INVERSE LIMITS AND PROFINITE GROUPS

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We discuss the inverse limit construction, and consider the special case of inverse limits of finite groups, which should best be considered as topological groups, and can be characterized by their topological properties. These are profinite groups, which arise naturally in infinite Galois theory.

No arguments in these notes are original in any way.

1. Inverse limits

Standard examples of inverse limits arise from sequences of groups, with maps between them: for instance, if we have the sequence $G_n = \mathbb{Z}/p^n\mathbb{Z}$ for $n \geq 0$, with the natural quotient maps $\pi_{n+1} : G_{n+1} \to G_n$, the inverse limit consists of tuples $(g_0, g_1, \dots) \in \prod_{n\geq 0} G_n$ such that $\pi_{n+1}(g_{n+1}) = g_n$ for all $n \geq 0$. This is a description of the p-adic integers $\mathbb{Z}_p$. It is clear that more generally if the G_n are any groups and $\pi_{n+1} : G_{n+1} \to G_n$ any homomorphisms, we can define a notion of the inverse limit group in the same way.

However, we will make a more general definition.

Definition 1.1. Suppose we have a set S, with a partial order on it. We say that S (with the partial order) is **directed** if given any $s_1, s_2 \in S$, there exists $s_3 \in S$ such that $s_1 \leq s_3$ and $s_2 \leq s_3$.

An **inverse system** of groups is a directed set S, together with groups G_s for every $s \in S$, as well as for all s_1, s_2 satisfying $s_1 \leq s_2$, a homomorphism $f_{s_1}^{s_2} : G_{s_2} \to G_{s_1}$. These homomorphisms must satisfy the conditions that $f_s^s = \mathrm{id}$ for all $s \in S$, and that for any $s_1 \leq s_2 \leq s_3$, we have

$$f_{s_1}^{s_2} \circ f_{s_2}^{s_3} = f_{s_1}^{s_3}.$$

Given an inverse system of groups, the **inverse limit** $\varprojlim_s G_s$ is the subgroup of $\prod_{s\in S} G_s$ consisting of elements $(g_s)_{s\in S}$ satisfying the condition that for all $s_1 \leq s_2$, we have $f_{s_1}^{s_2}(s_2) = s_1$.

Note that there are natural homomorphisms $\varprojlim_s G_s \to G_{s'}$ for every $s' \in S$. Using these maps, inverse limits satisfy a natural universal property, the proof of which we leave to the reader:

Proposition 1.2. *Let $\{G_s\}$ be an inverse system of groups, and H any group. Then composition with the maps $\varprojlim_s G_s \to G_{s'}$ induces an injection*

$$\mathrm{Hom}(H, \varprojlim_s G_s) \hookrightarrow \prod_{s\in S} \mathrm{Hom}(H, G_s),$$

with the image consisting precisely of tuples $(\varphi_s : H \to G_s)$ satisfying the condition that for all $s_1 \leq s_2$, we have

$$f_{s_1}^{s_2} \circ \varphi_{s_2} = \varphi_{s_1}.$$

2. Topological groups

We now describe how to combine topology and group theory:

Definition 2.1. A **topological group** is a set G with both the structure of a group, and of a topological space, such that the multiplication law $G \times G \to G$ and the inverse map $G \to G$ are continuous maps of topological spaces.

In the context of topological groups, we also require continuity for homomorphisms:

Definition 2.2. A **homomorphism** of topological groups is a homomorphism of the underlying groups which is continuous. An **isomorphism** of topological groups is a homomorphism of the underlying groups which is a homeomorphism.

Note that the definition of isomorphism is equivalent to the existence of an inverse (continuous) homomorphism, but is not equivalent to being a (continuous) bijective homomorphism, because the set-theoretic inverse need not be continuous in general. When we discuss subgroups in the context of topological groups, it is to be understood that the subgroup inherits the subset topology.

We can thus combine topological and group-theoretic notions freely, speaking for instance of closed subgroups, which often play an important role, particularly in profinite groups. We note that every group can be given the **discrete topology**, in which every subset is open. This is a very natural topology for certain groups, particularly finite groups, and it allows us to consider every group as a topological group, with usual homomorphisms and continuous homomorphisms being equivalent. However, when we deal with groups having other topologies, it is frequently useful to consider a variation of the usual notion of generators:

Definition 2.3. Given a topological group G, we say that a subset $S \subseteq G$ **topologically generates** G if the only closed subgroup of G containing S is G itself.

Inverse limits may also be constructed from families of topological groups:

Proposition 2.4. *Suppose we have an inverse system (G_s) of topological groups, where we require the homomorphisms in the system to be continuous. Then the inverse limit $\varprojlim_s G_s$ inherits a natural topological group structure as a subgroup of $\prod_{s \in S} G_s$, and the homomorphisms $\varprojlim_s G_s \to G_{s'}$ for each s' are homomorphisms of topological groups.*

We leave the verification of this as an exercise for the reader.

Before discussing profinite groups, we prove some basic results on topological groups.

Lemma 2.5. *Given an inverse system of Hausdorff topological groups G_s, the inverse limit $\varprojlim_s G_s$ is a closed subgroup of $\prod_{s \in S} G_s$. In particular, if all the G_s are Hausdorff and compact, then $\varprojlim_s G_s$ is Hausdorff and compact.*

Proof. The inclusion $\varprojlim_s G_s \subseteq \prod_{s \in S} G_s$ is determined as an intersection of conditions $f_{s_1}^{s_2}(g_{s_2}) = g_{s_1}$, so it suffices to show that each of these gives a closed subset. But we have continuous homomorphisms

$$\prod_{s \in S} G_s \to G_{s_1} \times G_{s_2} \stackrel{\mathrm{id} \times f_{s_1}^{s_2}}{\to} G_{s_1} \times G_{s_1},$$

and the desired condition is satisfied precisely by the preimage of the diagonal $\Delta \subseteq G_{s_1} \times G_{s_1}$. But since G_{s_1} is Hausdorff, Δ is closed, and we get the desired statement.

In particular, since products and subsets of Hausdorff spaces are Hausdorff, and products and closed subsets of compact spaces are compact, we get the remaining assertions as well. □

3. Profinite groups

We are now ready to define and explore the basic properties of profinite groups.

Definition 3.1. A **profinite group** is a topological group which is obtained as the inverse limit of a collection of finite groups, each given the discrete topology.

Without further comment, we will from now on assume that every finite group is equipped with the discrete topology. There is a natural method of associating a profinite group to any group G (without topology on it):

Definition 3.2. Given a group G, the **profinite completion** $\hat{G}$ of G is the inverse limit of G/H, where H runs over all normal subgroups of G of finite index.

Note that this is indeed an inverse system, since an intersection of two normal subgroups of finite index is a normal subgroup of finite index. Also, we have a natural homomorphism $G \to \hat{G}$, which is injective precisely when G has the property that for any non-trivial $g \in G$, there is some $H \trianglelefteq G$ of finite index such that $g \notin H$. We also have:

Proposition 3.3. *The image of G in $\hat{G}$ is a dense subgroup.*

Proof. Suppose we have a non-empty open subset $U \subseteq \hat{G}$; we want to show that there exists $g \in G$ whose image in $\hat{G}$ lies in U. Then by the definition of the product topology, U contains a non-empty subset of the form $\hat{G} \cap \prod_H U_H$, where all but finitely many of the U_H are equal to all of G/H (the remaining U_H are required to be open, but this is a vacuous condition since G/H has the discrete topology). We may therefore assume that U itself is of this form.

Let $H_1, \dots, H_n$ be the normal subgroups of finite index for which $U_H \neq G/H$. Then $H' := H_1 \cap \dots \cap H_n$ is a normal subgroup of finite index. Let $V_1, \dots, V_n \subseteq G/H'$ be the preimages of the U_{H_i} under $G/H' \twoheadrightarrow G/H_i$. We claim that if we let $V = \cap_i V_i$, then V is non-empty. But this follows from our hypothesis that $\hat{G} \cap \prod_H U_H$ is non-empty, since any element must have its image in G/H' contained in V by the definition of the inverse limit. Similarly, if we choose g mapping to V in G/H', we see from the definition of inverse limit that $g \in U$, as desired. □

Example 3.4. Let $\hat{\mathbb{Z}}$ be the inverse limit of $\{\mathbb{Z}/n\mathbb{Z}\}_n$, with homomorphisms given by the natural quotient maps $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ whenever $m|n$. Then the natural homomorphism $\mathbb{Z} \to \hat{\mathbb{Z}}$ has dense image, so we see that $\hat{\mathbb{Z}}$ is topologically generated by the image of $1 \in \mathbb{Z}$, so $\hat{\mathbb{Z}}$ is a topologically cyclic group.

Profinite completion has a universal property, the proof of which we leave to the reader:

Proposition 3.5. *Given any group G, and any profinite group H, any group homomorphism $G \to H$ (ignoring the topology on H) factors uniquely through a continuous homomorphism $\hat{G} \to H$.*

Warning 3.6. Note that although profinite completion has a universal property, it is not the case that for G a profinite group, the map $G \to \hat{G}$ is necessarily an isomorphism. This is closely related to the fact that for some profinite groups, the topology is not uniquely determined by the group structure. However, no such examples were known for topologically finitely generated groups, and it was an open question since at least 1980 whether it is true that every topologically finitely generated profinite group is isomorphic to its profinite completion. This was finally answered positively by Nikolov and Segal in a pair of papers in Annals of Mathematics which appeared 2007. This proof relies of the classification of finite simple groups, but most of the argument involves new and non-trivial ideas.

Profinite groups arise in infinite Galois theory, and not coincidentally, in the theory of algebraic fundamental groups. For an algebraic variety over $\mathbb{C}$, it turns out that the algebraic fundamental group is the profinite completion of the usual fundamental group. More broadly, all algebraic fundamental groups are profinite by construction, and in fact contain Galois groups as a special case.

We now discuss the topological structure of profinite groups in more detail. First, we have the following classification of profinite groups in topological terms:

Theorem 3.7. *A topological group is profinite if and only if it is Hausdorff, compact, and totally disconnected.*

Recall that a topological space is **totally disconnected** if the only (non-empty) connected subsets are one-point subsets, or equivalently, if for any two points there is an open and closed subset containing one but not the other. Our argument is adapted from *Profinite Groups*, by Ribes and Zalesskii. We begin with the following simple observation:

Proposition 3.8. *Let G be a compact group, and $H \subseteq G$ a subgroup. Then H is open if and only if H is closed and of finite index.*

Proof. Suppose H is open. Then H has finite index, since G is compact and every coset of an open subgroup is open. H is then also closed, since its complement is a union of cosets, each of which is open. Conversely, if H is closed and has finite index, then its complement is a finite union of closed cosets, and hence closed, so H is open. □

The most substantive step in the proof is the following:

Lemma 3.9. *Suppose that G is a Hausdorff, compact, totally disconnected topological group. Then if $\{H_s\}_{s\in S}$ is the set of open normal subgroups of G, we have $\cap_{s\in S} H_s = \{1\}$.*

Proof. We need to show that for every $g \in G$ with $g \neq 1$, there exists an open normal subgroup $H \trianglelefteq G$ not containing g. By total disconnectedness, there is some open and closed $U \subseteq G$ which contains 1 but not g; we we will construct the desired H inside of U. For the remainder of the proof, for any subset $T \subseteq G$, we denote by T^n the set of products of n elements from T, and by T^{-1} the set of inverses of elements of T. Now, consider $V = (G \smallsetminus U) \cap U^2$. Note that since U is compact, $U \times U \subseteq G \times G$ is compact, and so its image U^2 under the multiplication map is also compact. Since U is open, $G \smallsetminus U$ is closed and hence also compact, so we conclude that V is compact.

Now, suppose we have any $h \in U$. We claim we have $W_h, X_h \subseteq U$ open neighborhoods of h and 1 respectively, and such that $W_h X_h \subseteq (G \smallsetminus V) \cap U^2 \subseteq U$. Indeed, if we consider the multiplication map $G \times G \to G$, since $(h, 1)$ maps to h, which lies in $G \smallsetminus V$, and since V is compact, and hence closed, we have that $G \smallsetminus V$, and hence its preimage in $G \times G$, is an open subset containing $(h, 1)$, and hence containing some $W'_h \times X'_h$ with W'_h and X'_h open neighborhoods of h and 1. We can then set $W_h = W'_h \cap U$, and $X_h = X'_h \cap U$; we have $W_h X_h \subseteq G \smallsetminus V$ by construction, but also $W_h X_h \subseteq U^2$, so we obtain the desired claim.

The collection of W_h thus give an open cover of U, and since U is closed, there exist $h_1, \ldots, h_n \in U$ such that the W_{h_i} cover U. If we set $X = \cap X_{h_i}$, and $Y = X \cap X^{-1}$. Then Y is still an open neighborhood of 1, and contained in U, and we note that $UY = \cup_i W_{h_i} Y \subseteq \cup_i W_{h_i} X_{h_i} \subseteq U$. By induction, $UY^i \subseteq U$ for all $i \geq 1$, so we find that $H' = \cup_{i=1}^{\infty} Y^i$ is an open subgroup of G contained in U. Finally, if we set $H = \cap_{g \in G} g H' g^{-1}$, since H' is open, it has finite index, and so in fact there are only finitely many distinct conjugates subgroups $gH'g^{-1}$, so H is still open. We have therefore constructed the desired normal subgroup of G contained in U. □

We can now prove the theorem relatively easily.

Proof of theorem. First suppose that G is a profinite group. Then since the discrete topology on a finite group is compact, Hausdorff, and totally disconnected, we have by Lemma 2.5 that G is compact and Hausdorff. Furthermore, it is an easy topological exercise to see that products and subsets of totally disconnected spaces are totally disconnected, so we find that G is Hausdorff, compact, and totally disconnected.

Conversely, suppose that G is compact, Hausdorff, and totally disconnected. Then we claim that G is isomorphic to the inverse limit of all quotients by open normal subgroups, which we denote by H. Note that this forms an inverse system, since the intersection of any two open normal subgroups is again an open normal subgroup. By the proposition, any open subgroup has finite index, so our claim would prove the theorem. Of course, we have a natural map $G \to H$ by the universal property of the inverse limit, and both G and H are compact Hausdorff spaces. Therefore, closed subsets of G are compact and map to compact and hence closed subsets of H, and it is enough to check that this map is bijective and continuous.

Continuity is straightforward: we are only modding out by open subgroups, so any (necessarily finite) union of their cosets is still open, and each quotient map is thus continuous, which implies that the induced maps to the product and inverse limit are likewise continuous. We have already proved injectivity in Lemma 3.9. For surjectivity, let $(g_s G_s)_{s \in S}$ be an element of H, where the G_s are the open normal subgroups of G, and $g_s \in G$. Then by the proposition, each G_s and hence each $g_s G_s$ is a non-empty closed subset of G, and we want to see that $\cap_{s \in S} g_s G_s$ is non-empty. But suppose $\cap_{s \in S} g_s G_s = \emptyset$. Then because G is compact, there is some finite collection $s_1, \ldots, s_n \in S$ such that $\cap_{i=1}^n g_{s_i} G_{s_i} = \emptyset$. Since the G_s form an inverse system, there is some $s \in S$ with $s \geq s_i$ for $i = 1, \ldots, n$, and by definition of the inverse limit, we have $g_s G_s \subseteq \cap_{i=1}^n g_{s_i} G_{s_i}$. But G_s is non-empty, yielding a contradiction. □

Corollary 3.10. *Let G be a profinite group, and $H \subseteq G$ a subgroup. Then H is open if and only if H is closed and has finite index.*

Also, H is closed if and only if H is an intersection of open subgroups. H is normal and closed if and only if H is an intersection of open normal subgroups.

Proof. The first assertion follows immediately from the proposition and the fact that G is compact.

Next, it is clear that if H is an intersection of open subgroups, it is closed, and similarly in the normal case. For the converse, we first note that the easy direction of the theorem together with Lemma 3.9 proves that if $\{U_s\}$ is the set of all open normal subgroups of G, then $\cap_{s\in S}U_s = \{1\}$. We claim that if H is closed, we have

$$\cap_{s\in S}(HU_s) = H(\cap_{s\in S}U_s) = H,$$

which gives the desired expression for H as an intersection of open subgroups, normal when H is normal. It is clear that $H(\cap_{s\in S}U_s) \subseteq \cap_{s\in S}(HU_s)$, so we need to prove the opposite inclusion. Given $g \in \cap_{s\in S}(HU_s)$, we want to show that $g \in H(\cap_{s\in S}U_s)$, or equivalently $Hg \cap (\cap_{s\in S}U_s) \neq \emptyset$. But if this were not true, then by compactness we would have some $s_1, \ldots, s_n$ with $Hg \cap (\cap_{i=1}^n U_{s_i}) = \emptyset$. But $\cap_{i=1}^n U_{s_i}$ is itself an open normal subgroup of G, so $g \in H(\cap_{i=1}^n U_{s_i})$, by assumption, which is a contradiction. □