

# Hall Algebras - Bonn, Wintersemester 14/15

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# 1 Introduction

We outline the original context in which Hall algebras first appeared. Let  $p$  be a prime number, and let  $M$  be a finite abelian  $p$ -group. By the classification theorem for finitely generated abelian groups the group  $M$  decomposes into a direct sum of cyclic  $p$ -groups. Therefore, we have

$$M \cong \bigoplus_{i=1}^r \mathbb{Z}/(p^{\lambda_i})$$

where we may assume  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r$  so that the sequence

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r, 0, \dots)$$

is a *partition*, i.e., a weakly decreasing sequence of natural numbers with finitely many nonzero components. We call the partition  $\lambda$  the *type* of  $M$ . The association

$$M \mapsto \text{type of } M$$

provides a bijective correspondence between isomorphism classes of finite abelian  $p$ -groups and partitions.

Given partitions  $\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(s)}, \lambda$ , we define

$$g_{\mu^{(1)}\mu^{(2)}\dots\mu^{(s)}}^\lambda(p)$$

to be the number of flags

$$M = M_0 \supset M_1 \supset \dots \supset M_s \supset M_{s+1} = 0$$

such that  $M_{i-1}/M_i$  has type  $\mu^{(i)}$  where  $M$  is a fixed group of type  $\lambda$ . In this context, Philip Hall had the following insight:

**Theorem 1.1.** The numbers  $g_{\mu\nu}^\lambda(p)$  form the structure constants of a unital associative algebra with basis  $\{u_\lambda\}$  labelled by the set of all partitions. More precisely, the  $\mathbb{Z}$ -linear extension of the formula

$$u_\mu u_\nu = \sum_\lambda g_{\mu\nu}^\lambda(p) u_\lambda$$

defines a unital associative multiplication on the abelian group  $\bigoplus_\lambda \mathbb{Z}u_\lambda$ . Further, we have

$$u_{\mu^{(1)}} u_{\mu^{(2)}} \dots u_{\mu^{(s)}} = g_{\mu^{(1)}\mu^{(2)}\dots\mu^{(s)}}^\lambda(p) u_\lambda.$$

The resulting associative algebra is called *Hall's algebra of partitions*.

We compute some examples of products  $u_\mu u_\nu$ . Fixing an abelian  $p$ -group  $M$  of type  $\lambda$ , the number  $g_{\mu\nu}^\lambda(p)$  is the number of subgroups  $N \subset M$  such that  $N$  has type  $\nu$  and  $M/N$  has type  $\mu$ . In particular, we obtain that  $g_{\mu\nu}^\lambda(p)$  is nonzero if and only if  $M$  is an extension of a  $p$ -group  $N'$  of type  $\mu$  by a  $p$ -group  $N$  of type  $\nu$ .

(1) We compute

$$u_{(1)}u_{(1)} = g_{(1)(1)}^{(1,1)}(p)u_{(1,1)} + g_{(1)(1)}^{(2)}(p)u_{(2)}.$$

Further,  $g_{(1)(1)}^{(1,1)}(p)$  is the number of subgroups

$$N \subset M = \mathbb{Z}/(p) \oplus \mathbb{Z}/(p)$$

such that  $N \cong \mathbb{Z}/(p)$  and  $M/N \cong \mathbb{Z}/(p)$ . This coincides with the number of 1-dimensional subspaces in the  $\mathbb{F}_p$ -vector space  $(\mathbb{F}_p)^2$  (here  $\mathbb{F}_p$  denotes the field with  $p$  elements) of which there are  $p+1$ . The number  $g_{(1)(1)}^{(2)}(p)$  is the number of subgroups

$$N \subset M = \mathbb{Z}/(p^2)$$

such that  $N \cong \mathbb{Z}/(p)$  and  $M/N \cong \mathbb{Z}/(p)$ . Any such  $N$  must lie in the  $p$ -torsion subgroup of  $M$  which is  $p\mathbb{Z}/(p^2)$ . But  $p\mathbb{Z}/(p^2) \cong \mathbb{Z}/(p)$  and so  $N = p\mathbb{Z}/(p^2)$  which implies  $g_{(1)(1)}^{(2)}(p) = 1$ . In conclusion, we have

$$u_{(1)}u_{(1)} = (p+1)u_{(1,1)} + u_{(2)}.$$

(2) We compute

$$u_{(1,1)}u_{(1)} = g_{(1,1)(1)}^{(1,1,1)}(p)u_{(1,1,1)} + g_{(1,1)(1)}^{(2,1)}(p)u_{(2,1)}.$$

To compute  $g_{(1,1)(1)}^{(1,1,1)}(p)$  we have to determine the number of subgroups

$$N \subset M = \mathbb{Z}/(p) \oplus \mathbb{Z}/(p) \oplus \mathbb{Z}/(p)$$

such that  $N \cong \mathbb{Z}/(p)$  and  $M/N \cong \mathbb{Z}/(p) \oplus \mathbb{Z}/(p)$ . This is equivalent to counting 1-dimensional subspaces of  $\mathbb{F}_p^3$  of which there are  $p^2 + p + 1$ . Further, the number  $g_{(1,1)(1)}^{(2,1)}(p)$  is the number of subgroups

$$N \subset M = \mathbb{Z}/(p^2) \oplus \mathbb{Z}/(p)$$

such that  $N \cong \mathbb{Z}/(p)$  and  $M/N \cong \mathbb{Z}/(p) \oplus \mathbb{Z}/(p)$ . As above, the subgroup  $N$  must be contained in the  $p$ -torsion subgroup of  $M$  which is  $p\mathbb{Z}/(p^2) \oplus \mathbb{Z}/(p)$ . There are  $p+1$  such subgroups, but only one of them satisfies the condition  $M/N \cong \mathbb{Z}/(p) \oplus \mathbb{Z}/(p)$ :  $N = p\mathbb{Z}/(p^2)$ , contained in the first summand of  $M$ . Therefore, we have  $g_{(1,1)(1)}^{(2,1)}(p) = 1$  so that

$$u_{(1,1)}u_{(1)} = (p^2 + p + 1)u_{(1,1,1)} + u_{(2,1)}.$$

Note that, in the above examples, the structure constants  $g_{\mu\nu}^\lambda(p)$  are polynomial in  $p$ . Hall showed that this is true in general, and further, that the resulting polynomials have a very interesting relation to a class of symmetric functions called *Schur functions* which we will introduce later.

**Theorem 1.2.** (1) The numbers  $g_{\mu^{(1)}\mu^{(2)}\dots\mu^{(s)}}^\lambda(p)$  are polynomial in  $p$ .

(2) The leading terms  $c_{\mu\nu}^\lambda$  of the polynomials  $g_{\mu\nu}^\lambda(t)$  are the structure constants for the multiplication of Schur functions

$$s_\mu s_\nu = \sum_{\lambda} c_{\mu\nu}^\lambda s_\lambda.$$

The first goal of this course will be to prove Hall's results. The statement of Theorem 1.1 holds in a context more general than abelian  $p$ -groups which we will introduce next.

## 2 Proto-abelian categories and Hall algebras

### 2.1 Categorical preliminaries

Let  $\mathcal{C}$  be a category. A morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$  is called a *monomorphism* (or *monic*) if, for every pair of morphisms  $g : A \rightarrow X$ ,  $h : A \rightarrow X$ , we have

$$f \circ g = f \circ h \quad \Rightarrow \quad g = h.$$

We use the symbol  $\hookrightarrow$  to denote monomorphisms. A morphism  $f : X \rightarrow Y$  in  $\mathcal{C}$  is called an *epimorphism* (or *epic*) if, for every pair of morphisms  $g : Y \rightarrow A$ ,  $h : Y \rightarrow A$ , we have

$$g \circ f = h \circ f \quad \Rightarrow \quad g = h.$$

We use the symbol  $\twoheadrightarrow$  to denote epimorphisms. These notions are dual in the following sense: Introduce the opposite category  $\mathcal{C}^{\text{op}}$  with the same objects as  $\mathcal{C}$  but, for every pair of objects  $X, Y$ ,

$$\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$$

equipped with the apparent composition law. Then a morphism  $f : X \rightarrow Y$  is a monomorphism in  $\mathcal{C}$  if and only if the corresponding morphism in  $\mathcal{C}^{\text{op}}$  (from  $Y$  to  $X$ ) is an epimorphism.

**Example 2.1.** A morphism in the category of sets is a monomorphism (resp. epimorphism) if and only if it is injective (resp. surjective).

**Problem 2.2.** Show that the morphism  $\mathbb{Z} \rightarrow \mathbb{Q}$  in the category of rings is both a monomorphism and an epimorphism.

An object  $\emptyset$  in  $\mathcal{C}$  is called *initial* if, for every object  $X$ , there is a unique morphism

$$\emptyset \rightarrow X.$$

Dually, an object  $*$  in  $\mathcal{C}$  is called *final* if, for every object  $X$ , there is a unique morphism

$$X \rightarrow *.$$

An object is initial in  $\mathcal{C}$  if and only if it is final in  $\mathcal{C}^{\text{op}}$ . An object  $0$  of  $\mathcal{C}$  is called a *zero object* if it is both initial and final. The category  $\mathcal{C}$  is called *pointed* if it has a zero object. This is equivalent to the requirement that  $\mathcal{C}$  has an initial object  $\emptyset$ , a final object  $*$ , and the unique map

$$\emptyset \rightarrow *$$

is an isomorphism.

**Example 2.3.** The category of sets has an initial object given by the empty set. Any set with one element is a final object. The category of sets is therefore not pointed. The category of abelian groups is pointed since the zero group is both initial and final.

A commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow f \\ C & \xrightarrow{g} & D \end{array}$$

in  $\mathcal{C}$  is called *Cartesian* (or *pullback square*) if, for every pair of morphisms  $p : X \rightarrow B$ ,  $q : X \rightarrow C$  such that  $g \circ q = f \circ p$ , the dashed arrow in the diagram

$$\begin{array}{ccccc} X & & & & \\ & \searrow p & & & \\ & & A & \longrightarrow & B \\ & \swarrow q & \downarrow & & \downarrow f \\ & & C & \xrightarrow{g} & D \end{array}$$

can be filled in uniquely to make the diagram commute. Dually, a commutative square

$$\begin{array}{ccc} A & \xrightarrow{g'} & B \\ \downarrow f' & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in  $\mathcal{C}$  is called *coCartesian* (or *pushout square*) if, for every pair of morphisms  $p : B \rightarrow X$ ,  $q : C \rightarrow X$  such that  $q \circ f' = p \circ g'$ , the dashed arrow in the diagram

$$\begin{array}{ccccc} A & \xrightarrow{g'} & B & & \\ \downarrow f' & & \downarrow & & \searrow p \\ C & \longrightarrow & D & & \\ & & & \swarrow q & \\ & & & & X \end{array}$$

can be filled in uniquely to make the diagram commute.

**Proposition 2.4.** Let

$$\begin{array}{ccc} A & \xrightarrow{g'} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{g} & D \end{array}$$

be a pullback square in  $\mathcal{C}$  and assume that  $g$  is monic. Then  $g'$  is monic.

*Proof.* Follows in a straightforward way from the above definitions. □

We formulate the statement of the above proposition as: monomorphisms are stable under pullback. The dual argument, i.e., the same argument applied to  $\mathcal{C}^{\text{op}}$ , implies that epimorphisms are stable under pushout.

**Problem 2.5.** Give an example of a pushout square

$$\begin{array}{ccc} A & \xrightarrow{g'} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{g} & D \end{array}$$

in a category  $\mathcal{C}$  such that  $g'$  is monic but  $g$  is not monic, showing that monomorphisms are *not*, in general, stable under pushout. *Hint:* Consider the category of rings.

Let

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} \quad (2.6)$$

be a commutative square in a category  $\mathcal{C}$ . Assume that (2.6) is a pushout square. Then the defining universal property implies that the object  $D$  is uniquely determined (up to isomorphism) by the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

and we write  $D = B \amalg_A C$ . In the case when  $A$  is initial, we write  $D = B \amalg C$  and call  $D$  a *coproduct of  $B$  and  $C$* . Dually, assume that (2.6) is a pullback square. Then the object  $A$  is uniquely determined (up to isomorphism) by the diagram

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ C & \longrightarrow & D \end{array}$$

and we write  $A = B \times_D C$ . If  $D$  is final, we write  $A = B \times C$  and call  $A$  a *product of  $B$  and  $C$* .

Let

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & C \end{array} \quad (2.7)$$

be a commutative square in a pointed category  $\mathcal{C}$  where  $0$  denotes a zero object. If (2.7) is a pushout square then we call  $g$  (or sometimes  $C$ ) a *cokernel of  $f$* . If (2.7) is a pullback square then we call  $f$  (or sometimes  $A$ ) a *kernel of  $f$* .

## 2.2 Proto-abelian categories

Our goal in this section will be to introduce a certain class of categories to which Hall's associative multiplication law can be generalized.

**Definition 2.8.** A category  $\mathcal{C}$  is called *proto-abelian* if the following conditions hold.

- (1) The category  $\mathcal{C}$  is pointed.
- (2) (a) Every diagram in  $\mathcal{C}$  of the form

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

can be completed to a pushout square of the form

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ C & \hookrightarrow & D. \end{array}$$

- (b) Every diagram in  $\mathcal{C}$  of the form

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ C & \hookrightarrow & D \end{array}$$

can be completed to a pullback square of the form

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ C & \hookrightarrow & D. \end{array}$$

- (3) A commutative square in  $\mathcal{C}$  of the form

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ C & \hookrightarrow & D \end{array}$$

is a pushout square if and only if it is a pullback square.

**Proposition 2.9.** Let  $\mathcal{C}$  be a proto-abelian category.

- (1) The opposite category  $\mathcal{C}^{\text{op}}$  is proto-abelian.
- (2) Let  $I$  be a small category. The category  $\text{Fun}(I, \mathcal{C})$  of  $I$ -diagrams in  $\mathcal{C}$  is proto-abelian.

*Proof.* (1) is immediate, since in Definition 2.8, Conditions (2)(i) and (2)(ii) are dual, and Conditions (1) and (3) are self-dual. (2) follows since all conditions in Definition 2.8 can be verified *pointwise*: A morphism  $F \rightarrow G$  in  $\text{Fun}(I, \mathcal{C})$  is monic (resp. epic) if and only if, for every  $i \in I$ , the morphism  $F(i) \rightarrow G(i)$  in  $\mathcal{C}$  is monic. A square in  $\text{Fun}(I, \mathcal{C})$  is a pushout (resp. pullback) square if and only if, for every  $i \in I$ , the square in  $\mathcal{C}$  obtained by evaluating at  $i$  is a pushout (resp. pullback) square.  $\square$

**Proposition 2.10.** Let  $k$  be a field. The category  $\mathbf{Vect}_k$  of  $k$ -vector spaces is proto-abelian.

To show this we will use the following lemma.

**Lemma 2.11.** Given a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow g' \\ C & \xrightarrow{f'} & D \end{array} \quad (2.12)$$

in the category of  $k$ -vector spaces, the following are equivalent:

- (1) The square (2.12) is a pullback (resp. pushout) square.
- (2) The square

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \oplus C \\ \downarrow & & \downarrow \psi \\ 0 & \longrightarrow & D \end{array}$$

where  $\varphi = \begin{pmatrix} f \\ -g \end{pmatrix}$  and  $\psi = (g', f')$  is a pullback (resp. pushout) square.

- (3) The sequence

$$0 \longrightarrow A \xrightarrow{\varphi} B \oplus C \xrightarrow{\psi} D \longrightarrow 0$$

is left (resp. right) exact.

*Proof.* The equivalence of (1) and (2) follows directly from a comparison of the universal properties of pullback and pushout squares, respectively. The equivalence of (2) and (3) is immediate.  $\square$

*Proof.* (of Proposition 2.10) Property (1) is clear: the zero vector space  $\{0\}$  is a zero object. To show (2)(i), assume that we are given a diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \\ C & & \end{array}$$

of vector spaces. Using the notation from the lemma, we set  $D = \text{coker}(\varphi) = B \oplus C / \text{im}(\varphi)$ . We obtain a square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow g' \\ C & \xrightarrow{f'} & D \end{array}$$

which, by the lemma, is a pushout square. The morphism  $g'$  is epic, since epimorphisms are stable under pushout. It remains to show that  $f'$  is monic. By the lemma, we have a right exact sequence

$$0 \longrightarrow A \xrightarrow{\varphi} B \oplus C \xrightarrow{\psi} D \longrightarrow 0.$$

Since  $f = \pi_B \circ \varphi$  is monic, we deduce that  $\varphi$  is monic so that the sequence is exact. To show that  $f'$  is monic, it suffices to show that  $\ker(f') = 0$ . Let  $p : X \rightarrow C$  be a morphism such that  $f' \circ p = 0$ . We have to show that  $p = 0$ . We have a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \longrightarrow & B \oplus C & \longrightarrow & D \longrightarrow 0 \\
 & & \nwarrow \exists r & & \uparrow (0,p)^{\text{tr}} & & \nearrow 0 \\
 & & & & X & & 
 \end{array}$$

where the dashed arrow  $r$  exists by the left exactness of the sequence. We obtain  $f \circ r = 0$ , implying  $r = 0$ , and hence  $p = g \circ r = 0$ .

The argument for (2)(ii) is analogous. To obtain (3) we note that, given a pushout square of vector spaces as in (3), we have, by the lemma, a right exact sequence

$$0 \longrightarrow A \longrightarrow B \oplus C \longrightarrow D \longrightarrow 0$$

which is, as argued above, since  $A \hookrightarrow B$  is monic, also left exact. Therefore, the original square is also a pullback square. The reverse direction is analogous.  $\square$

**Remark 2.13.** The argumentation in the proof of Proposition 2.10 generalizes verbatim to the following categories

- finite dimensional vector spaces over  $k$ ,
- abelian groups,
- finite abelian groups,
- abelian  $p$ -groups,
- modules over a ring  $R$ ,
- ...

These are examples of *abelian categories*.

**Definition 2.14.** A category  $\mathcal{A}$  is called *abelian* if it has the following properties:

- (1)  $\mathcal{A}$  is pointed,
- (2)  $\mathcal{A}$  has products and coproducts,
- (3)  $\mathcal{A}$  has kernels and cokernels,
- (4) every monic is the kernel of its cokernel, and every epic is the cokernel of its kernel.

**Problem 2.15.** Show (or look up) that these properties imply that for every pair of objects  $X$  and  $Y$ , we have

- the set  $\text{Hom}_{\mathcal{A}}(X, Y)$  is equipped with an abelian groups structure which distributes over composition of morphisms,
- we have  $X \amalg Y \cong X \times Y =: X \oplus Y$ ,

and generalize Proposition 2.10 to show that every abelian category is proto-abelian.

We introduce a category  $\mathbf{Vect}_{\mathbb{F}_1}$  as follows. An object of  $\mathbf{Vect}_{\mathbb{F}_1}$  is a set  $K$  equipped with a marked point  $*$   $\in K$  (a *pointed set*). A morphism  $(K, *) \rightarrow (L, *)$  is a map  $f : K \rightarrow L$  of underlying sets such that  $f(*) = *$  and the restriction  $f|_{K \setminus f^{-1}(*)}$  is injective.

**Proposition 2.16.** The category  $\mathbf{Vect}_{\mathbb{F}_1}$  is proto-abelian.

*Proof.* Any singleton set is a zero object so that  $\mathbf{Vect}_{\mathbb{F}_1}$  is pointed. Given a diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \\ C & & \end{array}$$

in  $\mathbf{Vect}_{\mathbb{F}_1}$ , we first form the pushout in the category of sets

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow g' \\ C & \xrightarrow{f'} & B \amalg_A C \end{array}$$

where  $B \amalg_A C = B \cup C / \{f(a) \sim g(a), a \in A\}$ . It is straightforward to verify that the resulting pushout diagram in  $\mathbf{Set}$  in fact forms a pushout diagram in  $\mathbf{Vect}_{\mathbb{F}_1}$ . We explicitly verify that  $g'$  is monic (injective) and  $f'$  is epic (surjective). This implies (2)(i). The other properties follow similarly by explicit verification.  $\square$

## 2.3 Flags in proto-abelian categories

Let  $\mathcal{C}$  be a proto-abelian category and let  $N_1, N_2, \dots, N_s, M$  be objects of  $\mathcal{C}$ . We define a category  $\mathcal{F}_{N_1, \dots, N_s}^M(\mathcal{C})$  as follows. An object is given by a sequence

$$0 = M_s \xrightarrow{f_s} M_{s-1} \xrightarrow{f_{s-1}} \dots M_1 \xrightarrow{f_1} M_0 = M$$

such that  $\text{coker}(f_i) \cong N_i$ . A morphism between objects  $M_\bullet$  and  $M'_\bullet$  is given by a commutative diagram

$$\begin{array}{ccccccc} M_{s-1} & \hookrightarrow & M_{s-2} & \hookrightarrow & \dots & \hookrightarrow & M_1 & \hookrightarrow & M \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \text{id} \\ M'_{s-1} & \hookrightarrow & M'_{s-2} & \hookrightarrow & \dots & \hookrightarrow & M'_1 & \hookrightarrow & M. \end{array}$$

We refer to the isomorphism classes of the category  $\mathcal{F}_{N_1, \dots, N_s}^M(\mathcal{C})$  as *flags in  $M$  of type  $N_1, \dots, N_s$* .

**Remark 2.17.** We define a *subobject*  $A \subset M$  of  $M$  to be an equivalence class of monomorphisms  $A \hookrightarrow M$  where

$$A \hookrightarrow M \sim A' \hookrightarrow M \quad \Leftrightarrow \quad \text{there exists a commutative diagram} \quad \begin{array}{ccc} A & \hookrightarrow & M \\ \downarrow \cong & & \uparrow \\ A' & \hookrightarrow & M \end{array}$$

The collection of subobjects is naturally partially ordered:  $A \subset M \leq A' \subset M$  if there exists a commutative diagram

$$\begin{array}{ccc} A & \hookrightarrow & M \\ \downarrow & & \nearrow \\ A & \hookrightarrow & M \end{array}$$

for some (and hence all) representatives. Slightly abusing notation, we abbreviate this by writing

$$A \subset A' \subset M.$$

With this terminology, the isomorphism classes of the category  $\mathcal{F}_{N_1, \dots, N_s}^M(\mathcal{C})$  are in bijection with totally ordered chains

$$\{0 \subset M_{s-1} \subset M_{s-2} \subset \dots \subset M_1 \subset M \mid M_{i-1}/M_i \cong N_i\}.$$

**Remark 2.18.** An object of the category  $\mathcal{F}_{N_1, \dots, N_s}^M(\mathcal{C}^{\text{op}})$  can be identified with a sequence

$$0 = M_s \xleftarrow{f_s} M_{s-1} \xleftarrow{f_{s-1}} \dots M_1 \xleftarrow{f_1} M_0 = M$$

of epimorphisms in  $\mathcal{C}$  such that  $\ker(f_i) \cong N_i$ . We may thus refer to isomorphism classes in this category as coflags in  $\mathcal{C}$ .

A pushout (and hence pullback) square

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ C & \hookrightarrow & D \end{array}$$

in a proto-abelian category is called *biCartesian*. A biCartesian square of the form

$$\begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & C \end{array}$$

is called a *short exact sequence* or an *extension of  $C$  by  $A$* .

**Lemma 2.19.** Let  $\mathcal{C}$  be a proto-abelian category.

(1) Let  $\mathcal{C}_{/0}$  denote the category of diagrams

$$\{A \rightarrow 0\}$$

in  $\mathcal{C}$  where  $0$  is a fixed zero object. Then the forgetful functor

$$\mathcal{C}_{/0} \rightarrow \mathcal{C}$$

is an equivalence.

(2) Let  $\mathcal{D}$  denote the category of diagrams of the form

$$\left\{ \begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \\ C & & \end{array} \right\}$$

in  $\mathcal{C}$  and let  $\mathcal{D}^+$  denote the category of biCartesian squares

$$\left\{ \begin{array}{ccc} A & \hookrightarrow & B \\ \downarrow & & \downarrow \\ C & \hookrightarrow & D \end{array} \right\}$$

in  $\mathcal{C}$ . Then the forgetful functor

$$\mathcal{D}^+ \rightarrow \mathcal{D}$$

is an equivalence.

*Proof.* (1) The functor is essentially surjective, since for every object  $A$  there exists a map to 0. The functor is fully faithful, since the map  $A \rightarrow 0$  is unique, and hence for every morphism  $A \rightarrow A'$ , the diagram

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

commutes. To show (2) note that the forgetful functor is essentially surjective by Property (2)(i) of proto-abelian categories. Fully faithfulness is an immediate consequence of the universal property of pushout squares.  $\square$

**Corollary 2.20.** Let  $\mathcal{C}$  be a proto-abelian category. Let  $\mathcal{M}_3$  denote the category of diagrams of the form

$$\{A \hookrightarrow B \hookrightarrow C\}$$

in  $\mathcal{C}$  and let  $\mathcal{T}_3$  denote the category of diagrams of the form

$$\left\{ \begin{array}{ccccc} A & \hookrightarrow & B & \hookrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \hookrightarrow & A' & \hookrightarrow & B' \\ & & \downarrow & & \downarrow \\ & & 0 & \hookrightarrow & A'' \end{array} \right\}$$

where all squares are biCartesian. Then the forgetful functor  $\mathcal{T}_3 \rightarrow \mathcal{M}_3$  is an equivalence.

*Proof.* Iterate the arguments of Lemma 2.19 (1) and (2).  $\square$

**Remark 2.21.** By Corollary 2.20, there exists a (weak) inverse  $s : \mathcal{M}_3 \rightarrow \mathcal{T}_3$  of the forgetful functor. In particular, to any sequence  $A \hookrightarrow B \hookrightarrow C$  there is a naturally associated short exact sequence

$$\begin{array}{ccc} B/A & \hookrightarrow & C/A \\ \downarrow & & \downarrow \\ 0 & \hookrightarrow & C/B. \end{array}$$

In the context of abelian categories, this statement is known as the *Third isomorphism Theorem* which therefore generalizes to proto-abelian categories.

**Corollary 2.22.** Let  $\mathcal{C}$  be a proto-abelian category, and let  $N_1, \dots, N_s, M$  be objects in  $\mathcal{C}$ . Then there is an equivalence of categories

$$\mathcal{F}_{N_1, \dots, N_s}^M(\mathcal{C}) \simeq \mathcal{F}_{N_s, \dots, N_1}^M(\mathcal{C}^{\text{op}}).$$

*Proof.* Introduce the category  $\mathcal{T}_{N_1, \dots, N_s}^M(\mathcal{C})$  of diagrams of the form

$$\left\{ \begin{array}{ccc} M_{s,s-1} \hookrightarrow M_{s,s-2} \hookrightarrow \dots & & M_{s,1} \hookrightarrow M \\ \downarrow & \downarrow & \downarrow \\ 0 \hookrightarrow M_{s-1,s-2} \hookrightarrow \dots & & M_{s-1,1} \hookrightarrow M_{s-1,0} \\ & \downarrow & \downarrow \\ & 0 & \vdots \\ & \ddots & \vdots \\ & & M_{2,1} \hookrightarrow M_{2,0} \\ & & \downarrow \\ & & 0 \hookrightarrow M_{1,0} \end{array} \right\}$$

where all squares are biCartesian and  $M_{i,i-1} \cong N_i$ . There are natural forgetful functors from this category to  $\mathcal{F}_{N_1, \dots, N_s}^M(\mathcal{C})$  (resp.  $\mathcal{F}_{N_s, \dots, N_1}^M(\mathcal{C}^{\text{op}})$ ) obtained by forgetting everything but the top row (resp. the rightmost column). These functors are equivalences by an iterated application of the arguments of Lemma 2.19 (1) and (2) (resp. their dual versions).  $\square$

## 2.4 The Hall algebra

In this section, we define the Hall algebra of any proto-abelian category satisfying additional finiteness conditions.

Let  $\mathcal{C}$  be a proto-abelian category. Two extensions  $A \hookrightarrow B \twoheadrightarrow A'$  and  $A \hookrightarrow C \twoheadrightarrow A'$  of  $A'$  by  $A$  are called equivalent if there exists a commutative diagram

$$\begin{array}{ccccc} A & \hookrightarrow & B & \twoheadrightarrow & A' \\ \text{id} \downarrow & & \downarrow \cong & & \downarrow \text{id} \\ A & \hookrightarrow & C & \twoheadrightarrow & A'. \end{array}$$

We denote the set of equivalence classes of extensions of  $A'$  by  $A$  by  $\text{Ext}_{\mathcal{C}}(A', A)$ .

**Definition 2.23.** A proto-abelian category  $\mathcal{C}$  is called *finitary* if, for every pair of objects  $A, A'$ , the sets  $\text{Hom}_{\mathcal{C}}(A', A)$  and  $\text{Ext}_{\mathcal{C}}(A', A)$  have finite cardinality.

Given objects  $N_1, \dots, N_s, M$  in  $\mathcal{C}$ , we denote by

$$|\mathcal{F}_{N_1, \dots, N_s}^M(\mathcal{C})|$$

the number of isomorphism classes of objects in the category  $\mathcal{F}_{N_1, \dots, N_s}^M(\mathcal{C})$ .

**Lemma 2.24.** For objects  $L, M, N$  in  $\mathcal{C}$ , we have

$$|\mathcal{F}_{N,L}^M(\mathcal{C})| = \frac{|\{L \xrightarrow{f} M \mid \text{coker}(f) \cong N\}|}{|\text{Aut}(L)|} = \frac{|\{L \hookrightarrow M \twoheadrightarrow N \text{ short exact}\}|}{|\text{Aut}(L)||\text{Aut}(N)|} = \frac{|\{M \xrightarrow{g} N \mid \ker(g) \cong L\}|}{|\text{Aut}(N)|}$$

*Proof.* The three formulas are obtained by counting isomorphism classes in the categories  $\mathcal{F}_{N,L}^M(\mathcal{C})$ ,  $\mathcal{T}_{N,L}^M(\mathcal{C})$ , and  $\mathcal{F}_{L,N}^M(\mathcal{C}^{\text{op}})$ , which are equivalent by the proof of Corollary 2.22.  $\square$

**Theorem 2.25.** Let  $\mathcal{C}$  be a finitary proto-abelian category, and let

$$\text{Hall}(\mathcal{C}) = \bigoplus_{[M] \in \text{iso}(\mathcal{C})} \mathbb{Z}[M]$$

denote the free abelian group on the set of isomorphism classes of objects in  $\mathcal{C}$ . The bilinear extension of the formula

$$[N] \cdot [L] = \sum_{[M] \in \text{iso}(\mathcal{C})} |\mathcal{F}_{N,L}^M(\mathcal{C})| [M]$$

defines a unital associative multiplication law on  $\text{Hall}(\mathcal{C})$ .

*Proof.* To verify associativity, we have to show the equality

$$([N][L])[K] = [N]([L][K])$$

for every triple of objects  $K, L, N$  in  $\mathcal{C}$ . We have

$$([N][L])[K] = \sum_{[M]} \sum_{[P]} |\mathcal{F}_{P,K}^M(\mathcal{C})| |\mathcal{F}_{N,L}^P(\mathcal{C})| [M]$$

where

$$\begin{aligned} \sum_{[P]} |\mathcal{F}_{N,L}^P(\mathcal{C})| |\mathcal{F}_{P,K}^M(\mathcal{C})| &= \sum_{[P]} \frac{|\{M \xrightarrow{g} P \mid \ker(g) \cong K\}|}{|\text{Aut}(P)|} \frac{|\{P \xrightarrow{g'} N \mid \ker(g') \cong L\}|}{|\text{Aut}(N)|} \\ &= \sum_{[P]} \frac{|\{M \xrightarrow{g} P \xrightarrow{g'} N \mid \ker(g) \cong L, \ker(g') \cong K\}|}{|\text{Aut}(P)||\text{Aut}(N)|} \\ &= |\mathcal{F}_{K,L,N}^M(\mathcal{C}^{\text{op}})| \end{aligned}$$

and

$$[N]([L][K]) = \sum_{[M]} \sum_{[P]} |\mathcal{F}_{L,K}^P(\mathcal{C})| |\mathcal{F}_{N,P}^M(\mathcal{C})| [M]$$

where

$$\begin{aligned}
\sum_{[P]} |\mathcal{F}_{L,K}^P(\mathcal{C})| |\mathcal{F}_{N,P}^M(\mathcal{C})| &= \sum_{[P]} \frac{|\{K \xrightarrow{f} P \mid \text{coker}(f) \cong L\}|}{|\text{Aut}(K)|} \frac{|\{P \xrightarrow{f'} M \mid \text{coker}(f') \cong N\}|}{|\text{Aut}(P)|} \\
&= \sum_{[P]} \frac{|\{K \xrightarrow{f} P \xrightarrow{f'} M \mid \text{coker}(f) \cong L, \text{coker}(f') \cong N\}|}{|\text{Aut}(K)| |\text{Aut}(P)|} \\
&= |\mathcal{F}_{N,L,K}^M(\mathcal{C})|.
\end{aligned}$$

Therefore, associativity follows by Corollary 2.22. To show unitality, we claim that, for every object  $N$  of  $\mathcal{C}$ , we have

$$[N][0] = [0][N] = [N].$$

We use the formula

$$|\mathcal{F}_{N,0}^M(\mathcal{C})| = \frac{|\{0 \hookrightarrow M \twoheadrightarrow N \text{ short exact}\}|}{|\text{Aut}(0)| |\text{Aut}(N)|}$$

from Lemma 2.24. Given a short exact sequence

$$\begin{array}{ccc}
0 & \hookrightarrow & M \\
\downarrow & & \downarrow g \\
0 & \hookrightarrow & N
\end{array}$$

it follows that  $g$  is an isomorphism (since it is the pushout of an isomorphism). Therefore,  $|\mathcal{F}_{N,0}^M(\mathcal{C})|$  is nonzero if and only if  $[M] = [N]$  and

$$|\mathcal{F}_{N,0}^M(\mathcal{C})| = \frac{|\text{Isom}(M, N)|}{|\text{Aut}(N)|} = 1.$$

This shows  $[N][0] = [N]$ , the proof of  $[0][N] = [N]$  is analogous.  $\square$

**Problem 2.26.** Show that, for objects  $N_1, \dots, N_s$  of  $\mathcal{C}$ , we have

$$[N_1][N_2] \cdots [N_s] = \sum_{[M]} |\mathcal{F}_{N_1, N_2, \dots, N_s}^M(\mathcal{C})| [M].$$

## 2.5 First Examples: The categories $\mathbf{Vect}_{\mathbb{F}_q}$ and $\mathbf{Vect}_{\mathbb{F}_1}$

Let  $\mathbf{Vect}'_{\mathbb{F}_q}$  denote the category of finite dimensional vector spaces over the field  $\mathbb{F}_q$  where  $q$  is some prime power. The category  $\mathbf{Vect}'_{\mathbb{F}_q}$  is finitary proto-abelian, and we have

$$\text{Hall}(\mathbf{Vect}'_{\mathbb{F}_q}) = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}[\mathbb{F}_q^n]$$

with multiplication given by

$$[\mathbb{F}_q^n][\mathbb{F}_q^m] = g_{n,m}^{n+m}(q)[\mathbb{F}_q^{n+m}]$$

where

$$\begin{aligned}
g_{n,m}^{n+m}(q) &= |\mathcal{F}_{\mathbb{F}_q^n, \mathbb{F}_q^m}^{\mathbb{F}_q^{n+m}}(\mathbf{Vect}'_{\mathbb{F}_q})| \\
&= |\{V \subset \mathbb{F}_q^{n+m} \mid V \cong \mathbb{F}_q^m, \mathbb{F}_q^{n+m}/V \cong \mathbb{F}_q^n\}| \\
&= \frac{|\{\mathbb{F}_q^m \hookrightarrow \mathbb{F}_q^{n+m}\}|}{|\mathrm{GL}_n(\mathbb{F}_q)|} \\
&= \frac{(q^{n+m} - 1)(q^{n+m} - q) \cdots (q^{n+m} - q^{m-1})}{(q^m - 1)(q^m - q) \cdots (q^m - q^{m-1})} \\
&= \frac{(q^{n+m} - 1)(q^{n+m-1} - 1) \cdots (q^{n+1} - 1)}{(q^m - 1)(q^{m-1} - 1) \cdots (q - 1)} \\
&= \begin{bmatrix} n+m \\ m \end{bmatrix}_q.
\end{aligned}$$

The symbol appearing in the last line of the calculation denotes a  $q$ -binomial coefficient which is defined as follows: We first define the  $q$ -analog of a natural number  $n \in \mathbb{N}$  to be

$$[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}.$$

The  $q$ -factorial of  $n$  is defined as

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$$

and we finally set

$$\begin{bmatrix} n+m \\ m \end{bmatrix}_q = \frac{[n+m]_q!}{[m]_q! [n]_q!}.$$

We have an isomorphism of  $\mathbb{Z}$ -algebras

$$\begin{aligned}
\mathrm{Hall}(\mathbf{Vect}'_{\mathbb{F}_q}) &\xrightarrow{\cong} \mathbb{Z}[x, \frac{x^2}{[2]_q!}, \frac{x^3}{[3]_q!}, \dots] \subset \mathbb{Q}[x] \\
[\mathbb{F}_q^n] &\mapsto \frac{x^n}{[n]_q!}
\end{aligned}$$

so that  $\mathrm{Hall}(\mathbf{Vect}'_{\mathbb{F}_q})$  can be regarded as a  $q$ -analogue of a free divided power algebra on one generator.

On the other hand, consider the category  $\mathbf{Vect}'_{\mathbb{F}_1}$  defined as the full subcategory of  $\mathbf{Vect}_{\mathbb{F}_1}$  spanned by those pointed sets which have finite cardinality. The category  $\mathbf{Vect}'_{\mathbb{F}_1}$  is finitary proto-abelian. We have

$$\mathrm{Hall}(\mathbf{Vect}'_{\mathbb{F}_1}) = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}[\{*, 1, 2, \dots, n\}]$$

with multiplication

$$[\{*, 1, 2, \dots, n\}][\{*, 1, 2, \dots, m\}] = \lambda_{n,m}^{n+m} [\{*, 1, 2, \dots, n+m\}]$$

where

$$\begin{aligned}\lambda_{n,m}^{n+m} &= \frac{|\{\{*, 1, 2, \dots, m\} \hookrightarrow \{*, 1, 2, \dots, n+m\}\}|}{|S_m|} \\ &= \binom{n+m}{m}.\end{aligned}$$

We have an isomorphism of algebras

$$\begin{aligned}\text{Hall}(\mathbf{Vect}'_{\mathbb{F}_1}) &\xrightarrow{\cong} \mathbb{Z}[x, \frac{x^2}{2!}, \frac{x^3}{3!}, \dots] \subset \mathbb{Q}[x] \\ [\mathbb{F}_q^n] &\mapsto \frac{x^n}{n!}.\end{aligned}$$

Therefore, the algebra  $\text{Hall}(\mathbf{Vect}'_{\mathbb{F}_1})$  is the free divided power algebra on one generator. We have

$$\lambda_{n,m}^{n+m} = \lim_{q \rightarrow 1} g_{n,m}^{n+m}(q).$$

## 2.6 Statistical interpretation of $q$ -analogues

Let  $S$  be a finite set. A *statistic* on  $S$  is a function

$$f : S \longrightarrow \mathbb{N}.$$

Given a statistic  $f$ , we define the corresponding *partition function* to be

$$Z(q) = \sum_{s \in S} q^{f(s)}.$$

**Remark 2.27.** Evaluation of the partition function at  $q = 1$  yields the cardinality of the set  $S$  so that  $Z(q)$  can be interpreted as a  $q$ -analog of  $|S|$ . Note that, any  $q$ -analog obtained in this way from a statistic will therefore, by construction, be polynomial in  $q$ .

**Example 2.28.** (1) Consider the set  $S = \{1, \dots, n\}$ . We define a statistic on  $S$  via

$$f : S \longrightarrow \mathbb{N}, i \mapsto i - 1.$$

The corresponding partition function is

$$Z(q) = 1 + q + \dots + q^{n-1} = [n]_q.$$

(2) Consider the set  $S_n$  underlying the symmetric group on  $n$  letters. We define the *inversion statistic* on  $S_n$  as

$$\text{inv} : S_n \longrightarrow \mathbb{N}, \sigma = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma_1 & \sigma_2 & \dots & \sigma_n \end{pmatrix} \mapsto |\{(i, j) \mid i < j, \sigma_i > \sigma_j\}|$$

We claim that we have

$$\sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} = [n]_q!.$$

To show this, we interpret the summands in the expansion of the product

$$[n]_q! = 1(1+q)(1+q+q^2)\dots(1+q+\dots+q^{n-1}).$$

Given a summand  $q^a$  of the product, it must arise as a product  $q^{i_1}q^{i_2}\dots q^{i_n}$  with  $0 \leq i_k \leq k-1$ . We produce a corresponding permutation  $\sigma$  by providing an algorithm to write the list  $\sigma_1, \dots, \sigma_n$ . In step 1, we start by writing the number 1. In step 2, we write the number 2 to the left of 1 if  $i_1 = 1$  or to the right of 1 if  $i_1 = 0$ . At step  $k$ , there will be  $k-1$  numbers and we label the gaps between the numbers by  $0, \dots, k-1$  from right to left. We fill in the number  $k$  into the gap with label  $i_k$ . This algorithm produces a permutation  $\sigma$  with  $\text{inv}(\sigma) = a$ . The claim follows immediately from this construction.

(3) We define another statistic on the set  $S_n$ , called the *major index*, as

$$\text{maj} : S_n \longrightarrow \mathbb{N}, \sigma \mapsto \sum_{i \text{ such that } \sigma_i > \sigma_{i+1}} i.$$

As a homework problem, find a similar algorithm to the one for the inversion statistic which shows that

$$\sum_{\sigma \in S_n} q^{\text{maj}(\sigma)} = [n]_q!.$$

Statistics on the set  $S_n$  with partition function  $[n]_q!$  are called Mahonian, after Major Percy Alexander MacMahon who introduced the major index and showed that it is Mahonian.

Finally, we would like to find a statistical interpretation of the  $q$ -binomial coefficient. We will achieve this by defining a statistic on the set  $P(m, n+m)$  of subsets of  $\{1, \dots, n+m\}$  of cardinality  $m$ . A lattice path in the rectangle of size  $(n, m)$  is path in  $\mathbb{R}^2$  which begins at  $(0, 0)$ , ends at  $(n, m)$ , and is obtained by sequence of steps moving either one integer step to the east or to the north. Given  $K \in P(m, n+m)$ , we can construct a lattice path by the following rule: at step  $i$ , we move east if  $i \notin K$ , and north if  $i \in K$ . It is immediate that this construction provides a bijective correspondence between  $P(m, n+m)$  and lattice paths in the rectangle of size  $(n, m)$ . We now define the statistic as

$$a : P(m, n+m) \longrightarrow \mathbb{N}, K \mapsto a(K)$$

where  $a(K)$  denotes the area of the part of the rectangle of size  $(n, m)$  which lies above the lattice path corresponding to  $K$ .

**Proposition 2.29.** We have

$$\sum_{K \in P(m, n+m)} q^{a(K)} = \left[ \begin{matrix} n+m \\ m \end{matrix} \right]_q.$$

*Proof.* We prove this equality by showing that both sides satisfy the recursion

$$Q(m, n)(q) = q^n Q(m-1, n)(q) + Q(m, n-1).$$

On the left hand side, the term  $q^n Q(m-1, n)(q)$  is the contribution from subsets  $K$  such that  $n+m \in K$ , the term  $Q(m, n-1)$  is the contribution from subsets such that  $n+m \notin K$ . The right hand side satisfies the recursion by a straightforward calculation.  $\square$

In conclusion, we obtain that the structure constants of the Hall algebra of  $\mathbf{Vect}'_{\mathbb{F}_q}$  have a statistical interpretation and are hence polynomial in  $q$ . The value at  $q = 1$  is realized by the structure constants of the Hall algebra of  $\mathbf{Vect}'_{\mathbb{F}_1}$ .

## 2.7 Duality

Finally, we observe that the multiplication law for both the Hall algebra of  $\mathbf{Vect}'_{\mathbb{F}_q}$  and  $\mathbf{Vect}'_{\mathbb{F}_1}$  is commutative. We would like to give a conceptual explanation.

An *exact duality* on a proto-abelian category  $\mathcal{C}$  is an equivalence of categories

$$D : \mathcal{C}^{\text{op}} \longrightarrow \mathcal{C}$$

which preserves short exact sequences.

**Proposition 2.30.** Let  $\mathcal{C}$  be a finitary proto-abelian category which is equipped with an exact duality  $D$  such that, for every object  $N$ , we have  $D(N) \cong N$ . Then the Hall multiplication is commutative.

*Proof.* Note that, by Lecture 3, we always have an equivalence

$$\mathcal{F}_{N_1, \dots, N_s}^M(\mathcal{C}) \simeq \mathcal{F}_{N_s, \dots, N_1}^M(\mathcal{C}^{\text{op}}).$$

The duality induces another equivalence

$$\mathcal{F}_{N_s, \dots, N_1}^M(\mathcal{C}^{\text{op}}) \simeq \mathcal{F}_{N_s, \dots, N_1}^M(\mathcal{C}).$$

Hence, we have, in particular,

$$|\mathcal{F}_{N,L}^M(\mathcal{C})| = |\mathcal{F}_{L,N}^M(\mathcal{C})|$$

so that the Hall multiplication is commutative. □

**Example 2.31.** (1) The category  $\mathbf{Vect}'_{\mathbb{F}_q}$  is equipped with the exact duality  $V \mapsto V^* = \text{Hom}_{\mathbf{Vect}_{\mathbb{F}_q}}(V, \mathbb{F}_q)$  which satisfies  $V^* \cong V$ .

(2) The category  $\mathbf{Vect}'_{\mathbb{F}_1}$  is equipped with the exact duality  $K \mapsto K^* = \text{Hom}_{\mathbf{Vect}_{\mathbb{F}_1}}(K, \{1, *\})$ . Note that, in contrast to the vector spaces over  $\mathbb{F}_q$ , the dual  $K^*$  can be canonically identified with  $K$ .

**Problem 2.32.** Generalize Proposition 2.30 to show the following statement: Let  $\mathcal{C}$  be a finitary proto-abelian category equipped with an exact duality. Then the duality  $D$  induces an isomorphism of algebras  $\text{Hall}(\mathcal{C}) \xrightarrow{\cong} \text{Hall}(\mathcal{C})^{\text{op}}$ .

### 3 Hall's algebra and symmetric functions

Our goal in this section will be to analyze Hall's algebra of partitions: the Hall algebra of the category of finite abelian  $p$ -groups. To this end, we will have to understand some basic results about symmetric functions.

#### 3.1 Symmetric functions - Basics

Let  $\mathbb{Z}[x_1, \dots, x_n]$  denote the ring of polynomials with integer coefficients. The symmetric group  $S_n$  acts by permuting the variables. The polynomials which are invariant under this action are called *symmetric polynomials*. They form a ring which we denote by

$$\Lambda_n = \mathbb{Z}[x_1, \dots, x_n]^{S_n}.$$

The ring  $\Lambda_n$  is graded by total degree so that we have

$$\Lambda_n^k = \bigoplus_{k \geq 0} \Lambda_n^k.$$

Given a tuple  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$ , we obtain a monomial

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

For a partition  $\lambda$  of length  $l(\lambda) \leq n$ , we define

$$m_\lambda(x_1, \dots, x_n) = \sum_{\alpha} x^\alpha$$

where the sum ranges over all distinct permutations  $\alpha$  of  $(\lambda_1, \lambda_2, \dots, \lambda_n)$ . For example, we have

$$\begin{aligned} m_{(1,1,\dots,1)}(x_1, \dots, x_n) &= x_1 x_2 \cdots x_n \\ m_{(1,1,0,\dots,0)}(x_1, \dots, x_n) &= \sum_{i < j} x_i x_j \\ m_{(k,0,0,\dots,0)}(x_1, \dots, x_n) &= x_1^k + x_2^k + \cdots + x_n^k. \end{aligned}$$

It is immediate that the collection of polynomials  $\{m_\lambda(x_1, \dots, x_n) | l(\lambda) \leq n\}$  forms a  $\mathbb{Z}$ -basis of  $\Lambda_n$ . In particular, the set  $\{m_\lambda(x_1, \dots, x_n) | l(\lambda) \leq n, |\lambda| = k\}$  forms a  $\mathbb{Z}$ -basis of  $\Lambda_n^k$ .

Many statements and formulas involving symmetric polynomials hold independently of the number of variables  $n$ . To incorporate this, we introduce *symmetric functions* which formalize the notion of symmetric polynomials in countably many variables. Fix  $k \geq 0$  and consider the projective system of abelian groups

$$\cdots \longrightarrow \Lambda_{n+1}^k \xrightarrow{\rho_{n+1}} \Lambda_n^k \xrightarrow{\rho_n} \Lambda_{n-1}^k \longrightarrow \cdots$$

where the map  $\rho_{n+1} : \Lambda_{n+1}^k \longrightarrow \Lambda_n^k$  is obtained by sending  $x_{n+1}$  to 0. We denote the inverse limit of the projective system by

$$\Lambda^k = \varprojlim \Lambda_\bullet^k.$$

**Example 3.1.** Let  $\lambda$  be a partition and let  $n > l(\lambda)$ . Then we have

$$m_\lambda(x_1, \dots, x_{n-1}, 0) = m_\lambda(x_1, \dots, x_{n-1}).$$

Therefore, the sequence  $\{m_\lambda(x_1, \dots, x_n) | n > l(\lambda)\}$  defines an element of  $\Lambda^k$  which we denote by  $m_\lambda$ . We call the symmetric functions  $\{m_\lambda\}$  the *monomial symmetric functions*.

We finally define

$$\Lambda = \bigoplus_{k \geq 0} \Lambda^k$$

called the *ring of symmetric functions*. There is an apparent bilinear multiplication map  $\Lambda^k \times \Lambda^{k'} \longrightarrow \Lambda^{k+k'}$  which makes  $\Lambda$  a graded ring.

**Proposition 3.2.** Let  $k \geq 0$ . The set  $\{m_\lambda\}$  where  $\lambda$  ranges over all partitions of  $k$  forms a  $\mathbb{Z}$ -basis of  $\Lambda^k$ . In particular, the set  $\{m_\lambda\}$  where  $\lambda$  ranges over all partitions forms a basis of the ring  $\Lambda$  of symmetric functions.

*Proof.* Let  $n \geq k$  be natural numbers. For a partition  $\lambda$  of  $k$ , we have  $k \geq l(\lambda)$  and hence also  $n \geq l(\lambda)$ . Therefore, the collection  $\{m_\lambda(x_1, \dots, x_n)\}$  where  $\lambda$  runs over all partitions of  $k$  forms a basis of  $\Lambda_n^k$ . From this we deduce that the maps

$$\rho_{n+1}^k : \Lambda_{n+1}^k \longrightarrow \Lambda_n^k$$

are isomorphisms for  $n \geq k$  (they map the basis  $\{m_\lambda(x_1, \dots, x_{n+1})\}$  to the basis  $\{m_\lambda(x_1, \dots, x_n)\}$ ). In particular, the projection maps

$$\Lambda \longrightarrow \Lambda_n^k$$

which exhibit  $\Lambda$  as an inverse limit, are isomorphisms for  $n \geq k$ . In light of Example 3.1, this implies that  $\{m_\lambda\}$  where  $\lambda$  ranges over all partitions of  $k$  forms a basis of  $\Lambda$ .  $\square$

We introduce another family of symmetric functions. For  $r > 0$ , let

$$e_r = m_{(1^r)} = \sum_{i_1 < i_2 < \dots < i_r} x_{i_1} x_{i_2} \cdots x_{i_r}$$

and for a partition  $\lambda$ , we let

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_s}$$

where  $s = l(\lambda)$ . The symmetric functions  $\{e_\lambda\}$  are called *elementary symmetric functions*. Important for us will be the following result which is called the *fundamental theorem on symmetric functions*.

**Theorem 3.3.** The set  $\{e_\lambda\}$  forms a  $\mathbb{Z}$ -basis of  $\Lambda$ . In particular, the set  $\{e_1, e_2, \dots\}$  is algebraically independent and generates  $\Lambda$  as a ring so that

$$\Lambda = \mathbb{Z}[e_1, e_2, \dots]$$

is a polynomial ring in countably many variables.

For the proof we need to introduce some terminology for partitions. Given a partition  $\lambda$ , we introduce the conjugate partition  $\lambda'$  as

$$\lambda'_i = |\{j | \lambda_j \geq i\}|.$$

In terms of Ferrer diagrams, the diagram for  $\lambda'$  is simply obtained by transposing the diagram for  $\lambda$ . We introduce two orders on the set  $P_n$  of partitions of  $n$ :

- (1) The *lexicographic order*:  $\lambda \leq^l \mu$  if  $\lambda = \mu$  or the first nonzero difference  $\lambda_i - \mu_i$  is negative.
- (2) The *dominance order*:  $\lambda \leq^d \mu$  if

Observe that  $\lambda \leq^d \mu$  implies  $\lambda \leq^l \mu$ . Since the dominance order is the one appearing the most frequently, we also call it the *natural order* and simply denote it by  $\leq$ .

**Problem 3.4.** Show that the dominance order is not, in general, a total order on  $P_n$ .

As we will see, the following proposition immediately implies the fundamental theorem. By a  $\{0, 1\}$ -matrix we mean a matrix  $(A_{ij})$  where  $i$  and  $j$  range over all positive natural numbers and each entry is either 0 or 1.

**Proposition 3.5.** Let  $\lambda$  be a partition and let  $\lambda'$  be its conjugate.

- (1) We have

$$e_{\lambda'} = \sum_{\mu} a_{\lambda\mu} m_{\mu}$$

where  $a_{\lambda\mu}$  denotes the number of  $\{0, 1\}$ -matrices such that, for all  $i \geq 1$ , the sum over all entries in row  $i$  equals  $\lambda'_i$  and, for all  $j \geq 1$ , the sum over all entries in column  $j$  equals  $\mu_j$ .

- (2) We have  $a_{\lambda\lambda} = 1$ , and  $a_{\lambda\mu} = 0$  unless  $|\lambda| = |\mu|$  and  $\lambda \geq \mu$ .

Before we give the proof of the proposition, we argue how it implies the fundamental theorem.

**Remark 3.6.** A matrix  $(a_{\lambda\mu})$  with integer entries, indexed by the set of all partitions, which satisfies the conditions of part (2) of the proposition is called *upper unitriangular*. An upper unitriangular matrix  $(a_{\lambda\mu})$  is a block matrix with one block for every  $k \geq 1$  given by  $(a_{\lambda\mu})$  where  $\lambda$  and  $\mu$  range over all partitions of  $k$ . If we order the set of partitions of  $k$  in decreasing lexicographic order, then the corresponding block  $(a_{\lambda\mu})$  is an upper triangular matrix with entry 1 along the diagonal. We see that each such block has an inverse which is again an upper triangular matrix of the same kind. Therefore, the whole matrix  $(a_{\lambda\mu})$  has an inverse which is upper unitriangular, in particular, the inverse has entries in  $\mathbb{Z}$ .

By the remark, the matrix  $(a_{\lambda\mu})$  from Proposition 3.5 (2), where  $\lambda$  and  $\mu$  run over all partitions, is invertible over  $\mathbb{Z}$ . Since  $\{m_{\mu}\}$  forms a  $\mathbb{Z}$ -basis of  $\Lambda$ , it follows that  $\{e_{\lambda'}\}$  forms another  $\mathbb{Z}$ -basis of  $\Lambda$  such that  $(a_{\lambda\mu})$  is the transition matrix between the two bases. In particular, this implies the statement of the fundamental theorem.

It remains to provide a proof of the proposition:

*Proof.* (1) Let  $x^\mu$ , where  $\mu$  is a partition, be a monomial which appears in the product expansion of

$$e_{\lambda'} = e_{\lambda'_1} e_{\lambda'_2} \cdots e_{\lambda'_s}.$$

This means that  $x^\mu$  must be of the form

$$x^\mu = y_1 y_2 \cdots y_s \quad (3.7)$$

where  $y_i$  is a monomial term of  $e_{\lambda'_i}$ . We write each monomial  $y_i$  as

$$y_i = \prod_j x_j^{A_{ij}}$$

with  $A_{ij} \in \mathbb{N}$ . We then observe that the condition that  $y_i$  be a monomial term of  $e_{\lambda'_i}$  simply translates into the condition that the  $i$ th row of the matrix  $(A_{ij})$  has entries in  $\{0, 1\}$  and satisfies  $\sum_j A_{ij} = \lambda'_i$ . Similarly, the condition that equation (3.7) holds, i.e., that the variable  $x_j$  appears on the right-hand side with multiplicity  $\mu_j$ , translates into the condition that, for every  $j$ , the  $j$ th column of the matrix  $(A_{ij})$  sums to  $\mu_j$ . This implies that the number of terms  $x^\mu$  (and hence, since the result is symmetric, the number of terms  $m_\mu$ ) in  $e_{\lambda'}$  is exactly given by  $a_{\lambda\mu}$  implying part (1) of the proposition.

To show (2), first assume that  $(A_{ij})$  is a  $\{0, 1\}$ -matrix with row sums  $\lambda'$  and column sums  $\mu$  so that  $(A_{ij})$  has no *gaps*. Here a gap is a 0 entry in a row which is followed to the right by an entry 1 in the same row. The condition that  $(A_{ij})$  has no gaps means that the 1-entries of the matrix constitute a Ferrer diagram of the partition  $\lambda'$  whose transpose is  $\mu$ . This implies  $\lambda = \mu$ . Vice versa, it is easy to see that this “Ferrer” matrix is the unique  $\{0, 1\}$ -matrix with row sums  $\lambda'$  and column sums  $\lambda$ . Therefore, we obtain  $a_{\lambda\lambda} = 1$ . Now suppose that  $(A_{ij})$  is a  $\{0, 1\}$ -matrix with row sums  $\lambda'$  and column sums  $\mu$  which has a gap. Pick a row with a gap and swap the 0 forming the gap with the rightmost 1 in the same row, thus obtaining a new matrix  $(\tilde{A}_{ij})$ . The row sums of the new matrix have not changed. The sequence of column sums  $\alpha = (\alpha_1, \alpha_2, \dots)$  has changed, in particular, it may not form a partition. But, enlarging the definition of the dominance order from partitions to arbitrary sequences with entries in  $\mathbb{N}$ , it is immediate to verify  $\alpha \geq \mu$ . We obtain a modified matrix  $(\tilde{A}_{ij})$  with less gaps, row sums  $\lambda'$ , and column sums given by  $\alpha \geq \mu$ . If the matrix  $(\tilde{A}_{ij})$  has no gaps, then the above argument shows that  $\alpha = \lambda$ . Otherwise we iterate, producing a totally (dominance) ordered chain of sequences in  $\mathbb{N}$

$$\mu \leq \alpha \leq \cdots \leq \lambda$$

showing that  $\mu \leq \lambda$ . □

### 3.2 Hall’s algebra of partitions

We introduce Hall’s algebra of partitions in a context which is slightly more general than abelian  $p$ -groups: Let  $R$  be a discrete valuation ring, i.e., a principal ideal domain with exactly one nonzero maximal ideal  $\mathfrak{m}$ . We assume that the residue field  $k = R/\mathfrak{m}$  is finite of cardinality  $q$ . An  $R$ -module  $M$  is called *finite* if the set underlying  $M$  is finite. Since the residue field is finite, the condition on  $M$  to be finite is equivalent to  $M$  being finitely generated and  $\mathfrak{m}^r M = 0$  for some  $r > 0$ .

By the classification result for finitely generated modules over a PID, any finite  $R$ -module decomposes into a sum of cyclic  $R$ -modules

$$M \cong \bigoplus_i R/\mathfrak{m}^{\lambda_i}$$

where we may assume that  $\lambda = (\lambda_1, \lambda_2, \dots)$  forms a partition. Therefore, the isomorphism classes of finite  $R$ -modules are naturally labelled by partitions.

**Example 3.8.** (1) Let  $R = \mathbb{Z}_p$  denote the ring of  $p$ -adic integers. Then a finite  $\mathbb{Z}_p$ -module is a finite abelian group  $M$  such that  $p^r M = 0$  for some  $r > 0$ . Therefore, a finite  $\mathbb{Z}_p$ -module is simply a finite abelian  $p$ -group.

(2) Let  $R = \mathbb{F}_q[[t]]$  the ring of power series in  $t$  with coefficients in  $\mathbb{F}_q$ . A finite  $\mathbb{F}_q[[t]]$ -module is a finite dimensional  $\mathbb{F}_q$ -vector space  $V$  which is equipped with a nilpotent endomorphism  $T : V \rightarrow V, v \mapsto t.v$ .

We denote by  $R\text{-}\mathbf{mod}'$  the category of finite  $R$ -modules. The category  $R\text{-}\mathbf{mod}'$  is finitary abelian, and the Hall algebra has the underlying abelian group

$$\text{Hall}(R) = \text{Hall}(R\text{-}\mathbf{mod}') \cong \bigoplus_{\lambda} \mathbb{Z}[\bigoplus_i R/\mathfrak{m}^{\lambda_i}].$$

As for abelian  $p$ -groups, we introduce the symbol  $u_{\lambda}$  to denote  $[\bigoplus_i R/\mathfrak{m}^{\lambda_i}]$ . We will see later, that the structure constants of  $\text{Hall}(R)$  only depend on the cardinality  $q$  of the residue field. Before analyzing  $\text{Hall}(R)$  in more depth, we will try to understand the case  $q = 1$ .

### 3.3 The Hall algebra of $\mathbb{F}_1[[t]]$

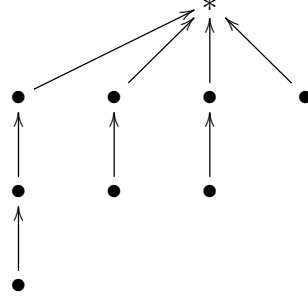
It is not immediately clear what the analogue of an abelian  $p$ -group for  $p = 1$  should be. Alternatively, we define in direct analogy with Example (2):

**Definition 3.9.** A *finite  $\mathbb{F}_1[[t]]$ -module* is an object of  $\mathbf{Vect}'_{\mathbb{F}_1}$  equipped with a nilpotent endomorphism.

**Proposition 3.10.** The isomorphism classes of finite  $\mathbb{F}_1[[t]]$ -modules are naturally labelled by the set of all partitions.

*Proof.* Let  $K$  be a finite  $\mathbb{F}_1[[t]]$ -module and denote by  $T : K \rightarrow K$  the corresponding nilpotent endomorphism. We introduce an oriented graph  $\Gamma$  as follows: the vertices are given by the elements of  $K$  and there is an edge from  $k$  to  $k'$  if  $T(k) = k'$ . The nilpotency condition means that, for every element  $k$ , we have  $T^r(k) = *$  for some  $r > 0$ . In other words, the graph  $\Gamma$  is a tree with root  $*$ . Note however, that since  $T$  is a morphism in the category  $\mathbf{Vect}'_{\mathbb{F}_1}$  it has to be injective on  $K \setminus T^{-1}(*)$ . This implies that the root  $*$  is the only branching point of the tree. Each element  $k$  of  $K$  which is not in the image of  $T$  defines a *branch*  $B_k = \{T^i(k) | i \geq 0\}$  of  $\Gamma$ . We define the length of  $B_k$  to be  $|B_k| - 1$ . We order the tuple of lengths of the branches of  $\Gamma$  in a weakly decreasing way to obtain a partition  $(\lambda_1, \lambda_2, \dots)$ . It is immediate that this construction establishes a natural bijection between isomorphism classes of finite  $\mathbb{F}_1[[t]]$ -module and partitions.  $\square$

**Example 3.11.** The finite  $\mathbb{F}_1[[t]]$ -module corresponding to the partition  $(3, 2, 2, 1, 0, \dots)$  is represented by the rooted tree

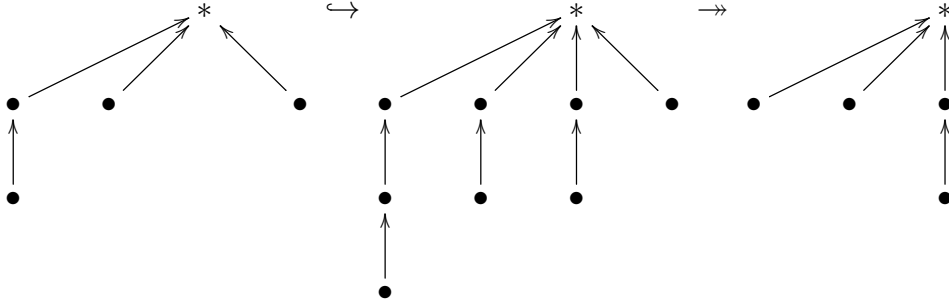


The category of finite  $\mathbb{F}_1[[t]]$ -modules is a finitary proto-abelian category. The abelian group underlying its Hall algebra is

$$\text{Hall}(\mathbb{F}_1[[t]]) = \bigoplus_{\lambda} \mathbb{Z}u_{\lambda}$$

where  $u_{\lambda}$  is a formal variable representing the isomorphism class of  $\mathbb{F}_1[[t]]$ -modules with branch lengths given by  $\lambda$ .

**Example 3.12.** An example of a short exact sequence of finite  $\mathbb{F}_1[[t]]$ -modules is given by



**Proposition 3.13.** The association  $K \mapsto \text{Hom}_{\mathbf{Vect}_{\mathbb{F}_1}}(K, \{1, *\})$  defines a duality functor  $D$  on the category of finite  $\mathbb{F}_1[[t]]$ -modules such that, for every object  $K$ , we have  $D(K) \cong K$ . In particular, the algebra  $\text{Hall}(\mathbb{F}_1[[t]])$  is commutative.

*Proof.* Given a finite  $\mathbb{F}_1[[t]]$ -module  $K$  with nilpotent endomorphism  $T$ , the dual  $D(T) = \text{Hom}(K, \{1, *\})$  is naturally equipped with a nilpotent endomorphism obtained by pre-composing with  $T$ . The corresponding oriented tree  $\Gamma_{D(T)}$  is obtained from  $\Gamma_T$  by removing  $*$ , reversing the orientation of all edges, and adding  $*$  as a root. This description implies that  $D(T)$  has the same branch lengths and is hence isomorphic to  $T$ .  $\square$

**Proposition 3.14.** Let  $\lambda$  be a partition and  $\lambda'$  its conjugate. We set  $s = l(\lambda')$ . Then, in  $\text{Hall}(\mathbb{F}_1[[t]])$ , we have

$$u_{(1^{\lambda'_1})} u_{(1^{\lambda'_2})} \cdots u_{(1^{\lambda'_s})} = \sum_{\mu} a_{\lambda\mu} u_{\mu} \quad (3.15)$$

where  $a_{\lambda\mu}$  denotes the number of  $\{0, 1\}$ -matrices with row sums  $\lambda'$  and column sums  $\mu$ .

*Proof.* The coefficient  $a_{\lambda\mu}$  is the number of flags

$$K = K_0 \supset K_1 \supset \cdots \supset K_{s-1} \supset K_s = \{*\}$$

where  $K$  is fixed of type  $\mu$  and  $K_{i-1}/K_i$  has type  $(1^{\lambda'_i})$ . We represent  $K$  as an oriented graph with branches labelled by  $1, \dots, s$ . Let  $K_1 \subset K$  be a submodule such that  $K/K_1$  has type  $(1^{\lambda'_1})$ . Then the set  $K \setminus K_1$  consists of exactly  $\lambda'_1$  elements which form the tips of pairwise disjoint branches. We may encode this in a vector  $v_1$  in  $\{0, 1\}^s$  where we mark those branches of  $K$  which contain a point in  $K \setminus K_1$  by 1 and all remaining branches by 0. note that the sum over all entries in  $v_1$  equals  $\lambda'_1$ . We repeat this construction for each  $K_i \subset K_{i-1}$  and organize the resulting vectors  $v_1, v_2, \dots, v_s$  as the rows of a matrix. By construction, this matrix has row sums  $\lambda'$  and column sums  $\mu$ . This construction establishes a bijection between flags of the above type and  $\{0, 1\}$ -matrices with row sums  $\lambda'$  and column sums  $\mu$ .  $\square$

**Corollary 3.16.** The set  $\{u_{(1^r)} \mid r > 0\}$  is algebraically independent and generates  $\text{Hall}(\mathbb{F}_1[[t]])$  as a  $\mathbb{Z}$ -algebra. In other words,

$$\text{Hall}(\mathbb{F}_1[[t]]) = \mathbb{Z}[u_{(1)}, u_{(1^2)}, \dots]$$

is a polynomial ring.

*Proof.* This is immediate from Proposition 3.5 and Remark 3.6: the matrix  $(a_{\lambda\mu})$  is upper unitriangular and hence invertible over  $\mathbb{Z}$ .  $\square$

**Corollary 3.17.** There is a  $\mathbb{Z}$ -algebra isomorphism

$$\varphi : \text{Hall}(\mathbb{F}_1[[t]]) \xrightarrow{\cong} \Lambda, \quad u_\lambda \mapsto m_\lambda$$

determined by  $\varphi(u_{(1^r)}) = m_{(1^r)}$ . We further have  $\varphi(u_\lambda) = m_\lambda$ .

*Proof.* The only statement requiring an argument is that  $\varphi$  satisfies  $\varphi(u_\lambda) = m_\lambda$ . The equations obtained by applying  $\varphi$  to (3.15), for all  $\lambda$ , uniquely determine the elements  $\varphi(u_\mu)$ , since the matrix  $(a_{\lambda\mu})$  is invertible over  $\mathbb{Z}$ . But, by Proposition 3.5, the elements  $\{m_\mu\}$  satisfy the very same set of equations so that we have  $\varphi(u_\mu) = m_\mu$ .  $\square$

### 3.4 Hall's algebra: a first analysis

We will analyze  $\text{Hall}(R)$ , where  $R$  is a discrete valuation ring with residue field  $k \cong \mathbb{F}_q$ , along the same lines as  $\text{Hall}(\mathbb{F}_1[[t]])$ .

We start with some preparatory remarks. Recall that the length  $l(M)$  of an  $R$ -module  $M$  is defined to be the length of a composition series of  $M$ . For example, the cyclic module  $R/\mathfrak{m}^n$  has composition series

$$R/\mathfrak{m}^n \supset \mathfrak{m}/\mathfrak{m}^n \supset \mathfrak{m}^2/\mathfrak{m}^n \supset \cdots \supset \mathfrak{m}^{n-1}/\mathfrak{m}^n \supset 0$$

so that we have  $l(R/\mathfrak{m}^n) = n$ . From this, we deduce that a finite module  $M$  of type  $\lambda$  has length given by  $l(M) = |\lambda|$ . Further, given a finite module  $M$ , the sequence  $(\mu_1, \mu_2, \dots)$  with

$$\mu_i = \dim_k(\mathfrak{m}^{i-1}M/\mathfrak{m}^iM)$$

is the conjugate partition of  $\lambda$ . Finally, recall that the length is additive in short exact sequences so that, for a submodule  $N \subset M$ , we have  $l(M) = l(N) + l(M/N)$ .

**Problem 3.18.** Show that  $\text{Hall}(R)$  is commutative by constructing an exact duality on the category of finite  $R$ -modules as follows. Let  $\pi$  be a local parameter of  $R$ , i.e., a generator of the maximal ideal  $\mathfrak{m}$ . Let  $E$  denote the direct limit of the diagram

$$R/\mathfrak{m} \xrightarrow{\pi} R/\mathfrak{m}^2 \xrightarrow{\pi} R/\mathfrak{m}^3 \xrightarrow{\pi} \dots$$

Show that the functor  $\text{Hom}_R(-, E)$  defines an exact duality such that, for every finite  $R$ -module  $M$ , we have  $\text{Hom}_R(M, E) \cong M$ .

We have the following analogue of the Proposition in Lecture 6.

**Proposition 3.19.** Let  $\lambda$  be a partition with conjugate  $\lambda'$  and set  $s = l(\lambda')$ . Then, in  $\text{Hall}(R)$ , we have

$$u_{(1^{\lambda'_1})} u_{(1^{\lambda'_2})} \cdots u_{(1^{\lambda'_s})} = \sum_{\mu} b_{\lambda\mu} u_{\mu} \quad (3.20)$$

where the matrix  $(b_{\lambda\mu})$  is upper unitriangular.

*Proof.* The coefficient  $b_{\lambda\mu}$  is the number of flags

$$M = M_0 \supset M_1 \supset M_2 \supset \cdots \supset M_{s-1} \supset M_s = 0$$

where  $M$  is fixed of type  $\mu$  and  $M_{i-1}/M_i$  is of type  $(1^{\lambda'_i})$ , i.e.,

$$M_{i-1}/M_i \cong \bigoplus_{\lambda'_i} R/\mathfrak{m}.$$

Given a flag of this kind, we must have  $\mathfrak{m}M_{i-1} \subset M_i$  and therefore  $\mathfrak{m}^i M \subset M_i$ . Using the additivity of length, this implies

$$l(M/\mathfrak{m}^i M) \geq l(M/M_i)$$

which by the above remarks translates into

$$\mu'_1 + \mu'_2 + \cdots + \mu'_{i-1} \geq \lambda'_1 + \lambda'_2 + \cdots + \lambda'_{i-1}$$

so that  $\mu' \geq \lambda'$  with respect to the dominance order. By Lemma 3.21 below, we have  $\lambda \leq \mu$  showing that the matrix  $(b_{\lambda\mu})$  is upper triangular. If  $\lambda = \mu$ , then, by the above argumentation, we must have  $\mathfrak{m}^i M = M_i$  which defines a unique flag of the above kind. Therefore  $b_{\lambda\lambda} = 1$  so that  $(b_{\lambda\mu})$  is upper unitriangular.  $\square$

**Lemma 3.21.** Let  $\lambda$  and  $\mu$  be partitions of  $n$ . Then we have

$$\lambda \geq \mu \iff \mu' \geq \lambda'.$$

*Proof.* It suffices to show one direction. Assume  $\mu' \not\geq \lambda'$ . Then there is an  $i \geq 1$  such that

$$\mu'_1 + \mu'_2 + \cdots + \mu'_j \geq \lambda'_1 + \lambda'_2 + \cdots + \lambda'_j$$

for  $1 \leq j \leq i-1$  but

$$\mu'_1 + \mu'_2 + \cdots + \mu'_i < \lambda'_1 + \lambda'_2 + \cdots + \lambda'_i. \quad (3.22)$$

This implies that we must have  $\lambda'_i > \mu'_i$ . We set  $l = \lambda'_i$  and  $m = \mu'_i$ . Since  $|\lambda'| = |\mu'|$ , equation (3.22) further implies that

$$\mu'_{i+1} + \mu'_{i+2} + \cdots > \lambda'_{i+1} + \lambda'_{i+2} + \cdots \quad (3.23)$$

We may write the left-hand side of (3.23) as

$$\mu'_{i+1} + \mu'_{i+2} + \cdots = \sum_{j=1}^m (\mu_j - i),$$

the number of boxes in the diagram of  $\mu$  which lie to the right of the  $i$ th column. Similarly, we have

$$\lambda'_{i+1} + \lambda'_{i+2} + \cdots = \sum_{j=1}^l (\lambda_j - i).$$

Therefore, from (3.23), we have

$$\sum_{j=1}^m (\mu_j - i) > \sum_{j=1}^l (\lambda_j - i) \geq \sum_{j=1}^m (\lambda_j - i).$$

But this implies

$$\mu_1 + \mu_2 + \cdots + \mu_m > \lambda_1 + \lambda_2 + \cdots + \lambda_m$$

so that  $\lambda \not\geq \mu$ . □

**Corollary 3.24.** The set  $\{u_{(1^r)} \mid r > 0\}$  is algebraically independent and generates  $\text{Hall}(R)$  as a  $\mathbb{Z}$ -algebra.

**Corollary 3.25.** There is a  $\mathbb{Z}$ -algebra isomorphism

$$\psi : \text{Hall}(R) \xrightarrow{\cong} \Lambda$$

determined by  $\psi(u_{(1^r)}) := m_{(1^r)}$ . In particular, the set  $\{\psi(u_\lambda)\}$  forms a  $\mathbb{Z}$ -basis of  $\Lambda$ .

The  $\mathbb{Z}$ -basis  $\{\varphi(u_\lambda)\}$  of  $\Lambda$  constructed via  $\text{Hall}(\mathbb{F}_1[[t]])$  in §3.3 coincides with the basis given by the monomial symmetric functions  $\{m_\lambda\}$ . As we will see later on, the symmetric functions  $\psi(u_\lambda)$  forming the  $\mathbb{Z}$ -basis of  $\Lambda$  from Corollary 3.25 are  $q$ -analogues of  $m_\lambda$  with highly interesting properties (up to some renormalization they are known as the *Hall-Littlewood functions*).

Our approach to understanding the symmetric functions  $\psi(u_\lambda)$  will be indirect: We will show how the coefficients  $b_{\lambda\mu}$  of (3.20) can be interpreted as  $q$ -analogs of the coefficients  $a_{\lambda\mu}$  of Lecture 6. To this end, we will introduce a statistic on the set of  $(0, 1)$ -matrices with row sums  $\lambda'$  and column sums  $\mu$ .

To be able to understand the result of this analysis, it is necessary to first learn a bit more about  $\Lambda$ .

### 3.5 More on symmetric functions

We have defined the  $\mathbb{Z}$ -bases  $\{m_\lambda\}$  and  $\{e_\lambda\}$  of  $\Lambda$  given by monomial symmetric functions and elementary symmetric functions, respectively. We will now introduce other natural  $\mathbb{Z}$ -bases of  $\Lambda$  indexed by partitions.

### 3.5.1 Complete symmetric functions

For  $r > 0$ , we define the *complete symmetric function*

$$h_r = \sum_{|\lambda|=r} m_\lambda,$$

which is simply the sum over all monomials of degree  $r$ . The family  $\{h_r\}$  has a concise generating series given by

$$H(t) = \sum_{r \geq 0} h_r t^r = \prod_{i \geq 1} (1 - x_i t)^{-1}.$$

In comparison, the family  $\{e_r\}$  has the generating series

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t)$$

so that we obtain

$$E(-t)H(t) = 1.$$

Comparing coefficients of  $t^n$ , for  $n > 0$ , yields the equation

$$h_n - e_1 h_{n-1} + e_2 h_{n-2} - \cdots + (-1)^n e_n = 0. \quad (3.26)$$

**Remark 3.27.** Note that the resulting system of equations defines recursions which uniquely determine  $\{h_r\}$  in terms of  $\{e_r\}$  and vice versa.

Since the set  $\{e_r\}$  is algebraically independent, we may define a  $\mathbb{Z}$ -algebra homomorphism

$$\omega : \Lambda \longrightarrow \Lambda, \quad e_r \mapsto h_r.$$

By Remark 3.27, applying  $\omega$  to the equations (3.26) implies that  $\omega(h_r) = e_r$  so that  $\omega$  is an involution, in particular, an isomorphism. We have therefore proven:

**Proposition 3.28.** The set  $\{h_\lambda\}$  of complete symmetric functions where

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots$$

is a  $\mathbb{Z}$ -basis of  $\Lambda$ .

### 3.5.2 Schur functions

We introduce yet another  $\mathbb{Z}$ -basis of  $\Lambda$  which will be most relevant for our analysis of  $\text{Hall}(R)$ : the set of *Schur functions*. Let  $\alpha \in \mathbb{N}^n$  and consider the polynomial

$$a_\alpha := \sum_{\sigma \in S_n} \text{sign}(\sigma) \sigma(x^\alpha) = \det(x_i^{\alpha_j})$$

obtained by *anti*-symmetrizing  $x^\alpha$ . We assume

$$\alpha_1 > \alpha_2 > \cdots > \alpha_n$$

so that  $\alpha = \lambda + \delta$  where  $\lambda$  is a partition and  $\delta = (n-1, n-2, \dots, 1, 0)$ . Note that  $a_\alpha$  is divisible in  $\mathbb{Z}[x_1, \dots, x_n]$  by the Vandermonde matrix

$$a_\delta = \det(x_i^{n-j}) = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

and we define the *Schur polynomial*

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\delta}}{a_\delta}.$$

**Example 3.29.** We have

$$\begin{aligned} s_{(1,1)}(x_1, x_2) &= \frac{\begin{vmatrix} x_1^2 & x_1 \\ x_2^2 & x_2 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}} = \frac{x_1^2 x_2 - x_1 x_2^2}{x_1 - x_2} = x_1 x_2 \\ &= m_{(1,1)}(x_1, x_2) \end{aligned}$$

and

$$\begin{aligned} s_{(3,1)}(x_1, x_2) &= \frac{\begin{vmatrix} x_1^4 & x_1 \\ x_2^4 & x_2 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}} = \frac{x_1^4 x_2 - x_1 x_2^4}{x_1 - x_2} = x_1^3 x_2 + x_1 x_2^3 + x_1^2 x_2^2 \\ &= m_{(3,1)}(x_1, x_2) + m_{(2,2)}(x_1, x_2). \end{aligned}$$

The set  $\{a_{\lambda+\delta}(x_1, \dots, x_n) | l(\lambda) \leq n\}$  forms a  $\mathbb{Z}$ -basis of the group  $A_n$  of antisymmetric polynomials in  $n$  variables. Every antisymmetric polynomial is divisible by  $a_\delta$  so that we obtain an isomorphism of abelian groups

$$\Lambda_n \xrightarrow{\cong} A_n$$

given by multiplication by  $a_\delta$ . Therefore, the set  $\{s_\lambda(x_1, \dots, x_n) | l(\lambda) \leq n\}$  forms a  $\mathbb{Z}$ -basis of  $\Lambda_n$ .

**Problem 3.30.** Verify that, for  $n > l(\lambda)$ , we have  $s_\lambda(x_1, \dots, x_n, 0) = s_\lambda(x_1, \dots, x_n)$ .

The sequence  $(s_\lambda(x_1, \dots, x_n) | n \geq l(\lambda))$  therefore defines an element  $s_\lambda$  of  $\Lambda$  which we call the *Schur function* corresponding to  $\lambda$ . By the above discussion, we obtain the following:

**Proposition 3.31.** The set  $\{s_\lambda\}$  of Schur functions forms a  $\mathbb{Z}$ -basis of  $\Lambda$ .

Since we have

$$\Lambda = \mathbb{Z}[e_1, e_2, \dots] = \mathbb{Z}[h_1, h_2, \dots]$$

we may express each  $s_\lambda$  as a polynomial in  $\{e_r\}$  and  $\{h_r\}$ . The precise formulas are as follows.

**Proposition 3.32** (Jacobi-Trudi). Let  $\lambda$  be a partition.

(1) For  $n \geq l(\lambda)$ , we have

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i, j \leq n}.$$

(2) For  $m \geq l(\lambda')$ , we have

$$s_\lambda = \det(e_{\lambda'_i - i + j})_{1 \leq i, j \leq m}.$$

*Proof.* To show (1) we work with  $n$  variables  $x_1, \dots, x_n$ . Denote by  $e_r^{(k)}$  the elementary symmetric polynomials in  $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n$ . They satisfy

$$E^{(k)}(t) = \sum_{r=0}^{n-1} e_r^{(k)} t^r = \prod_{i \neq k} (1 + x_i t).$$

On the other hand, we have

$$H(t) = \prod_{i=1}^n (1 - x_i t)^{-1} = \sum_{r \geq 0} h_r(x_1, \dots, x_n) t^r.$$

For  $\alpha \in \mathbb{N}^n$ , we compare coefficients of  $t^{\alpha_i}$  in the equality

$$H(t)E^{(k)}(-t) = (1 - x_k t)^{-1}$$

to obtain the equations

$$\sum_{j=1}^n h_{\alpha_i - n + j} (-1)^{n-j} e_{n-j}^{(k)} = x_k^{\alpha_i}.$$

We may express this set of equations as a matrix identity

$$H_\alpha M = A_\alpha$$

where  $H_\alpha = (h_{\alpha_i - n + j})_{i,j}$ ,  $M = ((-1)^{n-j} e_{n-j}^{(k)})_{j,k}$ , and  $A_\alpha = (x_k^{\alpha_i})_{i,k}$ . Note that, for  $\alpha = \rho = (n-1, n-2, \dots, 1, 0)$ , we have  $\det(H_\rho) = 1$  and hence  $\det(M) = \det(A_\rho) = a_\rho$  is given by the Vandermonde determinant. We therefore obtain the equation

$$\det(h_{\alpha_i - n + j}) a_\rho = a_\alpha \tag{3.33}$$

which, evaluated at  $\alpha = \lambda + \rho$ , gives the desired formula.

To show (2), we directly verify the determinantal identity

$$\det(h_{\lambda_i - i + j}) = \det(e_{\lambda'_i - i + j})$$

We introduce, for  $N > 0$ , the matrices

$$H = (h_{i-j})_{0 \leq i, j \leq N}$$

and

$$E = ((-1)^{i+j} e_{i-j})_{0 \leq i, j \leq N}.$$

We observe that  $\det(E) = \det(H) = 1$  and that  $H$  and  $E$  are inverse to one another (here we use the formula  $\sum_{r=0}^n (-1)^r e_r h_{n-r} = 0$  from Lecture 7). We now set  $p = l(\lambda)$ ,  $q = l(\lambda')$  and set  $N = p + q - 1$ . We now apply Lemma 3.34 below to rows

$$\alpha = (\lambda_i + p - i)_{1 \leq i \leq p}$$

and columns

$$\beta = (p - i)_{1 \leq i \leq p}$$

of the matrix  $H$ . The corresponding minor of  $H$  is

$$\min_{\alpha, \beta}(H) = \det(h_{\lambda_i - i + j}).$$

By Lemma 3.35, we have

$$\check{\alpha} = (p - 1 + j - \lambda'_j)_{1 \leq j \leq q}$$

and

$$\check{\beta} = (p - 1 + j)_{1 \leq j \leq q}$$

so that the corresponding cofactor of  $E^{\text{tr}}$  is given by

$$(-1)^{|\lambda'|} \min_{\check{\alpha}, \check{\beta}}(E^{\text{tr}}) = (-1)^{|\lambda'|} \det((-1)^{\lambda'_i} e_{\lambda'_i - i + j}).$$

The signs cancel, so that we obtain the desired formula by Lemma 3.34.  $\square$

**Lemma 3.34.** Let  $A, B$  be square matrices indexed by  $\{0, 1, \dots, N\}$  with  $\det(A) = \det(B) = 1$  and  $AB = I$ . Consider subsets

$$\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_p\} \subset \{0, 1, \dots, N\}$$

and

$$\beta = \{\beta_1, \beta_2, \dots, \beta_p\} \subset \{0, 1, \dots, N\}$$

of cardinality  $p < N + 1$ . We define the *minor*

$$\min_{\alpha, \beta}(A)$$

to be the determinant of the  $p$ -by- $p$  submatrix of  $A$  with rows  $\alpha$  and columns  $\beta$ . We define

$$\check{\alpha} = \{0, 1, \dots, N\} \setminus \alpha$$

and

$$\check{\beta} = \{0, 1, \dots, N\} \setminus \beta.$$

Then we have an equality

$$\min_{\alpha, \beta}(A) = (-1)^{|\check{\alpha}| + |\check{\beta}|} \min_{\check{\alpha}, \check{\beta}}(B^{\text{tr}}).$$

In plain words, the minors of  $A$  coincide with the complementary *cofactors* of  $B^{\text{tr}}$ .

*Proof.* Homework. *Hint:* Given an  $N + 1$ -dimensional vector space  $V$ , and  $0 < p < N + 1$ , the pairing

$$\bigwedge^p V \otimes \bigwedge^{N+1-p} V \longrightarrow \bigwedge^{N+1} V$$

given by the exterior product is nondegenerate so that we obtain an isomorphism

$$\bigwedge^p V \cong \text{Hom}(\bigwedge^{N+1-p} V, \bigwedge^{N+1} V)$$

Now analyze how this isomorphism is compatible with the automorphisms induced by  $A : V \rightarrow V$  on the various exterior powers of  $V$ . Choose bases to obtain the desired formula.  $\square$

**Lemma 3.35.** Let  $\lambda$  be a partition. Set  $p = l(\lambda)$  and  $q = l(\lambda')$ . Then the concatenation of the sequences

$$(\lambda_i + p - i)_{1 \leq i \leq p}$$

and

$$(p - 1 + j - \lambda'_j)_{1 \leq j \leq q}$$

is a permutation of  $\{0, 1, \dots, p + q - 1\}$ .

*Proof.* Consider the diagram of  $\lambda$  visualized as boxes in an integer grid rectangle of size  $p \times q$ . The lower boundary of the diagram forms a lattice path starting at the bottom left corner of the rectangle and ending at the top right corner consisting of horizontal and vertical steps of length 1. The total length of the lattice path is  $p + q$ . We now label the steps by  $0, 1, \dots, p + q - 1$  starting in the bottom left corner so that the first step is labelled 0, the second step 1,  $\dots$ , the last step (reaching the top right corner)  $p + q - 1$ . We now read off that the vertical steps are labelled by the set

$$\{\lambda_i + p - i\}_{1 \leq i \leq p}$$

while the horizontal steps are labelled by

$$\{p - 1 + j - \lambda'_j\}_{1 \leq j \leq q}.$$

□

We have the following immediate consequences of Proposition 3.32.

**Corollary 3.36.** (1) For a partition  $\lambda$ , we have

$$\omega(s_\lambda) = s_{\lambda'}$$

where  $\omega$  is the involution of  $\Lambda$  from Lecture 7.

$$(2) \quad s_{(n)} = h_n$$

$$(3) \quad s_{(1^n)} = e_n$$

### 3.5.3 Transition matrices

Our goal will be to understand the transition matrices between the  $\mathbb{Z}$ -bases  $\{m_\lambda\}$ ,  $\{e_\lambda\}$ ,  $\{h_\lambda\}$ , and  $\{s_\lambda\}$  of  $\Lambda$ .

Besides the involution  $\omega$  on  $\Lambda$ , we will utilize a scalar product on  $\Lambda$  which we now introduce. We define a bilinear form on  $\Lambda$  by setting

$$\langle h_\lambda, m_\mu \rangle := \delta_{\lambda\mu}$$

and extending this formula bilinearly. We will prove:

**Proposition 3.37.** We have  $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$  so that the Schur functions form an orthonormal basis of  $\Lambda$  with respect to the above form.

The statement will follow at once from the following lemma.

**Lemma 3.38.** Consider two sets  $x_1, x_2, \dots$  and  $y_1, y_2, \dots$  of independent variables. Then we have the formulas

(1)

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} h_{\lambda}(x) m_{\lambda}(y)$$

(2)

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y)$$

*Proof.* Both formulas follow from an explicit manipulation of the involved power series. To show (1), we have

$$\begin{aligned} \prod_{i,j} (1 - x_i y_j)^{-1} &= \prod_j H(y_j) \\ &= \prod_j \sum_{r \geq 0} h_r(x) y_j^r \\ &= \sum_{\alpha} h_{\alpha}(x) y^{\alpha} \end{aligned}$$

where the sum in the last line ranges over all *compositions*, i.e., sequences of natural numbers with finitely many nonzero terms. Here  $h_{\alpha} = h_{\alpha_1} h_{\alpha_2} \cdots$ , generalizing the definition for partitions. Finally, we may express the last line by summing over partitions to obtain

$$\sum_{\alpha} h_{\alpha}(x) y^{\alpha} = \sum_{\lambda} h_{\lambda}(x) m_{\lambda}(y)$$

To show (2), we work with variables  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$  and calculate

$$\begin{aligned} a_{\rho}(x) a_{\rho}(y) \prod_{i,j} (1 - x_i y_j)^{-1} &= a_{\rho}(x) a_{\rho}(y) \sum_{\alpha} h_{\alpha}(x) y^{\alpha} \\ &= a_{\rho}(x) \sum_{\alpha, \sigma \in S_n} h_{\alpha}(x) \text{sign}(\sigma) y^{\alpha + \sigma \cdot \rho} \\ &= \sum_{\beta} a_{\rho}(x) \sum_{\sigma} \text{sign}(\sigma) h_{\beta - \sigma \cdot \rho}(x) y^{\beta} \end{aligned}$$

where the sum ranges over all  $\beta \in \mathbb{N}^n$ . We now use the formula

$$a_{\rho} \det(h_{\beta_i - n + j}) = a_{\beta}$$

from the proof of the Jacobi-Trudi formulas in Lecture 8 which implies that the last line equals

$$\sum_{\beta} a_{\beta}(x) y^{\beta}.$$

Finally, we note that, since  $a_{\sigma \beta}(x) = \text{sign}(\sigma) a_{\beta}(x)$ , we may write this expression as a sum

$$\sum_{\lambda} a_{\lambda + \rho}(x) a_{\lambda + \rho}(y)$$

over all partitions of length  $\leq n$ . We obtain the claimed formula by letting  $n \rightarrow \infty$ .  $\square$

We prove Proposition 3.37.

*Proof.* We expand the Schur functions in terms of the bases  $\{h_\lambda\}$  and  $\{m_\lambda\}$  writing  $s_\lambda = \sum_\nu c_{\lambda\nu} h_\nu$  and  $s_\lambda = \sum_\nu d_{\lambda\nu} m_\nu$ . We write  $C = (c_{\lambda\nu})$  and  $D = (d_{\lambda\nu})$  for the corresponding matrices indexed by all partitions. Comparing coefficients of the formula

$$\sum_\lambda s_\lambda(x) s_\lambda(y) = \sum_\mu h_\mu(x) m_\mu(y)$$

we obtain

$$C^{\text{tr}} D = I. \quad (3.39)$$

On the other hand, we have

$$\langle s_\lambda, s_\mu \rangle = \sum_\nu c_{\lambda\nu} d_{\mu\nu}$$

which equals  $\delta_{\lambda\mu}$  since (3.39) implies  $CD^{\text{tr}} = I$ .  $\square$

As immediate consequences, we obtain:

**Corollary 3.40.** The pairing  $\langle -, - \rangle$  is symmetric, positive definite, and invariant under the involution  $\omega$ .

We will now use the various structures introduced on  $\Lambda$  to study the transition matrices between the bases  $\{m_\lambda\}$ ,  $\{e_\lambda\}$ ,  $\{h_\lambda\}$ , and  $\{s_\lambda\}$ . Given two  $\mathbb{Z}$ -bases  $\{u_\lambda\}$  and  $\{v_\lambda\}$  of  $\Lambda$ , we denote by  $M(u, v)$  the transition matrix from  $\{v_\lambda\}$  to  $\{u_\lambda\}$  so that

$$u_\lambda = \sum_\mu M(u, v)_{\lambda\mu} v_\mu.$$

**Example 3.41.** We have already computed the coefficients of the transition matrix  $M(e, m)$  from  $\{m_\lambda\}$  to  $\{e_\lambda\}$ : we have

$$M(e, m)_{\lambda\mu} = a_{\lambda'\mu},$$

the number of  $\{0, 1\}$ -matrices with row sums  $\lambda$  and columns sums  $\mu$ .

The following formulas are immediate from the definitions:

- $M(v, u) = M(u, v)^{-1}$
- $M(u, w) = M(u, v)M(v, w)$
- Suppose  $u'$  and  $v'$  are dual bases (with respect to  $\langle -, - \rangle$ ) of  $u$  and  $v$ , respectively. Then we have

$$M(u', v') = M(v, u)^{\text{tr}} = (M(u, v)^{-1})^{\text{tr}} =: M(u, v)^*.$$

- $M(\omega(u), \omega(v)) = M(u, v)$ .

We denote by

$$K = M(s, m)$$

the transition matrix between  $\{m_\lambda\}$  and  $\{s_\lambda\}$ . We further introduce the transposition matrix  $J$  with

$$J_{\lambda\mu} = \begin{cases} 1 & \lambda = \mu' \\ 0 & \text{else.} \end{cases}$$

Note that we have  $J = M(\omega(s), s)$ . Then we claim that all transition matrices among the bases  $\{m_\lambda\}$ ,  $\{e_\lambda\}$ ,  $\{h_\lambda\}$ , and  $\{s_\lambda\}$  can be expressed in terms of  $J$  and  $K$ :

- $M(s, h) = M(s', m') = M(s, m)^* = K^*$
- $M(s, e) = M(\omega(s), h) = M(\omega(s), s)M(s, h) = JK^*$

Explicit formulas for all transition matrices are now immediately obtained by forming compositions and inverses of these matrices.

The entries  $K_{\lambda\mu}$  of the matrix  $K$  are called *Kostka numbers* and admit an explicit combinatorial formula: Let  $\lambda$  be a partition and  $\alpha$  a composition. We define a *tableau of shape  $\lambda$  and weight  $\mu$*  to be a filling of the boxes in the diagram of  $\lambda$  by natural numbers such that the following conditions are satisfied:

- (1) the numbers in each row are weakly increasing,
- (2) the numbers in each column are strictly increasing,
- (3) the number  $i$  appears  $\alpha_i$  times.

**Example 3.42.** The diagram

1	1	1	4	5
2	3	3		
4				

represents a tableau of shape  $(5, 3, 1)$  and weight  $(3, 1, 2, 2, 1)$ .

**Theorem 3.43.** The entry  $K_{\lambda\mu}$  is given by the number of tableaux of shape  $\lambda$  and weight  $\mu$ .

**Example 3.44.** We have the formula

$$s_{(3)} = m_{(3)} + m_{(2,1)} + m_{(1,1,1)}$$

where the coefficients correspond to the tableaux

1	1	1
1	1	2
1	2	3

of shape  $(3)$ . As another example, we have

$$s_{(2,1)} = m_{(2,1)} + 2m_{(1,1,1)}$$

where the coefficient of  $m_{(2,1)}$  corresponds to the tableau

1	1
2	

and the coefficient of  $m_{(1,1,1)}$  corresponds to the tableaux

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array}.$$

The theorem follows immediately from the following combinatorial formula for Schur functions. We have

$$s_\lambda = \sum_T x^{\text{weight}(T)}$$

where the sum runs over all tableau of shape  $\lambda$ .

We only indicate the ideas of the proof, referring to MacDonald's book (Section I.5) for details:

- (1) We interpret a tableau of shape  $\lambda$  as a sequence

$$0 = \lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(n)} = \lambda$$

where the partition  $\lambda^{(i)}$  contains those boxes with label  $\leq i$ .

- (2) We introduce *skew Schur functions*  $s_{\mu/\nu}$  defined by the formula

$$\langle s_{\mu/\nu}, s_\lambda \rangle = \langle s_\mu, s_\nu s_\lambda \rangle.$$

- (3) We prove a formula

$$s_\lambda(x_1, \dots, x_n) = \sum_{0=\lambda^{(0)} \subset \dots \subset \lambda^{(n)}=\lambda} s_{\lambda^{(n)}/\lambda^{(n-1)}}(x_n) \cdots s_{\lambda^{(1)}/\lambda^{(0)}}(x_1)$$

where the sum ranges over all partitions coming from tableaux of shape  $\lambda$ .

- (4) Finally show  $s_{\lambda^{(i)}/\lambda^{(i-1)}}(x_i) = x_i^{\alpha_i}$  where  $\alpha$  denotes the weight of the corresponding tableau.

### 3.6 Zelevinsky's statistic

Let  $R$  be a DVR with residue field  $k \cong \mathbb{F}_q$ . We have shown that, in the Hall algebra of  $R$ , we have the formula

$$u_{(1^{\lambda'_1})} \cdots u_{(1^{\lambda'_s})} = \sum_{\mu} b_{\lambda\mu} u_{\mu}$$

where  $b_{\lambda\mu}$  is an upper unitriangular matrix. In this and the next lecture, we will show that  $b_{\lambda\mu}$  is the partition function of a statistic and deduce Hall's theorem. We introduce some terminology.

For compositions  $\alpha, \beta$ , we define an array  $A$  of shape  $\alpha$  and weight  $\beta$  to be a labelling of the squares of the diagram of  $\alpha$  such that the number  $i$  appears  $\beta_i$  times. We express an array as a function

$$A : \alpha \longrightarrow \mathbb{N}^+$$

where we consider (the diagram of)  $\alpha$  as a subset of  $\mathbb{N}^+ \times \mathbb{N}^+$ . We will extend  $A$  to all of  $\mathbb{N}^+ \times \mathbb{N}^+$  by letting elements in the complement of  $\alpha$  have value  $\infty$ . For  $x = (i, j) \in$

$\mathbb{N}^+ \times \mathbb{N}^+$ , we denote  $x^\rightarrow = (i, j + 1)$ . We call an array row-ordered (resp. row-strict) if, for every  $x \in \alpha$ ,  $A(x^\rightarrow) \geq A(x)$  (resp.  $A(x^\rightarrow) > A(x)$ ). Analogously, we define column-ordered and column-strict arrays. Further, we define a lexicographic order on  $\mathbb{N}^+ \times \mathbb{N}^+$  via

$$(i, j) < (i', j') \iff \text{either } j < j', \text{ or } j = j' \text{ and } i > i'.$$

Given a row-strict array  $A$  of shape  $\alpha$ , we define

$$d(A) = |\{(x, y) \in \alpha \times \alpha \mid y < x \text{ and } A(x) < A(y) < A(x^\rightarrow)\}|.$$

We now have the following statistical interpretation of the coefficients  $b_{\lambda\mu}$ :

**Theorem 3.45.** We have

$$b_{\lambda\mu} = \sum_A q^{d(A)}$$

where  $A$  ranges over all row-strict arrays of shape  $\mu$  and weight  $\lambda'$ .

We will provide a proof in the next lecture. In this lecture, we will analyze consequences of the theorem.

**Remark 3.46.** Given a row-strict array of shape  $\alpha$  and weight  $\beta$ , we introduce a  $\{0, 1\}$ -matrix which has entry 1 at the positions  $\{(i, A(x))\}$  where  $x = (i, j)$  runs over all  $x \in \alpha$ , and entry 0 elsewhere. It is immediate that this construction provides a bijection between row-strict arrays of shape  $\alpha$  and weight  $\beta$  and  $\{0, 1\}$ -matrices with row sums  $\alpha$  and column sums  $\beta$ . In particular, the theorem states that  $b_{\lambda\mu}$  can be interpreted as the partition function of the statistic  $d$  on the set of  $\{0, 1\}$ -matrices with row sums  $\mu$  and column sums  $\lambda'$  (or equivalently column sums  $\mu$  and row sums  $\lambda'$ ).

We introduce the polynomial  $a_{\lambda\mu}(t) = \sum_A t^{d(A)}$  so that  $b_{\lambda\mu} = a_{\lambda\mu}(q)$ .

**Proposition 3.47.** (1) The polynomial  $a_{\lambda\mu}(t)$  has nonnegative integral coefficients.

(2)  $a_{\lambda\mu}(1)$  is the number of  $\{0, 1\}$ -matrices with row sums  $\lambda'$  and column sums  $\mu$ .

(3)  $a_{\lambda\mu}(t) = 0$  unless  $\mu \leq \lambda$ . Moreover,  $a_{\lambda\lambda}(t) = 1$ .

*Proof.* (1) is obvious, and (2) is a consequence of the above remark. (3) We have  $a_{\lambda\mu}(1) = 0$  unless  $\mu \leq \lambda$  which implies the first statement by (1).  $a_{\lambda\lambda}(t) = 1$  follows from direct computation, using the fact that there is precisely one  $\{0, 1\}$ -matrix with row sums  $\lambda'$  and column sums  $\lambda$  and the corresponding array  $A$  satisfies  $d(A) = 0$ .  $\square$

Therefore, we obtain the following corollary of Theorem 3.45.

**Corollary 3.48.** The structure constants of  $\text{Hall}(R)$  are polynomial in  $q$ : There exist polynomials  $g_{\mu\nu}^\lambda(t)$  such that

$$u_\mu u_\nu = \sum_\lambda g_{\mu\nu}^\lambda(q) u_\lambda.$$

*Proof.* The structure constants with respect to the basis  $\{v_\lambda\} = \{u_{(1^{\lambda'_1})} \cdots u_{(1^{\lambda'_s})}\}$  are constant. The transition matrix from  $\{u_\lambda\}$  to  $\{v_\lambda\}$  is  $(a_{\lambda\mu}(q))$  which has polynomial entries in  $q$ . Further, since, by the above proposition, the matrix  $(a_{\lambda\mu}(t))$  is upper unitriangular, its inverse also has polynomial entries showing that the entries of  $(a_{\lambda\mu}(q))^{-1}$  are polynomial in  $q$ . Therefore, the structure constants with respect to  $\{u_\lambda\}$  must also be polynomial in  $q$ .  $\square$

The following proposition provides more detailed information on the polynomials  $a_{\lambda\mu}(t)$ . For a partition  $\lambda$ , we introduce

$$n(\lambda) = \sum_{i \geq 0} \binom{\lambda_i}{2}.$$

**Proposition 3.49.** For a row-strict array  $A$  of shape  $\mu$  and weight  $\lambda'$  let  $\tilde{d}(A)$  denote the number of pairs  $(x, y) \in \mu \times \mu$  such that  $y$  lies above  $x$  and  $A(x) < A(y) < A(x^\rightarrow)$ . Then we have

- (1)  $d(A) + \tilde{d}(A) = n(\mu) - n(\lambda)$
- (2)  $\tilde{d}(A) = 0$  if and only if  $A$  is column-ordered.

*Proof.* To show (1) we consider the following sets

$$\begin{aligned} D(A) &= \{(x, y) \in \mu \times \mu \mid y < x, A(x) \leq A(y) < A(x^\rightarrow)\} \\ N(\mu) &= \{(x, y) \in \mu \times \mu \mid y \text{ lies above } x\} \\ \tilde{D}(A) &= \{(x, y) \in N(\mu) \mid A(x) < A(y) < A(x^\rightarrow)\}. \end{aligned}$$

We have

$$\begin{aligned} |D(A)| &= d(A) + n(\lambda) \\ |N(\mu)| &= n(\mu) \\ |\tilde{D}(A)| &= \tilde{d}(A) \end{aligned}$$

We claim that there is a bijection

$$\varphi : D(A) \rightarrow N(\mu) \setminus \tilde{D}(A)$$

(this will imply (1)). Let  $(x, y) \in D(A)$ .  $x$  and  $y$  can not lie in the same row. We replace  $x$  by  $x'$  in the same row as  $x$  and the same column as  $y$ . If  $y$  lies above  $x'$  then we set  $\varphi(x, y) = (x', y)$ , else we set  $\varphi(x, y) = (y, x')$ . A case by case analysis shows that  $\varphi$  defines indeed a bijection between the above sets. To show (2) the if part is clear. Assume that  $A$  is not column-ordered. Then we choose a maximal column with a pair  $(x, y)$  such that  $y$  lies above  $x$ ,  $A(x) < A(y)$ . From the fact that the column to the right of  $x$  and  $y$  is ordered, it follows that  $A(y) < A(x^\rightarrow)$  so that  $(x, y)$  is an element of  $\tilde{D}(A)$ .  $\square$

As an immediate consequence we have:

**Corollary 3.50.** (1) The polynomial  $a_{\lambda\mu}(t)$  has degree  $\leq n(\mu) - n(\lambda)$ .

(2) The coefficient of  $t^{n(\mu)-n(\lambda)}$  equals  $K_{\mu'\lambda'}$ .

We renormalize

$$\tilde{a}_{\lambda\mu}(t) = t^{n(\mu)-n(\lambda)} a_{\lambda\mu}(t^{-1})$$

so that we have

- $\tilde{a}_{\lambda\mu}(1) = a_{\lambda\mu}$
- $\tilde{a}_{\lambda\mu}(0) = K_{\mu'\lambda'}$ .

We introduce the family  $\{P_\mu(x, t)\}$  of elements of  $\Lambda[t] = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}[t]$  defined via the formula

$$e_{\lambda'} = \sum_{\mu} \tilde{a}_{\lambda\mu}(t) P_\mu(x, t) \quad (3.51)$$

so that  $(\tilde{a}_{\lambda\mu}(t))$  is the transition matrix from the  $\mathbb{Z}[t]$ -basis  $\{P_\mu(x, t)\}$  to the  $\mathbb{Z}[t]$ -basis  $\{e_{\lambda'}\}$  of  $\Lambda[t]$ . The functions  $P_\lambda(x, t)$  are called *Hall-Littlewood symmetric functions*.

**Theorem 3.52.** We have

- (1)  $P_\lambda(x, 1) = m_\lambda$
- (2)  $P_\lambda(x, 0) = s_\lambda$
- (3) The isomorphism

$$\psi_q : \text{Hall}(R) \otimes \mathbb{Q} \longrightarrow \Lambda \otimes \mathbb{Q}$$

defined by  $\psi_q(u_{(1^r)}) = q^{-\binom{r}{2}} e_r$  satisfies

$$\psi_q(u_\lambda) = q^{-n(\lambda)} P(x, q^{-1}).$$

*Proof.* (1) follows from the fact that  $\tilde{a}_{\lambda\mu}(1) = a_{\lambda\mu}$  is the transition matrix from the basis  $\{m_\mu\}$  to  $\{e_{\lambda'}\}$ . (2) follows since  $K_{\mu'\lambda'}$  is the transition matrix from the basis  $\{s_\mu\}$  to  $\{e_{\lambda'}\}$  (see Lecture 9). Statement (3) follows from applying  $\psi_q$  to the formula

$$u_{(1^{\lambda'_1})} \cdots u_{(1^{\lambda'_s})} = \sum_{\mu} b_{\lambda\mu} u_\mu$$

and comparing with (3.51) evaluated at  $t = q^{-1}$ . □

Introducing the polynomials  $f_{\mu\nu}^\lambda(t)$  defined by

$$P_\mu(x, t) P_\nu(x, t) = \sum_{\lambda} f_{\mu\nu}^\lambda(t) P_\lambda(x, t)$$

we have corresponding statements about structure constants:

- (1)  $f_{\mu\nu}^\lambda(1)$  are the structure constants of  $\Lambda$  with respect to the basis  $\{m_\lambda\}$ .
- (2)  $f_{\mu\nu}^\lambda(0)$  are the structure constants of  $\Lambda$  with respect to  $\{s_\lambda\}$ .

(3) We have

$$g_{\mu\nu}^\lambda(t) = t^{n(\lambda)-n(\mu)-n(\nu)} f_{\mu\nu}^\lambda(t^{-1}).$$

which follow immediately from Theorem 3.52. In particular, we obtain Hall's theorem:

**Theorem 3.53.** The polynomials  $g_{\mu\nu}^\lambda(t)$  have degree  $\leq n(\lambda) - n(\mu) - n(\nu)$  and the coefficients of  $t^{n(\lambda)-n(\mu)-n(\nu)}$  are the structure constants of  $\Lambda$  with respect to the Schur basis  $\{s_\lambda\}$ .

We conclude our analysis of  $\text{Hall}(R)$  for a DVR  $R$  with  $k \cong \mathbb{F}_q$  by proving Zelevinsky's theorem. We prove a slightly stronger version which will allow for an inductive argument.

**Theorem 3.54.** Let  $\mu, \lambda$  be partitions and let  $\alpha \sim \mu$  be a composition which is a permutation of  $\mu$ . Then we have

$$b_{\lambda\mu} = \sum_A q^{d(A)}$$

where  $A$  ranges over all arrays of shape  $\alpha$  and weight  $\lambda'$ .

*Proof. Step 1.* We reformulate the formula in terms of sequences of compositions: For  $\alpha, \beta$  compositions, we write  $\beta \dashv \alpha$  if, for all  $i$ ,  $\alpha_i - 1 \leq \beta_i \leq \alpha_i$ . For  $\beta \dashv \alpha$ , we define

$$d(\alpha, \beta) = |\{(i, j) \mid \beta_i = \alpha_i, \beta_j = \alpha_j - 1, \text{ and } (j, \alpha_j) < (i, \alpha_i)\}|$$

We then observe that, given  $\alpha$  composition and  $\lambda$  partition with  $l(\lambda') \leq s$ , we have a natural bijection between

$$\{\text{row-strict arrays } A \text{ of shape } \alpha \text{ and weight } \lambda'\}$$

and

$$\{\text{sequences } 0 = \alpha^{(0)} \dashv \alpha^{(1)} \dashv \dots \dashv \alpha^{(s)} = \alpha \text{ with } |\alpha^{(i)}| - |\alpha^{(i-1)}| = \lambda'_i\}.$$

Under this correspondence, we have

$$d(A) = \sum_{i \geq 0} d(\alpha^{(i)}, \alpha^{(i-1)}).$$

**Step 2.** Induction: Assume that

$$u_{(1^{\lambda'_1})} \cdots u_{(1^{\lambda'_{s-1}})} = \sum_{\nu} \left( \sum_{0=\beta^{(0)} \dashv \dots \dashv \beta^{(s-1)}=\beta} \prod_{i \geq 1} q^{d(\beta^{(i)}, \beta^{(i-1)})} \right) u_{\nu}$$

where, for each  $\nu$ ,  $\beta$  is a fixed permutation of  $\nu$ . Then to show that

$$u_{(1^{\lambda'_1})} \cdots u_{(1^{\lambda'_s})} = \sum_{\mu} \left( \sum_{0=\alpha^{(0)} \dashv \dots \dashv \alpha^{(s)}=\alpha} \prod_{i \geq 1} q^{d(\alpha^{(i)}, \alpha^{(i-1)})} \right) u_{\mu}$$

where, for each  $\mu$ ,  $\alpha$  is a fixed permutation of  $\mu$ , it suffices to show

$$u_{\nu} u_{(1^r)} = \sum_{\mu} \left( \sum_{\beta \dashv \alpha, |\alpha| - |\beta| = r, \beta \sim \nu} q^{d(\alpha, \beta)} \right) u_{\mu}$$

where, for each  $\mu$ ,  $\alpha$  is any fixed permutation of  $\mu$ .

**Step 3.** In other words, we have to verify for the structure constant  $g_{\nu(1^r)}^\mu$  of  $\text{Hall}(R)$ , the formula

$$g_{\nu(1^r)}^\mu = \sum_{\beta} q^{d(\alpha, \beta)}$$

where  $\alpha$  is a fixed permutation of  $\mu$  and  $\beta$  runs through all permutations of  $\nu$  such that  $\beta \dashv \alpha$ , and  $|\alpha| - |\beta| = r$ . Recall that, for a fixed  $R$ -module  $M$  of type  $\mu$  the number  $g_{\nu(1^r)}^\mu$  is the number of submodules  $N \subset M$  such that  $N$  has type  $(1^r)$  and  $M/N$  has type  $\nu$ . In particular, we have  $\mathfrak{m}N = 0$ , so that  $N$  is a  $r$ -dimensional  $k$ -subspace of the  $k$ -vector space given by the socle  $S = \{x \in M \mid \mathfrak{m}x = 0\}$  of  $M$ . We denote by  $G_r(S)$  the set of all  $r$ -dimensional subspaces of  $S$ . For every choice of a basis  $\{v_i \mid i \in I\}$  of  $S$  together with a total order of  $I$ , the set  $G_r(S)$  is the disjoint union of *Schubert cells* defined as follows: We have one Schubert cell  $C_J$  for every  $r$ -subset  $J$  of  $I$ . The elements of  $C_J$  have coordinates  $(c_{ij} \in k \mid j \in J, i \in I \setminus J, i > j)$  where the subspace of  $S$  corresponding to a coordinate  $(c_{ij})$  has basis  $\{v_j + \sum_i c_{ij}v_i\}_{j \in J}$ .

**Problem 3.55.** Verify that  $G_r(S)$  is the disjoint union of the Schubert cells  $C_J$ .

Therefore, we have

$$|C_J| = q^{d(J)}$$

where  $d(J)$  is the number of pairs  $(i, j)$  such that  $j \in J, i \in I \setminus J, i > j$ . Suppose

$$M \cong \bigoplus_{i \in I} R/\mathfrak{m}^{\alpha_i}$$

We order  $I$  so that  $j < i$  iff  $(j, \alpha_j) < (i, \alpha_i)$ . Then we have a bijective correspondence between subsets  $J \subset I$  and compositions  $\beta \dashv \alpha$  (where  $\beta_i = \alpha_{i-1}$  iff  $i \in J$ ). Under this correspondence, we have  $d(J) = d(\alpha, \beta)$ . Further, for all  $k$ -subspaces  $N \subset S \subset M$  which lie in a fixed Schubert cell  $C_J$ , the quotient  $M/N$  has the same type  $\lambda \sim \beta$ . Therefore, only those Schubert cells so that  $M/N$  has type  $\nu$  contribute to the count and we obtain precisely the claimed formula.  $\square$

## 4 Hall algebras via groupoids

### 4.1 2-pullbacks

Let  $\mathcal{C}$  be a category. Consider a diagram

$$X \longrightarrow Z \longleftarrow Y$$

in  $\mathcal{C}$  and assume that the pullback  $X \times_Z Y$  exists. From the universal property of pullbacks, we deduce:

(A) An isomorphism

$$\begin{array}{ccccc} X & \longrightarrow & Z & \longleftarrow & Y \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ X' & \longrightarrow & Z' & \longleftarrow & Y' \end{array}$$

of diagrams in  $\mathcal{C}$  induces an isomorphism of pullbacks

$$X \times_Z Y \xrightarrow{\cong} X' \times_{Z'} Y'.$$

**Example 4.1.** Consider a diagram of categories

$$\mathcal{A} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B} \quad (4.2)$$

where  $F$  and  $G$  are functors. Then the pullback

$$\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$$

exists, and can be explicitly described as follows: The objects are given by pairs  $(a, b)$  where  $a$  is an object of  $\mathcal{A}$ , and  $b$  is an object of  $\mathcal{B}$ , so that  $F(a) = G(b)$ . A morphism  $(a, b) \rightarrow (a', b')$  is given by a pair of morphisms  $f : a \rightarrow a'$ , and  $g : b \rightarrow b'$  so that  $F(f) = G(g)$ .

According to (A), an isomorphism of diagrams of categories as in (4.2) induces an isomorphism of pullbacks. On the other hand, since we are typically interested in categories up to equivalence, we may hope that an *equivalence of diagrams* as in (4.2) induces an *equivalence* of pullbacks. Unfortunately, this fails:

**Example 4.3.** Let  $\mathcal{A}$  be a category and consider the diagram

$$\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \times \mathcal{A} \xleftarrow{\Delta} \mathcal{A}.$$

We have

$$\mathcal{A} \times_{\mathcal{A} \times \mathcal{A}} \mathcal{A} \cong \mathcal{A}.$$

Consider the commutative diagram of categories

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\Delta} & \mathcal{A} \times \mathcal{A} & \xleftarrow{\Delta} & \mathcal{A} \\ \downarrow \text{id} & & \downarrow \text{id} & & \downarrow F \\ \mathcal{A} & \xrightarrow{\Delta} & \mathcal{A} \times \mathcal{A} & \xleftarrow{G} & \underline{\text{Iso}}(\mathcal{A}) \end{array} \quad (4.4)$$

Here  $\underline{\text{Iso}}(\mathcal{A})$  denotes the category with objects given by isomorphisms  $a \xrightarrow{\cong} b$  and morphisms given by commutative diagrams

$$\begin{array}{ccc} a & \xrightarrow{\cong} & b \\ \downarrow & & \downarrow \\ a' & \xrightarrow{\cong} & b' \end{array},$$

the functor  $F$  is given by the assignment  $a \mapsto (a \xrightarrow{\text{id}} a)$  and the functor  $G$  maps  $a \xrightarrow{\cong} b$  to  $(a, b)$ . It is immediate that  $\Delta$  is fully faithful and essentially surjective so that (4.4) is an equivalence of diagrams. However, the induced map on pullbacks

$$\mathcal{A} \times_{\mathcal{A} \times \mathcal{A}} \mathcal{A} \rightarrow \mathcal{A} \times_{\mathcal{A} \times \mathcal{A}} \underline{\text{Iso}}(\mathcal{A})$$

is given by the functor

$$H : \mathcal{A} \rightarrow \underline{\text{Aut}}(\mathcal{A})$$

where  $\underline{\text{Aut}}(\mathcal{A})$  denotes the category with objects given by automorphisms  $a \xrightarrow{\cong} a$  and morphisms by commutative diagrams

$$\begin{array}{ccc} a & \xrightarrow{\cong} & a \\ \downarrow f & & \downarrow f \\ a' & \xrightarrow{\cong} & a' \end{array},$$

and  $H$  is given by the assignment  $a \mapsto \text{id}_a$ . In particular, the functor  $H$  is not an equivalence of categories as soon as  $\mathcal{A}$  has objects with nontrivial automorphisms.

Our solution to this problem will be to modify our concept of pullback. Given a diagram

$$\mathcal{A} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}.$$

of categories, we introduce the *2-pullback* to be the category

$$\mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B}$$

with objects given by triples  $(a, b, \varphi)$  where  $a \in \mathcal{A}$ ,  $b \in \mathcal{B}$  and  $\varphi : F(a) \xrightarrow{\cong} G(b)$ . A morphism  $(a, b, \varphi) \rightarrow (a', b', \varphi')$  is given by a pair of morphisms  $f : a \rightarrow a'$ ,  $g : b \rightarrow b'$  such that the diagram

$$\begin{array}{ccc} F(a) & \xrightarrow{\varphi} & G(b) \\ \downarrow F(f) & & \downarrow G(g) \\ F(a') & \xrightarrow{\varphi'} & G(b') \end{array}$$

commutes.

Note that we have a diagram of categories

$$\begin{array}{ccc} \mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B} & \xrightarrow{F'} & \mathcal{B} \\ G' \downarrow & \eta \nearrow & \downarrow G \\ \mathcal{A} & \xrightarrow{F} & \mathcal{C} \end{array}$$

where  $\eta : FG' \Rightarrow GF'$  is a natural isomorphism of functors. We call a diagram

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{F'} & \mathcal{B} \\ G' \downarrow & \eta \nearrow & \downarrow G \\ \mathcal{A} & \xrightarrow{F} & \mathcal{C} \end{array}$$

a *2-pullback diagram* if the functor  $\mathcal{X} \rightarrow \mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B}$  given by the assignment  $x \mapsto (G'(x), F'(x), \eta(x))$  is an equivalence.

**Proposition 4.5.** An equivalence of diagrams of functors

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} & \xleftarrow{G} & \mathcal{B} \\ R \downarrow \simeq & & T \downarrow \simeq & & S \downarrow \simeq \\ \mathcal{A}' & \xrightarrow{F'} & \mathcal{C}' & \xleftarrow{G'} & \mathcal{B}' \end{array}$$

induces an equivalence

$$H : \mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B} \longrightarrow \mathcal{A}' \times_{\mathcal{C}'}^{(2)} \mathcal{B}'$$

of 2-pullbacks.

*Proof.* The functor  $H$  is given by the assignment  $(a, b, \varphi) \mapsto (R(a), S(b), T(\varphi))$ . Fully faithfulness is verified immediately. To show that  $H$  is essentially surjective, let  $(a', b', \varphi')$  be an object in  $\mathcal{A}' \times_{\mathcal{C}'}^{(2)} \mathcal{B}'$ . Since  $R$  and  $S$  are essentially surjective, there exist  $a, b$  and isomorphisms  $R(a) \rightarrow a'$ ,  $S(b) \rightarrow b'$ . There is a unique isomorphism  $\psi : R(a) \rightarrow S(b)$  which makes the diagram

$$\begin{array}{ccc} R(a) & \xrightarrow{\psi} & S(b) \\ \downarrow & & \downarrow \\ a' & \xrightarrow{\varphi'} & b' \end{array}$$

commute. Since the functor  $T$  is fully faithful, there exists a (unique) isomorphism  $\varphi : a \rightarrow b$  such that  $T(\varphi) = \psi$ . The triple  $(a, b, \varphi)$  defines an object of  $\mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B}$  such that  $H(a, b, \varphi)$  is isomorphic to  $(a', b', \varphi')$ .  $\square$

**Remark 4.6.** Our definition of a 2-pullback square is slightly ad hoc. It can be shown that 2-pullback squares are characterized, up to equivalence, by a 2-universal property which is version of the usual universal property of the pullback in which all commutative diagrams only commute up to a specified natural isomorphism. We will not spell out the details.

**Problem 4.7.** Suppose that we have a diagram

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{G} & \mathcal{C} \\ R \downarrow & \lambda \nearrow & T \downarrow & \eta \nearrow & S \downarrow \\ \mathcal{A}' & \xrightarrow{F'} & \mathcal{B}' & \xrightarrow{G'} & \mathcal{C}' \end{array}$$

where both squares are 2-pullback squares. Show that the outer rectangle equipped with the natural isomorphism

$$G'F'R \xRightarrow{G' \circ \lambda} G'TF \xRightarrow{\eta \circ F} SGF$$

is a 2-pullback square.

It turns out that in some cases, ordinary pullback squares of categories are actually 2-pullback squares: A functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  is called *isofibration* if, for every object  $a \in \mathcal{A}$  and every isomorphism  $\varphi : F(a) \xrightarrow{\cong} b$  in  $\mathcal{B}$ , there exists an isomorphism  $\tilde{\varphi} : a \rightarrow a'$  in  $\mathcal{A}$  such that  $F(\tilde{\varphi}) = \varphi$ .

**Proposition 4.8.** Let

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow{F} & \mathcal{C} \end{array}$$

be a commutative square of categories and assume that  $F$  is an isofibration. Then the square is a 2-pullback square.

*Proof.* We may assume that  $\mathcal{X} = \mathcal{A} \times_{\mathcal{C}} \mathcal{B}$  and have to verify that the functor

$$H : \mathcal{A} \times_{\mathcal{C}} \mathcal{B} \rightarrow \mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B}$$

given by the assignment  $(a, b) \mapsto (a, b, \text{id} : F(a) \rightarrow G(b))$  is an equivalence of categories. Again fully faithfulness is immediate. To show that  $H$  is essentially surjective, let  $(a, b, \varphi)$  be an object of  $\mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B}$ . Since  $F$  is an isofibration, there exists a morphism  $\tilde{\varphi} : a \rightarrow a'$  such that  $F(\tilde{\varphi}) = \varphi$ . It is immediate to verify that we have  $H(a', b) \cong (a, b, \varphi)$ .  $\square$

**Example 4.9.** (1) The functor  $\underline{\text{Iso}}(\mathcal{A}) \rightarrow \mathcal{A}, (a \xrightarrow{\cong} b) \mapsto (a, b)$  is an isofibration. Therefore, the pullback square

$$\begin{array}{ccc} \underline{\text{Aut}}(\mathcal{A}) & \longrightarrow & \underline{\text{Iso}}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{A} \times \mathcal{A} \end{array}$$

is a 2-pullback square.

(2) By (1), the pullback square

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ \mathcal{A} & \longrightarrow & \mathcal{A} \times \mathcal{A} \end{array}$$

can not in general be a 2-pullback square, since otherwise, the equivalence of diagrams (4.4) would imply that the induced functor

$$\mathcal{A} \longrightarrow \underline{\text{Aut}}(\mathcal{A})$$

is an equivalence. We have seen that this is not the case as soon as  $\mathcal{A}$  has objects with nontrivial automorphisms. In agreement with the above proposition, the diagonal functor  $\mathcal{A} \rightarrow \mathcal{A} \times \mathcal{A}$  is not an isofibration.

**Lemma 4.10.** For any diagram

$$\mathcal{A} \xrightarrow{F} \mathcal{C} \xleftarrow{G} \mathcal{B}$$

of categories, there exists an equivalent diagram

$$\mathcal{A}' \xrightarrow{F'} \mathcal{C} \xleftarrow{G} \mathcal{B}$$

where  $F'$  is an isofibration.

*Proof.* We define  $\mathcal{A}'$  to be the category with objects given by triples  $(a, c, \varphi)$  where  $a \in \mathcal{A}$ ,  $c \in \mathcal{C}$ , and  $\varphi : F(a) \rightarrow c$  is an isomorphism. A morphism consists of pair of morphisms  $f : a \rightarrow a'$ ,  $g : c \rightarrow c'$ , such that the diagram

$$\begin{array}{ccc} F(a) & \xrightarrow{\varphi} & c \\ F(f) \downarrow & & \downarrow g \\ F(a') & \xrightarrow{\varphi'} & c' \end{array}$$

commutes. We define a functor  $i : \mathcal{A} \rightarrow \mathcal{A}'$  by the association  $a \mapsto (a, F(a), \text{id})$  and a functor  $F' : \mathcal{A}' \rightarrow \mathcal{C}$  by  $(a, c, \varphi) \mapsto c$ . It is immediate to verify that  $i$  is an equivalence so that we have an equivalence of diagrams

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{F} & \mathcal{C} & \xleftarrow{G} & \mathcal{B} \\ \downarrow i & & \downarrow \text{id} & & \downarrow \text{id} \\ \mathcal{A}' & \xrightarrow{F'} & \mathcal{C} & \xleftarrow{G} & \mathcal{B} \end{array} .$$

□

**Remark 4.11.** The above lemma shows that we may always compute 2-pullbacks as ordinary pullbacks if we are willing to replace the given diagram by an equivalent one.

## 4.2 Groupoids of flags

A *groupoid* is a category in which all morphisms are invertible.

**Example 4.12.** (1) Let  $G$  be a group. Then we can define a groupoid  $BG$  which has one object  $*$  and  $\text{Hom}(*, *) = G$  where the composition of morphisms is given by the group law.

(2) Let  $X$  be a topological space. Then we can define the fundamental groupoid  $\Pi(X)$  whose objects are the points of  $X$  and a morphism between  $x$  and  $y$  is a homotopy class of continuous paths connecting  $x$  to  $y$ .

(3) Let  $\mathcal{C}$  be a category. Then can form the *maximal groupoid*  $\mathcal{C}^\cong$  of  $\mathcal{C}$  by simply discarding all noninvertible morphisms in  $\mathcal{C}$ .

We introduce a family of groupoids which will be of central relevance for this section. Let  $\mathcal{C}$  be a proto-abelian category. Let  $\mathcal{X}_n = \mathcal{X}_n(\mathcal{C})$  denote the maximal groupoid in the category of diagrams

$$\begin{array}{ccc}
 0 \hookrightarrow A_{0,1} \hookrightarrow A_{0,2} \hookrightarrow \dots & A_{0,n-1} \hookrightarrow A_{0,n} & \\
 \downarrow & \downarrow & \\
 0 \hookrightarrow A_{1,2} \hookrightarrow \dots & A_{1,n-1} \hookrightarrow A_{1,n} & \\
 \downarrow & \downarrow & \\
 0 & \vdots & \vdots \\
 & \ddots & \\
 & A_{n-2,n-1} \hookrightarrow A_{n-2,n} & \\
 & \downarrow & \downarrow \\
 & 0 \hookrightarrow A_{n-1,n} & \\
 & \downarrow & \\
 & 0 & 
 \end{array} \tag{4.13}$$

in  $\mathcal{C}$  where  $0$  is a fixed zero object in  $\mathcal{C}$  and all squares are required to be biCartesian. A crucial observation is that the various groupoids  $\mathcal{X}_\bullet$  are related to one another: for example, for every  $0 \leq k \leq n$ , we have a functor

$$\partial_k : \mathcal{X}_n \rightarrow \mathcal{X}_{n-1}$$

obtained by omitting in the diagram (4.13) the objects in the  $k$ th row and  $k$ th column and forming the composite of the remaining morphisms. Similarly, for every  $0 \leq k \leq n$ , we have functors

$$\sigma_k : \mathcal{X}_n \rightarrow \mathcal{X}_{n+1}$$

given by replacing the  $k$ th row by two rows connected via identity maps and replacing the  $k$ th column by two columns connected via identity maps.

**Example 4.14.** In this section, we will focus on the family of groupoids  $\mathcal{X}_\bullet$ ,  $n \leq 3$ ,

$$\mathcal{X}_0 \rightleftarrows \mathcal{X}_1 \rightleftarrows \mathcal{X}_2 \rightleftarrows \mathcal{X}_3 \quad (4.15)$$

where we have indicated the functors  $\{\partial_k\}$  and  $\{\sigma_k\}$ . We have:

- The groupoid  $\mathcal{X}_0$  is the discrete groupoid  $\{0\}$  with one object.
- The groupoid  $\mathcal{X}_1$  can be identified with the maximal groupoid in the category  $\mathcal{C}$ . The functor  $\sigma_0 : \mathcal{X}_0 \rightarrow \mathcal{X}_1$  is given by  $0 \mapsto 0$ .
- The groupoid  $\mathcal{X}_2$  can be identified with the groupoid of short exact sequences in  $\mathcal{C}$ . The three functors  $\partial_* : \mathcal{X}_2 \rightarrow \mathcal{X}_1$  associate to a short exact sequence the three involved objects, respectively. The functor  $\sigma_0 : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  associates to an object  $A$ , the short exact sequence  $A \xrightarrow{\text{id}} A \rightarrow 0$ . The functor  $\sigma_1 : \mathcal{X}_1 \rightarrow \mathcal{X}_2$  associates to an object  $A$ , the short exact sequence  $0 \hookrightarrow A \xrightarrow{\text{id}} A$ .

**Proposition 4.16.** Let  $\mathcal{C}$  be a proto-abelian category.

- (1) The commutative squares of groupoids

$$\begin{array}{ccc} \mathcal{X}_3 & \xrightarrow{\partial_1} & \mathcal{X}_2 \\ \partial_3 \downarrow & & \downarrow \partial_2 \\ \mathcal{X}_2 & \xrightarrow{\partial_1} & \mathcal{X}_1 \end{array} \quad \begin{array}{ccc} \mathcal{X}_3 & \xrightarrow{\partial_2} & \mathcal{X}_2 \\ \partial_0 \downarrow & & \downarrow \partial_0 \\ \mathcal{X}_2 & \xrightarrow{\partial_1} & \mathcal{X}_1 \end{array} \quad (4.17)$$

are 2-pullback squares.

- (2) The commutative squares of groupoids

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\sigma_0} & \mathcal{X}_2 \\ \downarrow & & \downarrow \partial_0 \\ \mathcal{X}_0 & \xrightarrow{\sigma_0} & \mathcal{X}_1 \end{array} \quad \begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\sigma_1} & \mathcal{X}_2 \\ \downarrow & & \downarrow \partial_1 \\ \mathcal{X}_0 & \xrightarrow{\sigma_0} & \mathcal{X}_1 \end{array} \quad (4.18)$$

are 2-pullback squares.

*Proof.* (1) We denote by  $\mathcal{M}_n$  the maximal groupoid in the category

$$\{A_1 \hookrightarrow A_2 \hookrightarrow \dots \hookrightarrow A_n\}$$

of chains of monomorphisms in  $\mathcal{C}$ . By the argument of Corollary 2.20, the forgetful functor

$$\mathcal{X}_n \longrightarrow \mathcal{M}_n$$

is an equivalence of categories. Combining these functors, we obtain an equivalence between the left-hand square in (4.17) and the commutative square

$$\begin{array}{ccc} \mathcal{M}_3 & \longrightarrow & \mathcal{M}_2 \\ \downarrow & & \downarrow \\ \mathcal{M}_2 & \longrightarrow & \mathcal{M}_1 \end{array} \quad (4.19)$$

where the functors are given by

$$\begin{array}{ccccc} A_1 & \hookrightarrow & A_2 & \hookrightarrow & A_3 & \xrightarrow{\quad} & A_2 & \hookrightarrow & A_3 \\ & & \downarrow & & & & \downarrow & & \\ & & A_1 & \hookrightarrow & A_2 & \xrightarrow{\quad} & A_2 & & \end{array}$$

By Proposition 4.5, it suffices to verify that (4.19) is a 2-pullback square. Since all functors in the square are isofibrations, it suffices to verify that (4.19) is an ordinary pullback square. This follows by explicit verification.

To show that the right-hand square in (4.17) is a 2-pullback square, we apply the analogous argument using the category  $\mathcal{E}_n$  of chains of epimorphisms in  $\mathcal{C}$  instead of the category  $\mathcal{M}_n$ .

(2) We verify the statement for the left-hand square of (4.18). Since the functor  $\partial_0$  is an isofibration, it suffices to show that the canonical functor

$$(0, \sigma_0) : \mathcal{X}_1 \longrightarrow \mathcal{X}_0 \times_{\mathcal{X}_1} \mathcal{X}_2$$

is an equivalence. The right-hand side is the groupoid of short exact sequences with cokernel equal to 0. But such a short exact sequence is of the form

$$A \xrightarrow{\cong} B \twoheadrightarrow 0$$

so that the right-hand side is isomorphic to the groupoid of isomorphisms in  $\mathcal{C}$ . We now explicitly verify that the functor  $(0, \sigma_0)$  is an equivalence.  $\square$

Below, we will give a new construction of the Hall algebra of a proto-abelian category  $\mathcal{C}$  in which associativity will be an immediate consequence of the 2-pullback squares (4.17) and unitality will be a consequence of the 2-pullback squares (4.18).

**Remark 4.20.** A more systematic way to organize the collection of groupoids  $\mathcal{X}_\bullet$  is as follows. We define the category  $\Delta$  with objects given by the linearly ordered sets  $[n] = \{0, 1, \dots, n\}$  and morphisms given by maps of underlying sets which preserve the linear order  $\leq$ . Then the above functorialities assemble to a functor

$$\mathcal{X} : \Delta^{\text{op}} \longrightarrow \mathbf{Grpd}$$

with values in the category of groupoids. We will study this point of view in more detail below.

### 4.3 Groupoid cardinality and integrals

Let  $K, S$  be finite sets. We have

$$|K \amalg S| = |K| + |S|$$

and

$$|K \times S| = |K||S|$$

so that the categorical operations  $\amalg$  and  $\times$  yield upon application of  $|\cdot|$  the numerical operations of addition and multiplication. One may wonder if there is a categorical analogue of division (or subtraction).

**Example 4.21.** Consider the set  $K = \{1, 2, 3, 4\}$  equipped with the action of the cyclic group  $C_2 = \langle \tau \rangle$  of order 2 where  $\tau$  acts via the permutation (14)(23). Then we can form the orbit set  $K/C_2$  and have

$$|K/C_2| = 2 = |K|/|C_2|$$

so that categorical construction of forming the quotient yields division by 2 upon application of  $|\cdot|$ . However, this interpretation fails as soon as the group action has nontrivial stabilizers: Letting the group  $C_2$  act on the set  $S = \{1, 2, 3\}$  via the permutation (13), we obtain a quotient  $S/C_2$  of cardinality  $2 \neq \frac{3}{2}$ .

We will now define a notion of cardinality for groupoids which solves the issue of the example. From now on we use the notation  $\pi_0(\mathcal{A})$  for the set of isomorphism classes of objects in  $\mathcal{A}$ . A groupoid  $\mathcal{A}$  is called *finite* if

- (1) the set  $\pi_0(\mathcal{A})$  of isomorphism classes of objects is finite,
- (2) for every object  $a \in \mathcal{A}$ , the set of automorphisms of  $a$  is finite.

Given a finite groupoid  $\mathcal{A}$ , we introduce the *groupoid cardinality*

$$|\mathcal{A}| = \sum_{[a] \in \pi_0(\mathcal{A})} \frac{1}{|\text{Aut}(a)|}.$$

**Remark 4.22.** It is immediate from the definition, that groupoid cardinality is invariant under equivalences of finite groupoids.

**Example 4.23.** Any finite set  $K$  can be interpreted as a *discrete* groupoid with  $K$  as its set of objects and morphisms given by identity morphisms only. The groupoid cardinality of the discrete groupoid associated with  $K$  agrees with the cardinality of the set  $K$ .

Let  $K$  be a finite set equipped with a right action of a finite group  $G$ . We define the *action groupoid*  $K//G$  to have  $K$  as its set of objects and morphisms between two elements  $k$  and  $k'$  given by elements  $g \in G$  such that  $k.g = k'$ .

**Proposition 4.24.** We have

$$|K//G| = |K|/|G|.$$

*Proof.* The set of isomorphism classes  $\pi_0(K//G)$  can be identified with the set of orbits of the action of  $G$  on  $K$ . The automorphism group of an object  $k$  of  $K//G$  coincides with the stabilizer group  $G_k = \{g \in G \mid k.g = k\} \subset G$ . The orbit of an element  $k$  under  $G$  can be identified with the quotient set  $G/G_k$  of cardinality  $|G|/|G_k|$ . We compute

$$\begin{aligned} |K//G| &= \sum_{[k] \in \pi_0(K//G)} \frac{1}{|G_k|} \\ &= \frac{1}{|G|} \sum_{[k] \in \pi_0(K//G)} \frac{|G|}{|G_k|} \\ &= \frac{|K|}{|G|} \end{aligned}$$

where the last equality follows since the disjoint union of the orbits yields the set  $K$ .  $\square$

**Proposition 4.25.** Let  $\{*\}$  denote the discrete groupoid with one object and let

$$\begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{B} \\ \downarrow & \eta \nearrow & \downarrow \\ \{*\} & \longrightarrow & \mathcal{C} \end{array}$$

be a 2-pullback square of groupoids such that  $\mathcal{A}$  and  $\mathcal{C}$  are finite, and further  $|\pi_0(\mathcal{C})| = 1$ . Then  $\mathcal{B}$  is finite and we have

$$|\mathcal{B}| = |\mathcal{A}||\mathcal{C}|.$$

*Proof.* We start by noting that  $\pi_0(\mathcal{B})$  is finite, since the map  $\pi_0(\mathcal{A}) \rightarrow \pi_0(\mathcal{B})$  is surjective (which follows immediately from  $|\pi_0(\mathcal{C})| = 1$ ) and  $\mathcal{A}$  is finite. As a first step, we use Lemma 4.10 and the fact that groupoid cardinality is invariant under equivalences, so that we may assume that the diagram is a commutative pullback diagram of the form

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{G} & \mathcal{B} \\ \downarrow & & \downarrow F \\ \{*\} & \longrightarrow & \mathcal{C} \end{array}$$

where  $F$  is an isofibration. As for any groupoid, we have a decomposition

$$\mathcal{B} \cong \coprod_{[b] \in \pi_0(\mathcal{B})} \mathcal{B}(b)$$

where  $\mathcal{B}(b)$  denotes the full subcategory of  $\mathcal{B}$  consisting of all objects isomorphic to  $b$ . Similarly, we have a decomposition

$$\mathcal{A} \cong \coprod_{[b] \in \pi_0(\mathcal{B})} \mathcal{A}(b)$$

where  $\mathcal{A}(b)$  denotes the full subcategory of  $\mathcal{A}$  consisting of objects  $a \in \mathcal{A}$  such that  $G(a)$  is isomorphic to  $b$ . We obtain, for every  $[b] \in \pi_0(\mathcal{B})$ , a pullback square

$$\begin{array}{ccc} \mathcal{A}(b) & \longrightarrow & \mathcal{B}(b) \\ \downarrow & & \downarrow F|_{\mathcal{B}(b)} \\ \{*\} & \longrightarrow & \mathcal{C} \end{array}$$

where  $F|_{\mathcal{B}(b)}$  is an isofibration. It suffices to show

$$|\mathcal{C}||\mathcal{A}(b)| = |\mathcal{B}(b)|$$

since this implies

$$|\mathcal{C}||\mathcal{A}| = \sum_{[b] \in \pi_0(\mathcal{B})} |\mathcal{C}||\mathcal{A}(b)| = \sum_{[b] \in \pi_0(\mathcal{B})} |\mathcal{B}(b)| = |\mathcal{B}|$$

where we note that the sums are finite since  $\pi_0(\mathcal{B})$  is finite. In other words, we may assume that  $|\pi_0(\mathcal{B})| = 1$  (and, by assumption,  $|\pi_0(\mathcal{C})| = 1$ ). In this case, since  $F$  is an isofibration, we may find an object  $b \in \mathcal{B}$  such that  $F(b) = c$  where  $c$  is the image of  $*$  in  $\mathcal{C}$ . Therefore, we have an equivalence of diagrams

$$\begin{array}{ccccc} \{*\} & \longrightarrow & BG & \longleftarrow & BH \\ \downarrow \text{id} & & \downarrow \simeq & & \downarrow \simeq \\ \{*\} & \longrightarrow & \mathcal{C} & \longleftarrow & \mathcal{B} \end{array}$$

where  $G = \text{Aut}_{\mathcal{C}}(c)$ ,  $H = \text{Aut}_{\mathcal{B}}(b)$  and the vertical functors map the base point of  $BG$  (resp.  $BH$ ) to  $c$  (resp.  $b$ ). Any functor  $BH \rightarrow BG$  corresponds to a group homomorphism  $H \rightarrow G$  which we denote by  $\varphi$ . We now explicitly compute the 2-pullback

$$\{*\} \times_{BG} BH.$$

The objects are given by triples  $(*, *_{BH}, g : *_{BG} \rightarrow *_{BG})$  where  $g \in G$  so that the set of objects can be identified with  $G$ . A morphism from  $g$  to  $g'$  corresponds to a morphism  $h : *_{BH} \rightarrow *_{BH}$  such that the diagram

$$\begin{array}{ccc} *_{BG} & \xrightarrow{g} & *_{BG} \\ \downarrow \text{id} & & \downarrow \varphi(h) \\ *_{BG} & \xrightarrow{g'} & *_{BG} \end{array}$$

commutes. This condition is equivalent to  $\varphi(h)g = g'$ . This groupoid can be identified with the action groupoid of the  $H$ -action on  $G$  given by  $g.h = \varphi(h^{-1})g$ . At this point, we deduce that the group  $H$  must be finite, since otherwise, the action would have infinite stabilizers which contradicts the finiteness (by assumption) of the action groupoid. This implies that the groupoid  $\mathcal{B}$  is finite. We further have

$$|\{*\} \times_{BG} BH| = |G/H| = \frac{|G|}{|H|}.$$

which implies the claimed formula

$$|BH| = |BG||\{*\} \times_{BG} BH|.$$

□

Given a set  $K$  and a function  $\varphi : K \rightarrow \mathbb{Q}$  with finite support, we can introduce the integral

$$\int_K \varphi = \sum_{k \in K} \varphi(k).$$

If  $K$  is finite, then we have  $\int_K \mathbb{1} = |K|$  where  $\mathbb{1}$  denotes the constant function on  $K$  with value 1. We give a generalization to groupoids.

Given a groupoid  $\mathcal{A}$ , we define  $\mathcal{F}(\mathcal{A})$  to be the  $\mathbb{Q}$ -vector space of functions  $\varphi : \text{ob } \mathcal{A} \rightarrow \mathbb{Q}$  which are

- (1) constant on isomorphism classes,
- (2) nonzero on only finitely many isomorphism classes.

We call  $\mathcal{A}$  *locally finite* if every connected component  $\mathcal{A}(a)$  is finite. Given a locally finite groupoid  $\mathcal{A}$  and  $\varphi \in \mathcal{F}(\mathcal{A})$ , we define the *groupoid integral*

$$\int_{\mathcal{A}} \varphi = \sum_{[a] \in \pi_0(\mathcal{A})} \frac{\varphi(a)}{|\text{Aut}(a)|}.$$

Note that, if  $\mathcal{A}$  is finite, then we have  $\int_{\mathcal{A}} \mathbb{1} = |\mathcal{A}|$ .

We further introduce a relative version of the groupoid integral given by integration along the fibers: A functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  of groupoids is called

- (1) *finite* if every 2-fiber of  $F$  is finite,
- (2) *locally finite* if, for every  $a \in \mathcal{A}$ , the restriction of  $F$  to  $\mathcal{A}(a)$  is finite,
- (3)  $\pi_0$ -*finite* if the induced map of sets  $\pi_0(\mathcal{A}) \rightarrow \pi_0(\mathcal{B})$  has finite fibers.

Given a locally finite functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  and a function  $\varphi \in \mathcal{F}(\mathcal{A})$ , we define the *pushforward*  $F_! \varphi \in \mathcal{F}(\mathcal{B})$  via

$$F_! \varphi(b) := \int_{\mathcal{A}_b} \varphi|_{\mathcal{A}_b}$$

where  $\mathcal{A}_b$  is the 2-fiber of  $F$  at  $b$  and  $\varphi|_{\mathcal{A}_b}$  denotes the pullback of  $\varphi$  along the natural functor  $\mathcal{A}_b \rightarrow \mathcal{A}$ . We obtain a  $\mathbb{Q}$ -linear map

$$F_! : \mathcal{F}(\mathcal{A}) \longrightarrow \mathcal{F}(\mathcal{B}).$$

**Example 4.26.** For a locally finite groupoid  $\mathcal{A}$  the constant functor  $F : \mathcal{A} \rightarrow \{*\}$  is locally finite and we have  $F_! \varphi(*) = \int_{\mathcal{A}} \varphi$ .

Given a  $\pi_0$ -finite functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  and a function  $\varphi \in \mathcal{F}(\mathcal{B})$ , we define the *pullback*  $F^* \varphi \in \mathcal{F}(\mathcal{A})$  via

$$F^* \varphi(a) := \varphi(F(a)).$$

We obtain a  $\mathbb{Q}$ -linear map

$$F^* : \mathcal{F}(\mathcal{B}) \longrightarrow \mathcal{F}(\mathcal{A}).$$

The central properties of the pullback and pushforward operations are captured in the following Proposition.

**Proposition 4.27.** (1) *Functoriality.*

- (a) Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  and  $G : \mathcal{B} \rightarrow \mathcal{C}$  be  $\pi_0$ -finite functors of groupoids. Then the composite  $G \circ F$  is  $\pi_0$ -finite and we have

$$(G \circ F)^* = F^* \circ G^*.$$

- (b) Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  and  $G : \mathcal{B} \rightarrow \mathcal{C}$  be locally finite functors of groupoids. Then the composite  $G \circ F$  is locally finite and we have

$$(G \circ F)_! = G_! \circ F_!.$$

(2) *Base change.* Let

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{F'} & \mathcal{B} \\ G' \downarrow & \nearrow & \downarrow G \\ \mathcal{A} & \xrightarrow{F} & \mathcal{C} \end{array} \quad (4.28)$$

be a 2-pullback square with  $F$  locally finite and  $G$   $\pi_0$ -finite. Then  $F'$  is locally finite,  $G'$  is  $\pi_0$ -finite, and we have

$$(F')_! \circ (G')^* = G^* \circ F_!.$$

*Proof.* (1) The statements of (i) are immediate. We show (ii). By restricting to connected components, we may assume that  $\mathcal{A}, \mathcal{B}$  are connected,  $F, G$  are finite, and have to show that  $G \circ F$  is finite. To show this, consider the diagram

$$\begin{array}{ccccc} \mathcal{A}_b & \longrightarrow & \{b\} & & \\ \downarrow & \nearrow & \downarrow & & \\ \mathcal{A}_c & \longrightarrow & \mathcal{B}_c & \longrightarrow & \{c\} \\ \downarrow & \nearrow & \downarrow & \nearrow & \downarrow \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{G} & \mathcal{C} \end{array} \quad (4.29)$$

where  $c \in \mathcal{C}$ ,  $b \in \mathcal{B}_c$ , and where all squares are 2-pullback squares. It follows from Problem 4.7 that  $\mathcal{A}_c$  is the 2-fiber of  $G \circ F$  over  $c$ , and  $\mathcal{A}_b$  is the 2-fiber of  $F$  over the image of  $b$  in  $\mathcal{B}$ . For the connected component  $\mathcal{B}_c(b)$  of  $\mathcal{B}_c$ , we further have a restricted 2-pullback square

$$\begin{array}{ccc} \mathcal{A}_b & \longrightarrow & \{b\} \\ \downarrow & \nearrow & \downarrow \\ \mathcal{A}_c(b) & \longrightarrow & \mathcal{B}_c(b) \end{array} \quad (4.30)$$

where we use the notation from the proof of Proposition 4.25. Therefore, by Proposition 4.25, the groupoid  $|\mathcal{A}_c(b)|$  is finite. Since we have the finite decomposition

$$\mathcal{A}_c = \coprod_{[b] \in \pi_0(\mathcal{B}_c)} \mathcal{A}_c(b)$$

we deduce that  $\mathcal{A}_c$  is finite which, by definition, implies that  $G \circ F$  is finite.

To show functoriality, note that both sides of the formula

$$(G \circ F)_! = G_! \circ F_!$$

are  $\mathbb{Q}$ -linear so that we may reduce to the case  $\varphi = \mathbb{1}_{[a]}$  where  $a \in \mathcal{A}$ . Therefore, we may assume that  $\mathcal{A}, \mathcal{B}$  are connected,  $F, G$  finite, and  $\varphi = \mathbb{1}$ . In this case, we compute

$$(G \circ F)_! \mathbb{1}(c) = \sum_{[a] \in \pi_0(\mathcal{A}_c)} \frac{1}{|\text{Aut}(a)|} = |\mathcal{A}_c|.$$

On the other hand, we have

$$\begin{aligned} G_!(F_! \varphi)(c) &= \sum_{[b] \in \pi_0(\mathcal{B}_c)} \frac{F_! \mathbb{1}(i(b))}{|\text{Aut}_{\mathcal{B}_c}(b)|} \\ &= \sum_{[b] \in \pi_0(\mathcal{B}_c)} |\mathcal{A}_b| |\mathcal{B}_c(b)| \\ &= \sum_{[b] \in \pi_0(\mathcal{B}_c)} |\mathcal{A}_c(b)| \\ &= |\mathcal{A}_c| \end{aligned}$$

where  $i : \mathcal{B}_b \rightarrow \mathcal{B}$  denotes the natural functor, and we apply Proposition 4.25 to (4.30).

(2) Replacing  $\mathcal{A}$  by  $\mathcal{A}(a)$  and  $\mathcal{X}$  by  $\mathcal{X}(a)$ , we may assume that  $\mathcal{A}$  is connected and  $F$  is finite. We will first show that in this case  $F'$  is finite, in particular, locally finite. Given  $b \in \mathcal{B}$ , we augment the 2-pullback diagram (4.28) to form

$$\begin{array}{ccc} \mathcal{X}_b & \longrightarrow & \{b\} \\ \downarrow & \nearrow & \downarrow \\ \mathcal{X} & \xrightarrow{F'} & \mathcal{B} \\ G' \downarrow & \nearrow & \downarrow G \\ \mathcal{A} & \xrightarrow{F} & \mathcal{C} \end{array} \tag{4.31}$$

By Problem 4.7, the exterior rectangle is a 2-pullback square so that  $\mathcal{X}_b$  is a 2-fiber of  $F'$  over  $G(b)$  which is finite since  $F$  is finite by assumption.

To show that  $G'$  is  $\pi_0$ -finite, we have to show (still assuming  $\mathcal{A}$  to be connected) that  $\pi_0(\mathcal{X})$  is finite. We have an equivalence

$$\begin{aligned} \mathcal{X} &\simeq \mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B} \\ &\simeq \mathcal{A} \times_{\mathcal{C}}^{(2)} \coprod_{[b] \in \pi_0(\mathcal{B})} \mathcal{B}(b) \\ &\simeq \coprod_{[b] \in \pi_0(\mathcal{B}) | G([b]) = F([a])} \mathcal{A} \times_{\mathcal{C}}^{(2)} \mathcal{B}(b) \\ &\simeq \coprod_{[b] \in \pi_0(\mathcal{B}) | G([b]) = F([a])} \mathcal{A} \times_{\mathcal{C}(F(a))}^{(2)} \mathcal{B}(b) \end{aligned}$$

where  $a$  is any object of  $\mathcal{A}$ . Since the  $G$  is  $\pi_0$ -finite, the resulting decomposition of  $\mathcal{X}$  has only finitely many components. It therefore suffices to show that each of these components has finitely many isomorphism classes so that we have reduced to the case

when  $\mathcal{B}$  and  $\mathcal{C}$  (and  $\mathcal{A}$  are connected). In other words, given group homomorphisms  $\varphi : G \rightarrow H$ , and  $\psi : G' \rightarrow H$ , we have to show that

$$\mathcal{X} = BG \times_{BH}^{(2)} BG'$$

has finitely many isomorphism classes, assuming that the map  $\mathcal{B}\varphi : BG \rightarrow BH$  is finite. As we have seen, the 2-fiber of  $\mathcal{B}\varphi : BG \rightarrow BH$  is the action groupoid corresponding to the  $G$ -action on  $H$  via  $\varphi$ . In particular, we have that  $\pi_0(H//G)$  is finite so that the action has finitely many orbits. A direct computation shows that  $\mathcal{X}$  is given by the action groupoid of the action of  $G \times G'$  on  $H$  via  $h.(g, g') = \varphi(g)^{-1}h\psi(g')$ . Since the  $G$ -action on  $H$  has finitely many orbits, this action also has finitely many orbits implying  $\pi_0(\mathcal{X})$  finite.

To show the base change formula, note that we may assume  $\varphi = \mathbb{1}$ . Using the diagram (4.31), we compute

$$(F')_! \circ (G')^* \mathbb{1}(b) = |\mathcal{X}_b| = G^* \circ F_! \mathbb{1}(b)$$

since  $\mathcal{X}_b$  is both the 2-fiber of  $F'$  over  $b$  and the 2-fiber of  $F$  over  $G(b)$ . □

## 4.4 The Hall algebra

Recall that, to a proto-abelian category  $\mathcal{C}$  we have associated the collection  $\mathcal{X}_\bullet(\mathcal{C})$  of groupoids of flags in  $\mathcal{C}$  which are related by functors

$$\partial_i : \mathcal{X}_n \rightarrow \mathcal{X}_{n-1}$$

for  $0 \leq i \leq n$ . More generally, for every  $n$ , and a subset  $I \subset \{0, 1, \dots, n\}$  of cardinality  $m + 1$ , we obtain a functor

$$\mathcal{X}_n \rightarrow \mathcal{X}_m$$

which is obtained by selecting only those rows and columns of a diagram in  $\mathcal{X}_n$  whose indices lie in  $I$ .

Given groupoids  $\mathcal{A}$  and  $\mathcal{B}$ , we have a canonical identification

$$\mathcal{F}(\mathcal{A}) \otimes_{\mathbb{Q}} \mathcal{F}(\mathcal{B}) \cong \mathcal{F}(\mathcal{A} \times \mathcal{B}) \quad (4.32)$$

which we leave implicit in what follows.

**Theorem 4.33.** Let  $\mathcal{C}$  be a finitary proto-abelian category. Consider the diagram

$$\begin{array}{ccc} \mathcal{X}_2 & \xrightarrow{F} & \mathcal{X}_1 \\ \downarrow G & & \\ \mathcal{X}_1 \times \mathcal{X}_1 & & \end{array}$$

of groupoids of flags where  $G = (\partial_0, \partial_2)$  and  $F = \partial_1$ . Then  $F$  is locally finite,  $G$  is  $\pi_0$ -finite, and the  $\mathbb{Q}$ -linear map

$$\mu : \mathcal{F}(\mathcal{X}_1 \times \mathcal{X}_1) \xrightarrow{F_! \circ G^*} \mathcal{F}(\mathcal{X}_1)$$

defines an associative algebra structure on the  $\mathbb{Q}$ -vector space  $\mathcal{F}(\mathcal{X}_1)$ .

*Proof.* The condition on the  $\text{Ext}^1$ -sets of  $\mathcal{C}$  to be finite implies that the functor  $G$  is  $\pi_0$ -finite. Given a pair of objects  $(A, A')$  of  $\mathcal{C}$ , consider the restriction of  $F$  to the subgroupoid  $\mathcal{X}_2(A, A') \subset \mathcal{X}_2$  of short exact sequences in  $\mathcal{C}$  whose kernel (resp. cokernel) is isomorphic to  $A$  (resp.  $A'$ ). Since  $F$  is an isofibration, we may compute the 2-fiber of the restriction as a strict fiber. We have a pullback square

$$\begin{array}{ccc} (\mathcal{F}_{A', A}^B)^{\cong} & \longrightarrow & \{B\} \\ \downarrow & & \downarrow \\ \mathcal{X}_2(A, A') & \longrightarrow & \mathcal{X}_1 \end{array}$$

The groupoid  $(\mathcal{F}_{A', A}^B)^{\cong}$  is the maximal subgroupoid in the category of flags in  $B$  of type  $A', A$  introduced in §4.2. It is discrete and has finitely many isomorphism classes which implies the local finiteness of  $F$ .

Consider the diagram

$$\begin{array}{ccc}
 & & \mathcal{X}_2 \xrightarrow{F} \mathcal{X}_1 . \\
 & & \downarrow G \\
 \mathcal{X}_2 \times \mathcal{X}_1 & \xrightarrow{F \times \text{id}} & \mathcal{X}_1 \times \mathcal{X}_1 \\
 \downarrow G \times \text{id} & & \\
 \mathcal{X}_1 \times \mathcal{X}_1 \times \mathcal{X}_1 & & 
 \end{array} \tag{4.34}$$

The map  $\mathcal{F}(\mathcal{X}_1 \times \mathcal{X}_1 \times \mathcal{X}_1) \rightarrow \mathcal{F}(\mathcal{X}_1)$  given by the composite

$$F_! \circ G^* \circ (F \times \text{id})_! \circ (F \times \text{id})^*$$

is, via (4.32), identified with  $\mu \circ (\mu \otimes \text{id})$ . We may extend the diagram (4.34) to the diagram

$$\begin{array}{ccccc}
 \mathcal{X}_3 & \longrightarrow & \mathcal{X}_2 & \xrightarrow{F} & \mathcal{X}_1 . \\
 \downarrow & & \downarrow G & & \\
 \mathcal{X}_2 \times \mathcal{X}_1 & \xrightarrow{F \times \text{id}} & \mathcal{X}_1 \times \mathcal{X}_1 & & \\
 \downarrow G \times \text{id} & & & & \\
 \mathcal{X}_1 \times \mathcal{X}_1 \times \mathcal{X}_1 & & & & 
 \end{array}$$

where the two components of  $\mathcal{X}_3 \rightarrow \mathcal{X}_2 \times \mathcal{X}_1$  correspond to the subsets  $\{0, 1, 2\}$  and  $\{2, 3\}$  of  $\{0, 1, 2, 3\}$  while the functor  $\mathcal{X}_3 \rightarrow \mathcal{X}_2$  corresponds to the subset  $\{0, 1, 3\}$ . We have a chain of natural functors composite of the natural functors

$$\mathcal{X}_3 \xrightarrow{F_1} (\mathcal{X}_2 \times \mathcal{X}_1) \times_{\mathcal{X}_1 \times \mathcal{X}_1}^{(2)} \mathcal{X}_2 \xrightarrow{F_2} \mathcal{X}_2 \times_{\mathcal{X}_1}^{(2)} \mathcal{X}_2$$

where  $F_2$  is an equivalence by direct verification and the composite  $F_2 \circ F_1$  is an equivalence by Proposition 4.16. Therefore, the functor  $F_1$  is an equivalence which shows that the top left square is a 2-pullback square. Therefore, by Proposition 4.27, the map  $\mu \circ (\mu \otimes \text{id})$  equals to  $T_! \circ R^*$  where

$$R : \mathcal{X}_3 \rightarrow \mathcal{X}_1 \times \mathcal{X}_1 \times \mathcal{X}_1$$

corresponds to the subsets  $\{0, 1\}$ ,  $\{1, 2\}$  and  $\{2, 3\}$  of  $\{0, 1, 2, 3\}$  while the functor

$$S : \mathcal{X}_3 \rightarrow \mathcal{X}_1$$

corresponds to  $\{0, 3\}$ . An analogous argument, where the other pullback square of Proposition 4.16 is used, shows that  $\mu \circ (\text{id} \otimes \mu)$  can also be identified with  $T_! \circ R^*$ , thus showing associativity.  $\square$

**Proposition 4.35.** The resulting associative algebra is isomorphic to  $\text{Hall}(\mathcal{C})_{\mathbb{Q}}^{\text{op}}$ .

*Proof.* We have a natural  $\mathbb{Q}$ -linear isomorphism

$$\text{Hall}(\mathcal{C})_{\mathbb{Q}} \longrightarrow \mathcal{F}(\mathcal{X}_1)$$

determined by  $[A] \mapsto \mathbb{1}_{[A]}$ . From the calculation of the local 2-fibers of  $F$  in the proof of the above Theorem, it follows that the value of the function

$$F_! \circ G^*(\mathbb{1}_{([A],[A'])})$$

at  $B \in \mathcal{X}_1$  is given by the groupoid cardinality of the groupoid

$$\mathcal{F}_{A',A}^B(\mathcal{C})$$

of flags in  $B$  of type  $A', A$ . This groupoid is discrete so that we have

$$|\mathcal{F}_{A',A}^B(\mathcal{C})| = \pi_0(\mathcal{F}_{A',A}^B(\mathcal{C})).$$

But these numbers are precisely the structure constants of  $\text{Hall}(\mathcal{C})^{\text{op}}$  which proves the claim.  $\square$

**Remark 4.36.** The proof of Proposition 4.35 shows that the 2-fibers of the functor

$$F : \mathcal{X}_2 \rightarrow \mathcal{X}_1$$

are discrete. This means that we could replace  $\mathcal{F}(\mathcal{A})$  by the abelian group  $\mathcal{F}(\mathcal{A})_{\mathbb{Z}}$  of finitely supported functions taking values in  $\mathbb{Z}$  instead of  $\mathbb{Q}$ . We can pushforward functions in  $\mathcal{F}(\mathcal{A})_{\mathbb{Z}}$  along locally finite functors whose 2-fibers are further required to be discrete. In this context, the construction of Theorem 4.33 recovers the integral Hall algebra of  $\mathcal{C}$ . The reason for working with  $\mathcal{F}(\mathcal{A})$  is that, below, we will be interested in pushing forward along functors which do *not* have discrete 2-fibers.

Analyzing the proof of Theorem 4.33, we observe that there seem to be two separate ingredients which imply the associativity of the Hall algebra

- (1) The properties of the family  $\mathcal{X}_{\bullet}$  of groupoids of flags captured in Proposition 4.16.
- (2) The properties of the association  $\mathcal{A} \rightarrow \mathcal{F}(\mathcal{A})$  captured in Proposition 4.27.

Our next goal is to turn this observation into a mathematical statement. To this end, we need to introduce some terminology.

## 4.5 Monoidal categories and lax monoidal functors

A *monoidal category* is a category  $\mathcal{C}$  equipped with the following data:

- a functor

$$\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

called *tensor product*,

- an object  $I$  called *unit*,
- for every triple of objects  $A, B, C$ , an isomorphism

$$\alpha_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C),$$

called *associator*,

- for every object  $A$ , isomorphisms

$$\lambda_A : I \otimes A \xrightarrow{\cong} A$$

and

$$\rho_A : A \otimes I \xrightarrow{\cong} A$$

called *left (resp. right) unitor*.

These data are required to satisfy the following conditions:

- The isomorphisms  $\alpha_{A,B,C}$ ,  $\lambda_A$ , and  $\rho_A$  are required to be functorial in their arguments.
- For every quadrupel  $(A, B, C, D)$  of objects, the *MacLane pentagon*

$$\begin{array}{ccc}
 & ((A \otimes B) \otimes C) \otimes D & \\
 \alpha_{A,B,C} \otimes \text{id}_D \swarrow & & \searrow \alpha_{A \otimes B, C, D} \\
 (A \otimes (B \otimes C)) \otimes D & & (A \otimes B) \otimes (C \otimes D) \\
 \downarrow \alpha_{A, B \otimes C, D} & & \downarrow \alpha_{A, B, C \otimes D} \\
 A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\text{id}_A \otimes \alpha_{B, C, D}} & A \otimes (B \otimes (C \otimes D))
 \end{array}$$

commutes.

- For every pair of objects  $(A, B)$ , the diagram

$$\begin{array}{ccc}
 (A \otimes I) \otimes B & \xrightarrow{\alpha_{A, I, B}} & A \otimes (I \otimes B) \\
 \rho_A \otimes \text{id}_B \searrow & & \swarrow \text{id}_A \otimes \lambda_B \\
 & A \otimes B &
 \end{array}$$

commutes.

**Example 4.37.** (1) The tensor product  $\otimes_k$  on the category of vector spaces over a field  $k$  can be extended to a monoidal structure.

(2) Any category with products can be given a monoidal structure with  $A \otimes B = A \times B$ . We call this monoidal structure the *Cartesian monoidal structure*.

(3) Consider the discrete category  $\{*\}$  with one object. This category carries a monoidal structure by letting  $* \otimes * = *$  and all structural isomorphisms given by the identity map.

Let  $\mathcal{C}, \mathcal{D}$  be monoidal categories. A *lax monoidal functor* from  $\mathcal{C}$  to  $\mathcal{D}$  consists of

- a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$ ,
- a natural transformation  $\gamma_{A,B} : F(A) \otimes F(B) \rightarrow F(A \otimes B)$  making the diagram

$$\begin{array}{ccc}
 (F(A) \otimes F(B)) \otimes F(C) & \xrightarrow{\quad} & F(A) \otimes (F(B) \otimes F(C)) \\
 \gamma_{A,B} \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes \gamma_{B,C} \\
 F(A \otimes B) \otimes F(C) & & F(A) \otimes F(B \otimes C) \\
 \gamma_{A \otimes B, C} \downarrow & & \downarrow \gamma_{A, B \otimes C} \\
 F((A \otimes B) \otimes C) & \xrightarrow{F(\alpha_{A,B,C})} & F(A \otimes (B \otimes C))
 \end{array}$$

commute,

- a morphism  $e : I_{\mathcal{D}} \rightarrow F(I_{\mathcal{C}})$  making the diagrams

$$\begin{array}{ccccc}
 F(A) \otimes I_{\mathcal{D}} & \xrightarrow{\text{id} \otimes e} & F(A) \otimes F(I_{\mathcal{C}}) & \xrightarrow{\gamma_{A, I_{\mathcal{C}}}} & F(A \otimes I_{\mathcal{C}}) \\
 & \searrow \rho_{F(A)} & & \swarrow F(\rho_A) & \\
 & & F(A) & & \\
 \\ 
 I_{\mathcal{D}} \otimes F(A) & \xrightarrow{e \otimes \text{id}} & F(I_{\mathcal{C}}) \otimes F(A) & \xrightarrow{\gamma_{I_{\mathcal{C}}, A}} & F(I_{\mathcal{C}} \otimes A) \\
 & \searrow \lambda_{F(A)} & & \swarrow F(\lambda_A) & \\
 & & F(A) & & 
 \end{array}$$

commute.

A lax monoidal functor is called *monoidal* if the natural transformation  $\gamma_{A,B}$  is an isomorphism.

**Example 4.38.** Let  $\mathcal{C}$  be a monoidal category and let  $F : \{*\} \rightarrow \mathcal{C}$  be a lax monoidal functor. Unravelling the definition, we obtain the following data:

- An object  $A = F(*)$  in  $\mathcal{C}$ .

- Morphisms  $\mu : A \otimes A \rightarrow A$  and  $e : I \rightarrow A$  which satisfy

$$\begin{array}{ccc}
(A \otimes A) \otimes A & \xrightarrow{\alpha_{A,A,A}} & A \otimes (A \otimes A) \\
\mu \otimes \text{id} \downarrow & & \downarrow \text{id} \otimes \mu \\
A \otimes A & & A \otimes A \\
\mu \downarrow & \xrightarrow{\text{id}} & \mu \downarrow \\
A & & A
\end{array}$$
  

$$\begin{array}{ccccc}
A \otimes I & \xrightarrow{\text{id} \otimes e} & A \otimes A & \xrightarrow{\mu} & A \\
& \searrow \rho_A & & \swarrow \text{id} & \\
& & A & & 
\end{array}$$
  

$$\begin{array}{ccccc}
I \otimes A & \xrightarrow{e \otimes \text{id}} & A \otimes A & \xrightarrow{\mu} & A \\
& \searrow \lambda_A & & \swarrow \text{id} & \\
& & A & & 
\end{array}$$

We call  $(A, \mu, e)$  an *algebra object* in  $\mathcal{C}$ .

**Remark 4.39.** Since the composite of two lax monoidal functors is naturally lax monoidal, we observe: Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a lax monoidal functor of monoidal categories and  $A$  and algebra object in  $\mathcal{C}$ . Then  $F(A)$  is naturally an algebra object in  $\mathcal{D}$ .

**Example 4.40.** An algebra object in the category of sets, equipped with the Cartesian monoidal structure, is a monoid. There is a monoidal functor  $F : (\mathbf{Set}, \times) \rightarrow (\mathbf{Vect}_k, \otimes_k)$  which assigns to a set  $K$  the free vector space on  $K$ . The algebra object  $F(M)$  corresponding to a monoid  $M$  is the *monoid algebra* of  $M$ . For example,  $F(\mathbb{N})$  is isomorphic to the polynomial algebra in one variable.

**Remark 4.41.** Let  $(\mathcal{C}, \otimes)$  be a monoidal category. Then the opposite category  $\mathcal{C}^{\text{op}}$  is equipped with a natural monoidal structure. We refer to algebra objects in  $(\mathcal{C}^{\text{op}}, \otimes)$  as *coalgebra objects* in  $(\mathcal{C}, \otimes)$ .

## 4.6 Spans of groupoids and the abstract Hall algebra

We introduce the category  $\text{Span}(\mathbf{Grpd})$  of *spans in groupoids*. The objects are given by (small) groupoids. The set of morphisms between groupoids  $\mathcal{A}, \mathcal{B}$ , is defined to be

$$\text{Hom}(\mathcal{A}, \mathcal{B}) = \left\{ \begin{array}{ccc} & \mathcal{X} & \\ \swarrow & & \searrow \\ \mathcal{A} & & \mathcal{B} \end{array} \right\} / \sim$$

where two *spans*  $\mathcal{A} \leftarrow \mathcal{X} \rightarrow \mathcal{B}$  and  $\mathcal{A} \leftarrow \mathcal{X}' \rightarrow \mathcal{B}$  are considered equivalent if there exists a diagram

$$\begin{array}{ccccc} & & \mathcal{X} & & \\ & \swarrow & & \searrow & \\ \mathcal{A} & & & & \mathcal{B} \\ & \nwarrow & \downarrow F & \nearrow & \\ & & \mathcal{X}' & & \end{array}$$

where  $F$  is an equivalence. The composition of morphisms is given by forming 2-pullbacks: Given morphisms  $f : \mathcal{A} \rightarrow \mathcal{B}$  and  $g : \mathcal{B} \rightarrow \mathcal{C}$  in  $\text{Span}(\mathbf{Grpd})$ , we represent them by spans  $\mathcal{A} \leftarrow \mathcal{X} \rightarrow \mathcal{B}$  and  $\mathcal{B} \leftarrow \mathcal{Y} \rightarrow \mathcal{C}$  and form the diagram

$$\begin{array}{ccccc} & & \mathcal{Z} & & \\ & \swarrow & & \searrow & \\ & \mathcal{X} & \Rightarrow & \mathcal{Y} & \\ & \swarrow \quad \searrow & & \swarrow \quad \searrow & \\ \mathcal{A} & & \mathcal{B} & & \mathcal{C} \end{array}$$

where the square is a 2-pullback square. We then define the composite  $g \circ f$  to be the morphism from  $\mathcal{A}$  to  $\mathcal{C}$  represented by the span

$$\begin{array}{ccc} & \mathcal{Z} & \\ \swarrow & & \searrow \\ \mathcal{A} & & \mathcal{C} \end{array}$$

It follows from the invariance properties of 2-pullbacks that this operation is well-defined.

We define a monoidal structure on  $\text{Span}(\mathbf{Grpd})$  as follows: The tensor product is defined on objects via  $\mathcal{A} \otimes \mathcal{B} = \mathcal{A} \times \mathcal{B}$  and on morphisms via

$$\begin{array}{ccc} \begin{array}{ccc} & \mathcal{X} & \\ \swarrow & & \searrow \\ \mathcal{A} & & \mathcal{B} \end{array} & \otimes & \begin{array}{ccc} & \mathcal{X}' & \\ \swarrow & & \searrow \\ \mathcal{A}' & & \mathcal{B}' \end{array} \\ & & = & \begin{array}{ccc} & \mathcal{X} \times \mathcal{X}' & \\ \swarrow & & \searrow \\ \mathcal{A} \times \mathcal{A}' & & \mathcal{B} \times \mathcal{B}' \end{array} \end{array}$$

Consider the natural functor

$$i : \mathbf{Grpd} \longrightarrow \text{Span}(\mathbf{Grpd})$$

which is the identity on objects and maps a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  to the morphism represented by the span

$$\begin{array}{ccc} & \mathcal{A} & \\ \text{id} \swarrow & & \searrow F \\ \mathcal{A} & & \mathcal{B}. \end{array}$$

We now define the unit, associator, and unitors to be the image under  $i$  of unit, associator, and unitors of the cartesian monoidal structure on **Grpd**. The functoriality of  $i$  implies that all coherence conditions are satisfied.

Finally, we define a subcategory of  $\text{Span}^f(\mathbf{Grpd}) \subset \text{Span}(\mathbf{Grpd})$  with the same objects but morphisms given by spans

$$\begin{array}{ccc} & \mathcal{X} & \\ G \swarrow & & \searrow F \\ \mathcal{A} & & \mathcal{B}. \end{array}$$

such that  $F$  locally finite and  $G$   $\pi_0$ -finite. The composition of such spans is well-defined by Proposition 4.27.

**Theorem 4.42.** Let  $\mathcal{C}$  be a proto-abelian category and let  $\mathcal{X}_\bullet$  be the corresponding groupoids of flags in  $\mathcal{C}$ .

- (1) The morphisms in  $\text{Span}(\mathbf{Grpd})$  represented by the spans

$$\mu : \begin{array}{ccc} & \mathcal{X}_2 & \\ G \swarrow & & \searrow F \\ \mathcal{X}_1 \times \mathcal{X}_1 & & \mathcal{X}_1 \end{array}$$

and

$$e : \begin{array}{ccc} & \mathcal{X}_0 & \\ \text{id} \swarrow & & \searrow 0 \mapsto 0 \\ \mathcal{X}_0 & & \mathcal{X}_1 \end{array}$$

make  $\mathcal{X}_1$  an algebra object in  $\text{Span}(\mathbf{Grpd})$ . We call  $(\mathcal{X}_1, \mu, e)$  the *abstract Hall algebra* of  $\mathcal{C}$ .

- (2) Assume  $\mathcal{C}$  is finitary. Then  $(\mathcal{X}_1, \mu, e)$  defines an algebra object in  $\text{Span}^f(\mathbf{Grpd}) \subset \text{Span}(\mathbf{Grpd})$ .
- (3) The association  $\mathcal{A} \mapsto \mathcal{F}(\mathcal{A})$  extends to a monoidal functor

$$\mathcal{F} : \text{Span}^f(\mathbf{Grpd}) \rightarrow \mathbf{Vect}_{\mathbb{Q}}.$$

For finitary  $\mathcal{C}$ , the resulting algebra object  $\mathcal{F}(\mathcal{X}_1, \mu, e)$  in  $\mathbf{Vect}_{\mathbb{Q}}$  is isomorphic to the (opposite) Hall algebra of  $\mathcal{C}$ .

*Proof.* (1) We have to verify  $\mu \circ (\mu \otimes \text{id}) = \mu \circ (\text{id} \otimes \mu)$ . To compute the left-hand side, we consider the commutative diagram

$$\begin{array}{ccccc}
 \mathcal{X}_3 & \longrightarrow & \mathcal{X}_2 & \xrightarrow{F} & \mathcal{X}_1 \\
 \downarrow & & \downarrow G & & \\
 \mathcal{X}_2 \times \mathcal{X}_1 & \xrightarrow{F \times \text{id}} & \mathcal{X}_1 \times \mathcal{X}_1 & & \\
 \downarrow G \times \text{id} & & & & \\
 \mathcal{X}_1 \times \mathcal{X}_1 \times \mathcal{X}_1 & & & & 
 \end{array}$$

We claim that the left-hand square is a pullback square: There is a sequence of natural functors

$$\mathcal{X}_3 \longrightarrow (\mathcal{X}_2 \times \mathcal{X}_1) \times_{\mathcal{X}_1 \times \mathcal{X}_1}^{(2)} \mathcal{X}_2 \longrightarrow \mathcal{X}_2 \times_{\mathcal{X}_1}^{(2)} \mathcal{X}_1$$

where the composite is an equivalence by using the first 2-pullback square of Proposition 4.16 and the second functor is an equivalence by direct verification. It follows that the first functor is an equivalence which shows the claim. Therefore, we have

$$\begin{array}{ccc}
 & \mathcal{X}_3 & \\
 \mu \circ (\mu \otimes \text{id}) = & \swarrow R & \searrow T \\
 & \mathcal{X}_1 \times \mathcal{X}_1 \times \mathcal{X}_1 & \mathcal{X}_1
 \end{array}$$

where  $R$  is given by the subsets  $\{0, 1\}$ ,  $\{1, 2\}$ ,  $\{2, 3\}$  of  $\{0, 1, 2, 3\}$  and  $T$  by the subset  $\{0, 3\}$ . Similarly, using the second 2-pullback square of Proposition 4.16, we obtain that

$$\begin{array}{ccc}
 & \mathcal{X}_3 & \\
 \mu \circ (\text{id} \otimes \mu) = & \swarrow R & \searrow T \\
 & \mathcal{X}_1 \times \mathcal{X}_1 \times \mathcal{X}_1 & \mathcal{X}_1
 \end{array}$$

so that we deduce associativity. To show right unitality  $\mu \circ (e \otimes \text{id}) = \rho_{\mathcal{X}_1}$ , we consider the diagram

$$\begin{array}{ccccc}
 \mathcal{X}_1 & \xrightarrow{R} & \mathcal{X}_2 & \xrightarrow{F} & \mathcal{X}_1 \\
 \downarrow & & \downarrow G & & \\
 \mathcal{X}_1 \times \mathcal{X}_0 & \longrightarrow & \mathcal{X}_1 \times \mathcal{X}_1 & & \\
 \downarrow & & & & \\
 \mathcal{X}_1 \times \mathcal{X}_0 & & & & 
 \end{array}$$

where  $R$  is the functor which associates to an object  $A$  of  $\mathcal{X}_1$  the short exact sequence

$$\begin{array}{ccc}
 A & \xrightarrow{\text{id}} & A \\
 & & \downarrow \\
 & & 0.
 \end{array}$$

We claim that the top right square is a 2-pullback square which, by a similar argument as above, reduces to the first 2-pullback square of Proposition 4.16(2). Right unitality follows from the second 2-pullback square of Proposition 4.16(2) by an analogous argument.

(2) This is immediate from the proof of Theorem 4.33.

(3) We define a functor  $\mathcal{F} : \text{Span}(\mathbf{Grpd}) \rightarrow \mathbf{Vect}_{\mathbb{Q}}$  by associating to a groupoid  $\mathcal{A}$  the vector space  $\mathcal{F}(\mathcal{A})$  and to a span

$$\begin{array}{ccc} & \mathcal{X} & \\ G \swarrow & & \searrow F \\ \mathcal{A} & & \mathcal{B}. \end{array}$$

the linear map  $F_! \circ G^* : \mathcal{F}(\mathcal{A}) \rightarrow \mathcal{F}(\mathcal{B})$ . We define

$$\gamma_{\mathcal{A}, \mathcal{B}} : \mathcal{F}(\mathcal{A}) \otimes \mathcal{F}(\mathcal{B}) \rightarrow \mathcal{F}(\mathcal{A} \times \mathcal{B})$$

to be the isomorphism given by associating to  $\varphi \otimes \psi$  the function  $(a, b) \mapsto \varphi(a)\psi(b)$ . Further, the unit morphism

$$e : \mathbb{Q} \rightarrow \mathcal{F}(\{*\})$$

is given by mapping  $q$  to  $* \mapsto q$ . All coherence conditions are easy to verify.  $\square$

## 4.7 Green's theorem

Green's theorem states that, under certain assumptions on an abelian category  $\mathcal{C}$ , we can introduce a *coproduct* on the Hall algebra  $\text{Hall}(\mathcal{C})$  making it a bialgebra up to a certain twist. In this section, we will use the abstract Hall algebra introduced in §4.6 to provide a proof of this statement.

Let  $\mathcal{C}$  be a proto-abelian category and  $\mathcal{X}_\bullet$  the corresponding groupoids of flags. Instead of considering the span

$$\mu : \begin{array}{ccc} & \mathcal{X}_2 & \\ G \swarrow & & \searrow F \\ \mathcal{X}_1 \times \mathcal{X}_1 & & \mathcal{X}_1 \end{array} \quad (4.43)$$

which represents the multiplication on the abstract Hall algebra, we may form the *reverse* span

$$\Delta : \begin{array}{ccc} & \mathcal{X}_2 & \\ F \swarrow & & \searrow G \\ \mathcal{X}_1 & & \mathcal{X}_1 \times \mathcal{X}_1 \end{array} \quad (4.44)$$

Also taking into account the reverse  $c$  of the unit morphism  $e$ , it is immediate that  $(\mathcal{X}_1, \Delta, c)$  forms a *coalgebra object* in  $\text{Span}(\mathbf{Grpd})$  (cf. Remark 4.41). The proof consists of reading all span diagrams involved in the proof of Theorem 4.42 in reverse direction.

Given an associative  $k$ -algebra  $A$ , equipped with a coproduct  $\Delta : A \rightarrow A \otimes A$ , we may ask if multiplication and comultiplication are compatible in the sense

$$\Delta(ab) = \Delta(a)\Delta(b). \quad (4.45)$$

In other words, introducing on  $A \otimes A$  the algebra structure  $(a \otimes b)(a' \otimes b) = aa' \otimes bb'$ , we ask if the coproduct  $\Delta$  is a homomorphism of algebras.

**Example 4.46.** Let  $G$  be a group and let

$$k[G] = \bigoplus_{g \in G} kg$$

denote the group algebra over the field  $k$ . The  $k$ -linear extension of the formula  $\Delta(g) = g \otimes g$  defines a coproduct

$$\Delta : k[G] \rightarrow k[G] \otimes k[G]$$

on  $k[G]$ . It is immediate to verify (4.45).

We address the analogous compatibility question for the abstract Hall algebra.

### 4.7.1 Squares, frames, and crosses

To analyze whether or not the equation (4.45) holds for the abstract Hall algebra, we explicitly compute both sides using (4.43) and (4.44). The left-hand side is given by the

composite

$$\begin{array}{ccccc}
 + & \longrightarrow & \mathcal{X}_2 & \xrightarrow{G} & \mathcal{X}_1 \times \mathcal{X}_1 \\
 \downarrow & & \downarrow F & & \\
 \mathcal{X}_2 & \xrightarrow{F} & \mathcal{X}_1 & & \\
 \downarrow G & & & & \\
 \mathcal{X}_1 \times \mathcal{X}_1 & & & & 
 \end{array}$$

which yields

$$\begin{array}{ccc}
 & + & \\
 L \swarrow & & \searrow R \\
 \mathcal{X}_1 \times \mathcal{X}_1 & & \mathcal{X}_1 \times \mathcal{X}_1
 \end{array} \tag{4.47}$$

where  $+$  denotes the groupoid of *exact crosses* in  $\mathcal{C}$ : diagrams

$$\begin{array}{ccccc}
 & B & & & \\
 & \downarrow & & & \\
 A' & \longrightarrow & B' & \longrightarrow & C' \\
 & & \downarrow & & \\
 & & B'' & & 
 \end{array}$$

consisting of two exact sequences  $\mathcal{C}$  with common middle term. The functor  $L$  associates to such a cross the pair of objects  $(B, B'')$ , the functor  $R$  assigns the pair  $(A', C')$ . Note that we can compute the 2-pullback  $+$  as an ordinary pullback since the functor  $F$  is an isofibration. The right-hand side of (4.45) is given by

$$\begin{array}{ccccc}
 \square & \longrightarrow & \mathcal{X}_2 \times \mathcal{X}_2 & \xrightarrow{F \times F} & \mathcal{X}_1 \times \mathcal{X}_1 \\
 \downarrow & & \downarrow P & & \\
 \mathcal{X}_2 \times \mathcal{X}_2 & \xrightarrow{G \times G} & \mathcal{X}_1 \times \mathcal{X}_1 \times \mathcal{X}_1 \times \mathcal{X}_1 & & \\
 \downarrow F \times F & & & & \\
 \mathcal{X}_1 \times \mathcal{X}_1 & & & & 
 \end{array}$$

where the functor  $P$  assigns to a pair  $(A \rightarrow A' \rightarrow A'', C \rightarrow C' \rightarrow C'')$  of short exact sequences the 4-tupel  $(A, C, A'', C'')$  of objects in  $\mathcal{X}_1$ . The composite is represented by the span

$$\begin{array}{ccc}
 & \square & \\
 M \swarrow & & \searrow N \\
 \mathcal{X}_1 \times \mathcal{X}_1 & & \mathcal{X}_1 \times \mathcal{X}_1
 \end{array} \tag{4.48}$$

where  $\square$  denotes the groupoid of *exact frames* in  $\mathcal{C}$ : diagrams

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & & & \downarrow \\ A' & & & & C' \\ \downarrow & & & & \downarrow \\ A'' & \longrightarrow & B'' & \longrightarrow & C'' \end{array}$$

where the two complete rows and columns are short exact sequences. The functor  $M$  assigns to such a frame the pair  $(B, B'')$  while the functor  $N$  assigns the pair  $(A', C')$ . To compare the groupoids  $+$  and  $\square$ , we introduce another groupoid: the groupoid  $\boxplus$  of *exact 3-by-3 squares* given by commutative diagrams in  $\mathcal{C}$  of the form

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' \\ \downarrow & & \downarrow & & \downarrow \\ A'' & \longrightarrow & B'' & \longrightarrow & C'' \end{array}$$

where all rows and columns are required to be short exact sequences. We obtain a commutative diagram

$$\begin{array}{ccccc} & & + & & \\ & \swarrow & \uparrow & \searrow & \\ \mathcal{X}_1 \times \mathcal{X}_1 & \longleftarrow & \boxplus & \longrightarrow & \mathcal{X}_1 \times \mathcal{X}_1 \\ & \swarrow & \downarrow & \searrow & \\ & & \square & & \end{array}$$

of groupoids. In what follows, we will assume that the category  $\mathcal{C}$  is *abelian*.

**Lemma 4.49.** The forgetful functor  $F : \boxplus \rightarrow +$  is an equivalence of groupoids.

*Proof.* We claim that there is a commutative diagram of groupoids

$$\begin{array}{ccc} \boxplus & \xrightarrow{F} & + \\ & \searrow P \quad \nearrow Q & \\ & \diamond & \end{array}$$

where  $\diamond$  denotes the groupoid of diagrams of the form

$$\begin{array}{ccccc} A & \longrightarrow & B & & \\ \downarrow & & \downarrow & & \\ A' & \longrightarrow & B' & \longrightarrow & C' \\ & & \downarrow & & \downarrow \\ & & B'' & \longrightarrow & C'' \end{array}$$

where the middle row and column are short exact, the top-left square is a pullback square and the bottom-right square is a pushout square, and the functors  $P$  and  $Q$  are forgetful functors. To show that  $P$  exists, we verify that the top-left square of any exact 3-by-3 square is pullback (the fact that the bottom-right square is pushout is the dual statement). Applying the snake lemma to the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A' & \longrightarrow & A' \oplus B & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 \end{array} \quad (4.50)$$

and using the exactness of

$$0 \longrightarrow A \longrightarrow B \longrightarrow C'$$

we obtain the exact sequence

$$0 \longrightarrow A \longrightarrow A' \oplus B \longrightarrow B'$$

which shows our claim by Lemma 2.11. By the argument of Lemma 2.19, it is clear that the functor  $Q$  is an equivalence: essential surjectivity follows from the existence of pullbacks and pushouts, and fully faithfulness follows from the universal properties which these constructions enjoy. Similarly, using the universal property of kernels and cokernels, it is clear that the functor  $P$  is fully faithful. To show that  $P$  is essentially surjective, we have to show that every object of  $\Diamond$  can be extended to an exact 3-by-3 square. To this end, it suffices to verify that, given an object of  $\Diamond$ , the sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C' \longrightarrow C'' \longrightarrow 0 \quad (4.51)$$

and

$$0 \longrightarrow A \longrightarrow A' \longrightarrow B'' \longrightarrow C'' \longrightarrow 0 \quad (4.52)$$

are exact. Indeed, given the exactness of (4.51), we may fill in the top-right corner of a 3-by-3 square using the cokernel of  $A \rightarrow B$  which is also a kernel of  $C' \rightarrow C''$ . Similarly, we can fill in the bottom-left corner using the exactness of (4.52). We show exactness of the first sequence, the argument for the exactness of the second one being analogous. Applying the snake lemma to (4.50), but now using the exactness of

$$0 \longrightarrow A \longrightarrow A' \oplus B \longrightarrow B'$$

implies the exactness of

$$0 \longrightarrow A \longrightarrow B \longrightarrow C'. \quad (4.53)$$

Similarly, applying the snake lemma to the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & B & \longrightarrow & B' & \longrightarrow & B'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C' & \longrightarrow & B'' \oplus C' & \longrightarrow & B'' \longrightarrow 0 \end{array}$$

and using the exactness of

$$B' \longrightarrow B'' \oplus C' \longrightarrow C'' \longrightarrow 0$$

we obtain the exactness of

$$B \longrightarrow C' \longrightarrow C'' \longrightarrow 0. \quad (4.54)$$

Combining (4.53) and (4.54), we obtain the exact sequence (4.51) as claimed.  $\square$

Using Lemma 4.49, we may replace (4.47) by the span

$$\begin{array}{ccc} & \boxplus & \\ \swarrow & & \searrow \\ \mathfrak{X}_1 \times \mathfrak{X}_1 & & \mathfrak{X}_1 \times \mathfrak{X}_1 \end{array}$$

which represents the same morphism in  $\text{Span}(\mathbf{Grpd})$ . We obtain a diagram

$$\begin{array}{ccc} & \boxplus & \\ \swarrow & & \searrow \\ \mathfrak{X}_1 \times \mathfrak{X}_1 & & \mathfrak{X}_1 \times \mathfrak{X}_1 \\ & \searrow & \swarrow \\ & \square & \end{array}$$

of groupoids. If the forgetful functor  $\pi : \boxplus \rightarrow \square$  were an equivalence, this would imply the compatibility of the multiplication and comultiplication for the abstract Hall algebra. As it turns out, this is *not* the case – the situation is more subtle. We will analyze the discrepancy of the functor  $\pi$  from being an equivalence by calculating its 2-fibers. Note that,  $\pi$  being an isofibration, we may calculate the 2-fibers as ordinary fibers of  $\pi$ . Suppose the frame  $f$  is given by the diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & & & \downarrow \\ A' & & & & C' \\ \downarrow & & & & \downarrow \\ A'' & \longrightarrow & B'' & \longrightarrow & C'' \end{array}$$

The fiber  $\boxplus_f$  is the groupoid of diagrams of the form

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & Y & \longrightarrow & C' \\ \downarrow & & \downarrow & & \downarrow \\ A'' & \longrightarrow & B'' & \longrightarrow & C'' \end{array}$$

with morphisms inducing the identity on the fixed outer frame  $f$ . Note that associated to the frame  $f$ , there is a long exact sequence

$$\xi_f : 0 \longrightarrow A \longrightarrow A' \amalg_A B \longrightarrow B'' \times_{C''} C' \longrightarrow C'' \longrightarrow 0 \quad (4.55)$$

given as the Baer sum (see §4.7.2) of the two outer long exact sequences of the frame.

**Definition 4.56.** Let

$$\xi : 0 \longrightarrow Q \longrightarrow R \longrightarrow S \longrightarrow T \longrightarrow 0$$

be an exact sequence in an abelian category  $\mathcal{A}$ . We introduce a corresponding groupoid  $\mathcal{T}\text{riv}(\xi)$  of commutative diagrams of the form

$$\begin{array}{ccccccc} Q & \longrightarrow & R & \xrightarrow{\quad} & S & \longrightarrow & T \\ & & & \searrow & \nearrow & & \\ & & & Y & & & \end{array} \quad (4.57)$$

such that the natural maps  $Y/Q \rightarrow S$  and  $Y/R \rightarrow T$  are isomorphisms. The morphisms in  $\mathcal{T}\text{riv}(\xi)$  are given by isomorphisms of diagrams which induce the identity on  $\xi$ .

**Lemma 4.58.** There is an equivalence of groupoids

$$\boxplus_f \longrightarrow \mathcal{T}\text{riv}(\xi_f).$$

*Proof.* It is clear that, given an exact 3-by-3 square, we obtain a canonical diagram of the form (4.57). To show that this association yields a well-defined functor, we have to verify that the sequences

$$0 \longrightarrow A \longrightarrow Y \longrightarrow B'' \times_{C''} C' \longrightarrow 0 \quad (4.59)$$

and

$$0 \longrightarrow A' \amalg_A B \longrightarrow Y \longrightarrow C'' \longrightarrow 0 \quad (4.60)$$

are exact. Applying the snake lemma to the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A' \oplus B & \longrightarrow & Y \oplus Y & \longrightarrow & B'' \oplus C' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Y & \xrightarrow{\text{id}} & Y & \longrightarrow & 0 \longrightarrow 0 \end{array}$$

yields the exact sequence

$$0 \longrightarrow A \longrightarrow Y \longrightarrow B'' \oplus C'$$

which we may extend to the exact sequence

$$0 \longrightarrow A \longrightarrow Y \longrightarrow B'' \oplus C' \longrightarrow C'' \longrightarrow 0$$

using that the bottom-right square in any exact 3-by-3 square is a pushout square. Since the kernel of the map  $B'' \oplus C' \rightarrow C''$  is  $B'' \times_{C''} C'$ , we obtain the exactness of (4.59). The exactness of (4.60) follows similarly.

Since the fully faithfulness of the functor is clear, it remains to show that it is essentially surjective. From an object in  $\mathcal{T}\text{riv}(\xi_f)$ , we obtain a canonical 3-by-3-square. It remains to verify that the involved sequences

$$0 \longrightarrow A' \longrightarrow Y \longrightarrow C' \longrightarrow 0$$

and

$$0 \longrightarrow B \longrightarrow Y \longrightarrow B'' \longrightarrow 0$$

are exact. Applying the snake lemma to the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \longrightarrow & Y & \longrightarrow & C'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C & \longrightarrow & C' & \longrightarrow & C'' \longrightarrow 0 \end{array}$$

and using the exactness of

$$0 \longrightarrow A' \longrightarrow R \longrightarrow C \longrightarrow 0$$

as a Yoneda pushout (see §4.7.2) of the exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

we obtain the first exact sequence. The second exact sequence is obtained by the symmetric argument.  $\square$

It turns out that the groupoid  $\mathcal{T}\text{riv}(\xi)$  has a beautiful interpretation in the context of Yoneda's theory of extensions. We review some aspects of this theory.

#### 4.7.2 Yoneda's theory of extensions

Let  $\mathcal{C}$  be an abelian category and let  $A, B$  be objects in  $\mathcal{C}$ . An  $n$ -extension of  $B$  by  $A$  is an exact sequence

$$\xi : 0 \longrightarrow A \longrightarrow X_{n-1} \longrightarrow X_{n-2} \longrightarrow \dots \longrightarrow X_0 \longrightarrow B \longrightarrow 0$$

in  $\mathcal{C}$ . Given another extension

$$\xi' : 0 \longrightarrow A \longrightarrow X'_{n-1} \longrightarrow X'_{n-2} \longrightarrow \dots \longrightarrow X'_0 \longrightarrow B \longrightarrow 0$$

we say that  $\xi$  and  $\xi'$  are *Yoneda equivalent* if there exists a commutative diagram

$$\begin{array}{ccccccc} & & X_{n-1} & \longrightarrow & X_{n-2} & \longrightarrow & \dots \longrightarrow X_0 \\ & \nearrow & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & A & \longrightarrow & Y_{n-1} & \longrightarrow & Y_{n-2} \longrightarrow \dots \longrightarrow Y_0 \longrightarrow B \longrightarrow 0 \\ & \searrow & \downarrow & & \downarrow & & \downarrow \\ & & X'_{n-1} & \longrightarrow & X'_{n-2} & \longrightarrow & \dots \longrightarrow X'_0 \end{array} \quad (4.61)$$

with exact rows.

**Problem 4.62.** Show that Yoneda equivalence defines an equivalence relation on the set of  $n$ -extensions of  $B$  by  $A$ .

We denote by  $\text{Ext}^n(B, A)$  the set of equivalence classes of  $n$ -extensions of  $B$  by  $A$ . The association

$$(B, A) \mapsto \text{Ext}^n(B, A)$$

is functorial in both arguments: Given a morphism  $f : A \rightarrow A'$ , and an extension  $\xi$  of  $B$  by  $A$  as above, we obtain an  $n$ -extension

$$f_*(\xi) : 0 \rightarrow A' \rightarrow X_{n-1} \amalg_A A' \rightarrow X_{n-2} \rightarrow \dots \rightarrow X_0 \rightarrow B \rightarrow 0$$

of  $B$  by  $A'$  called the *Yoneda pushout* of  $\xi$  along  $f$ .

**Problem 4.63.** Show that the sequence  $f_*(\xi)$  is exact. Further show that, for fixed  $B$ , this construction defines a functor  $\mathcal{C} \rightarrow \mathbf{Set}$ ,  $A \mapsto \text{Ext}^n(B, A)$ .

Dually, given a morphism  $g : B' \rightarrow B$ , we obtain an  $n$ -extension

$$g^*(\xi) : 0 \rightarrow A \rightarrow X_{n-1} \rightarrow X_{n-2} \rightarrow \dots \rightarrow X_0 \times_B B' \rightarrow B' \rightarrow 0$$

of  $B'$  by  $A$  called the *Yoneda pullback* of  $\xi$  along  $g$ . Further, the set  $\text{Ext}^n(B, A)$  is equipped with an addition law: Given two extensions  $\xi$  and  $\xi'$  as above, we first define

$$\xi \oplus \xi' : 0 \rightarrow A \oplus A \rightarrow X_{n-1} \oplus X'_{n-1} \rightarrow \dots \rightarrow X_0 \oplus X'_0 \rightarrow B \oplus B \rightarrow 0$$

and then the *Baer sum*

$$\xi + \xi' = (\Delta_B)^*((\nabla_A)_*(\xi \oplus \xi'))$$

where  $\Delta_B : B \rightarrow B \oplus B$  and  $\nabla_A : A \oplus A \rightarrow A$  denote diagonal and codiagonal, respectively. We will see below that the Baer sum defines an abelian group structure on the set  $\text{Ext}^n(B, A)$ .

**Example 4.64.** In the case  $n = 2$ , we obtain

$$\xi + \xi' : 0 \rightarrow A \rightarrow X_1 \amalg_A X'_1 \rightarrow X_0 \times_B X'_0 \rightarrow B \rightarrow 0$$

which is the operation used to produce the exact sequence (4.55).

Due to the complicated nature of the equivalence relation (4.61), it is hard to decide whether two given extensions  $\xi$  and  $\xi'$  are equivalent, let alone to compute the group  $\text{Ext}^n(B, A)$ . The situation simplifies greatly if the abelian category  $\mathcal{A}$  has enough projectives which we assume from now on (what follows can alternatively be done via dual arguments assuming that  $\mathcal{A}$  has enough injectives).

Given an extension

$$\xi : 0 \rightarrow A \rightarrow X_{n-1} \rightarrow X_{n-2} \rightarrow \dots \rightarrow X_0 \rightarrow B \rightarrow 0$$

we choose a projective resolution

$$\dots \rightarrow P_1 \rightarrow P_0 \rightarrow B$$

of  $B$ . Using the projectivity of the objects  $P_i$  we can construct a lift of the identity map  $B \rightarrow B$  to a morphism of complexes

$$\begin{array}{ccccccc} \dots & \longrightarrow & P_{n+1} & \xrightarrow{d} & P_n & \xrightarrow{d} & \dots \longrightarrow P_1 \xrightarrow{d} P_0 \longrightarrow B \\ & & \downarrow & & \downarrow f_n & & \downarrow f_1 & \downarrow f_0 & \downarrow \text{id} \\ \dots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & \dots \longrightarrow X_1 \longrightarrow X_0 \longrightarrow B \end{array} \quad (4.65)$$

so that we obtain an element  $f_n \in \text{Hom}(P_n, A)$  satisfying  $f_n \circ d = 0$ . This element therefore defines a  $n$ -cocycle in the complex  $\text{Hom}(P_\bullet, A)$ .

**Lemma 4.66.** The association  $\xi \mapsto f_n$  defines a bijection

$$\text{Ext}^n(B, A) \xrightarrow{\cong} H^n(\text{Hom}(P_\bullet, A))$$

which carries the Baer sum to the sum given by the natural abelian group structure on the right hand side. In particular, the set  $\text{Ext}^n(B, A)$  equipped with the Baer sum forms an abelian group.

*Proof.* Homework. □

Since  $\text{Ext}^n(B, A)$  forms an abelian group, there exists a distinguished equivalence class of  $n$ -extensions which are *trivial* in the sense that they represent the neutral element. We will now provide a detailed study of trivial extensions in the cases  $n = 1$  and  $n = 2$ .

Let

$$\xi : 0 \longrightarrow A \xrightarrow{i} X_0 \longrightarrow B \longrightarrow 0$$

be an extension of  $B$  by  $A$ . A *splitting* of  $\xi$  is a morphism  $s : X_0 \rightarrow A$  such that  $si = \text{id}_A$ . We denote by  $\text{Split}(\xi)$  the set of splittings of  $\xi$  which we will now analyze explicitly: Fix a projective resolution  $P_\bullet$  of  $B$ , and a lift of  $\text{id} : B \rightarrow B$  as in (4.65). In particular, we obtain a corresponding cocycle  $f_1 \in \text{Hom}(P_1, A)$ . Consider the differential

$$d : \text{Hom}(P_0, A) \rightarrow \text{Hom}(P_1, A).$$

**Proposition 4.67.** There is a canonical bijection of sets

$$d^{-1}(f_1) \xrightarrow{\cong} \text{Split}(\xi).$$

In particular,

- (1) A splitting exists if and only if the class of  $\xi$  in  $\text{Ext}^1(B, A)$  is trivial.
- (2) If the class of  $\xi$  is trivial, then the set of different splittings admits a simply transitive action of the abelian group  $\text{Hom}(B, A)$ .

*Proof.* The diagram (4.65) induces a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_1/\text{im } P_2 & \longrightarrow & P_0 & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow \bar{f}_1 & & \downarrow f_0 & & \downarrow \text{id} \\ 0 & \longrightarrow & A & \longrightarrow & X_0 & \longrightarrow & B \longrightarrow 0 \end{array}$$

with exact rows. Forming the pushout of the top-left square, we obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & A \amalg_{P_1} P_0 & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow g & & \downarrow \text{id} \\ 0 & \longrightarrow & A & \longrightarrow & X_0 & \longrightarrow & B \longrightarrow 0 \end{array}$$

with exact rows so that, by the snake lemma, the morphism  $g$  is an isomorphism. Therefore, splittings of  $\xi$  are canonically identified with splittings of the short exact sequence

$$0 \longrightarrow A \longrightarrow A \amalg_{P_1} P_0 \longrightarrow B \longrightarrow 0. \quad (4.68)$$

We now provide the claimed bijection. Let  $\varphi \in \text{Hom}(P_0, A)$  such that  $\varphi \circ d = f_1$ . Then we obtain a morphism

$$A \amalg_{P_1} P_0 \longrightarrow A, (a, p) \mapsto a + \varphi(p)$$

which defines a splitting of (4.68). Vice versa, given a splitting  $s$ , we pull back via the canonical morphism  $P_0 \rightarrow A \amalg_{P_1} P_0$  to obtain a morphism  $\varphi : P_0 \rightarrow A$  which, by the relations defining the pushout, satisfies  $\varphi \circ d = f_1$ . It is immediate to verify that these two assignments define inverse maps.  $\square$

We will now provide an analogous point of view on trivial 2-extensions. Let

$$\xi : 0 \longrightarrow A \longrightarrow X_1 \longrightarrow X_0 \longrightarrow B \longrightarrow 0$$

be a 2-extension of  $B$  by  $A$ . A *trivialization* of  $\xi$  is a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & X_1 & \xrightarrow{\quad} & X_0 & \longrightarrow & B \\ & & & \searrow & \nearrow & & \\ & & & & Y & & \end{array} \quad (4.69)$$

such that the natural maps  $Y/A \rightarrow S$  and  $Y/X_1 \rightarrow B$  are isomorphisms. Note that, in contrast to the case  $n = 1$ , where the collection of splittings forms a *set*, the collection of trivializations naturally organizes into a *groupoid*: the groupoid  $\mathcal{T}\text{riv}(\xi)$  introduced in Definition 4.56. We will now argue that the groupoid  $\mathcal{T}\text{riv}(\xi)$  is the  $n = 2$  analogue of the set  $\mathcal{S}\text{plit}(\xi)$ : Fix a projective resolution  $P_\bullet$  of  $B$ , and a lift of  $\text{id} : B \rightarrow B$  as in (4.65). We obtain a corresponding cocycle  $f_2 \in \text{Hom}(P_2, A)$  and consider the complex

$$\text{Hom}(P_0, A) \xrightarrow{d} \text{Hom}(P_1, A) \xrightarrow{d} \text{Hom}(P_2, A).$$

**Proposition 4.70.** There is a canonical equivalence of groupoids

$$T : d^{-1}(f_2) // \text{Hom}(P_0, A) \xrightarrow{\simeq} \mathcal{T}\text{riv}(\xi)$$

where the left-hand side denotes the action groupoid corresponding to the action of the abelian group  $\text{Hom}(P_0, A)$  on  $d^{-1}(f_2)$  via  $(t, g) \mapsto t + g \circ d$ . In particular,

- (1) A trivialization exists, i.e.,  $\mathcal{T}\text{riv}(\xi) \neq \emptyset$ , if and only if the class of  $\xi$  in  $\text{Ext}^2(B, A)$  is trivial.
- (2) Assume that the class of  $\xi$  is trivial. Then
  - (i) The set of isomorphism classes  $\pi_0(\mathcal{T}\text{riv}(\xi))$  is acted upon simply transitively by the group  $\text{Ext}^1(B, A)$ .
  - (ii) The automorphism group of any object of  $\mathcal{T}\text{riv}(\xi)$  is isomorphic to the abelian group  $\text{Hom}(B, A)$ .

*Proof.* Let  $t$  be an object of the action groupoid, i.e., an element  $t \in \text{Hom}(P_1, A)$  such that  $t \circ d = f_2$ . From the chosen diagram

$$\begin{array}{ccccccccc} P_3 & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & B \\ \downarrow & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow \text{id} \\ 0 & \longrightarrow & A & \xrightarrow{i} & X_1 & \longrightarrow & X_0 & \longrightarrow & B \end{array}$$

we obtain a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & P_1/\text{im } P_2 & \longrightarrow & P_0 & \longrightarrow & B \longrightarrow 0 \\
\downarrow & & \downarrow f_1 - i \circ t & & \downarrow f_0 & & \downarrow \text{id} \\
A & \xrightarrow{i} & X_1 & \longrightarrow & X_0 & \longrightarrow & B \longrightarrow 0
\end{array} \tag{4.71}$$

with exact rows. We form the pushout

$$Y_t := X_1 \amalg_{P_1/\text{im } P_2} P_0 \cong X_1 \amalg_{P_1} P_0$$

to obtain a commutative diagram

$$\begin{array}{ccccc}
& & Y_t & & \\
& \nearrow & & \searrow & \\
A & \xrightarrow{i} & X_1 & \longrightarrow & X_0 \longrightarrow B.
\end{array} \tag{4.72}$$

The sequence

$$0 \longrightarrow X_1 \longrightarrow Y_t \longrightarrow B \longrightarrow 0$$

is a Yoneda pushout of the top exact sequence in (4.71) and therefore exact. We further obtain from (4.72) a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & X_1/A & \longrightarrow & Y_t/A & \longrightarrow & B \longrightarrow 0 \\
& & \downarrow \text{id} & & \downarrow & & \downarrow \text{id} \\
0 & \longrightarrow & X_1/A & \longrightarrow & X_0 & \longrightarrow & B \longrightarrow 0
\end{array}$$

where the top row is exact by the third isomorphism theorem, and the bottom row is trivially exact. The snake lemma implies that  $Y_t/A \rightarrow X_0$  is an isomorphism so that the diagram (4.72) defines an object of  $\mathcal{T}\text{riv}(\xi)$ . This defines the functor  $T$  on objects. Given a morphism between objects  $t$  and  $t'$ , i.e., an element  $g \in \text{Hom}(P_0, A)$  such that  $t' = t + g \circ d$ , we obtain an induced morphism  $Y_t \rightarrow Y_{t'}$  via the formula

$$X_1 \amalg_{P_1}^t P_0 \longrightarrow X_1 \amalg_{P_1}^{t'} P_0, (x, p) \mapsto (x + i(g(p)), p)$$

This association defines  $T$  on morphisms. To show that  $T$  is essentially surjective, let

$$\begin{array}{ccccc}
& & Y & & \\
& \nearrow & & \searrow & \\
A & \xrightarrow{i} & X_1 & \longrightarrow & X_0 \longrightarrow B.
\end{array}$$

be an object of  $\mathcal{T}\text{riv}(\xi)$ . We consider the diagram

$$\begin{array}{ccccc}
P_1 & & & & P_0 \\
\downarrow f_1 & & & \swarrow & \downarrow \\
& & Y & & \\
& \nearrow & & \searrow & \\
X_1 & \longrightarrow & & \longrightarrow & X_0
\end{array}$$

and use the projectivity of  $P_0$  to obtain a morphism  $P_0 \rightarrow Y$  making the diagram commute. Note that the differential  $d : P_1 \rightarrow P_0$  does not necessarily make the diagram commute. However, using the exact sequence

$$0 \longrightarrow A \longrightarrow Y \longrightarrow X_0 \longrightarrow 0$$

we may find a homomorphism  $t : P_1 \rightarrow A$  such that the diagram

$$\begin{array}{ccc} P_1 & \xrightarrow{d} & P_0 \\ f_1 - i \circ t \downarrow & \nearrow & \downarrow \\ X_1 & \longrightarrow & X_0 \end{array}$$

$Y$

does commute. It follows that  $\text{im } P_2 \subset P_1$  lies in the kernel of  $f_1 - i \circ t$  such that we obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_1 / \text{im } P_2 & \longrightarrow & P_0 & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow & \searrow & \downarrow & & \downarrow \\ & & f_1 - i \circ t & & Y & & \\ & & \downarrow & \nearrow & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & X_1 & \longrightarrow & X_0 \longrightarrow B \longrightarrow 0 \end{array}$$

We thus obtain a canonical morphism  $Y_t \rightarrow Y$  which fits in a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & Y_t & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow & & \downarrow \text{id} \\ 0 & \longrightarrow & X_1 & \longrightarrow & Y & \longrightarrow & B \longrightarrow 0 \end{array}$$

with exact rows so that it must be an isomorphism by the snake lemma. This concludes the argument for the essential surjectivity of  $T$ . Fully faithfulness can be seen as follows: Consider a morphism

$$\varphi : Y_t \longrightarrow Y_{t'}$$

which defines a morphism in  $\mathcal{T}\text{riv}(\xi)$ . Let  $\iota : P_0 \rightarrow Y_t$  and  $\iota' : P_0 \rightarrow Y_{t'}$  be the canonical morphisms. Then the difference  $\varphi \circ \iota - \iota' : P_0 \rightarrow Y_{t'}$  composes to the zero map into  $X_0$  and hence factors uniquely through a morphism  $g : P_0 \rightarrow A$ . It is immediate to verify that this construction provides an inverse to the map

$$\text{Hom}(t, t') \longrightarrow \text{Hom}_{\mathcal{T}\text{riv}(\xi)}(Y_t, Y_{t'})$$

given by the functor  $T$ . □

### 4.7.3 Proof of Green's theorem

In the previous sections we have seen that the compatibility

$$\Delta(ab) = \Delta(a)\Delta(b) \quad (4.73)$$

of multiplication and comultiplication fails for the abstract Hall algebra since the forgetful functor

$$\pi : \boxplus \rightarrow \square$$

from exact 3-by-3 squares to exact frames is not an equivalence. However, the language of groupoids gives us a precise measure for the failure of (4.73): the 2-fibers of the functor  $\pi$ . As we have seen in Lemma 4.58, the 2-fiber  $\boxplus_f$  over a fixed frame  $f$  is given by the groupoid  $\text{Triv}(\xi_f)$  of trivializations of the 2-extension  $\xi_f$  obtained as the Baer sum of the two 2-extensions of  $C''$  by  $A$  which form the frame  $f$ .

We will now make sufficient assumptions on the category  $\mathcal{C}$  so that we can work around the fact that  $\pi$  is not an equivalence after passing from groupoids to functions by means of the monoidal functor

$$\mathcal{F} : \text{Span}^f(\mathbf{Grpd}) \longrightarrow \mathbf{Vect}_{\mathbb{Q}}.$$

Namely, we will assume that the abelian category  $\mathcal{C}$  is

- (1) *finitary* in the sense of Definition 2.23,
- (2) *cofinitary*: every object of  $\mathcal{C}$  has only finitely many subobjects,
- (3) *hereditary*: for every pair of objects  $A, B$  of  $\mathcal{C}$ , we have  $\text{Ext}^i(A, B) \cong 0$  for  $i > 1$ .

We have seen that the condition on  $\mathcal{C}$  to be finitary implies that the abstract Hall algebra defines an algebra object in  $\text{Span}^f(\mathbf{Grpd}) \subset \text{Span}(\mathbf{Grpd})$ .

**Proposition 4.74.** Let  $\mathcal{C}$  be a finitary abelian category.

- (1) The condition on  $\mathcal{C}$  to be cofinitary implies that the object  $(\mathcal{X}_1, \Delta, c)$  defines a coalgebra object in  $\text{Span}^f(\mathbf{Grpd})$ .
- (2) The condition on  $\mathcal{C}$  to be hereditary implies that all 2-fibers of  $\pi$  are nonempty.

*Proof.* Homework. □

Consider the commutative diagram

$$\begin{array}{ccccc}
 \boxplus & & & & \\
 \searrow \pi & \nearrow R' & & & \\
 & \square & \xrightarrow{R} & \mathcal{X}_2 \times \mathcal{X}_2 & \xrightarrow{F \times F} \mathcal{X}_1 \times \mathcal{X}_1 \\
 \searrow L' & \downarrow L & & \downarrow P & \\
 & \mathcal{X}_2 \times \mathcal{X}_2 & \xrightarrow{G \times G} & \mathcal{X}_1 \times \mathcal{X}_1 \times \mathcal{X}_1 \times \mathcal{X}_1 & \\
 & \downarrow F \times F & & & \\
 & \mathcal{X}_1 \times \mathcal{X}_1 & & & 
 \end{array}$$

The failure of the equality (4.73) after passing to functions is given by

$$(R')_!(L')^* \stackrel{?}{\neq} (R)_!(L)^*.$$

We compute the left-hand side explicitly: letting

$$\varphi = \mathbb{1}_{(A \rightarrow B \rightarrow C, A'' \rightarrow B'' \rightarrow C'')} \in \mathcal{F}(\mathcal{X}_2 \times \mathcal{X}_2),$$

we have

$$(R')_!(L')^*(\varphi) = R_!\pi_!\pi^*L^*(\varphi) = \frac{|\mathrm{Ext}^1(C'', A)|}{|\mathrm{Hom}(C'', A)|} R_!L^*(\varphi). \quad (4.75)$$

The last equality follows from Lemma 4.76 below since, using Proposition 4.70 together with the assumption on  $\mathcal{C}$  to be hereditary, we have

$$|\boxplus_f| = |\mathcal{T}\mathrm{riv}(\xi_f)| = \frac{|\mathrm{Ext}^1(C'', A)|}{|\mathrm{Hom}(C'', A)|}.$$

**Lemma 4.76.** Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a functor which is both  $\pi_0$ -finite and locally finite. Let  $b$  be an object of  $\mathcal{B}$ . Then, for every  $\varphi \in \mathcal{F}(\mathcal{B})$ , we have

$$(F_!F^*\varphi)(b) = |\mathcal{A}_b|\varphi(b)$$

so that the effect of  $F_!F^*$  on the function  $\varphi$  is given by rescaling with the groupoid cardinalities of the 2-fibers of  $F$ .

*Proof.* This follows immediately from the definitions. □

To compensate for the rescaling factor

$$\frac{|\mathrm{Ext}^1(C'', A)|}{|\mathrm{Hom}(C'', A)|}$$

appearing in (4.75), we use the following modifications:

- (1) Instead of the comultiplication  $\Delta$ , we use the comultiplication represented by the span

$$\Delta' : \begin{array}{ccc} & \mathcal{X}_2 & \\ F \swarrow & & \searrow G' \\ \mathcal{X}_1 & & \mathcal{X}_1 \times \mathcal{X}_1 \end{array}$$

where  $G'$  assigns to a short exact sequence  $A \rightarrow B \rightarrow C$  the pair of objects  $(C, A)$  (instead of  $(A, C)$ ).

- (2) Letting  $H = \mathcal{F}(\mathcal{X}_1, \mu, e)$  we define on  $H \otimes H$  the *twisted* algebra structure

$$\mu^t : (H \otimes H) \otimes (H \otimes H) \longrightarrow H \otimes H$$

given by setting

$$(\mathbb{1}_A \otimes \mathbb{1}_B)(\mathbb{1}_{A'} \otimes \mathbb{1}_{B'}) := \frac{|\mathrm{Ext}^1(A', B)|}{|\mathrm{Hom}(A', B)|} (\mathbb{1}_A \mathbb{1}_{A'}) \otimes (\mathbb{1}_B \mathbb{1}_{B'}).$$

**Problem 4.77.** Show that the formula in (2) defines an associative algebra structure on  $H \otimes H$ .

We arrive at the main result of this section.

**Theorem 4.78** (Green). Let  $\mathcal{C}$  be a finitary, cofinitary, and hereditary abelian category and consider the datum  $H = \mathcal{F}(\mathcal{X}_1, \mu, e, \Delta', c)$ . Then the diagram

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\mu} & H \\ \downarrow \Delta' \otimes \Delta' & & \downarrow \Delta' \\ (H \otimes H) \otimes (H \otimes H) & \xrightarrow{\mu^t} & H \otimes H \end{array}$$

commutes.

**Remark 4.79.** A more conceptual way to interpret the result of the theorem is given as follows. Given an abelian category  $\mathcal{C}$ , as usual assumed to be small, we introduce the *Grothendieck group*  $K(\mathcal{C})$  to be the free abelian group on the set of isomorphism classes  $\{[A]\}$  of objects modulo the subgroup generated by  $\{[A] - [B] + [C]\}$  where  $A \rightarrow B \rightarrow C$  runs over all short exact sequences in  $\mathcal{C}$ . The Hall algebra  $H(\mathcal{C})$  is naturally an algebra object in the category of  $K(\mathcal{C})$ -graded vector spaces. Assuming that  $\mathcal{C}$  is hereditary and finitary, the formula

$$\chi(A, B) = \frac{|\mathrm{Ext}^1(A, B)|}{|\mathrm{Hom}(A, B)|}$$

yields a well-defined group homomorphism

$$K(\mathcal{C}) \otimes_{\mathbb{Z}} K(\mathcal{C}) \rightarrow (\mathbb{Q} \setminus \{0\})^{\times}.$$

This allows us to introduce a *braiding*

$$b_{V,W} : V \otimes W \xrightarrow{\cong} W \otimes V, \quad v \otimes w \mapsto \chi(B, A)w \otimes v$$

on the monoidal category  $\mathbf{Vect}_{\mathbb{Q}}^{K(\mathcal{C})}$  where  $v \otimes w$  denotes a homogeneous element of degree  $([A], [B])$ . Now we can say that  $H$  defines a bialgebra in the *braided* monoidal category of  $K(\mathcal{C})$ -graded vector spaces.

#### 4.7.4 Example

Note, that we have investigated the compatibility of product and coproduct without determining an explicit formula for the coproduct. We now provide a formula and analyze the compatibility of multiplication and comultiplication of product and coproduct for the category  $\mathbf{Vect}'_{\mathbb{F}_q}$  of finite dimensional  $\mathbb{F}_q$ -vector spaces.

Let  $\mathcal{C}$  be a finitary and cofinitary abelian category. For objects  $A, A'$  in  $\mathcal{C}$ , we introduce the groupoid  $\mathcal{E}xt(A', A)$  with objects given by short exact sequences

$$0 \longrightarrow A \longrightarrow X \longrightarrow A' \longrightarrow 0$$

in  $\mathcal{C}$  and morphisms given by diagrams

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & A' \\ \downarrow \text{id} & & \downarrow \cong & & \downarrow \text{id} \\ A & \longrightarrow & X' & \longrightarrow & A' \end{array}$$

We denote by  $\mathcal{E}\text{xt}(A', A)^B$  the full subgroupoid of  $\mathcal{E}\text{xt}(A', A)$  consisting of those short exact sequences where  $X \cong B$ . With this notation, we have

$$\Delta'(\mathbb{1}_{[B]}) = (G')^! F^*(\mathbb{1}_{[B]}) = \sum_{[A'], [A]} |\mathcal{E}\text{xt}(A', A)^B| \mathbb{1}_{[A']} \otimes \mathbb{1}_{[A]}.$$

**Example 4.80.** For the category of finite dimensional  $\mathbb{F}_q$ -vector spaces we have

$$\mathcal{F}(\xi_1) \cong \bigoplus_{n \geq 0} \mathbb{Q} \mathbb{1}_n$$

where we set  $\mathbb{1}_n := \mathbb{1}_{[\mathbb{F}_q^n]}$ . Further, we have seen

$$\mathbb{1}_n \mathbb{1}_m = \begin{bmatrix} n+m \\ m \end{bmatrix}_q \mathbb{1}_{m+n}.$$

We compute

$$\begin{aligned} \Delta'(\mathbb{1}_n) &= \sum_{k+l=n} |\mathcal{E}\text{xt}(\mathbb{F}_q^k, \mathbb{F}_q^l)^{\mathbb{F}_q^n}| \mathbb{1}_k \otimes \mathbb{1}_l \\ &= \sum_{k+l=n} q^{-kl} \mathbb{1}_k \otimes \mathbb{1}_l. \end{aligned}$$

Thus, we have

$$\Delta'(\mathbb{1}_m \mathbb{1}_n) = \begin{bmatrix} n+m \\ n \end{bmatrix}_q \sum_{x+y=n+m} q^{-xy} \mathbb{1}_x \otimes \mathbb{1}_y$$

and

$$\begin{aligned} \Delta'(\mathbb{1}_m) \Delta'(\mathbb{1}_n) &= \left( \sum_{k+l=m} q^{-kl} \mathbb{1}_k \otimes \mathbb{1}_l \right) \left( \sum_{r+s=n} q^{-rs} \mathbb{1}_r \otimes \mathbb{1}_s \right) \\ &= \sum_{n=r+s, m=k+l} q^{-kl-rs-r'l} \mathbb{1}_{k+r} \otimes \mathbb{1}_{l+s} \\ &= \sum_{n+m=x+y} q^{-xy} \sum_{k \leq n} q^{k(n-l)} \begin{bmatrix} x \\ k \end{bmatrix}_q \begin{bmatrix} y \\ n-k \end{bmatrix}_q. \end{aligned}$$

The compatibility of product and coproduct up to twist therefore amounts to the formula

$$\begin{bmatrix} n+m \\ n \end{bmatrix}_q = \sum_{k \leq n} q^{k(n-k)} \begin{bmatrix} x \\ k \end{bmatrix}_q \begin{bmatrix} y \\ n-k \end{bmatrix}_q$$

where  $x+y=n+m$ .

## 5 Hall monoidal categories

### 5.1 Simplicial objects and Segal conditions

We have seen that, given a proto-abelian category  $\mathcal{C}$ , the associated collection of groupoids of flags  $\mathcal{X}_\bullet$  can be used to form the abstract Hall algebra. In this section, we would like to abstract the features of the family  $\mathcal{X}_\bullet$  which enable us to do this.

We begin by systematically describing the structure underlying  $\mathcal{X}_\bullet$ . To this end, we introduce the *simplex category*  $\Delta$  whose objects are denoted by  $[n]$ ,  $n \geq 0$ . A morphism from  $[m]$  to  $[n]$  is given by a map of sets

$$\{0, 1, \dots, m\} \longrightarrow \{0, 1, \dots, n\}$$

which preserves the linear order  $\leq$ .

**Remark 5.1.** For convenience reasons, we sometimes implicitly replace the simplex category by the larger category of *all* finite nonempty linearly ordered sets. This is harmless since every such a set is isomorphic to some standard ordinal  $[n]$  via a unique isomorphism. For example, we write  $\{0, 2\} \rightarrow \{0, 1, 2\}$  to refer to the map  $d_1 : [1] \rightarrow [2]$ .

Let  $\mathcal{C}$  be a category. A functor

$$X : \Delta^{\text{op}} \longrightarrow \mathcal{C}$$

is called *simplicial object in  $\mathcal{C}$* . Given a simplicial object  $X$ , we obtain, for every  $0 \leq k \leq n$ , a morphism

$$\partial_k : X_n \longrightarrow X_{n-1}$$

given by the image under  $X$  of the morphism

$$[n-1] \rightarrow [n], i \mapsto \begin{cases} i & \text{for } i < k, \\ i+1 & \text{for } i \geq k \end{cases}$$

which omits  $k$ . We call  $\partial_k$  the *kth face map*. Further, we have, for every  $0 \leq k \leq n$ , a morphism

$$\sigma_k : X_n \longrightarrow X_{n+1}$$

given by the image under  $X$  of the morphism

$$[n+1] \rightarrow [n], i \mapsto \begin{cases} i & \text{for } i < k, \\ i-1 & \text{for } i \geq k \end{cases}$$

which repeats  $k$ . We call  $\sigma_k$  the *kth degeneracy map*.

**Example 5.2.** Let  $X$  be a topological space. We introduce a functor

$$\Delta \longrightarrow \mathbf{Top}, [n] \mapsto \mathbb{R}\{0, 1, \dots, n\}$$

where  $\mathbb{R}\{0, 1, \dots, n\}$  denotes the free real vector space on the set  $\{0, 1, \dots, n\}$  considered as a topological space. Given  $n \geq 0$ , we define the subspace

$$|\Delta^n| = \{(t_0, t_1, \dots, t_n) \mid 0 \leq t_i \leq 1, \sum t_i = 1\} \subset \mathbb{R}\{0, 1, \dots, n\}$$

called the *geometric  $n$ -simplex*. It is easy to verify that the above functor restricts to

$$|\Delta^\bullet| : \Delta \longrightarrow \mathbf{Top}, [n] \mapsto |\Delta^n|.$$

We now define the simplicial set

$$\mathrm{Sing}_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}, [n] \mapsto \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X)$$

where the functoriality is given by pulling back along  $|\Delta^\bullet|$ . The resulting simplicial set is called the *singular simplicial set of  $X$* . The initial idea of this construction was to provide a combinatorial approach to homotopy theory.

**Example 5.3.** Let  $\mathcal{C}$  be a category. We introduce a functor

$$\Delta \longrightarrow \mathbf{Cat}, [n] \mapsto \langle n \rangle$$

where  $\langle n \rangle$  is the linearly ordered set  $\{0, 1, \dots, n\}$  interpreted as a category with a unique morphism from  $i$  to  $j$  if  $i \leq j$ . The simplicial set

$$\mathrm{N}_\bullet(\mathcal{C}) : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}, [n] \mapsto \mathrm{Fun}(\langle n \rangle, \mathcal{C})$$

is called the *nerve of  $\mathcal{C}$* .

**Proposition 5.4.** Let  $\mathcal{C}$  be a proto-abelian category. Then the collection  $\mathcal{X}_\bullet$  of groupoids of flags forms a simplicial object in  $\mathbf{Grpd}$ .

*Proof.* In §4.2, we have already described the functors  $\partial_k$  and  $\sigma_k$ . It is straightforward to generalize this functoriality to arbitrary morphisms  $f : [m] \rightarrow [n]$  in  $\Delta$ : From a diagram in  $\mathcal{X}_n$ , we obtain a diagram in  $\mathcal{X}_m$  by omitting all objects lying in rows or columns whose indices are not in the image of  $f$ , composing the remaining arrows, or introduce identity maps in between those rows and columns whose indices lie in a nontrivial fiber of  $f$ . It is clear from the construction that this provides a functor.  $\square$

The nerve of a category is a simplicial set satisfies a special property:

**Definition 5.5.** A simplicial set  $K$  is called *Segal* if, for every  $0 < i < n$ , the square

$$\begin{array}{ccc} K_{\{0,1,\dots,n\}} & \longrightarrow & K_{\{i,i+1,\dots,n\}} \\ \downarrow & & \downarrow \\ K_{\{0,1,\dots,i\}} & \longrightarrow & K_{\{i\}}, \end{array}$$

where we use the notation from Remark 5.1, is a pullback square.

**Problem 5.6.** Show that the nerve of a category is a Segal simplicial set. Vice versa, show that any Segal simplicial set is the nerve of a category.

It turns out that the simplicial groupoid of flags in a proto-abelian category satisfies a 2-dimensional variant of the Segal condition which we introduce next. Let  $X_\bullet$  be a simplicial object in  $\mathbf{Grpd}$ .

- (1) Consider a planar  $n+1$ -gon  $P$  with vertices labelled cyclically by the set  $\{0, 1, \dots, n\}$ . Let  $i < j$  be the vertices of a diagonal of  $P$  which subdivides the polygon into two polygons with labels  $\{0, 1, \dots, i, j, j+1, \dots, n\}$  and  $\{i, i+1, \dots, j\}$ . We obtain a corresponding commutative square of groupoids

$$\begin{array}{ccc} \mathcal{X}_{\{0,1,\dots,n\}} & \longrightarrow & \mathcal{X}_{\{0,1,\dots,i,j,\dots,n\}} \\ \downarrow & & \downarrow \\ \mathcal{X}_{\{i,i+1,\dots,j\}} & \longrightarrow & \mathcal{X}_{\{i,j\}}. \end{array} \quad (5.7)$$

- (2) For every  $0 \leq i < n$ , there is commutative square

$$\begin{array}{ccc} \mathcal{X}_{\{0,1,\dots,n-1\}} & \longrightarrow & \mathcal{X}_{\{i\}} \\ \downarrow \sigma_i & & \downarrow \\ \mathcal{X}_{\{0,1,\dots,n\}} & \longrightarrow & \mathcal{X}_{\{i,i+1\}} \end{array} \quad (5.8)$$

where  $\sigma_i$  denotes the  $i$ th degeneracy map.

**Definition 5.9.** A simplicial object  $X_\bullet$  in  $\mathbf{Grpd}$  is called *2-Segal* if, for every polygonal subdivision, the corresponding square (5.7) is 2-pullback square. If, in addition, all squares (5.8) are 2-pullback squares then we call  $X_\bullet$  *unital*.

**Proposition 5.10.** Let  $\mathcal{C}$  be a proto-abelian category. Then the simplicial groupoid  $\mathcal{X}_\bullet$  of flags in  $\mathcal{C}$  is unital 2-Segal.

*Proof.* We note that all maps in the diagram (5.7) are isofibrations so that it suffices to check that the squares are ordinary pullback squares. The functor

$$\mathcal{X}_{\{0,1,\dots,n\}} \longrightarrow \mathcal{X}_{\{0,1,\dots,i,j,\dots,n\}} \times_{\mathcal{X}_{\{i,j\}}} \mathcal{X}_{\{i,i+1,\dots,j\}}$$

is a forgetful functor which forgets those objects in the diagram (4.13) whose indices  $(x, y)$  correspond to diagonals of  $P_n$  which cross the diagonal  $(i, j)$ . But these objects can be filled back in by forming pullbacks or pushouts, using the axioms of a proto-abelian category. We leave the verification of (5.8) to the reader.  $\square$

In conclusion, we have arrived at the insight that the construction of the Hall algebra can be split into two steps:

$$\text{proto-abelian categories} \xrightarrow{\mathcal{X}_\bullet(-)} \text{unital 2-Segal groupoids} \xrightarrow{\mathcal{F}(-)} \text{associative algebras}$$

This raises questions:

- (1) Are there other examples of unital 2-Segal groupoids which lead to interesting associative algebras?
- (2) Are there alternatives for  $\mathcal{F}(-)$ ?
- (3) Only the lowest 2-Segal conditions corresponding to the two triangulations of the square play a role when applying  $\mathcal{F}(-)$ . What is the relevance of the higher 2-Segal conditions corresponding to subdivisions of a more general planar polygons?

## 5.2 Colimits and Kan extensions

We recall some basics of the theory of colimits. For details we refer to [ML98]. Let  $I, \mathcal{C}$  be categories and let  $\varphi : I \rightarrow \mathcal{C}$  be a functor. Consider the diagonal functor

$$\Delta : \mathcal{C} \longrightarrow C^I, x \mapsto \Delta(x)$$

where  $\Delta(x)$  is the constant  $I$ -diagram with value  $x$ . A *cone over  $\varphi$*  is a pair  $(x, \eta : \varphi \Rightarrow \Delta(x))$  where  $x$  is an object of  $\mathcal{C}$ , called the *vertex of the cone* and  $\eta$  is a natural transformation. A cone  $(c, \eta)$  over  $\varphi$  is called *colimit cone* if it has the following universal property: for every cone  $(x, \nu)$  over  $\varphi$ , there exists a unique morphism  $f : c \rightarrow x$  such that  $\nu = \Delta(f)\eta$ . We refer to the vertex of a colimit cone over  $\varphi$  as the *colimit of  $\varphi$* .

**Example 5.11.** (1) Consider  $I = \emptyset$  and  $\varphi : I \rightarrow \mathcal{C}$  the empty functor. A cone over  $\varphi$  is an object in  $\mathcal{C}$  and a colimit cone is an initial object.

(2) Let  $I$  be the discrete category with two objects. A functor  $\varphi : I \rightarrow \mathcal{C}$  corresponds to a pair of objects  $(a, b)$ . A cone over  $\varphi$  is a diagram of the form

$$a \longrightarrow x \longleftarrow b.$$

The cone is a colimit if it exhibits  $x \cong a \amalg b$  as the coproduct of  $a$  and  $b$ .

(3) Let  $I$  be the category with objects  $0, 1, 2$  and morphisms  $0 \rightarrow 1, 0 \rightarrow 2$ , in addition to the identity morphisms. A functor  $\varphi : I \rightarrow \mathcal{C}$  is a diagram

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \\ c & & \end{array}$$

and a cone over  $\varphi$  is a commutative square

$$\begin{array}{ccc} a & \longrightarrow & b \\ \downarrow & & \downarrow \\ c & \longrightarrow & x. \end{array}$$

The cone is a colimit cone if the square is a pushout square.

(4) Let  $I$  be the category with objects  $0, 1$  and two morphisms  $f : 0 \rightarrow 1, g : 0 \rightarrow 1$  in addition to identity morphisms. A colimit of an  $I$ -diagram  $\varphi$  is called *coequalizer* of the corresponding morphisms  $\varphi(f)$  and  $\varphi(g)$ .

(5) Let  $I$  be a set, considered as a discrete category. A functor  $\varphi : I \rightarrow \mathcal{C}$  is a set of objects  $\{a_i | i \in I\}$ . A colimit is called the *coproduct* of this set and denoted by  $\amalg_{i \in I} a_i$ .

**Proposition 5.12.** Suppose a category  $\mathcal{C}$  has all small coproducts and coequalizers. Then  $\mathcal{C}$  has all small colimits.

*Proof.* Given a diagram  $\varphi : I \rightarrow \mathcal{C}$ , we can identify a cone over  $\varphi$  with a cone over the diagram

$$\coprod_{f \in \text{Mor}(I)} \varphi(s(f)) \rightrightarrows \coprod_{i \in I} \varphi(i) \quad (5.13)$$

where the upper arrow is induced by the maps  $\text{id} : \varphi(s(f)) \rightarrow \varphi(s(f))$  and the lower arrow is induced by the maps  $\varphi(f) : \varphi(s(f)) \rightarrow \varphi(t(f))$ . Here  $s(f)$  and  $t(f)$  denote the source and target of the morphism  $f$ , respectively. In particular, a colimit cone over (5.13), which exists by assumption, determines a colimit cone over  $\varphi$ .  $\square$

**Remark 5.14.** We have a dual theory of limits: a limit of a diagram  $\varphi : I \rightarrow \mathcal{C}$  is defined to be a colimit of the opposite diagram  $\varphi^{\text{op}} : I^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$ .

Recall that an adjunction

$$F : \mathcal{C} \longleftrightarrow \mathcal{D} : G$$

between categories  $\mathcal{C}$  and  $\mathcal{D}$  is a pair of functors  $(F, G)$  equipped, for all  $x \in \mathcal{C}$  and  $y \in \mathcal{D}$ , with an isomorphism

$$\text{Hom}_{\mathcal{D}}(F(x), y) \cong \text{Hom}_{\mathcal{C}}(x, G(y))$$

which is functorial in  $x$  and  $y$ . It is immediate from the definitions that, if  $I, \mathcal{C}$  are categories such that every  $I$ -diagram in  $\mathcal{C}$  has a colimit, then the association  $\varphi \mapsto \text{colim}(\varphi)$  is functorial and we obtain an adjunction

$$\text{colim} : \mathcal{C}^I \longleftrightarrow \mathcal{C} : \Delta. \quad (5.15)$$

Note that the diagonal functor  $\Delta : \mathcal{C} \rightarrow \mathcal{C}^I$  can be interpreted as the pullback functor along the constant functor  $I \rightarrow *$ . We will now define a relative variant of the concept of colimit: we replace  $I \rightarrow *$  by a functor  $F : I \rightarrow J$  and will generalize the adjunction (5.15) to an adjunction

$$F_! : \mathcal{C}^I \longleftrightarrow \mathcal{C}^J : F^* \quad (5.16)$$

Just as for colimits we first define  $F_!\varphi$  for a given  $I$ -diagram  $\varphi : I \rightarrow \mathcal{C}$ .

**Definition 5.17.** Let  $F : I \rightarrow J$  be a functor and  $\varphi : I \rightarrow \mathcal{C}$  be an  $I$ -diagram in a category  $\mathcal{C}$ . A *left extension of  $\varphi$  along  $F$*  is a pair  $(\psi, \nu)$  consisting of a functor  $\psi : J \rightarrow \mathcal{C}$  and a natural transformation  $\nu : \varphi \Rightarrow F^*\psi$ . A *left Kan extension*  $(F_!\varphi, \eta)$  is a left extension which is universal in the following sense: for every left extension  $(\psi, \nu)$  there is a unique natural transformation  $\xi : F_!\varphi \rightarrow \psi$  such that  $\nu = F^*(\xi) \circ \eta$ .

Again, it follows immediately from the definitions that, if every  $I$ -diagram  $\varphi$  admits a left Kan extension along  $F : I \rightarrow J$ , then we have an adjunction as in (5.16).

We now address the question of how to decide whether Kan extensions exist and, if so, how we can compute them. Given  $F : I \rightarrow J$  and  $j \in J$ , we define the *comma category*  $I \downarrow j$  to have objects given by pairs  $(i, f)$  where  $i \in I$  and  $f : F(i) \rightarrow j$  is a morphism in  $J$ . A morphism  $(i, f) \rightarrow (i', f')$  is given by a morphism  $g : i \rightarrow i'$  such that  $f = f' \circ F(g)$ .

**Theorem 5.18.** Let  $F : I \rightarrow J$  and  $\varphi : I \rightarrow \mathcal{C}$  be functors. Assume that  $\mathcal{C}$  has small colimits. Then the left Kan extension  $F_!\varphi$  of  $\varphi$  along  $F$  exists and is on an object  $j \in J$  given by the formula

$$F_!\varphi(j) \cong \text{colim } \varphi|_{I \downarrow j}$$

where the  $\varphi|_{I \downarrow j}$  denotes the pullback of  $\varphi$  along the functor  $I \downarrow j \rightarrow I, (i, f) \mapsto i$ .

*Proof.* [Mac95] □

**Example 5.19.** Let  $H \subset G$  be a subgroup. We have a corresponding functor of groupoids  $F : BH \rightarrow BG$ . Let  $\mathcal{C}$  be the category of vector spaces over a field  $k$ . A functor  $\varphi : BH \rightarrow \mathcal{C}$  corresponds to a representation of the group  $H$  on a vector space  $V$ . The representation  $F_! \varphi : BG \rightarrow \mathcal{C}$  is known as the *induced representation of  $V$  along  $H \subset G$*  given explicitly as follows: let  $\{g_i\}$  be a chosen set of representatives of the quotient set  $G/H$ . Then the underlying vector space of  $F_! \varphi$  is given by

$$\bigoplus_{[g_i] \in G/H} V_{g_i}$$

where each  $V_{g_i} = V$ . An element  $g \in G$  acts by mapping  $v \in V_{g_i}$  to  $h.v \in V_{g_j}$  where  $gg_i = g_j h$ .

### 5.3 Pull-push for functors

Given a groupoid  $\mathcal{A}$ , we denote by  $\text{Fun}(\mathcal{A})$  the category of functors from  $\mathcal{A}$  to the category  $\mathbf{Vect}_{\mathbb{C}}$  of finite dimensional complex vector spaces which are nonzero on only finitely many isomorphism classes of  $\mathcal{A}$ . Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  be a functor of groupoids. We have:

- if  $F$  is  $\pi_0$ -finite, then we have a corresponding pullback functor

$$F^* : \text{Fun}(\mathcal{B}) \longrightarrow \text{Fun}(\mathcal{A}), \varphi \mapsto \varphi \circ F.$$

- if  $F$  is locally finite, then we have a pushforward functor

$$F_! : \text{Fun}(\mathcal{A}) \longrightarrow \text{Fun}(\mathcal{B})$$

which is defined as a left Kan extension functor. By the pointwise formula for Kan extensions, we have

$$F_!(\varphi)(b) = \text{colim}_{\mathcal{A}_b} \varphi|_{\mathcal{A}_b}$$

where  $\mathcal{A}_b$  denotes the 2-fiber of  $F$  over  $b$ .

These operations satisfy the following compatibility conditions (in analogy to Proposition 4.27).

**Proposition 5.20.** (1) *Functoriality.*

- (a) Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  and  $G : \mathcal{B} \rightarrow \mathcal{C}$  be  $\pi_0$ -finite functors of groupoids. Then we have

$$(G \circ F)^* = F^* \circ G^*.$$

- (b) Let  $F : \mathcal{A} \rightarrow \mathcal{B}$  and  $G : \mathcal{B} \rightarrow \mathcal{C}$  be locally finite functors of groupoids. Then we have a canonical isomorphism

$$(G \circ F)_! \cong G_! \circ F_!.$$

(2) *Base change.* Let

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{F'} & \mathcal{B} \\ G' \downarrow & \nearrow & \downarrow G \\ \mathcal{A} & \xrightarrow{F} & \mathcal{C} \end{array} \quad (5.21)$$

be a 2-pullback square with  $F$  locally finite and  $G$   $\pi_0$ -finite. Then we have a canonical isomorphism

$$(F')_! \circ (G')^* \cong G^* \circ F_!.$$

Let  $\mathcal{C}$  be a finitary proto-abelian category. Given a span of groupoids

$$\begin{array}{ccc} & \mathcal{X} & \\ L \swarrow & & \searrow R \\ \mathcal{A} & & \mathcal{B} \end{array}$$

with  $L$   $\pi_0$ -finite and  $R$  locally finite, we obtain a corresponding functor

$$R_! \circ L^* : \text{Fun}(\mathcal{A}) \longrightarrow \text{Fun}(\mathcal{B}).$$

Applying this to the span

$$\begin{array}{ccc} & \mathcal{X}_2 & \\ G \swarrow & & \searrow F \\ \mathcal{X}_1 \times \mathcal{X}_1 & & \mathcal{X}_1 \end{array}$$

yields a functor  $\text{Fun}(\mathcal{X}_1 \times \mathcal{X}_1) \rightarrow \text{Fun}(\mathcal{X}_1)$  which we precompose with the pointwise tensor product  $\text{Fun}(\mathcal{X}_1) \times \text{Fun}(\mathcal{X}_1) \rightarrow \text{Fun}(\mathcal{X}_1 \times \mathcal{X}_1)$  to obtain

$$\otimes : \text{Fun}(\mathcal{X}_1) \times \text{Fun}(\mathcal{X}_1) \longrightarrow \text{Fun}(\mathcal{X}_1).$$

Further, from the span

$$\begin{array}{ccc} & \mathcal{X}_0 & \\ \text{id} \swarrow & & \searrow \sigma_0 \\ \mathcal{X}_0 & & \mathcal{X}_1 \end{array}$$

we obtain a functor  $\text{Fun}(\mathcal{X}_0) \longrightarrow \text{Fun}(\mathcal{X}_1)$  which we evaluate on  $k$  to obtain

$$I \in \text{Fun}(\mathcal{X}_1).$$

**Theorem 5.22.** Let  $\mathcal{C}$  be a finitary proto-abelian category. The datum  $(\text{Fun}(\mathcal{X}_1), \otimes, I)$  naturally extends to a monoidal structure on the category  $\text{Fun}(\mathcal{X}_1)$ .

*Proof.* We sketch the basic idea of the proof, the main point being the derivation of MacLane's pentagon. There are five 2-Segal conditions involving  $\mathcal{X}_4$ , corresponding to the five possible subdivisions of a planar pentagon. We denote by  $\mathcal{P}_{ij}$  the subdivision  $0 \leq i < j \leq 4$  of the pentagon. We further introduce the notation

$$\mathcal{X}(\mathcal{P}_{ij}) = \mathcal{X}_{\{0, \dots, i, j, \dots, 4\}} \times_{\mathcal{X}_{\{i, j\}}} \mathcal{X}_{\{i, \dots, j\}} \quad (5.23)$$

for the corresponding pullback so that, for every subdivision, we have, by the 2-Segal property, an equivalence

$$\mathcal{X}_4 \longrightarrow \mathcal{X}(\mathcal{P}_{ij}).$$

For example, we have

$$\mathcal{X}(\mathcal{P}_{13}) = \mathcal{X}_{\{0,1,3,4\}} \times_{\mathcal{X}_{\{1,3\}}} \mathcal{X}_{\{1,2,3\}} \cong \mathcal{X}_3 \times_{\mathcal{X}_1} \mathcal{X}_2$$

Similarly, we label the five different triangulations  $\mathcal{T}_{ij,kl}$  of the pentagon via their internal edges  $i \rightarrow j$  and  $k \rightarrow l$ . We use the notation  $\mathcal{X}(\mathcal{T}_{ij,kl})$  analogous to (5.23) so that we have, for example,

$$\mathcal{X}(\mathcal{T}_{13,14}) = \mathcal{X}_{\{0,1,4\}} \times_{\mathcal{X}_{\{1,4\}}} \mathcal{X}_{\{1,3,4\}} \times_{\mathcal{X}_{\{1,3\}}} \mathcal{X}_{\{1,2,3\}} \cong \mathcal{X}_2 \times_{\mathcal{X}_1} \mathcal{X}_2 \times_{\mathcal{X}_1} \mathcal{X}_2.$$

We obtain a commutative diagram of groupoids

in which, by the various 2-Segal conditions, all functors are equivalences.

The basic idea of the argument is now as follows: Let  $\varphi, \psi, \xi$  be objects in  $\text{Fun}(\mathcal{X}_1)$ . We have a diagram of groupoids

$$\mathcal{X}_{\{0,1,2\}} \times_{\mathcal{X}_{\{0,2\}}} \mathcal{X}_{\{0,2,3\}} \longleftarrow \mathcal{X}_{\{0,1,2,3\}} \longrightarrow \mathcal{X}_{\{0,1,3\}} \times_{\mathcal{X}_{\{1,3\}}} \mathcal{X}_{\{1,2,3\}}$$

where, by the lowest 2-Segal conditions corresponding to the two triangulations of a square, both functors are equivalences. They are responsible for isomorphisms

$$(\varphi \otimes \psi) \otimes \xi \xleftarrow[\cong]{\alpha_1} \varphi \otimes \psi \otimes \xi \xrightarrow[\cong]{\alpha_2} \varphi \otimes (\psi \otimes \xi)$$

which we use to define the associator of the monoidal structure as  $\alpha = \alpha_2 \circ \alpha_1^{-1}$ . Here, the middle term  $\varphi \otimes \psi \otimes \xi$  is defined as a pull-push along the span

$$\mathcal{X}_{\{0,1\}} \times \mathcal{X}_{\{1,2\}} \times \mathcal{X}_{\{2,3\}} \longleftarrow \mathcal{X}_{\{0,1,2,3\}} \longrightarrow \mathcal{X}_{\{0,3\}}.$$

Given four objects  $\varphi, \psi, \xi, \varepsilon$ , the commutative diagram (5.24) is responsible for a commutative diagram of isomorphisms

$$(5.25)$$

Along the boundary of (5.25), we extract the commutative MacLane pentagon for the tensor product on  $\text{Fun}(\mathcal{X}_1)$ . To make this argument formally precise is somewhat tedious: it is best done by interpreting monoidal structures in terms of Grothendieck fibrations over the category  $\Delta^{\text{op}}$ .  $\square$

We call the monoidal category  $(\text{Fun}(\mathcal{X}_1), \otimes, I)$  the *Hall monoidal category*  $\text{Hall}^\otimes(\mathcal{C})$  of  $\mathcal{C}$ .

**Remark 5.26.** The construction of the Hall monoidal category can be understood as an instance of *Day convolution* [Day74]: From the simplicial groupoid of flags we can construct a promonoidal structure on the groupoid  $\mathcal{X}_1$  which is then turned into a monoidal one by passing to functors.

We discuss two examples:

### 5.3.1 $\text{Hall}^\otimes(\mathbf{Vect}_{\mathbb{F}_1})$

We restrict attention to the skeleton  $\mathcal{C} \subset \mathbf{Vect}_{\mathbb{F}_1}$  consisting of the standard pointed sets  $\{*, 1, \dots, n\}$ ,  $n \geq 0$ . Thus, we have

$$\mathcal{X}_1(\mathcal{C}) \simeq \coprod_{n \geq 0} BS_n$$

so that an object of  $\text{Hall}^\otimes(\mathcal{C})$  is given by a sequence  $(\rho_n)_{n \geq 0}$  of representations of  $S_n$  in  $\mathbf{Vect}_{\mathcal{C}}$  where only finitely many representations are nonzero. We have

$$(\rho_n)_{n \geq 0} \cong \bigoplus_{n \geq 0} \rho_n$$

where  $\rho_n$  is interpreted as an object of  $\text{Hall}^\otimes(\mathcal{C})$  which is zero on all groupoids  $BS_m$ ,  $m \neq n$ . The tensor product of  $\text{Hall}^\otimes(\mathcal{C})$  is additive so that it suffices to describe  $\rho_n \otimes \rho_m$ . This is obtained by pull-push along the span of groupoids

$$\mathcal{X}_1 \times \mathcal{X}_1 \longleftarrow \mathcal{X}_2 \longrightarrow \mathcal{X}_1$$

which factors through the pull-push along the span

$$BS_n \times BS_m \xleftarrow{\cong} B(S_n \times S_m) \longrightarrow BS_{n+m}.$$

This is obtained by restricting to the subgroupoid of  $\mathcal{X}_2$  spanned by a fixed chosen short exact sequence

$$\{*, 1, \dots, n\} \hookrightarrow \{*, 1, \dots, n+m\} \twoheadrightarrow \{*, 1, \dots, m\}. \quad (5.27)$$

and noting that this choice determines an embedding of the automorphism group  $S_n \times S_m$  of (5.27) into the automorphism group  $S_{n+m}$  of  $\{*, 1, \dots, n+m\}$ . The tensor product  $\rho_n \otimes \rho_m$  in  $\text{Hall}^\otimes(\mathcal{C})$  is therefore given by the induced representation of the external tensor product  $\rho_n \boxtimes \rho_m$  along the embedding  $S_n \times S_m \subset S_{n+m}$ .

The resulting monoidal category plays an important role in classical representation theory. It is canonically monoidally equivalent to the category of *polynomial functors*: functors  $F : \mathbf{Vect}_{\mathbb{C}} \rightarrow \mathbf{Vect}_{\mathbb{C}}$  satisfying the following condition

- for every collection of morphisms  $f_i : V \rightarrow W$ ,  $1 \leq i \leq n$ , between fixed vector spaces, the expression  $F(\lambda_1 f_1 + \dots + \lambda_r f_r)$ ,  $\lambda_i \in \mathbb{C}$ , is a function polynomial with coefficients in  $\text{Hom}(F(V), F(W))$ .

The polynomial functor corresponding to the representation  $\rho_n$  is given by

$$F : \mathbf{Vect}_{\mathbb{C}} \rightarrow \mathbf{Vect}_{\mathbb{C}}, V \mapsto (X_n \otimes V^{\otimes n})^{S_n}.$$

The Hall monoidal structure induces the structure of an associative algebra on the Grothendieck group  $K_0(\text{Hall}^\otimes(\mathcal{C}))$ . Using the interpretation via polynomial functors we can canonically identify  $K_0(\text{Hall}^\otimes(\mathcal{C}))$  with the algebra  $\Lambda$  of symmetric functions. Under this identification, the basis given by isomorphism classes of irreducible representations gets identified with the basis of  $\Lambda$  given by the Schur functions. Therefore, we obtain yet another Hall algebraic construction of the algebra of symmetric functions which naturally exhibits an interesting basis. For a detailed exposition of this theory (due to Schur) we refer the reader to [Mac95].

### 5.3.2 $\text{Hall}^\otimes(\mathbf{Vect}_{\mathbb{F}_q})$

We consider the skeleton  $\mathcal{C} \subset \mathbf{Vect}_{\mathbb{F}_q}$  consisting of the standard objects  $\mathbb{F}_q^n$  so that we have

$$\mathcal{X}_1(\mathcal{C}) \simeq \coprod_{n \geq 0} B \text{GL}_n(\mathbb{F}_q).$$

An object of  $\text{Hall}^\otimes(\mathcal{C})$  is therefore given by a sequence  $(\rho_n)_{n \geq 0}$  of representations of  $\text{GL}_n(\mathbb{F}_q)$  in  $\mathbf{Vect}_{\mathbb{C}}$  with only finitely many nonzero components. The tensor product  $\rho_n \otimes \rho_m$  is obtained by pull-push along the span of groupoids

$$\mathcal{X}_1 \times \mathcal{X}_1 \longleftarrow \mathcal{X}_2 \longrightarrow \mathcal{X}_1$$

which factors through the pull-push along the span

$$BGL_n(\mathbb{F}_q) \times BGL_m(\mathbb{F}_q) \xleftarrow{\cong} BP_{n,m}(\mathbb{F}_q) \longrightarrow BGL_{n+m}(\mathbb{F}_q)$$

where  $P_{n,m}$  is the parabolic subgroup of  $GL_{n+m}$  given by the automorphism group of a fixed short exact sequence

$$\mathbb{F}_q^n \hookrightarrow \mathbb{F}_q^{n+m} \twoheadrightarrow \mathbb{F}_q^m.$$

Therefore, the tensor product  $\rho_n \otimes \rho_m$  in  $\text{Hall}^\otimes(\mathcal{C})$  is given by first pulling back the representation of  $GL_n(\mathbb{F}_q) \times GL_m(\mathbb{F}_q)$  along  $P_{n,m} \rightarrow GL_n(\mathbb{F}_q) \times GL_m(\mathbb{F}_q)$  and then forming the induced representation along  $P_{n,m} \subset GL_{n+m}(\mathbb{F}_q)$ .

Green [Gre55, Mac95] has developed a  $q$ -analog of Schur's theory which uses the associative algebra given by the Grothendieck group of  $\text{Hall}^\otimes(\mathbf{Vect}_{\mathbb{F}_q})$  to construct all irreducible characters of the groups  $GL_n(\mathbb{F}_q)$ . The monoidal category  $\text{Hall}^\otimes(\mathbf{Vect}_{\mathbb{F}_q})$  itself features in the work of Joyal-Street [JS95] who explain that the commutativity of Green's algebra comes from a (partial) braided structure.

## 6 Derived Hall algebras via $\infty$ -groupoids

The idea of constructing the Hall algebra via the simplicial groupoid of flags  $\mathcal{X}_\bullet$  is a very flexible one. We explain how it can be adopted to construct Hall algebras of derived categories or, more generally, stable  $\infty$ -categories. This section is a translation of [Toë06] (also cf. [Ber13]) into the language of  $\infty$ -categories which makes the analogy to proto-abelian categories immediate. We use [Lur09] as a standard reference.

### 6.1 Coherent diagrams in differential graded categories

An  $\infty$ -category  $\mathcal{C}$  is a simplicial set such that, for every  $0 < i < n$  and every  $\Lambda_i^n \rightarrow \mathcal{C}$ , there exists a commutative diagram

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{\quad} & \mathcal{C} \\ \downarrow & \nearrow & \\ \Delta^n & & . \end{array}$$

The arrow  $\Lambda_i^n \rightarrow \mathcal{C}$  represents the boundary of an  $n$ -simplex with  $i$ th face removed, called an *inner horn* in  $\mathcal{C}$ , and the condition asks that it can be filled to a full  $n$ -simplex in  $\mathcal{C}$ .

**Example 6.1.** The nerve of a small category provides an example of an  $\infty$ -category where every inner horn has a *unique* filling. This corresponds to the fact that every  $n$ -tuple of composable morphisms has a unique composite. In a general  $\infty$ -category the composite is not required to be unique. However, the totality of all horn filling conditions encodes that it is unique up to a coherent system of homotopies.

**Example 6.2.** Given  $\infty$ -categories  $\mathcal{C}, \mathcal{D}$ , we define the simplicial set  $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$  of *functors from  $\mathcal{C}$  to  $\mathcal{D}$*  given by the internal hom in the category of simplicial sets. Then  $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$  is an  $\infty$ -category.

**Example 6.3.** A *differential graded (dg) category*  $T$  is a category enriched over the monoidal category of complexes of abelian groups. The collection of dg categories organizes into a category  $\mathrm{dgc}at$  with morphisms given by enriched functors. Following [Lur11], we associate to  $T$  an  $\infty$ -category called the dg nerve of  $T$ .

We associate to the  $n$ -simplex  $\Delta^n$  a dg category  $\mathrm{dg}(\Delta^n)$  with objects given by the set  $\{0, 1, \dots, n\}$ . The graded  $\mathbb{Z}$ -linear category underlying  $\mathrm{dg}(\Delta^n)$  is freely generated by the morphisms

$$f_I \in \mathrm{dg}(\Delta^n)(i_-, i_+)^{-m}$$

where  $I$  runs over the subsets  $\{i_- < i_m < i_{m-1} < \dots < i_1 < i_+\} \subset \{0, 1, \dots, n\}$ ,  $m \geq 0$ . On these generators, the differential is given by the formula

$$df_I = \sum_{1 \leq j \leq m} (-1)^j (f_{I \setminus \{i_j\}} - f_{\{i_j < \dots < i_m < i_+\}} \circ f_{\{i_- < i_1 < \dots < i_j\}})$$

and extended to all morphisms by the  $\mathbb{Z}$ -linear Leibniz rule. We have  $d^2 = 0$  on generators and therefore on all morphisms. The dg categories  $\mathrm{dg}(\Delta^n)$ ,  $n \geq 0$ , assemble to form a cosimplicial object in  $\mathrm{dgc}at$  which allows us to define the dg nerve of  $T$

$$\mathrm{N}_{\mathrm{dg}}(T) = \mathrm{Hom}_{\mathrm{dgc}at}(\mathrm{dg}(\Delta^\bullet), T).$$

It is shown in [Lur11, 1.3.1.10] that  $\mathrm{N}_{\mathrm{dg}}(T)$  is in fact an  $\infty$ -category.

It is instructive to analyze the low-dimensional simplices of the dg nerve  $N_{\text{dg}}(T)$ :

- The 0-simplices are the objects of  $T$ .
- A 1-simplex in  $N_{\text{dg}}(T)$  is a morphism  $f_{\{0,1\}} : a_0 \rightarrow a_1$  of degree 0 which is closed, i.e.,  $df = 0$ .
- A 2-simplex in  $N_{\text{dg}}(T)$  is given by objects  $a_0, a_1, a_2$ , closed morphisms  $f_{\{0,1\}} : a_0 \rightarrow a_1$ ,  $f_{\{1,2\}} : a_1 \rightarrow a_2$ ,  $f_{\{0,2\}} : a_0 \rightarrow a_2$ , and a morphism  $f_{\{0,1,2\}} : a_0 \rightarrow a_2$  of degree  $-1$  which satisfies

$$df_{\{0,1,2\}} = f_{\{0,2\}} - f_{\{1,2\}} \circ f_{\{0,1\}}$$

so that we obtain a triangle in  $T$  which commutes up to the chosen homotopy  $f_{\{0,1,2\}}$ . A key point here is that we do not simply require the triangle to commute up to homotopy, but the homotopy is part of the data forming the triangle.

- A 3-simplex in  $N_{\text{dg}}(T)$  involves the data of the four boundary 2-simplices as above and, in addition, a morphism  $f_{\{0,1,2,3\}} : a_0 \rightarrow a_3$  of degree  $-2$  such that

$$df_{\{0,1,2,3\}} = f_{\{0,1,3\}} - f_{\{2,3\}} \circ f_{\{0,1,2\}} - f_{\{0,2,3\}} + f_{\{1,2,3\}} \circ f_{\{0,1\}}.$$

We can interpret this data as follows: The boundary of a 3-simplex in  $N_{\text{dg}}(T)$  encodes two homotopies between  $f_{\{0,3\}}$  and the composite  $f_{\{2,3\}} \circ f_{\{1,2\}} \circ f_{\{0,1\}}$  given by  $f_{\{0,1,3\}} + f_{\{1,2,3\}} \circ f_{\{0,1\}}$  and  $f_{\{0,2,3\}} + f_{\{2,3\}} \circ f_{\{0,1,2\}}$ , respectively. To obtain a full 3-simplex in  $N_{\text{dg}}(T)$  we have to provide the homotopy  $f_{\{0,1,2,3\}}$  between these homotopies.

• ...

The passage from a dg category  $T$  to the  $\infty$ -category  $N_{\text{dg}}(T)$  allows us (and forces us) to systematically consider diagrams in  $T$  which commute up to specified coherent homotopy: Let  $I$  be a category and  $N(I)$  its nerve. We define a *coherent  $I$ -diagram in  $T$*  to be a functor, i.e., a map of simplicial sets

$$N(I) \rightarrow N_{\text{dg}}(T).$$

**Example 6.4.** Consider the category  $I$  given by the universal commutative square:  $I$  has four objects 1, 2, 3, 4, morphisms  $f_1 : 1 \rightarrow 2$ ,  $f_2 : 2 \rightarrow 4$ ,  $f_3 : 1 \rightarrow 3$ ,  $f_4 : 3 \rightarrow 4$  subject to the relation  $f_2 \circ f_1 = f_4 \circ f_3$ . An  $I$ -coherent diagram in  $T$  consists of

$$\begin{array}{ccc} a_1 & \xrightarrow{f_1} & a_2 \\ f_3 \downarrow & \searrow^{h_1} & \downarrow f_2 \\ a_3 & \xrightarrow{f_4} & a_4 \end{array}$$

$g$  (diagonal arrow from  $a_1$  to  $a_4$ )

where the morphisms  $f_1, f_2, f_3, f_4$  and  $g$  are closed of degree 0, and we have  $dh_1 = g - f_2 \circ f_1$ ,  $dh_2 = g - f_4 \circ f_3$ .

One of the main advantage of homotopy coherent diagrams over homotopy commutative ones is the existence of a good theory of limits. We give an example.

**Example 6.5.** Let  $\mathcal{A}$  be an abelian category with enough projectives and consider the dg category  $\mathbf{Ch}^-(\mathcal{A}_{\text{proj}})$  of bounded-above cochain complexes of projective objects in  $\mathcal{A}$ . We define the *bounded-above derived  $\infty$ -category of  $\mathcal{A}$*  as the dg nerve  $\mathcal{D}^-(\mathcal{A}) := N_{\text{dg}}(\mathbf{Ch}^-(\mathcal{A}_{\text{proj}}))$ . The ordinary bounded-above derived category is obtained by passing to the homotopy category  $h(\mathcal{D}^-(\mathcal{A}))$  which is defined as the ordinary category obtained by identifying homotopic morphisms. Consider an edge  $f : X \rightarrow Y$  in  $\mathcal{D}^-(\mathcal{A})$ , i.e., a morphism between bounded-above complexes of projectives  $X$  and  $Y$ . Consider the cone of  $f$ , i.e., the complex with

$$\text{cone}(f)_n = X^{n+1} \oplus Y^n$$

and differential given by

$$d = \begin{pmatrix} -d_X & 0 \\ f & d_Y \end{pmatrix}.$$

We obtain a coherent square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \searrow^h & \downarrow i \\ 0 & \longrightarrow & \text{cone}(f) \end{array} \quad (6.6)$$

in  $\mathcal{D}^-(\mathcal{A})$  where

$$i : Y \rightarrow \text{cone}(f), y \mapsto (0, y)$$

and

$$h : X \rightarrow \text{cone}(f), x \mapsto (-x, 0)$$

so that we have  $dh = 0 - i \circ f$ . The key fact is that the diagram (6.6) is a pushout diagram in the  $\infty$ -category  $\mathcal{D}^-(\mathcal{A})$  so that the cone is characterized by a universal property. This statement becomes wrong if we pass from  $\mathcal{D}^-(\mathcal{A})$  to the *homotopy category*  $h(\mathcal{D}^-(\mathcal{A}))$ : the image of the square (6.6) commutes up to unspecified homotopy, but this data is, in general, insufficient to characterize  $\text{cone}(f)$  by a universal property. As an extreme case, consider the cone of the zero morphism  $X \rightarrow 0$  which is the translation  $X[1]$ . The coherent square

$$\begin{array}{ccc} X & \xrightarrow{0} & 0 \\ \downarrow & \searrow^h & \downarrow i \\ 0 & \longrightarrow & X[1] \end{array}$$

involves the map

$$h : X \rightarrow X[1], x \mapsto x$$

considered as a self-homotopy of 0. For precise definitions and proofs, we refer the reader to [Lur11, 1.3.2].

## 6.2 Stable $\infty$ -categories

We take for granted the existence of a theory of limits for  $\infty$ -categories (cf. [Lur09]).

**Definition 6.7.** An  $\infty$ -category  $\mathcal{C}$  is called *stable* if the following conditions hold:

- (1) The  $\infty$ -category  $\mathcal{C}$  is pointed.

(2) (a) Every diagram in  $\mathcal{C}$  of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ C & & \end{array}$$

can be completed to a pushout square of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D. \end{array}$$

(b) Every diagram in  $\mathcal{C}$  of the form

$$\begin{array}{ccc} & & B \\ & & \downarrow \\ C & \longrightarrow & D \end{array}$$

can be completed to a pullback square of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D. \end{array}$$

(3) A square in  $\mathcal{C}$  of the form

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a pushout square if and only if it is a pullback square.

**Remark 6.8.** Note that these axioms correspond precisely to the axioms defining a proto-abelian category, except that we drop the conditions on horizontal (resp. vertical) morphisms to be monic (resp. epic).

**Example 6.9.** The bounded-above derived  $\infty$ -category  $\mathcal{D}^-(\mathcal{A})$  of an abelian category with enough projectives is stable.

**Remark 6.10.** It can be shown (cf. [Lur11]) that the homotopy category of a stable  $\infty$ -category  $\mathcal{C}$  has a canonical triangulated structure. The stable  $\infty$ -category  $\mathcal{C}$  can be regarded as an *enhancement* of  $\mathrm{h}\mathcal{C}$  with better properties such as the existence of functorial cones.

To generalize the simplicial groupoid of flags  $\mathcal{X}_\bullet$  to stable  $\infty$ -categories we need to understand what the  $\infty$ -categorical analog of a groupoid is: An  $\infty$ -groupoid is an  $\infty$ -category  $\mathcal{C}$  whose homotopy category is a groupoid.

**Example 6.11.** (1) The nerve of a groupoid is an  $\infty$ -groupoid.

(2) Given an  $\infty$ -category  $\mathcal{C}$ , let  $\mathcal{C}^\simeq$  denote the simplicial subset of  $\mathcal{C}$  consisting of only those simplices whose edges become isomorphisms in the homotopy category  $\mathrm{h}\mathcal{C}$ . Then  $\mathcal{C}^\simeq$  is an  $\infty$ -groupoid called the *maximal  $\infty$ -groupoid in  $\mathcal{C}$* .

(3) Let  $X$  be a topological space. We define the singular simplicial set  $\mathrm{Sing}(X)$  with  $n$ -simplices

$$\mathrm{Sing}(X)_n := \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X)$$

given by continuous morphisms from a geometric  $n$ -simplex to  $X$ . Then  $\mathrm{Sing}(X)$  is an  $\infty$ -groupoid. The functor  $\mathrm{Sing}$  from topological spaces to simplicial sets has a left adjoint given by the geometric realization  $|K|$  of a simplicial set. The pair of functors  $(|-|, \mathrm{Sing})$  defines a Quillen equivalence between suitably defined model categories of topological spaces and  $\infty$ -groupoids so that, from the point of view of homotopy theory, the two concepts are equivalent (see, e.g., [Lur09]).

### 6.3 The $S$ -construction and the derived Hall algebra

With these concepts at hand, we can adapt the theory of Section 4 from proto-abelian categories to stable  $\infty$ -categories. We define the analog of the simplicial groupoid of flags  $\mathcal{X}_\bullet$  for a stable  $\infty$ -category  $\mathcal{C}$  as follows: Let  $n \geq 0$ . We introduce the category  $T^n = \mathrm{Fun}([1], [n])$  where we interpret the linearly ordered sets  $[1]$  and  $[n]$  as categories. A functor

$$N(T^n) \rightarrow \mathcal{C}$$

is simply a coherent version of a triangular diagram of shape (4.13). We define

$$\mathcal{X}_n \subset \mathrm{Fun}(N(T^n), \mathcal{C})^\simeq$$

to be the  $\infty$ -groupoid of coherent diagrams so that

- (1) the diagonal objects are 0,
- (2) all squares are pushout squares and hence, by stability, also pullback squares.

Using Example 6.11(3), we typically interpret the resulting simplicial  $\infty$ -groupoid  $\mathcal{X}_\bullet$  as a simplicial space.

**Remark 6.12.** In other words, besides the idea to use coherent diagrams, the only substantial modification in comparison to the case when  $\mathcal{C}$  is proto-abelian is to allow arbitrary chains of morphisms as opposed to flags given by chains of monomorphisms.

**Theorem 6.13** ([DK12]). Let  $\mathcal{C}$  be a stable  $\infty$ -category. The simplicial space  $\mathcal{X}_\bullet(\mathcal{C})$  is 2-Segal.

**Remark 6.14.** For the simplicial groupoids of flags in proto-abelian categories, the pullback conditions on the squares (5.7) and (5.8) have to be interpreted in the 2-category of groupoids: the squares have to be 2-pullback squares. In the context of Theorem 6.13, the pullback conditions have to be interpreted in the  $\infty$ -category of  $\infty$ -groupoids. Using the equivalence between  $\infty$ -groupoids and topological spaces this can be made quite explicit: the squares have to be homotopy pullback squares.

The construction of the Hall algebra of a stable  $\infty$ -category from  $\mathcal{X}_\bullet(\mathcal{C})$  can also be adapted to our new context: given a topological space  $X$ , we pass to the vector space  $\mathcal{F}(X)$  of functions  $\varphi : X \rightarrow \mathbb{Q}$  which are constant along connected components and only supported on finitely many connected components. All definitions of Section 4.3 admit natural generalizations to this context. The central idea is to replace the groupoid cardinality

$$|\mathcal{A}| = \sum_{[a] \in \pi_0(\mathcal{A})} \frac{1}{|\mathrm{Aut}(a)|}$$

by the homotopy cardinality

$$|X| = \sum_{[x] \in \pi_0(X)} \frac{1}{|\pi_1(X, x)|} \frac{|\pi_2(X, x)|}{1} \frac{1}{|\pi_3(X, x)|} \cdots$$

as introduced by Baez-Dolan [BD01], and 2-pullbacks by homotopy pullbacks. In particular, we obtain natural pushforward and pullback operations for maps  $X \rightarrow Y$  of topological spaces satisfying suitable finiteness conditions. Assume that the stable  $\infty$ -category  $\mathcal{C}$  is *finitary*: for every pair of objects  $X, Y$ , the groups  $\mathrm{Hom}(X, Y[i])$  of morphisms in the homotopy category are finite and non-zero for only finitely many  $i$ . Then we may apply the constructions of Section 4.3, adapted to the current situation, to obtain the Hall algebra of  $\mathcal{C}$ .

**Example 6.15.** Let  $\mathcal{A}$  be a finitary abelian category of finite global dimension with enough projective objects. Then the dg nerve  $\mathcal{D}^b(\mathcal{A})$  of the full dg subcategory of  $\mathbf{Ch}^-(\mathcal{A}_{\mathrm{proj}})$  consisting of those complexes with bounded cohomology objects is finitary. In this example, we obtain the derived Hall algebra as defined in [Toë06].

In complete analogy to [Toë06], we obtain an explicit description of the structure constants of the derived Hall algebra of a stable  $\infty$ -category  $\mathcal{C}$ . Given objects  $X, Y, Z$ , we have

$$g_{X,Y}^Z = \frac{|\mathrm{Hom}(X, Z)_Y| \prod_{i>0} |\mathrm{Hom}(X[i], Z)|^{(-1)^i}}{|\mathrm{Aut}(X)| \prod_{i>0} |\mathrm{Hom}(X[i], X)|^{(-1)^i}}$$

where  $\mathrm{Hom}$  denotes the morphisms in the homotopy category of  $\mathrm{h}\mathcal{C}$  of  $\mathcal{C}$  and  $\mathrm{Hom}(X, Z)_Y$  denotes the subset given by those morphisms whose cone is isomorphic to  $Y$ . Note that the structure constants only depend on the triangulated category  $\mathrm{h}\mathcal{C}$

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