Weak Measurements: Wigner-Moyal and Bohm in a New Light.

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Weak Measurements.

1. New type of quantum measurement. [Aharonov, and Vaidman, Phys. Rev., 41, (1990) 11-19].

2. What do we measure? Weak values defined by $A_{W(\psi,\phi)} = \dfrac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle} \in \mathbb{C}$

3. Will show that $P_{W(\psi,x)}$ is related to the Bohm momentum

   $P^2_{W(\psi,x)}$ is related to the Bohm energy and quantum potential.

   [Leavens, Found. Phys., 35 (2005) 469-91]

4. Will obtain expressions for weak values for spin-1/2 particles using Clifford algebras.

5. Will indicate how to combine the Clifford with the Moyal algebra.

6. Discuss what may lie behind the formalism-- Process.

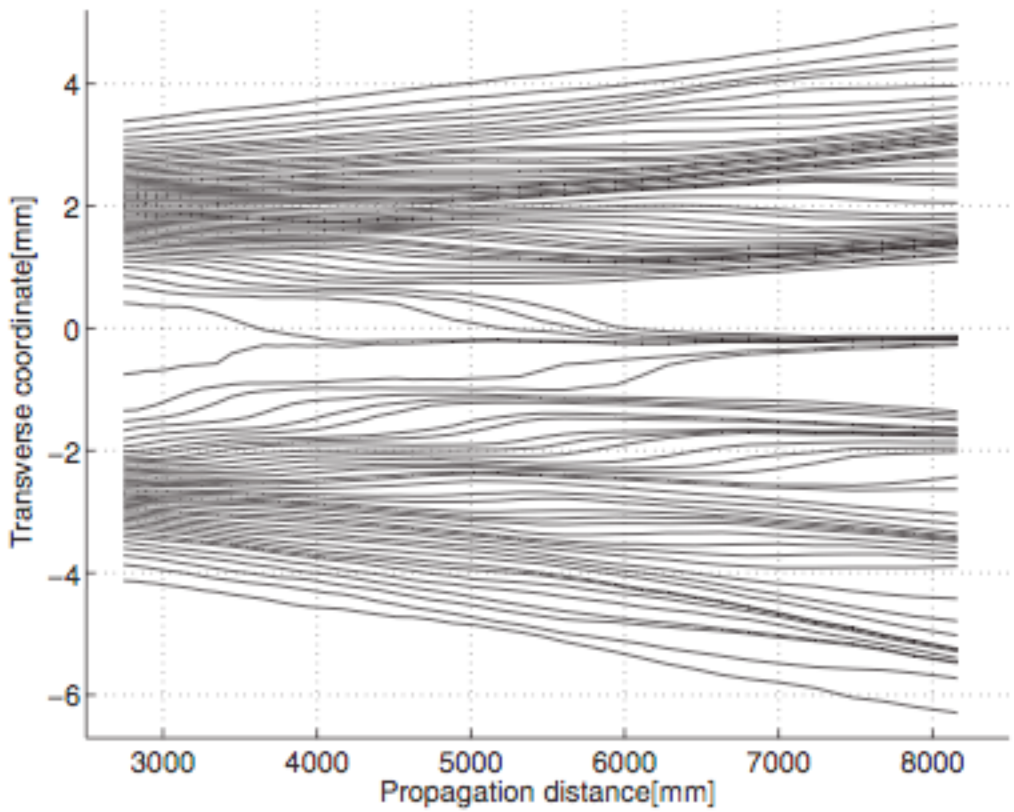
Acknowledgements

Ernst Binz, Maurice de Gosson, Bob Callaghan and David Robson.

Weak measurements.

Why the interest?

Photon 'trajectories'.

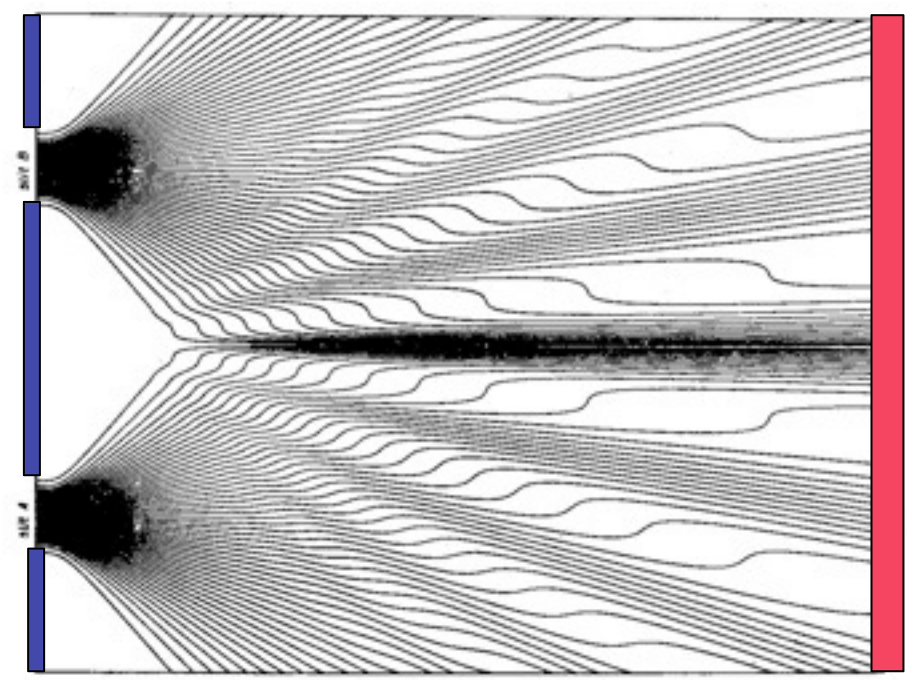


Experimental--Photons.

[Kocsis, Braverman,Ravets,Stevens,Mirin, Shalm, Steinberg, *Science* **332**, 1170 (2011)]

[Prosser, IJTP , 15, (1976) 169]

Schrödinger particle 'trajectories



Theory--Schrödinger particle.

[Philippidis, Dewdney and Hiley, Nuovo Cimento **52B**, 15-28 (1979)]

Real part of Schrödinger equation

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + Q + V = 0$$

$$E_B = -\frac{\partial S}{\partial t} \qquad P_B = \nabla S$$

Weak Measurement: the principle.

$$H_I = \lambda(t)\hat{A}_s\hat{B}_d \qquad |\text{initial}\rangle = |\psi_s\rangle|\Psi_d\rangle \qquad |\text{final}\rangle = e^{i\int \lambda(t)\hat{A}_s\hat{B}_d dt}|\psi_s\rangle|\Psi_d\rangle$$

$$\int \lambda(t)dt = 1 \qquad |\text{final}\rangle = e^{i\hat{A}_s b}|\psi_s\rangle|\Psi_d\rangle$$

Form the transition probability amplitude $\quad T_{\phi-\psi} = \langle b_d|\langle\phi_s|\text{final}\rangle \quad \langle b_d|\hat{A}|\Psi_d\rangle = \hat{A}\Psi_d(b)$

$$T_{\phi-\psi} = \langle\phi_s|e^{i\hat{A}b}|\psi_s\rangle\Psi_d(b) = \sum_{n=0}\frac{(ib)^n}{n!}\langle\phi_s|\hat{A}^n|\psi_s\rangle\Psi_d(b) = \langle\phi_s|\psi_s\rangle\sum_{n=0}\frac{(ib)^n}{n!}\frac{\langle\phi_s|\hat{A}^n|\psi_s\rangle}{\langle\phi_s|\psi_s\rangle}\Psi_d(b)$$

$$T_{\phi-\psi} = \langle\phi_s|\psi_s\rangle\left[e^{ib\langle A\rangle_W} + \underbrace{\sum_{n=2}\frac{(ib)^n}{n!}[\langle A^n\rangle_W - \langle A\rangle_W^n]}_{\text{small}}\right]\Psi_d(b)$$

[Duck, Stevenson and Sudarshan Phys. Rev, 40 (1989) 2112-7]

Post select with $b = x \qquad T_{\phi-\psi} = \langle\phi_s|\psi_s\rangle e^{ix\langle A\rangle_W}\Psi_d(x)$

Choose $\quad \Psi_d(x) = \exp\left[\frac{-x^2}{4(\Delta x)^2}\right]$ and take the imaginary part of $\langle A\rangle_W$ we find

$$T_{\phi-\psi} \propto e^{-x\langle A\rangle_{IW}}\exp\left[\frac{-x^2}{4(\Delta x)^2}\right] \propto \exp\frac{-\left[x + 2(\Delta x)^2\langle A\rangle_{IW}\right]^2}{4(\Delta x)^2}$$

Centre of Gaussian in x-space shifted by amount $\propto \langle A\rangle_{IW}$

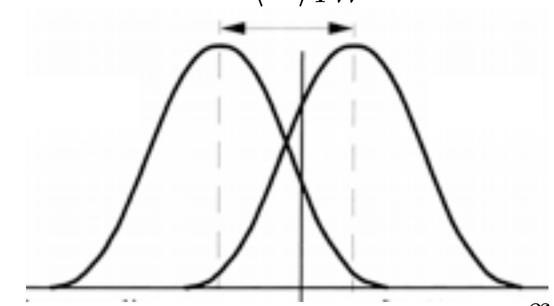
Centre of Gaussian in p-space shifted by amount $\propto \langle A\rangle_{RW}$

Weak Experiment cont.

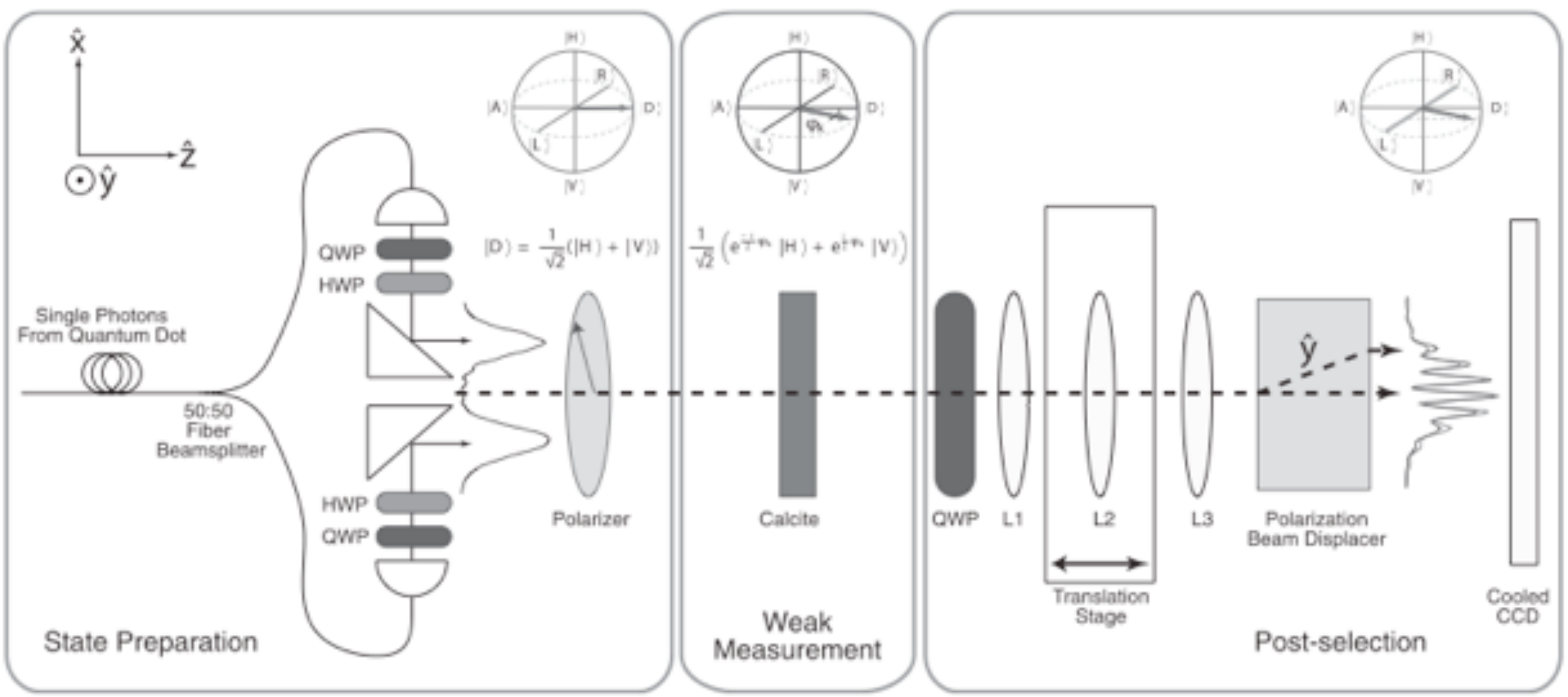
von Neumann measurement. Single measurement ---- collapse of the wave function. $\sum_j \Psi_j \to \Psi_r$

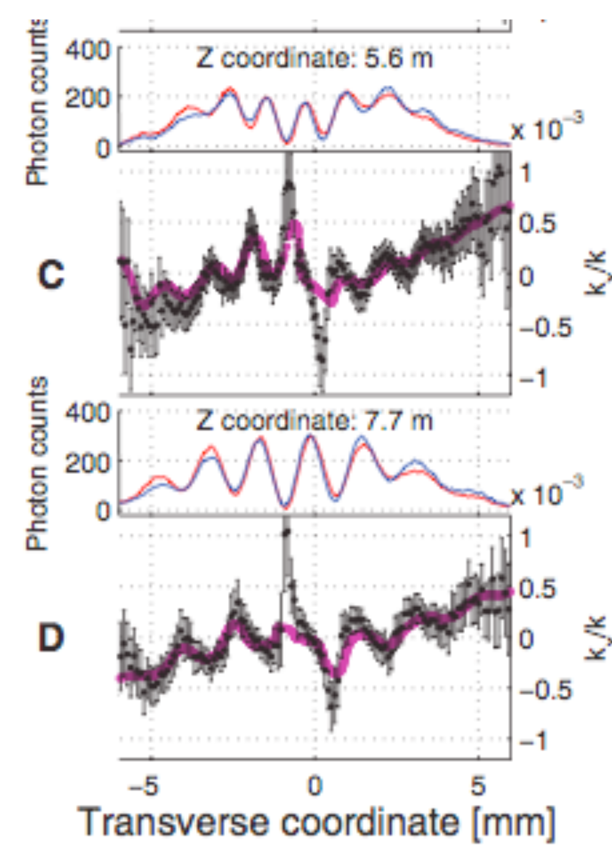
Weak measurement.

Statistical measurement producing a phase shift in the distribution of final results.



Actual experimental arrangement for photon trajectories.





$$|\Psi\rangle = |D\rangle_{\text{polarizer}} |\psi\rangle_{\text{path}} \qquad H = g\hat{P}_x \hat{S}_1 \qquad \hat{S}_1 = (|H\rangle\langle H| - |V\rangle\langle V|)/2$$

[Kocsis, Braverman,Ravets,Stevens,Mirin, Shalm, Steinberg,

Science **332**, 1170 (2011)]

Weak values. $\frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}$

How do they appear in the formalism?

[Hiley quant-ph/1111.6536]

$$\langle\psi|A|\psi\rangle = \sum \langle\psi|\phi_j\rangle\langle\phi_j|A|\psi\rangle$$

where $|\phi_j\rangle$ form a complete orthonormal set.

Then

Post select

$$\langle\psi|A|\psi\rangle = \sum \langle\psi|\phi_j\rangle \left(\frac{\langle\phi_j|\psi\rangle}{\langle\phi_j|\psi\rangle}\right)\langle\phi_j|A|\psi\rangle = \sum \rho_j \frac{\langle\phi_j|A|\psi\rangle}{\langle\phi_j|\psi\rangle}$$

Weak value.

Special case:

If $A|\phi_j\rangle = a_j|\phi_j\rangle$

$$\langle\psi|A|\psi\rangle = \sum \rho_j a_j$$

Well known result!

Eigenvalue!

Remember $\frac{\langle\phi|A|\psi\rangle}{\langle\phi|\psi\rangle}$ is a complex number.

But what does it mean in general?

It is clearly a transition probability amplitude.

Weak values when $\hat{P}$ is involved.

Consider $$\langle x|\hat{P}|\psi(t)\rangle = \int \langle x|\hat{P}|x'\rangle\langle x'|\psi(t)\rangle dx' = -i\nabla_x\psi(x,t)$$

Write $\psi(x,t) = R(x,t)e^{iS(x,t)}$ then

$$\frac{\langle x|\hat{P}|\psi(t)\rangle}{\langle x|\psi(t)\rangle} = \nabla_x S(x,t) - i\nabla_x\rho(x,t)/2\rho(x,t) \qquad \text{with} \quad \rho(x,t) = |\psi(x,t)|^2$$

Bohm momentum. osmotic momentum.

Bohm velocity: $$v_B(x,t) = \nabla_x S(x,t)/m = \frac{1}{2m}\left[\frac{\langle\psi(t)|\overleftarrow{\hat{P}}|x\rangle}{\langle\psi(t)|x\rangle} + \frac{\langle x|\overrightarrow{\hat{P}}|\psi(t)\rangle}{\langle x|\psi(t)\rangle}\right]$$

Osmotic velocity: $$v_O(x,t) = \frac{1}{2m}\frac{\nabla_x\rho(x,t)}{\rho(x,t)} = \frac{1}{2m}\left[\frac{\langle\psi(t)|\overleftarrow{\hat{P}}|x\rangle}{\langle\psi(t)|x\rangle} - \frac{\langle x|\overrightarrow{\hat{P}}|\psi(t)\rangle}{\langle x|\psi(t)\rangle}\right]$$

In terms of one expression

$$[\![P_W]\!]_\pm = \left[\frac{\langle\psi(t)|\overleftarrow{\hat{P}}|x\rangle}{\langle\psi(t)|x\rangle} \pm \frac{\langle x|\overrightarrow{\hat{P}}|\psi(t)\rangle}{\langle x|\psi(t)\rangle}\right] = \frac{\psi^*(x,t)\overleftarrow{\hat{P}}_x\psi(x,t) \pm \psi^*(x,t)\overrightarrow{\hat{P}}_x\psi(x,t)}{\rho(x,t)}.$$

More simply for Schrödinger

$$[\![P_W]\!]_\pm = \frac{i}{\rho(x,t)}[(\nabla_x\psi^*(x,t))\psi(x,t) \mp \psi^*(x,t)(\nabla_x\psi(x,t))]$$

then

$$-i\rho[\![P_{xW}]\!]_+ = [\nabla_x\psi^*(x)]\psi(x) - \psi^*(x)[\nabla_x\psi(x)] = \psi^*(x)\overleftrightarrow{\nabla}_x\psi(x)$$

$$-i\rho[\![P_{xW}]\!]_- = [\nabla_x\psi^*(x)]\psi(x) + \psi^*(x)[\nabla_x\psi(x)] = \nabla_x(\psi^*(x)\psi(x)) = \nabla_x\rho(x).$$

Remark 1: Relation to Nelson.

Mean forward derivative: $$Dx(t) = \lim_{\Delta t \to 0^+} E_i \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

[Nelson, Phys. Rev., **150**, (1966), 1079-1085.]

Mean backward derivative: $$D_* x(t) = \lim_{\Delta t \to 0^+} E_i \frac{x(t) - x(t-\Delta t)}{\Delta t}$$

With these construct a forward velocity $b(x,t)$ and backward velocity $b_*(x,t)$:

$$[b(x,t) + b_*(x,t)]/2 = v_B(x,t) = \frac{\nabla_x S(x,t)}{m}$$

[Bohm and Hiley, Phys. Reps, **172**, (1989), 92-122.]

$$b(x,t) - b_*(x,t) = v_O(x,t) = \frac{1}{2m} \frac{\nabla_x \rho(x,t)}{\rho(x,t)}$$

Remark 2: Relation to Energy-Momentum Tensor.

$$T^{\mu\nu} = -\left\{\frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi)}\partial^\nu\psi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi^*)}\partial^\nu\psi^*\right\}$$

Take the Schrödinger Lagrangian: $\mathcal{L} = -\frac{1}{2m}\nabla\psi^*\cdot\nabla\psi + \frac{i}{2}[\psi^*(\partial_t\psi) - (\partial_t\psi^*)\psi] - V\psi^*\psi.$

and find

$$T^{0\mu} = -\frac{i}{2}[(\partial^\mu\psi^*)\psi - \psi^*(\partial^\mu\psi)] = \frac{i}{2}[\psi^*\overleftrightarrow{\partial}^\mu\psi] = -\rho\partial^\mu S$$

Recalling that

$$P_B = \nabla S \qquad \text{and} \qquad E_B = -\partial_t S$$

Then we find

$$\rho P_{jB} = \rho\partial_j S = -T^{0j} = -\frac{\rho}{2}[\![P_{jW}]\!]_+$$

Bohm momentum.

$$\rho E_B = -\rho\partial_t S = -T^{00} = -\frac{\rho}{2}[\![P_{tW}]\!]_+$$

Bohm energy.

This generalises to the Pauli and Dirac particles

[Hiley and Callaghan, Fond. Phys **42** (2012) 192-208, math-ph:1011.4031 and 1011.4033]

The Bohm kinetic energy.

$$\frac{1}{2}[\![P_W^2]\!]_+ = (\nabla_x S(x))^2 - \frac{\nabla_x^2 R(x)}{R(x)} = P_B^2 + Q.$$

Quantum potential

$$\frac{1}{2i}[\![P_W^2]\!]_- = \nabla_x^2 S(x) + \left(\frac{\nabla_x\rho(x)}{\rho(x)}\right)\nabla_x S(x).$$

[Leavens, Found. Phys., 35 (2005) 469-91]

[Wiseman, New J. Phys., 9 (2007) 165-77.]

Weak values from Moyal algebra.

Multiplication of phase space functions defined by

[Moyal, Proc. Camb. Phil. Soc. 45, (1949), 99-123].
[Baker, Phys. Rev. 6 (1958) 2198-2206.]

$$a(x,p)\star b(x,p) \qquad \text{where} \qquad \star = \exp\left[\frac{i}{2}(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_x)\right]$$

Let $f(x,p)$ be the density matrix in (x, p) representation, viz. $f(x,p) = \int \rho(x-y/2; x+y/2)e^{-ipy}dy$

$$p\star f(x,p) = \left(p - \frac{i}{2}\overrightarrow{\partial}_x\right)f(x,p) \qquad \text{and} \qquad f(x,p)\star p = f(x,p)\left(p + \frac{i}{2}\overleftarrow{\partial}_x\right)$$

Now form

$$[p, f(x,p)]_{BB} := \frac{(f\star p + p\star f)}{2} = pf(x,p)$$ **Baker bracket**

$$[p, f(x,p)]_{MB} := \frac{(f\star p - p\star f)}{i} = \nabla_x f(x,p)$$ **Moyal bracket** $\qquad [x,p]_{MB} = 1$

To make contact with configuration space we must form a marginal;

$$\int [p, f(x,p)]_{BB}dp = \int pf(x,p)dp = \rho(x)\overline{\overline{p}}(x)$$ **Moyal momentum**

Using $\rho(x)\overline{\overline{p^n}}(x) = \left(\frac{1}{2i}\right)^n\left[\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^n \psi(x_1)\psi^\star(x_2)\right]_{x_1=x_2=x}$ (A1.1) we find $\overline{\overline{p}}(x) = \nabla_x S(x)$ (A1.6)

$$P_B(x) = \int [p, f(x,p)]_{BB}dp/\rho(x)$$ **Bohm Momentum**

Finally

$$\frac{\nabla_x\rho(x)}{2\rho(x)} = \left[\int [p, f(x,p)]_{MB}dp\right]/\rho(x)$$ **Osmotic Momentum**

Kinetic energy.

Form

$$p^2 \star f(x,p) = [p^2 - ip\overrightarrow{\nabla}_x - \tfrac{1}{4}\overrightarrow{\nabla}_x^{\,2}]f(x,p). \qquad \text{and} \qquad f(x,p)\star p^2 = f(x,p)[p^2 + ip\overleftarrow{\nabla}_x - \tfrac{1}{4}\overleftarrow{\nabla}_x^{\,2}]$$

Now form the Baker bracket

$$[p^2, f(x,p)]_{BB} = p^2 f(x,p) - \tfrac{1}{4}\nabla_x^2 f(x,p).$$

We need to find the marginal

$$\int [p^2, f(x,p)]_{BB} dp = \rho(x)\,\overline{\overline{p^2}}\,(x) - \frac{1}{4}\nabla_x^2 \rho(x).$$

with

$$\overline{\overline{p^2}}\,(x) = (\nabla_x S(x))^2 + \frac{1}{4}\frac{\nabla_x^2\rho(x)}{\rho(x)} - \frac{\nabla_x^2 R(x)}{R(x)}$$

So that

Bohm Kinetic Energy

$$\int [p^2, f(x,p)]_{BB} dp/\rho(x) = (\nabla_x S(x))^2 - \nabla_x^2 R(x)/R(x) = P_B^2 + Q.$$

[Remember m=1/2.]

Quantum Potential

To complete the story, the Moyal bracket gives

$$\int [p^2, F(x,p)]_{MB} dp/\rho(x) = \nabla_x^2 S(x) + \left(\frac{\nabla_x \rho(x)}{\rho(x)}\right)\nabla_x S(x).$$

[Leavens, Found. Phys., 35 (2005) 469-91]

Summary so far.

We are interested in the values that can be found experimentally using weak measurements.

We have seen how these weak values are related to the Bohm momentum, Bohm energy and the quantum potential

$$\rho P_{jB} = \rho \partial_j S = -T^{0j}\,, \quad \rho E_B = -\rho \partial_t S = -T^{00}$$

The separation of the real from the imaginary parts of weak values achieved by forming brackets

$$[\![P_W]\!]_\pm = \left[\frac{\langle\psi(t)|\overleftarrow{\widehat{P}}|x\rangle}{\langle\psi(t)|x\rangle} \pm \frac{\langle x|\overrightarrow{\widehat{P}}|\psi(t)\rangle}{\langle x|\psi(t)\rangle} \right]$$

Then, for example

$$P_B = -\tfrac{1}{2}\,[\![P_{xW}]\!]_+ \qquad \text{and} \qquad E_B = -\tfrac{1}{2}\,[\![P_{tW}]\!]_+$$

In the Moyal algebra these brackets are replaced by the Baker and Moyal brackets

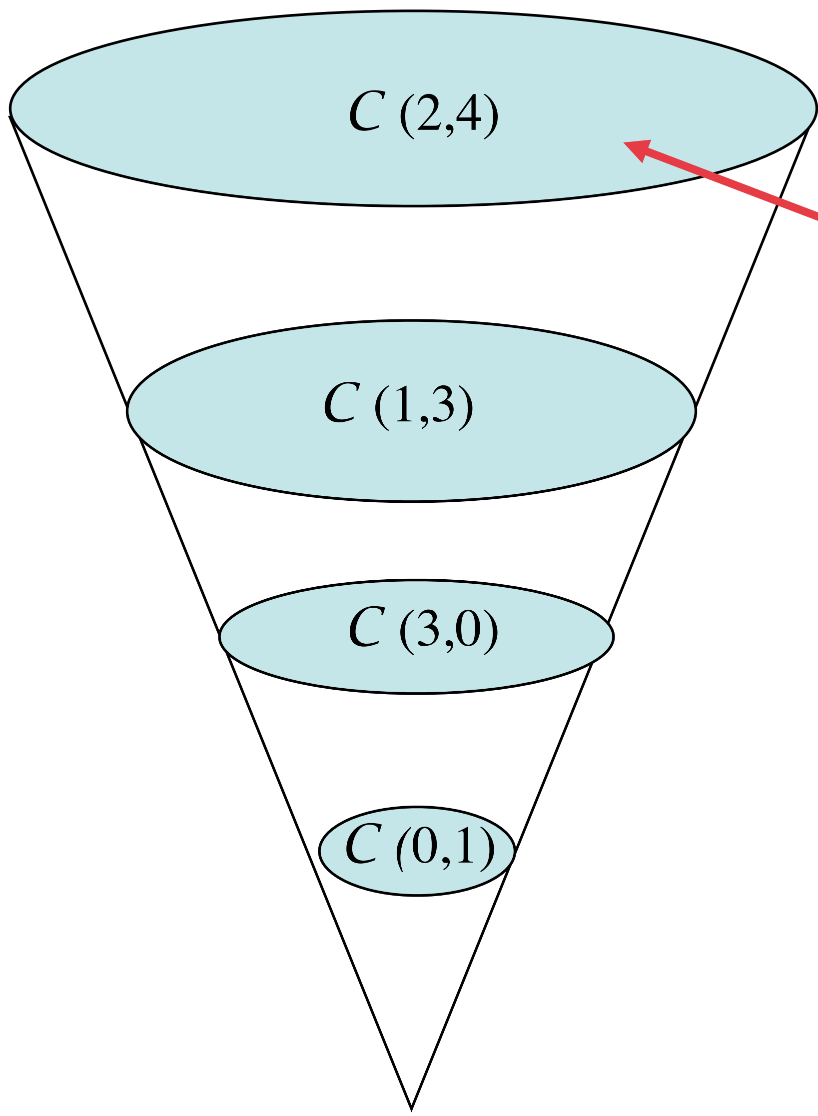
$$[\![P_W]\!]_+ \Rightarrow \;[p, f(x,p)]_{BB} = \frac{(f \star p + p \star f)}{2} \qquad \text{and} \qquad [\![P_W]\!]_- \Rightarrow \;[p, f(x,p)]_{MB} = \frac{(f \star p - p \star f)}{i}$$

What about spin and relativity?

Spin and Clifford Algebras.

We could use standard approach with matrices. Neater to use Clifford algebras.

One single mathematical structure with a natural hierarchy



	Generating elements.
Conformal	$\{1, e_0, e_1, e_2, e_3, e_4, e_5\}$
Twistors	$\{\omega, \pi\}$
Dirac	$\{1, e_0, e_1, e_2, e_3\}$
	$\{1, \gamma_0, \gamma_1, \gamma_2, \gamma_3\}$
Pauli	$\{1, e_1, e_2, e_3\}$
	$\{1, \sigma_1, \sigma_2, \sigma_3\}$
Schrödinger Klein-Gordon.	$\{1, e_1\}$
	$\{1, i\}$

General element of algebra $A(x) = \sum g(x) e_K$ where $e_K = e_i \circ e_j \circ \cdots \circ e_n$ and $i < j < \cdots < n$

Clifford product $e_i \circ e_j + e_j \circ e_i = 2g_{ij}$

[Hiley, and Callaghan, Found. Phys. 10.1007/s10701-011-9558-z; Maths-ph: 1011.4031 and Maths-ph:1011.4033.]

Replacement of kets and bras by elements of the algebra.

Clifford reversion

Replace $\langle x|\psi\rangle$ by $\Phi_L(x) = \phi_L(x)\epsilon$ Replace $\langle \psi|x\rangle$ by $\widetilde{\Phi}_L(x) = \epsilon\widetilde{\phi}_L(x)$

element of min left ideal idempotent $\epsilon^2 = \epsilon$

Pauli

$$\langle x|\Psi_P\rangle \rightarrow \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} \qquad \Phi_L(x) = \left[g_0(x) + \sum g_i(x)e_{jk}\right]\epsilon \qquad i,j,k \text{ cyclic} \qquad g \in \mathbb{R}$$

$$2g_0 = (\psi_1^* + \psi_1) \quad 2e_{123}g_3 = (\psi_1^* - \psi_1)$$
$$2g_2 = (\psi_2^* + \psi_2) \quad 2e_{123}g_1 = (\psi_2^* - \psi_2)$$
$$\epsilon = (1+e_3)/2$$

Dirac

$$\langle x|\Psi_D\rangle \rightarrow \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \psi_3(x)\\ \psi_4(x)\end{pmatrix} \qquad \Phi_L(x) = \left[g_0(x) + \sum g_i(x)e_{\mu\nu} + g_5(x)e_5\right]\epsilon' \qquad \mu < \nu$$

$$\epsilon' = (1+\gamma_0)/2$$

$$\psi_1 = g_0 - ib;\ \psi_2 = -d - ic;\ \psi_3 = h - ig_5;\ \psi_4 = f + ig$$

Complete by adding Schrödinger

$$\langle x|\Psi_s\rangle \rightarrow \psi(x) \qquad \Phi_L(x) = [g_0(x) + g_1(x)e]$$

$$\epsilon = 1$$

$$2g_0 = (\psi^* + \psi) \quad 2eg_1 = (\psi^* - \psi)$$

[Hiley, and Callaghan, Found. Phys. 10.1007/s10701-011-9558-z; Maths-ph: 1011.4031 and Maths-ph:1011.4033.]

Bohm momentum and energy for Pauli particle.

$P_B = -\frac{1}{2}\,[\![P_W]\!]_+$ now becomes

$$\rho P_B(x) = -i\Phi_L(x)\overleftrightarrow{\nabla}_x\widetilde{\Phi}_L(x) = -i\left[(\nabla_x\Phi_L(x))\,\widetilde{\Phi}_L(x) - \Phi_L(x)\left(\nabla_x\widetilde{\Phi}_L(x)\right)\right]$$

$\Phi_L(x) = \phi_L(x)\epsilon$ we choose the idempotent $\epsilon = (1 + e_3)/2$

Pauli current = convection part + rotation part

The Bohm momentum comes from the convection part using $\Phi_L(x) = \left[g_0(x) + \sum g_i(x)e_{jk}\right]\epsilon$

$$\rho P_B = -e_{123}[g_0\nabla_x g_3 - g_3\nabla_x g_0 + g_2\nabla_x g_1 - g_1\nabla_x g_2].$$

Using the conversion $g(x) \to \psi(x)$ and $\psi_j(x) = R_j(x)e^{iS_j(x)}$

$$\rho P_B(x) = \rho_1(x)\nabla_x S_1(x) + \rho_2(x)\nabla_x S_2(x) = -\frac{1}{2}\,[\![P_W]\!]_+$$ **Bohm Momentum**

$$\rho E_B(x) = \rho_1(x)\partial_t S_1(x) + \rho_2(x)\partial_t S_2(x) = -\frac{1}{2}\,[\![P_{tW}]\!]_+$$ **Bohm Energy**

Bohm, Schiller and Tiomno spin-1/2 model.

$$P_B = (\nabla S + \cos\theta\nabla\phi)/2 \qquad\qquad E_B = -(\partial_t S + \cos\theta\partial_t\phi)/2$$

They are the same if you use

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\cos(\theta/2)\exp(i\phi/2)\\ i\sin(\theta/2)exp(-i\phi/2)\end{pmatrix}\exp(i\psi/2)$$

[Bohm, Schiller, and Tiomno, Nuovo Cim. Supp. **1**, (1955) 48-66 and 67-91].

Bohm kinetic energy for spin-1/2 particle.

Then as shown in Hiley and Callaghan, the Bohm kinetic energy is

$$-m[\![P_W^2]\!]_+ = P_B^2(x) + [2(\nabla_x W(x)\cdot S(x)) + W^2(x)] = P_B^2 + Q.$$

Spin of particle $S = i(\phi_L e_3 \widetilde{\phi}_L)$ and $\rho W = \nabla_x(\rho S)$

[Hiley, and Callaghan, Found. Phys. **42**, (2012) 192 and Maths-ph: 1011.4031]

BST values:

$$Q = -\frac{1}{2m}\frac{\nabla^2 R}{R} + [(\nabla\theta)^2 + \sin^2\theta(\nabla\phi)^2]/8m$$

[Bohm, Schiller, and Tiomno, Nuovo Cim. Supp. **1**, (1955) 48-66 and 67-91].

Spin in the Moyal Algebra.

Central to the the Clifford algebra approach is the Clifford density element.

$$\rho(x) = \Phi_L(x)\widetilde{\Phi}_L(x)$$

Then

$$< A >= Tr[A\rho(x)] = Tr[A\Phi_L(x)\widetilde{\Phi}_L(x)] = Tr[\Phi_L(x)A\widetilde{\Phi}_L(x)] \;\; = \langle\psi|\hat{A}|\psi\rangle$$

Generalise $$\rho(x_1, x_2) = \Phi_L(x_1)\Xi_R(x_2)$$

Now we are in a position extend this to phase space using a Wigner-Moyal transformation

$$\rho_M(x,p) = F(x,p) = 2\pi\int \Phi_L(x_1)\Xi_R(x_2)e^{ipy}dy$$ **Cross-Wigner function.**

where $x = (x_2 + x_1)/2$ and $y = x_2 - x_1$

[de Gosson, Symplectic Metods in Harmonic Analysis]

Recall for Pauli $$\Phi_L(x) = \left[g_0(x) + \sum g_i(x)e_{jk}\right]\epsilon$$ **Replaces spinor ket.**

$$\Xi_R(x) = \left[G_0(x) - \sum G_i(x)e_{jk}\right]\epsilon$$ **Replaces spinor bra.**

Use these expressions in the cross-Wigner function.

Details of Spin in the Moyal Algebra.

Easier to use the momentum representation. $$F(x,p) = \left(\frac{1}{2\pi}\right)\int \Xi_L(p_1)\widetilde{\Xi}_L(p_2)e^{-ix\Delta p}d\Delta p.$$

Now with $p = (p_2 + p_1)/2$ and $\Delta p = p_2 - p_1$

and $$\Xi_L(p) = \xi(p)\epsilon = \left[\gamma_0(p) + \sum \gamma_K(p)e_K\right]\epsilon,$$ where the $\gamma(p)$s are Fourier transforms of the $g(x)$s.

Again choose $\epsilon = (1 + e_3)/2,$ taking just the 1 term and then define $2F(x,p) := \int \xi_L(p_2)\ 1\ \widetilde{\xi}_L(p_1)e^{-ix\Delta p}d\Delta p$

So that $$2F(x,p) = \int [\gamma_0(p_2)\gamma_0(p_1) + \gamma_1(p_2)\gamma_1(p_1) + \gamma_2(p_2)\gamma_2(p_1) + \gamma_3(p_2)\gamma_3(p_1)]e^{-ix\Delta p}d\Delta p.$$

Now form $$p \star F(x,p) = pF(x,p) - \frac{i}{2}\nabla_x F(x,p)$$ and $$F(x,p) \star p = pF(x,p) + \frac{i}{2}\nabla_x F(x,p)$$

Baker bracket $[p, F]_{BB} = pF(x,p),$

Moyal bracket $[p, F]_{MB} = \nabla_x F(x,p).$

Use the Moyal relation
$$F(x,p) = \frac{1}{2\pi}\iint M(p_1,p_2)\delta\left(p - \frac{p_2 - p_1}{2}\right)e^{ix(p_2-p_1)}dp_1dp_2$$

where $$M(p_1,p_2) = \tfrac{1}{2}\left[\phi_0(p_1)\phi_0^*(p_2) + \phi_1(p_1)\phi_1^*(p_2) + \phi_2(p_1)\phi_2^*(p_2) + \phi_3(p_1)\phi_3^*(p_2)\right]$$

so that we find
$$\int [p, F]_{BB}dp = \rho_1(x)\partial_x S_1(x) + \rho_2(x)\partial_x S_2(x).$$ **Baker bracket<--> Bohm momentum.**

$$\int [p^2, F(x,p)]_{BB}dp = \underbrace{\rho_1(x)(\nabla_x S_1(x))^2 + \rho_2(x)(\nabla_x S_2(x))^2}_{\textbf{Bohm KE}} - \underbrace{R_1(x)\nabla_x^2 R_1(x) - R_2(x)\nabla_x^2 R.}_{\textbf{Quantum Potential}}$$

Time Development Equations.

$\star$-ganvalues $\quad H(x,p)\star f(x,p) = E_1 f(x,p) \quad$ or $\quad f(x,p)\star H(x,p) = E_2 f(x,p)$

Time development

$$H\star f = \frac{i}{2\pi}\int \psi^*(x_2)[\partial_t\psi(x_1)]e^{ipy}dy \quad \text{and} \quad f\star H = \frac{-i}{2\pi}\int [\partial_t\psi^*(x_2)]\psi(x_1)e^{ipy}dy$$

NB
$x = (x_2 + x_1)/2$
$y = (x_2 - x_1)$

Difference

$$\frac{\partial f}{\partial t} + [f,H]_{MB} = 0$$

In the limit $O(\hbar) \rightarrow$ **Classical Liouville eqn.**

Sum

$$\frac{i}{\pi}\int \psi^*(x_2)\overleftrightarrow{\partial_t}\psi(x_1)e^{ipy}dy + [f,H]_{BB} = 0$$

$$f(x,p)\mathcal{E}_B(x,p) = \frac{i}{2\pi}\int \psi^*(x_2)\overleftrightarrow{\partial_t}\psi(x_1)e^{ipy}dy$$

In the limit $O(\hbar) \rightarrow$ $\quad \frac{\partial S}{\partial t} + H = 0$ $\quad$ **Hamilton-Jacobi eqn.**

Take the marginal as before

$$\frac{\partial f(x,p)}{\partial t} + [f,H]_{MB} = 0$$ is equivalent to $$\frac{\partial \rho(x)}{\partial t} + [\rho,H]_- = 0$$ **Quantum Liouville eqn.**

$$2f(x,p)\mathcal{E}_B(x,p) + [f,H]_{BB} = 0$$ is equivalent to $$2\rho(x)E_B(x) + [\rho,H]_+ = 0$$

If $\quad H = \frac{p^2(x)}{2m} + V(x) \quad$ then $\quad \frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + Q + V = 0$ $\quad$ **Quantum Hamilton-Jacobi eqn.**

[Bohm, Phys. Rev., **85** (1952) 166-179; and **85** (1952), 180-193\.

Summary of Algebraic Time Evolution Equations.

$$\frac{\partial F}{\partial t} + [F, H]_{MB} = 0$$

$$2\frac{\partial S}{\partial t}F + [F, H]_{BB} = 0$$

Moyal algebra

Phase space

$$i\frac{\partial \rho}{\partial t} + [\rho, H]_{-} = 0$$

$$2\frac{\partial S}{\partial t}\rho + [\rho, H]_{+} = 0$$

Quantum algebra

Configuration space

Notice in the general form there is no quantum potential.

The QP appears ONLY in a representation

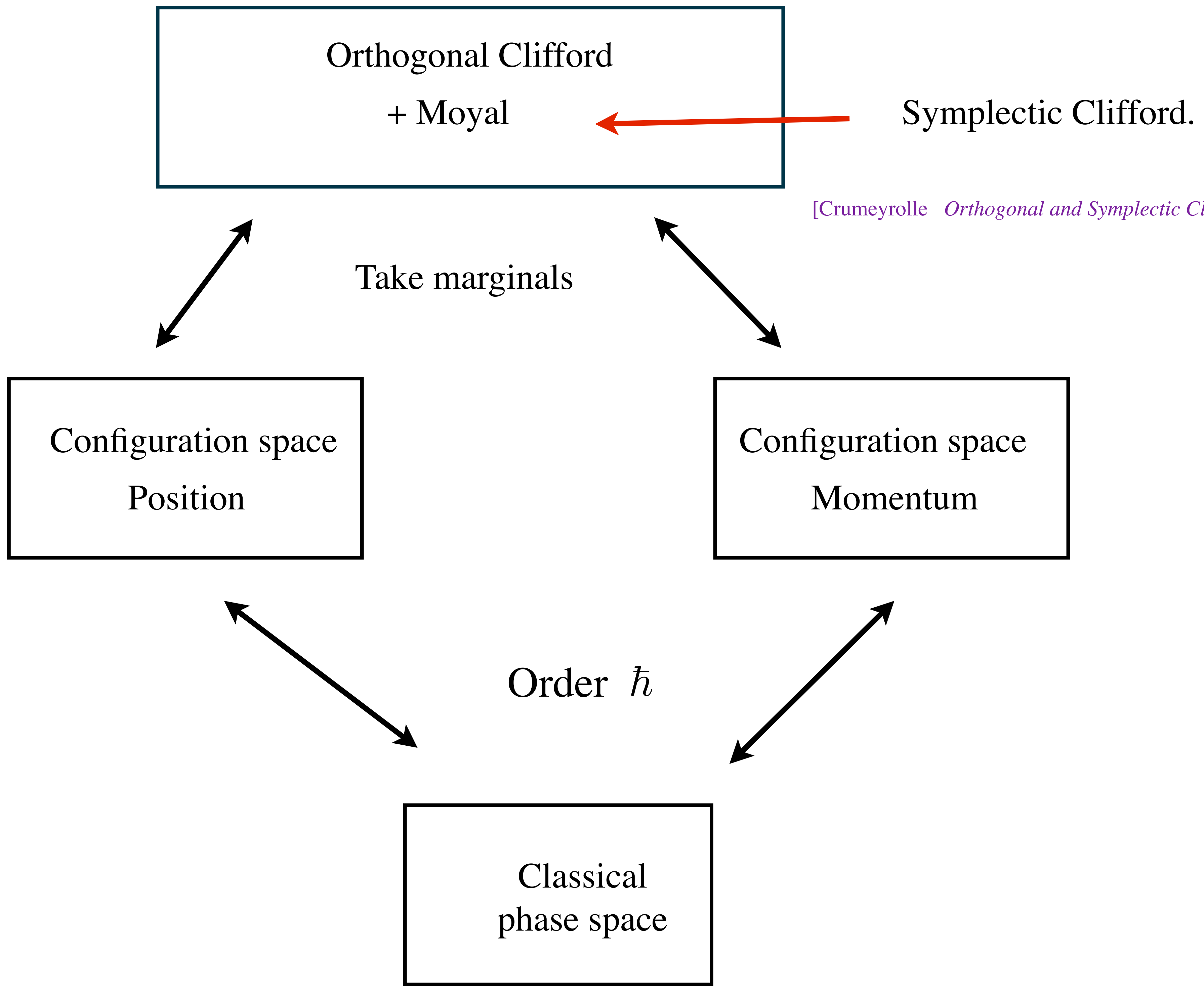
In the *x*-representation you get Bohm's $Q(x)$.

In the *p*-representation you get another QP--$Q(p)$

In Moyal take the *x*-marginal.

[M. R. Brown & B. J. Hiley, quant-ph/0005026]

Mathematical Structure.



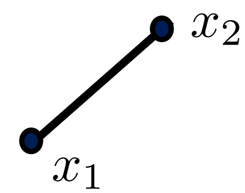
[Crumeyrolle *Orthogonal and Symplectic Clifford Algebras*, (1990)]

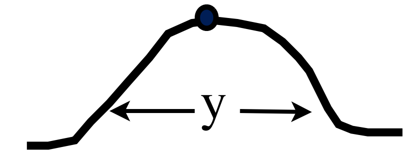
Physics behind these Algebras?

Basic structure $\rho_{\phi\psi}(x_1, x_2) \rightarrow f(x, p)$

Transition $\Rightarrow$ Process

"Phase" space



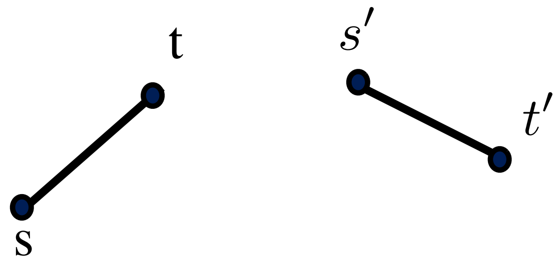


$$\rho_M(x, p) = 2\pi \int \Phi_L(x_1)\Xi_R(x_2)e^{ipy}dy$$

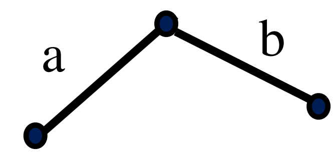
Quantum particle not a 'rock' but a 'blob'.

Structure process.

[Bohm, Proc. Int. Conf. on Elementary Particles, Kyoto, 252-287, (1965)]



Only combine if $t = s'$



groupoid

orthogonal and symplectic

product $\Rightarrow$ **order of succession** and the **order of co-existence** $\Rightarrow$ addition gives the algebra.

Combine Clifford and Moyal algebras $\Rightarrow$ Non-commutative geometry

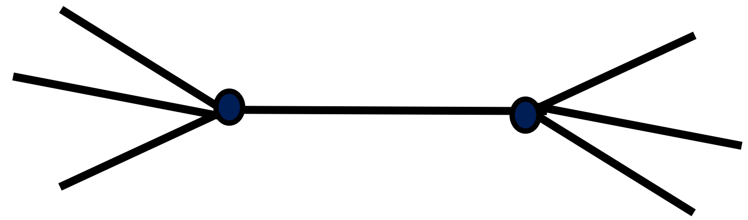
Orthogonal Clifford and symplectic Clifford algebras.

[Hiley, Lecture Notes in Physics, vol. 813, pp. 705-750, Springer (2011)].

[Lizzi, Non-commutative spaces, Springer Lecture Notes in Physics 774, 2009]

Consequences of Non-commutative Structure.

Changes from both sides



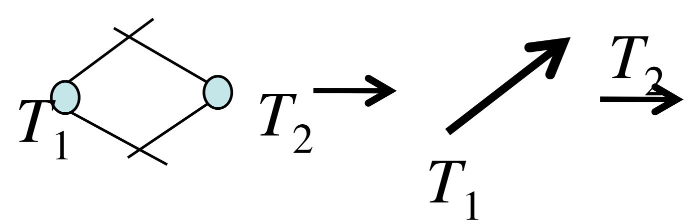
Inner automorphism TAT^{-1}

Evolution in τ $\quad A_\tau = T(\tau)A_{\tau_0}T(\tau)^{-1}$

If $T = \exp[iH\tau]$ then for small τ

$$i\frac{(A_\tau - A_0)}{\tau} = (HA_0 - A_0H) \quad \Rightarrow \quad i\dot{A} = [A, H] \qquad \text{Heisenberg eqn.}$$

Just Bohm's 'folding' and 'unfolding'.



$[T_1, T_2]$

[Hiley, Lecture Notes in Physics, vol. 813, pp. 705-750, Springer (2011)].

Overarching Philosophy.

[Bohm, D., *Wholeness and the Implicate Order*, 1980.]

The deeper structure gives rise to a non-commutative phase space geometry.

In this context non-commutativity $\Rightarrow$ not all orders can be made explicit together.

Implicate order $\Rightarrow$ algebra

Project out explicate orders

$\psi(x)$ or $\phi(p)$ Quantum world

Classical world