-point of the solution $p(t;\, p^0)$ is an equilibrium. In the case in which the set of all equilibria is finite, the quasi-stability of the process (3) implies the global stability. The function $\Phi(p)$ satisfying the condition (*b*) will be referred to as a *Lyapunov function* with respect to the process (3).

Let us consider the case in which the excess demand function $z(p) = [z_0(p), z_1(p), \ldots, z_n(p)]$ satisfies the *weak axiom of revealed preference*[3] at equilibrium :

$$z_0(p) + \sum_{i=1}^{n} \bar{p}_i z_i\,(p) > 0, \tag{4}$$

for any equilibrium $\bar{p}$ and any nonequilibrium $p$.

Then *the simultaneous tâtonnement process* (3) *is globally stable, provided the equilibrium vector* $\bar{p}$ *is uniquely determined* ;

$$f_i(t) = \beta_i\,[z_i(p(t))\,],\ \beta_i > 0 \textit{ sufficiently small, } i = 1, \ldots, n;$$

*and* $z_i(p)$ *are continuously twice differentiable with a nonsingular matrix*

$$\left[\sum_{i=0}^{n} p_i \frac{\partial^2 z_i}{\partial p_j\, \partial p_k}\right]_{j,\,k} \quad \textit{at the equilibrium } \bar{p}.$$

In fact, as will be rigorously shown in the Appendix, the weak axiom of revealed preference implies that the function

$$\Phi(p) = \tfrac{1}{2} \sum_{i=1}^{n} (p_i - \bar{p}_i)^2 \tag{5}$$

is a Lyapounov function; and the Stability Theorem above may be applied.

The weak axiom of revealed preference at equilibrium is satisfied if, e.g., all the commodities are strongly gross substitutes[4] or there is a community utility function.

[1] We are concerned with the so called Lyapunov second method. See Lyapunov [6] or Malkin [7]. The first application of the Lyapunov second method to the economic analysis was done by Clower and Bushaw [4].

[2] The present Stability Theorem is a difference equation analogue of the one which was proved in Uzawa [12] for systems of differential equations. For the proof, see the Appendix.

[3] The weak axiom of revealed preference which was originally introduced by Samuelson [9] and Wald [13] is usually formulated as follows :
(4)′ For any two price vectors $p^1$ and $p^2$,
$$p^1\, z(p^1) \geqq p^1\, z(p^2),\ z(p^1) \neq z(p^2) \text{ imply } p^2\, z(p^1) > p^2\, z(p^2).$$
It is easily seen that (4)′ implies (4).

[4] See Arrow, Block, and Hurwicz [1], Lemma 5, p. 90.

We shall next be concerned with the *successive tâtonnement* process which was introduced by Walras in [14], pp. 170-2. The process under consideration consists of a price adjustment which successively clears the markets of commodities.

Let $p(0) = [p_1(0), \ldots, p_n(0)]$ be a price vector initially announced. Secretary of Market first pays attention to the market of commodity 1, and determines the price $p_1(1)$ of commodity 1 at the next stage so that there will be no excess demand at price vector $[p_1(1), p_2(0), \ldots, p_n(0)]$. In other words, he solves the equation

$$z_1[p_1, p_2(0), \ldots, p_n(0)] = 0,$$

with respect to $p_1$. He then goes on to consider the market of commodity 2 and determines the price $p_2(1)$ of commodity 2 by solving the equation

$$z_2[p_1(1), p_2, p_3(0), \ldots, p_n(0)] = 0,$$

with respect to $p_2$. The prices $p_3(1), \ldots, p_n(1)$ are determined successively.

In general, the price $p_j(t+1)$ of commodity $j$ at stage $t+1$ is determined so as to satisfy the equation

$$z_j[p_1(t+1), \ldots, p_{j-1}(t+1), p_j(t+1), p_{j+1}(t), \ldots, p_n(t))] = 0,$$
$$t = 0, 1, 2, \ldots; j = 1, \ldots, n.$$

In taking into consideration the requirement that prices should be non-negative, we may modify the above process as follows :

*$p_j(t+1)$ is determined so as to satisfy the relation that*[1]

$$z_j[p_1(t+1), \ldots, p_j(t+1), p_{j+1}(t), \ldots, p_n(t)] \begin{cases} = 0, \text{ if } p_j(t+1) > 0, \\ \leqq 0, \text{ if } p_j(t+1) = 0, \end{cases} \tag{6}$$
$$t = 0, 1, 2, \ldots; j = 1, \ldots, n.$$

In order for the successive *tâtonnement* process to be possible, it is required that, for any non-negative $p_1(t+1), \ldots, p_{j-1}(t+1), p_{j+1}(t), \ldots, p_n(t)$, the relation (6) should have a unique non-negative solution $p_j(t+1)$.

It is evident that a price vector $p(t) = [p_1(t), \ldots, p_n(t)]$ is left unchanged by the successive *tâtonnement* process if and only if $p(t)$ is an equilibrium price vector.

We shall now consider the stability of the process defined by (6). The Stability Theorem remains valid for the successive *tâtonnement* process (6).

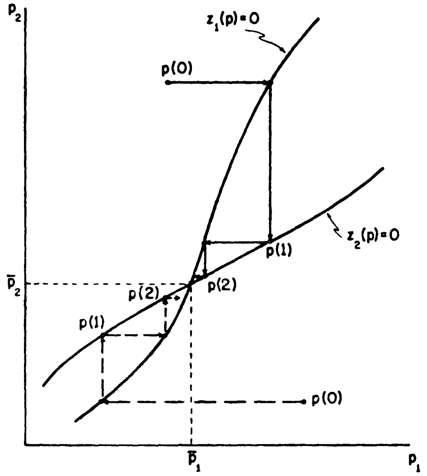
Let us first consider the case in which all commodities are strongly gross substitutes; i.e., for any commodity $j$, including the *numéraire*, the excess demand function $z_j(p_1, \ldots, p_n)$ is a strictly increasing continuous function of prices of commodities other than $j$. In this case, the equilibrium price vector $\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)$ is uniquely determined and is positive.

We have the following theorem :

*If all commodities are strongly gross substitutes, the successive tâtonnement process* (6) *is globally stable.*

[1] The system of difference equations (6) defines an iterative method for solving systems of equations which is known as the Gauss-Seidel method in numerical analysis. The case handled by Seidel [11] is the one in which $z(p)$ is linear, $[\partial z_i/\partial p_j]_{i,j}$ is symmetrical, and $\partial z_i/\partial p_i$ are all positive. Another formulation of the successive *tâtonnement* process was discussed by Professor M. Morishima in an unpublished paper.

For the case : $n = 2$, the global stability of the process (6) may be easily seen from the Figure. The general case will be proved in the Appendix.




We next consider the stability of the process (15) for the case in which the followin g two conditions are satisfied :

(7) For any fixed $p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n$, the equation

$$z_j(p_1, \ldots, p_{j-1}, p_j, p_{j+1}, \ldots, p_n) = 0$$

has at most one non-negative solution with respect to $p_j$, and

$$z_j(p_1, \ldots, p_{j-1}, p_j^*, p_{j+1}, \ldots, p_n) < 0, \quad \text{for sufficiently large } p_j^*.$$

(8) There is a differentiable function $\Phi(p)$ defined for all positive price vectors $p$ such that, for some *positive* functions $\lambda_1(p), \ldots, \lambda_n(p)$,

$$\frac{\partial \Phi}{\partial p_j} = - \lambda_j(p)\, z_j(p), \quad j = 1, \ldots, n.$$

The condition (8) is satisfied if, e.g., there is a community utility function or the excess demand function has a symmetric matrix of partial derivatives.

In this case, *the successive tâtonnement process* (6) *is quasi-stable*. The function $\Phi(p)$ satisfying the condition (8) becomes, as will be shown in the Appendix, a Lyapunov function for the process (6).

We thirdly consider the *two-commodity* case in which the condition (7) is satisfied. *The successive tâtonnement process* (6) *then is trivially stable*. In fact, the relation (6) for $t = 0$ is reduced to

$$z_1[p_1(1)] = 0,$$

which, by the Walras law, implies that $p_1(1)$ is an equilibrium price of commodity 1.

It may be finally noted that the above three cases, together with the dominant diagonal case, exhaust the cases essentially for which the stability of the Samuelson process—a differential equation formulation of the successive *tâtonnement* process—is known.[1]

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[1] See Samuelson [10], part II; Arrow and Hurwicz [3]; Arrow, Block, and Hurwicz [1].

## APPENDIX

### 1. *Proof of the Existence Theorem*

We first normalize price vectors such that the sum of prices of all commodities is one. Let $P$ be the set of all price vectors thus normalized :

$$P = \{p = (p_0, p_1, \ldots, p_n): p_i \geqq 0, i = 0, 1, \ldots, n, \text{ and } \sum_{i=0}^{n} p_i = 1\}$$

The simultaneous *tâtonnement* process, induced on the set $P$ of normalized price vectors, defines the following mapping $p \longrightarrow T(p) = (T_0(p), \ldots, T_n(p))$ :

$$(9) \qquad T_i(p) = \frac{1}{\lambda(p)} \max \{0, p_i + \beta\, z_i(p)\}, \quad i = 0, 1, \ldots, n, \ p \in P,$$

where $\beta$ is a positive number and

$$\lambda(p) = \sum_{i=0}^{n} \max \{0, p_i + \beta\, z_i(p)\}.$$

It is evident that

$$\lambda(p) > 0,$$

and $T(p)$ is a continuous mapping from $P$ into itself. Hence, by applying the Brouwer fixed-point theorem,[1] we know that there exists a normalized price vector $\bar{p} = (\bar{p}_0, \ldots, \bar{p}_n)$ such that

$$\bar{p} = T(\bar{p}).$$

which, in view of (9), may be written

$$(10) \qquad \lambda(\bar{p})\bar{p}_i = \max \{0, \bar{p}_i + \beta\, z_i(\bar{p})\}, \quad i = 0, 1, \ldots, n.$$

Multiplying (10) by $p_i$ and summing over $i = 0, 1, \ldots, n$, we get

$$\sum_{i=0} \lambda(\bar{p})\, \bar{p}_i^2 = \sum_{i=0}^{n} \bar{p}_i^2 + \sum_{i=0}^{n} \bar{p}_i\, z_i(\bar{p}),$$

which, by the Walras law, implies

$$\lambda(\bar{p}) = 1.$$

Hence, $\bar{p}$ is an equilibrium price vector. Q.E.D.

[1] Cf., e.g., Lefschetz [5], p. 117.

## 2. *Proof of the Stability Theorem*

Let $p(t)$ be the solution to the process (3) with an initial price vector $p(0)$ which is arbitrarily given, and let $p^*$ be any limit-point of $p(t)$, as $t$ tends to infinity; i.e.

$$p^* = \lim_{\nu \longrightarrow \infty} p(t_\nu),$$

for some subsequence $\{t_\nu\}$. Consider the solution $p^*(t)$ to the process (3) with initial price vector $p^*$ and define the function $\varphi^*(t)$ by

$$\varphi^*(t) = \Phi[p^*(t)].$$

Since $\varphi(t) = \Phi[p(t)]$ is nonincreasing and $\{p(t)\}$ is bounded, $\lim_{t \longrightarrow \infty} \varphi(t)$ exists, and is equal to, say, $\varphi^*$ :

$$\varphi^* = \lim_{t \longrightarrow \infty} \varphi(t).$$

On the other hand, since the solution $p[t; p(0)]$ to the system (3) is continuous and unique with respect to initial position $p(0)$, we have

$$p^*(t) = p(t; p^*) = \lim_{\nu \longrightarrow \infty} p[t; p(t_\nu)]$$

$$= \lim_{\nu \longrightarrow \infty} p[t + t_\nu; p(0)].$$

Hence, we have

$$\varphi^*(t) = \Phi(p^*(t)) = \lim_{\nu \longrightarrow \infty} \Phi(p(t+t_\nu)) = \lim_{\nu \longrightarrow \infty} \varphi(t+t_\nu) = \varphi^*, \text{ for all } t,$$

which, by the condition ($b$), shows that $p^* = p^*(0)$ is an equilibrium. Q.E.D.

## 3. *Stability of the Simultaneous Tâtonnement Process: The Weak Axiom Case*

We may without loss of generality assume that $f_i(t) = \beta z_i(p(t))$, $i = 1, \ldots, n$, and $\beta$ is a sufficiently small positive number.

For the solution $p(t) = p[t; p(0)]$ to the system (3) with an arbitrarily given initial price vector $p(0)$, we consider the function $\varphi(t)$ defined by

$$\varphi(t) = \Phi[p(t)], \quad t = 0, 1, 2, \ldots .$$

We first show that

$$\varphi(t+1) \leqq \varphi(t) - \beta \left\{ 2[z_0(t) + \sum_{i=1}^{n} \bar{p}_i z_i(t)] - \beta \sum_{i=1}^{n} z_i^2(t) \right\}, \tag{11}$$

where $z(t) = z[p(t)]$.

From the relation (3), we have

$$(12) \qquad p_i^2(t+1) \leqq [p_i(t) + \beta\, z_i(t)]^2 = p_i^2(t) + 2\,\beta\, p_i(t)\, z_i(t) + \beta^2 z_i^2(t), \qquad i = 1, \ldots, n.$$

Summing (12) over $i = 1, \ldots, n$, we get

$$(13) \qquad \sum_{i=1}^{n} p_i^2(t+1) \leqq \sum_{i=1}^{n} p_i^2(t) + 2\beta \sum_{i=1}^{n} p_i(t)\, z_i(t) + \beta^2 \sum_{i=1}^{n} z_i^2(t).$$

Substituting the Walras law (1) into (13), we have

$$(14) \qquad \sum_{i=1}^{n} p_i^2(t+1) \leqq \sum_{i=1}^{n} p_i^2(t) - 2\beta\, z_0(t) + \beta^2 \sum_{i=1}^{n} z_i^2(t).$$

On the other hand, multiplying (3) by $\bar{p}_i$ and summing over $i = 1, \ldots, n$, we get

$$(15) \qquad \sum_{i=1}^{n} \bar{p}_i p_i(t+1) \geqq \sum_{i=1}^{n} \bar{p}_i p_i(t) + \beta \sum_{i=1}^{n} \bar{p}_i\, z_i(t).$$

Subtracting two times (15) from (14) and adding $\sum_{i=1}^{n} \bar{p}_i^2$, we can derive the inequality (11).

Let $S$ be the set of the commodities such that, at equilibrium $\bar{p}$, they have negative excess demand :

$$S = \{i : z_i(\bar{p}) < 0\}.$$

Equilibrium prices of the commodities in $S$ are zero :

$$(16) \qquad \bar{p}_i = 0, \text{ for } i \in S.$$

We shall now show that, if $\beta$ is a sufficiently small positive number, there exists $t_0$ such that

$$p_i(t) = 0, \text{ for all } t \geqq t_0,\ i \in S.$$

Let $\varepsilon$ be a positive number such that

$$(17) \qquad z_i(p) \leqq -c < 0,$$

for all $i \in S$ and any price vector $p$ such that $\Phi(p) \leqq \varepsilon$, where $c$ is a positive number. Since the excess demand function $z(p)$ is continuous, it is possible to choose a positive number $\varepsilon$ which satisfies (17). We may assume without loss of generality that

$$\varepsilon \leqq \varphi(0) = \Phi[p(0)].$$

Let $\beta$ be any positive number smaller than the following two numbers :

$$(18) \qquad \inf_{\Phi(p) \leqq \frac{\varepsilon}{2}} \sqrt{\frac{\varepsilon/2}{\sum_{i=1}^{n} z_i^2(p)}}, \qquad \inf_{\frac{\varepsilon}{2} \leqq \Phi(p) \leqq \Phi[p(0)]} \frac{2[z_0(p) + \sum_{i=1}^{n} p_i\, z_i(p)]}{\sum_{i=1}^{n} z_i^2(p)}.$$

The weak axiom (4) and the continuity of the excess demand function $z(p)$ now imply that the two numbers defined by (18) are *positive*, so that the choice of $\beta$ is possible. Then, by the inequality (11), we have

$$(19) \qquad \varphi(t+1) \leqq \varphi(t) + \frac{\varepsilon}{2} \leqq \varepsilon, \quad \text{if } \varphi(t) \leqq \frac{\varepsilon}{2},$$

and

$$\varphi(t+1) < \varphi(t), \text{ if } \frac{\varepsilon}{2} < \varphi(t) \leqq \varphi(0), \qquad t = 0, 1, 2, \ldots \tag{20}$$

The inequalities (19) and (20) imply that there exists $t_1$ such that

$$\varphi(t) \leqq \varepsilon, \text{ for any } t \geqq t_1. \tag{21}$$

Otherwise, $\varphi(t)$ is strictly decreasing and the solution $p(t)$ is bounded, while no limit-point of $p(t)$ is the equilibrium, thus contradicting the Stability Theorem.

The relation (3) together with (17) and (21) implies that there exists $t_0$ for which (16) is satisfied.

The relation (16) and an argument similar to the one by which we derived (11) lead to the following :

$$\varphi(t+1) \leqq \varphi(t) - \beta \left\{ 2[z_0(t) + \sum_{i=1}^{n} \bar{p}_i z_i(t)] - \beta \sum_{i \notin S} z_i^2(t) \right\}. \tag{22}$$

We shall now show that

$$\inf_{0 < \Phi(p) \leqq \frac{\varepsilon}{2}} \frac{[z_0(p) + \sum_{i=1}^{n} \bar{p}_i z_i(p)]}{\sum_{i \notin S} z_i^2(p)} \tag{23}$$

is *positive.*

Expanding the numerator and the denominator of (23) in the Taylor series at the equilibrium $\bar{p}$, and noting the Walras law (1), we get

$$z_0(p) + \sum_{i=1}^{n} \bar{p}_i z_i(p) = \sum_{j,k=1}^{n} a_{jk}(p_j - \bar{p}_j)(p_k - \bar{p}_k) + 0[\Phi(p)],$$

$$\sum_{i \notin S} z_i^2(p) = \sum_{j,k=1}^{n} b_{jk}(p_j - \bar{p}_j)(p_k - \bar{p}_k) + 0[\Phi(p)],$$

where

$$a_{jk} = \left( \frac{\partial^2 z_0}{\partial p_j \, \partial p_k} + \sum_{i=1}^{n} \bar{p}_i \frac{\partial^2 z_i}{\partial p_j \, \partial p_k} \right)_{\bar{p}},$$

$$b_{jk} = \left( \sum_{i \notin S} \frac{\partial z_i}{\partial p_j} \frac{\partial z_i}{\partial p_k} \right)_{\bar{p}}, \qquad j, k = 1, \ldots, n,$$

and $0[\Phi(p)]$ correspond to terms such that

$$\lim_{\Phi(p) \longrightarrow 0} \frac{0[\Phi(p)]}{\Phi(p)} = 0.$$

The weak axiom together with the non-singularity of $(a_{jk})_{j,\,k\,=\,1,\,\ldots,\,n}$ implies, however, that the matrix $(a_{jk})_{j,\,k\,=\,1,\,\ldots,\,n}$ is *positive definite.* Hence

$$\lim_{\Phi(p) \longrightarrow 0} \frac{z_0(p) + \sum_{i=1}^{n} \bar{p}_i z_i(p)}{\sum_{i \notin S} z_i^2(p)} > 0.$$

which implies that the number (23) is positive.

Now let $\beta$ be any positive number smaller than the numbeis defined by (18) and (23). Then, by the relation (22), we have that

$$\varphi(t+1) < \varphi(t),$$

whenever $p(t)$ is not an equilibrium.

Applying the Stability Theorem, we have the stability of the process (3). Q.E.D.

4. *Stability of the Successive Tâtonnement Process: The Gross Substitute Case*

Consider the functions $\Lambda(p)$ and $\lambda(p)$ defined by

$$\Lambda(p) = \max\left\{1, \frac{p_1}{\bar{p}_1}, \ldots, \frac{p_n}{\bar{p}_n}\right\}, \tag{24}$$

$$\lambda(p) = \min\left\{1, \frac{p_1}{\bar{p}_1}, \ldots, \frac{p_n}{\bar{p}_n}\right\}, \tag{25}$$

We shall show that, for any solution $p(t)$ to the process (6),

$$\Lambda[p(t+1)] \leqq \Lambda[p(t)], \tag{26}$$

with strict inequality unless $\Lambda[p(t)] = 1$,

and

$$\lambda[p(t+1)] \geqq \lambda[(t)], \tag{27}$$

with strict inequality unless $\lambda[p(t)] = 1$.

Since the relation (27) is proved similarly to (26), we shall give a proof for (26).

In order to prove (26), it suffices to prove the following :

$$\begin{aligned} \Lambda[p_1(t+1), \ldots, p_j(t+1), p_{j+1}(t), \ldots . p_n(t)] \\ \leqq \Lambda[p_1(t+1), \ldots, p_{j-1}(t+1), p_j(t), \ldots, p_n(t)], \end{aligned} \tag{28}$$

with strict inequality unless $\Lambda[p_1(t+1), \ldots, p_{j-1}(t+1), p_j(t), \ldots, p_n(t)] = 1$,
$t = 0, 1, 2\ ;\ j = 1, \ldots, n.$

Gross substitutability, homogeneity of order zero, and the Walras law (1) imply that[1]

$$z_j[p_1(t+1), \ldots, p_{j-1}(t+1), p_j, p_{j+1}(t), \ldots, p_n(t)] \leqq 0, \tag{29}$$

whenever

$$\frac{p_j}{\bar{p}_j} \geqq \Lambda[p_1(t+1), \ldots, p_{j-1}(t+1), p_j(t), \ldots, p_n(t)],$$

[1] See Arrow, Block, and Hurwicz [1], Lemma 3, p. 89.

and, if $\Lambda[p_1(t+1), \ldots, p_{j-1}(t+1), p_j(t), \ldots, p_n(t)] > 1$, (29) holds with strict inequality. But, by gross substitutability and the Walras Law (1), $z_j[p_1(t+1), \ldots, p_{j-1}(t+1), p_j, p_{j+1}(t), \ldots, p_n(t)]$ is strictly decreasing with respect to $p_j$. Hence, (29) implies that

$$\text{(30)} \qquad \frac{p_j(t+1)}{\bar{p}_j} \leqq \Lambda[p_1(t+1), \ldots, p_{j-1}(t+1), p_j(t), \ldots, p_n(t)],$$

with strict inequality if $\Lambda[p_1(t+1), \ldots, p_{j-1}(t+1), p_j(t), \ldots, p_n(t)] > 1$.

The inequality (28) is easily implied by (30).

The relations (26) and (27) in particular show that the solution $p(t)$ to the process (6) is in a bounded set

$$\left\{ p = (p_1, \ldots, p_n) : \lambda(0) \leqq \frac{p_j}{\bar{p}_j} \leqq \Lambda(0), \quad j = 1, \ldots, n \right\}$$

of positive price vectors.

Since $\Lambda(p) = \lambda(p) = 1$ if and only if $p$ is an equilibrium, the relations (26) and (27) show that the function $\Phi(p)$ defined by

$$\Phi(p) = \Lambda(p) - \lambda(p),$$

is a Lyapanov function for the process (6). Q.E.D.

5. *Stability of the Successive Tâtonnement Process; The Quasi-Integrable Case*

Let $\Phi(p)$ be the function satisfying (8) and $\psi_j(p_j)$ be the function defined by

$$\text{(31)} \qquad \psi_j(p_j) = \Phi[p_1(t+1), \ldots, p_{j-1}(t+1), p_j, p_{j+1}(t), \ldots, p_n(t)].$$

The defining relation (6), together with (8), shows that $p_j(t+1)$ is a solution to the equation

$$\text{(32)} \qquad \psi_j(p_j) \begin{cases} = 0, & \text{if } p_j > 0, \\ \geqq 0, & \text{if } p_j = 0. \end{cases}$$

The relations (7) and (32) show that $p_j(t+1)$ uniquely minimizes $\psi_j(p_j)$ subject to $p_j \geqq 0$. Hence, in particular, we have

$$\text{(33)} \qquad \psi_j[p_j(t+1)] \leqq \psi_j[p_j(t)],$$

with strict inequality if $p_j(t+1) \neq p_j(t)$.

By (31), we may rewrite (33) as follows :

$$\text{(34)} \qquad \begin{aligned} &\Phi[p_1(t+1), \ldots, p_j(t+1), p_{j+1}(t), \ldots, p_n(t)] \\ &\qquad \leqq \Phi[p_1(t+1), \ldots, p_{j-1}(t+1), p_j(t), \ldots, p_n(t)], \end{aligned}$$

with strict inequality if $p_j(t+1) \neq p_j(t)$.

By summing (34) over $j = 1, \ldots, n$, we get

$$\Phi[p(t+1)] \leqq \Phi[p(t)],$$

with strict inequality if $p(t+1) \neq p(t)$.

Hence, the function $\Phi(p)$ is a Lyapounov function ; and applying the Stability Theorem, we know that the process (6) is quasi-stable in the present case.

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