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Ellipsoidal Figures of Equilibrium—An Historical Account*

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1. Newton

The study of the gravitational equilibrium of homogeneous uniformly rotating masses began with Newton's investigation on the figure of the earth (*Principia*, Book III, Propositions XVIII–XX). Newton showed that the effect of a small rotation on the figure *must* be in the direction of making it slightly oblate; and, further, that the equilibrium of the body will demand a simple proportionality between the *effect* of rotation, as measured by the ellipticity,

$$\epsilon = \frac{\text{equatorial radius} - \text{polar radius}}{\text{the mean radius}}, \tag{1}$$

and its *cause*, as measured by

$$\begin{aligned} m &= \frac{\text{centrifugal acceleration at the equator}}{\text{mean gravitational acceleration on the surface}} \\ &= \frac{\Omega^2 R}{GM/R^2} = \frac{\Omega^2 R^3}{GM}, \end{aligned} \tag{2}$$

where G denotes the constant of gravitation and M is the mass of the body. More precisely, Newton established the relation

$$\epsilon = \tfrac{5}{4} m \tag{3}$$

in case the body is homogeneous. The arguments by which Newton derived this relation are magisterial; and they are worth recalling.

Newton imagined a hole of unit cross-section drilled from a point on the equator to the center of the earth and a similar hole drilled from the pole to the center; and he further imagined that the "canals" so constructed were filled with

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a fluid (see Figure 1, after Newton's original illustration in the *Principia*). From the fact that the fluid in the canals will be in equilibrium, Newton concludes that the "weights" of the equatorial and the polar columns of the fluid must be equal. However, along the equator the acceleration due to gravity is "diluted" by the centrifugal acceleration; and since both these accelerations in a homogeneous body vary from the center proportionately with the distance, the "dilution factor" remains constant and is given by its value at the boundary, namely m.

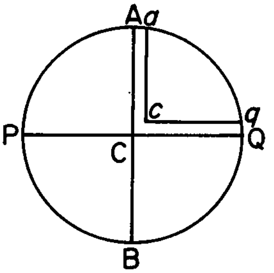

Figure 1

If a denotes the equatorial radius, the weight of the equatorial column is given by

(4) $$\text{weight of equatorial column} = \tfrac{1}{2}a g_{\text{equator}}(1 - m) ,$$

where g_{equator} is the acceleration due to gravity at the equator. Similarly, if b denotes the polar radius,

(5) $$\text{weight of polar column} = \tfrac{1}{2}b g_{\text{pole}} .$$

And since the two weights must be equal,

(6) $$a g_{\text{equator}}(1 - m) = b g_{\text{pole}} .$$

But for a slightly oblate body Newton knew that

(7) $$\frac{g_{\text{pole}}}{g_{\text{equator}}} = 1 + \tfrac{1}{5}\epsilon + O(\epsilon^2) .$$

Equations (6) and (7) and the definition of ϵ $(= 1 - b/a)$ now give

(8) $$1 - m = (1 - \epsilon)(1 + \tfrac{1}{5}\epsilon) + O(\epsilon^2) = 1 - \tfrac{4}{5}\epsilon + O(\epsilon^2) ;$$

and Newton's relation (3) follows.

It was known already in Newton's time that

(9) $$m = \tfrac{1}{290} .$$

Therefore, Newton concluded that if the earth were homogeneous, it should be oblate with an ellipticity

$$\epsilon = \tfrac{5}{4}\tfrac{1}{290} \simeq \tfrac{1}{230}. \tag{10}$$

This prediction of Newton was contrary to the astronomical evidence of the time and "two generations of the best astronomical observers formed in the school of the Cassini's struggled in vain against the authority and reasoning of Newton" (I. Todhunter [1], page 100). The opposing ideas of Newton and Cassini are strikingly illustrated in the accompanying old caricature (Figure 2). However,

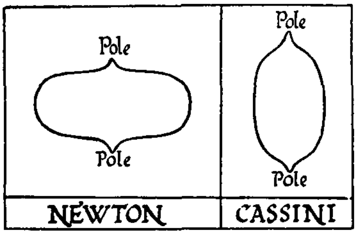

Figure 2

geodetic measurements made in Lapland by Maupertius and Clairaut (1738) afforded data which conclusively showed the flattening of the earth at the poles. As Todhunter has written ([1], page 100), "The success of the arctic expedition may be ascribed in great measure to the skill and energy of Maupertius; and his fame was widely celebrated. The engravings of the period represent him in the costume of a Lapland Hercules having a fur cap over his eyes; with one hand he holds a club and with the other he compresses the terrestrial globe." And Voltaire, then Maupertius' friend, congratulated him warmly for having "aplati les poles et les Cassini." Later Maupertius and Voltaire became involved in a heroic-comic controversy and Voltaire wrote

"Vouz avez confirmé dans les lieux pleins d'ennui
Ce que Newton connut sans sortir de chez lui."

We know now that the actual ellipticity of the earth ($\sim\frac{1}{294}$) is substantially smaller than Newton's predicted value ($\sim\frac{1}{230}$); and this discrepancy is interpreted in terms of the inhomogeneity of the earth.

2. Maclaurin.

The next advance (1742) in the theory was due to Maclaurin who generalized Newton's result to the case when the ellipticity caused by the rotation cannot be considered small.

Maclaurin had solved earlier the problem of the attraction of an oblate spheroid at an internal point; and he had shown in particular that the acceleration due to gravity at the equator and the poles have the values

$$(11)\qquad \begin{aligned} g_{\text{equator}} &= 2\pi G\rho a \frac{\sqrt{1-e^2}}{e^3}\left[\sin^{-1} e - e\sqrt{1-e^2}\,\right], \\ g_{\text{pole}} &= 4\pi G\rho a \frac{\sqrt{1-e^2}}{e^3}\left[e - \sqrt{1-e^2}\sin^{-1} e\right], \end{aligned}$$

where ρ is the density of the spheroid, a its semi-major axis, and e its eccentricity. And since both the centrifugal acceleration in the equatorial plane and the acceleration due to gravity vary linearly with the coordinates, Newton's argument applies to this case equally well and we can write

$$(12)\qquad \begin{aligned} (g_{\text{equator}} - a\Omega^2) &= g_{\text{pole}}\sqrt{1-e^2}, \\ \Omega^2 &= \frac{g_{\text{equator}} - g_{\text{pole}}\sqrt{1-e^2}}{a}. \end{aligned}$$

Inserting the expressions for g_{equator} and g_{pole} from equations (11), we obtain Maclaurin's formula

$$(13)\qquad \frac{\Omega^2}{\pi G\rho} = \frac{\sqrt{1-e^2}}{e^3}\,2(3-2e^2)\sin^{-1} e - \frac{6}{e^2}(1-e^2).$$

Maclaurin realized that the foregoing derivation does not establish that a rapidly rotating mass will necessarily take the figure of an oblate spheroid. But he did show: "1) that the force which results from the attraction of the spheroid and those extraneous powers compounded together acts always in a right line perpendicular to the surface of the spheroid, 2) that the columns of the fluid sustain or balance each other at the center of the spheroid, and 3) that any particle in the spheroid is impelled equally in all directions."

To appreciate the foregoing qualifications of Maclaurin, one must remember that there was as yet no theory of hydrostatic equilibrium which provided *sufficient* conditions; so Maclaurin had to content himself with showing that all the conditions which had been recognized as *necessary* for equilibrium were satisfied. Considering then, the state of knowledge in his time, one can only admire Maclaurin's achievement in deriving the exact relation (13). And as Todhunter remarks ([1], page 175), "Maclaurin well deserves the association of his name with that of the great master in the inscription which records that he was appointed professor of mathematics at Edinburgh *ipso Newtono suadente*."

A remarkable feature of Maclaurin's relation was noticed by Thomas Simpson (1743): for any angular velocity less than a certain maximum value there are two and only two possible "oblata". This result is noteworthy in that we cannot deduce from the fact of a small equatorial angular velocity that the spheroid departs only slightly from a sphere; for as $\Omega^2 \to 0$, we have two solutions: a solution which, indeed, leads to a spheroid of small eccentricity and a second solution which leads to a highly flattened spheroid. It is generally believed that d'Alembert was the first to notice this feature of Maclaurin's solution; but as Todhunter has remarked ([1], page 181): "although d'Alembert may have first explicitly published the statement, yet Simpson gives a table which distinctly implies the fact."

3. Jacobi

For nearly a century after Maclaurin's discovery of the spheroids (known after his name) it was believed that they represent the only admissible solution to the problem of the equilibrium of uniformly rotating homogeneous masses. The supposed generality of Maclaurin's solution was never questioned even though Lagrange in his *Mecanique Celeste* (1811) considered formally the possibility of ellipsoids with unequal axes satisfying the requirements of equilibrium. However, after obtaining two governing equations, in which the two equatorial axes occur symmetrically (see equation (17) below), Lagrange infers that the two axes must be equal even though only the *sufficiency* (not the *necessity*) could be concluded. Jacobi (1834) [3] recognized the inadequacy of Lagrange's demonstration[1] as he remarked, "One would make a grave mistake if one supposed that the spheroids of revolution are the only admissible figures of equilibrium even under the restrictive assumption of second degree surfaces." In making this last statement, Jacobi refers to the fact that while Maclaurin's solution provides, in the limit $\Omega^2 \to 0$, two solutions, one with $e \to 0$ and another with $e \to 1$, Legendre had shown that if one supposes that the figure is nearly spherical so that the attraction at a point on its surface can be expanded in powers of the departure from sphericity, then one obtains only the first of the two solutions "not in any approximation but with absolute geometrical rigor". According to Jacobi, the conclusion one must draw from Legendre's demonstration is that figures of equilibrium may exist that cannot be surmised from what one can establish in the limit of spherical figures. And Jacobi concludes "in fact a simple consideration shows that ellipsoids with three unequal axes can very well be figures of equilibrium; and that one can assume an ellipse of arbitrary shape for the equatorial section and determine the third axis (which is also the least of the three axes) and the angular velocity of rotation such that the ellipsoid is a figure of equilibrium."

The existence of these ellipsoids of Jacobi can be established and the relations governing them can be determined by a simple extension of Newton's original argument.

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[1] Rather as Dirichlet [4] states in his *Gedächtnissrede auf Carl Gustav Jacob Jacobi*, Jacobi's suspicion was aroused by the qualification "necessary" in an account of Lagrange's considerations by the author of a "well-known textbook".

At the time Jacobi made his discovery, it was known that the components of the attraction, g_i, $i = 1, 2, 3$, along the directions of the principal axes of an ellipsoid can be expressed in the manner

$$g_i = -2\pi G\rho A_i x_i , \tag{14}$$

where

$$A_i = a_1 a_2 a_3 \int_0^\infty \frac{du}{(a_i^2 + u)\Delta} , \tag{15}$$

and

$$\Delta^2 = (a_1^2 + u)(a_2^2 + u)(a_3^2 + u) . \tag{16}$$

The formulae for the components of the attraction in the foregoing forms were apparently first derived by Gauss (1813) and by Rodrigues (1815), independently. However, in the less symmetrical forms in which one generally writes them for purposes of reducing them to the standard elliptic integrals of the two kinds, they were known much earlier: they are (as Legendre has said) effectively included in Maclaurin's writings; but explicitly, for ellipsoids with three unequal axes, they occur for the first time[2] in Laplace's *Theorie du Mouvement et de la Figure Elliptique des Planetes* (1784).

Returning to the extension of Newton's argument to the case of tri-axial ellipsoids, we may imagine that three "canals" are drilled along the directions of the three principal axes from the surface to the center and further that they are all filled with a fluid. From the equilibrium of the fluid in the three canals, we may infer the equality of the weights of the three columns (per unit cross-section). We thus have

$$2A_1 a_1^2 - \frac{\Omega^2}{\pi G\rho} a_1^2 = 2A_2 a_2^2 - \frac{\Omega^2}{\pi G\rho} a_2^2 = 2A_3 a_3^2 . \tag{17}$$

These relations require (if $a_1 \neq a_2 \neq a_3$)

$$\frac{\Omega^2}{\pi G\rho} = 2\,\frac{A_1 a_1^2 - A_2 a_2^2}{a_1^2 - a_2^2} = 2a_1 a_2 a_3 \int_0^\infty \frac{u\,du}{(a_1^2 + u)(a_2^2 + u)\Delta} . \tag{18}$$

And we also have the purely geometrical condition

$$A_1 - \frac{a_3^2}{a_1^2} A_3 = A_2 - \frac{a_3^2}{a_2^2} A_3 , \tag{19}$$

[2] As Todhunter has pointed out ([1], page 417), the formulae themselves appear in the writing of d'Alembert though "he deliberately rejects them . . . this is perhaps the strangest of all his (d'Alembert's) strange mistakes." And with regard to Laplace's derivation, Todhunter says ([2], page 32) "thus Laplace values and appropriates the treasure which d'Alembert deliberately threw away."

or

$$a_1^2 a_2^2 \frac{A_2 - A_1}{a_1^2 - a_2^2} = a_3^2 A_3 . \tag{20}$$

This last relation explicitly has the form

$$a_1^2 a_2^2 \int_0^\infty \frac{du}{(a_1^2 + u)(a_2^2 + u)\Delta} = a_3^2 \int_0^\infty \frac{du}{(a_3^2 + u)\Delta} . \tag{21}$$

Equations (18) and (21), in *exactly* these forms, are given in Jacobi's paper. And as Jacobi further states, for any assigned a_1 and a_2, equation (21) allows a solution for a_3 which satisfies the inequality

$$\frac{1}{a_3^2} > \frac{1}{a_1^2} + \frac{1}{a_2^2} , \tag{22}$$

and that when $a_1 = a_2$ equations (18) and (21) determine a configuration common to the spheroidal and the ellipsoidal sequences.

Referring to this discovery of Jacobi, Thomson, and Tait in their *Natural Philosophy* ([5], Volume II, page 530) say "this curious theorem was discovered by Jacobi in 1834 and seems, simple as it is, to have been enunciated by him as a challenge to the French Mathematicians." In Todhunter's "History" there is no reference to Jacobi having issued a "challenge". But Todhunter ([2], page 381) does refer to a communication by Poisson to the French Academy on November 24, 1834 and states "Poisson begins by referring to a letter recently sent by Jacobi to the French Academy in which two results were enunciated. One was what we call Jacobi's theorem, namely, that an ellipsoid is a possible form of relative equilibrium for a rotating fluid; the other related to the attraction of a heterogeneous ellipsoid · · · Poisson's note related to the second result."

4. Meyer and Liouville

In his short and brief paper on the subject, Jacobi did not seriously examine the relationship of his ellipsoids to the Maclaurin spheroids. C. O. Meyer (1842) was the first to do so. Meyer's principal result was to show that the Jacobian sequence "bifurcates" (in the later terminology of Poincaré) from the Maclaurin sequence at the point where the eccentricity $e = 0.81267$. This result can be readily deduced from Jacobi's equations (18) and (21). Thus, by setting $a_1 = a_2$ in these equations we obtain the relations

$$\frac{\Omega^2}{\pi G \rho} = 2 a_1^2 a_3 \int_0^\infty \frac{u\, du}{(a_1^2 + u)^3 \sqrt{a_3^2 + u}} , \tag{23}$$

and

$$a_1^4\int_0^\infty \frac{du}{(a_1^2+u)^3\sqrt{a_3^2+u}} = a_3^2\int_0^\infty \frac{du}{(a_1^2+u)(a_3^2+u)^{3/2}}, \tag{24}$$

where $\Omega^2/\pi G\rho$ on the left-hand side of equation (23) must now be identified with Maclaurin's function (13). It can be shown that both equations (23) and (24) are simultaneously satisfied when

$$e = 0.81267, \quad \text{where } \Omega^2/\pi G\rho = 0.37423\,.$$

Since it is known that the maximum value of $\Omega^2/\pi G\rho$ along the Maclaurin sequence is 0.4493, it follows that for $\Omega^2/\pi G\rho < 0.37423$ there are three equilibrium figures possible: two Maclaurin spheroids and one Jacobi ellipsoid; for $0.4493 > \Omega^2/\pi G\rho > 0.3743$ only the Maclaurin figures are possible; and finally, for $\Omega^2/\pi G\rho > 0.4493$ no equilibrium figures are possible. This enumeration of the different possibilities is due to Meyer.

In 1846 Liouville restated Meyer's result using the angular momentum, instead of the angular velocity, as the variable; and he showed that while the angular momentum increases from zero to infinity along the Maclaurin sequence, the Jacobian figures are possible only for angular momenta exceeding a certain value (namely, that at the point of bifurcation along the Maclaurin sequence).

5. Dirichlet, Dedekind, and Riemann

The fact that no figures of equilibrium are possible for uniformly rotating bodies when the angular velocity exceeds a certain limit raises the question: What happens when the angular velocity exceeds this limit? Dirichlet addressed himself to this question during the winter of 1856–57; and though he included this topic in his lectures on partial differential equations in July 1857, he did not publish any detailed account of his investigations during his lifetime. Dirichlet's results were collated from some papers he left and were edited for publication by Dedekind [6]. "In this paper," as Riemann wrote, "Dirichlet opened up an entirely new approach to investigations bearing on the homologous motions of self-gravitating ellipsoids in a most remarkable way. The further development of this beautiful discovery has a particular interest to the mathematician even apart from its relevance to the forms of heavenly bodies which initially instigated these investigations."

The precise problem which Dirichlet considered in his paper is the following: Under what conditions can one have a configuration which, at every instant, has an ellipsoidal figure and in which the motion, in an inertial frame, is a linear function of the coordinates? Dirichlet formulated the general equations governing this problem (in a Lagrangian framework) and solved them in detail for the case when the bounding surface is a spheroid of revolution. Dirichlet did not seriously investigate the figures of equilibrium admissible under the general circumstances of his formulation. In the latter context, Dedekind ([7] in an addendum to Dirichlet's paper) proved explicitly the following theorem (though, as Riemann remarks, it is already implicit in Dirichlet's equations): *Let a homogeneous ellipsoid with semi-axes* a_1, a_2, *and* a_3 *be in gravitational equilibrium with a prevalent motion whose components, resolved along the instantaneous directions of the principal axes of the ellipsoid and in an inertial frame, are given by*

$$\mathbf{u}^{(0)} = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} \begin{vmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{vmatrix} = \mathbf{A} \begin{vmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{vmatrix}; \tag{25}$$

then the same ellipsoid will also be a figure of equilibrium if the prevalent motion is that derived from the transposed matrix $\mathbf{A}^\dagger$, *i.e.* $\mathbf{u}^{(0)\dagger}$ *given by*

$$\mathbf{u}^{(0)\dagger} = \begin{vmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{vmatrix} \begin{vmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{vmatrix} = \mathbf{A}^\dagger \begin{vmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{vmatrix}. \tag{26}$$

We shall call the configuration with the motion derived from $\mathbf{A}^\dagger$ as the *adjoint* of the configuration with the motion derived from $\mathbf{A}$.

Dedekind considered in particular the configurations which are congruent to the Jacobi ellipsoids and are their adjoints in the sense we have defined.

The motion of a Jacobi ellipsoid rotating uniformly with an angular velocity Ω about the x_3-axis can be represented in the manner

$$\mathbf{u}^{(0)} = \begin{vmatrix} 0 & -\Omega a_2 & 0 \\ \Omega a_1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{vmatrix}. \tag{27}$$

The motion in the adjoint configuration will be given by

$$\mathbf{u}^{(0)\dagger} = \begin{vmatrix} 0 & \Omega a_1 & 0 \\ -\Omega a_2 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} x_1/a_1 \\ x_2/a_2 \\ x_3/a_3 \end{vmatrix}, \tag{28}$$

or, in terms of components,

$$u_1^{(0)\dagger} = \frac{\Omega a_1}{a_2} x_2, \qquad u_2^{(0)\dagger} = -\frac{\Omega a_2}{a_1} x_1, \qquad u_3^{(0)\dagger} = 0; \tag{29}$$

and this motion clearly satisfies the condition

$$\mathbf{u}^{(0)\dagger} \cdot \operatorname{grad} \left(\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \right) = 0 \tag{30}$$

required for the preservation of the ellipsoidal boundary. Also, the motion represented by (29) is one of uniform vorticity

$$\zeta = -\Omega\left(\frac{a_2}{a_1} + \frac{a_1}{a_2}\right) = -\Omega\,\frac{a_1^2 + a_2^2}{a_1 a_2}. \tag{31}$$

These *ellipsoids of Dedekind*, while they are congruent to the Jacobi ellipsoids, are stationary in an inertial frame and they maintain their ellipsoidal figures by the internal motions which prevail. (Lamb erroneously attributes to Love the discovery of this relation between the Jacobi and the Dedekind ellipsoids.) It is also clear that the ellipsoids of Dedekind bifurcate from the Maclaurin spheroids at the same point that the Jacobi ellipsoids do.

The complete solution to the problem of the stationary figures admissible under Dirichlet's general assumptions was given by Riemann [8] in a paper of remarkable insight and power. Riemann first shows that under the restriction of motions which are linear in the coordinates, the most general type of motion compatible with an ellipsoidal figure of equilibrium consists of a superposition of a uniform rotation $\boldsymbol{\Omega}$ and internal motions of a uniform vorticity $\boldsymbol{\zeta}$ (in the rotating frame). More precisely he showed that ellipsoidal figures of equilibrium are possible only under the following three circumstances: (a) the case of uniform rotation with no internal motions, (b) the case when the directions of $\boldsymbol{\Omega}$ and $\boldsymbol{\zeta}$ coincide with a principal axis of the ellipsoids, and (c) the case when the directions of $\boldsymbol{\Omega}$ and $\boldsymbol{\zeta}$ lie in a principal plane of the ellipsoid. Case (a) leads to the sequences of Maclaurin and Jacobi. Case (b) leads to sequences of ellipsoids along which the ratio $f = \zeta/\Omega$ remains constant (the Jacobian and the Dedekind sequences are special cases of these general "Riemann sequences" for $f = 0$ and ∞, respectively). And finally, case (c) leads to three other classes of ellipsoids. Riemann wrote down the equations governing the equilibrium of these ellipsoids and specified their domain of occupancy in the a_1, a_2, a_3-space. (A more detailed description of the properties of these ellipsoids will be found in the Epilogue.) Riemann also sought to determine the stability of these ellipsoids by an energy criterion. But his criterion, as has recently been shown by Lebovitz [9], is erroneous and Riemann's conclusions, with the notable exception of those pertaining to the Maclaurin and the Riemann sequences for $f \geqq -2$, are false.

While Riemann's paper made an impressive start towards the solution of Dirichlet's general problem, it left a large number of questions unanswered. Indeed, even the relation of Riemann's ellipsoids to the Maclaurin spheroids which they adjoin was left obscure. Nevertheless these questions were to remain unanswered for more than a hundred years. The reason for this total neglect must, in part, be attributed to a spectacular discovery by Poincaré (see Section 6 below) which channeled all subsequent investigations along directions which appeared rich with possibilities; but the long quest it entailed turned out, in the end, to be after a chimera.

6. Poincaré and Cartan

The investigations relating to the equilibrium and the stability of ellipsoidal figures of equilibrium, for which Dirichlet and Riemann had laid such firm foundations, took an unexpected turn (from which it was not to be diverted for the next seventy-five years) when Poincaré [10] discovered in 1885 that along the Jacobian sequence a point of bifurcation occurs similar to the one along the Maclaurin sequence and that even as the Jacobian sequence branches off from the Maclaurin sequence, a new sequence of pear-shaped configurations branches off from the Jacobian sequence. This result of Poincaré is equivalent to the statement (in current terminology) that along the Jacobian sequence there is a point where the ellipsoid allows a neutral mode of oscillation belonging to the third harmonics. A corollary which was also enunciated by Poincaré is that along the Jacobian sequence there must be further points of bifurcation where the Jacobian ellipsoid allows a neutral mode of oscillation belonging to the fourth, fifth, and higher harmonics. And Poincaré conjectured "that the bifurcation of the pear-shaped body leads onward stably and continuously to a planet attended by a satellite, the bifurcation into the fourth zonal harmonic probably leads unstably to a planet with a satellite on each side, that into the fifth harmonics to a planet with two satellites on one and one on the other and so on" (Darwin). It was further conjectured by Darwin that one may look for the origin of the double stars in similar instabilities; the "fission theory" of the origin of double stars arose in this fashion. The grand mental panorama that was thus created was so intoxicating that those who followed Poincaré were not to recover from its pursuit. In any event, Darwin, Liapounoff, and Jeans spent years of effort towards the substantiation of these conjectures; and so single minded was the pursuit[3] that one did not even linger to investigate the stability of the Maclaurin spheroids and the Jacobi ellipsoids from a direct analysis of normal modes. Finally, in 1924 Cartan [11], [12] established that the Jacobi ellipsoid becomes unstable at its first point of bifurcation and behaves in this respect differently from the Maclaurin spheroid which, in the absence of any dissipative mechanism, is stable on either side of the point of bifurcation where the Jacobian sequence branches off.

And at this point the subject quietly went into a coma.

Epilogue

The subject of the allowed ellipsoidal figures of equilibrium of homogeneous masses and their stability, left incomplete by Riemann, has now been completed

[3] For example, the question whether along the Dedekind sequence a neutral point occurs similar to the one along the congruent Jacobian sequence does not appear to have been considered or even raised.

(Chandrasekhar [13], [14], and [15]). This is clearly not the place to attempt even an outline of the methods by which the various aspects of the problem have been solved (however, for a brief account, see Chandrasekhar and Lebovitz [16]); we shall content ourselves by simply exhibiting the results of the analysis. Figure 3 depicts the domains of occupancy of the ellipsoidal figures in the a_2/a_1, a_3/a_1-plane, and Figures 4 and 5 present results of the stability analysis.

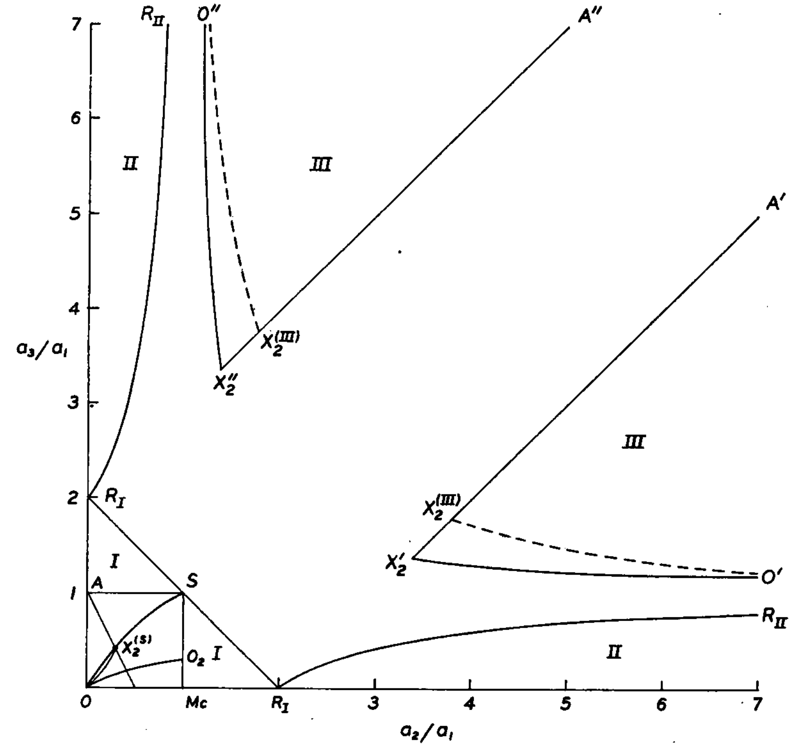

Figure 3. The domain of occupancy of the Riemann ellipsoids in the a_2/a_1, a_3/a_1-plane.

The stable part of the Maclaurin sequence is represented by the segment O_2S on the line $a_1 = a_2$. At O_2 the Maclaurin spheroid becomes unstable by overstable oscillations.

The Riemann ellipsoids of type S (for which the directions of rotation and vorticity coincide with the x_3-axis) are included between the selfadjoint sequences represented by SO and O_2O. Along the arc $X_2^{(S)}O$ the Riemann ellipsoids of type S become unstable by an odd mode of oscillation belonging to the second harmonics.

The Riemann ellipsoids, in which the directions of rotation and vorticity do not coincide but lie in the a_2, a_3-plane, are of three types—I, II, and III—with the domains of occupancy shown. Type I ellipsoids adjoin the Maclaurin sequence and are bounded on one side (SR_{I}) by a selfadjoint sequence. Along the locus $R_{\text{I}}R_{\text{II}}$, which limits the domain of occupancy of the type II ellipsoids, the pressure is zero. And along the loci $X_2'O'$ and $X_2''O''$, limiting the domain of occupancy of the type III ellipsoids, the directions of $\mathbf{\Omega}$ and $\boldsymbol{\zeta}$ coincide with one of the principal axes (the a_3-axis in the case $a_2 > a_1$ and the a_2-axis in the case $a_2 < a_1$). The locus $X_2'O'$ (for the case $a_2 > a_1$) is transformed into $X_2^{(S)}O$ if the roles of a_1 and a_2 are interchanged; and simultaneously the domain of occupancy $A'X_2'O'$ similarly becomes transformed into the domain $AX_2^{(S)}O$. The dotted curve $X_2^{(\mathrm{III})}O'$ defines the locus of configurations, among the type III ellipsoids, that are marginally overstable by a mode of oscillation belonging to the second harmonics.

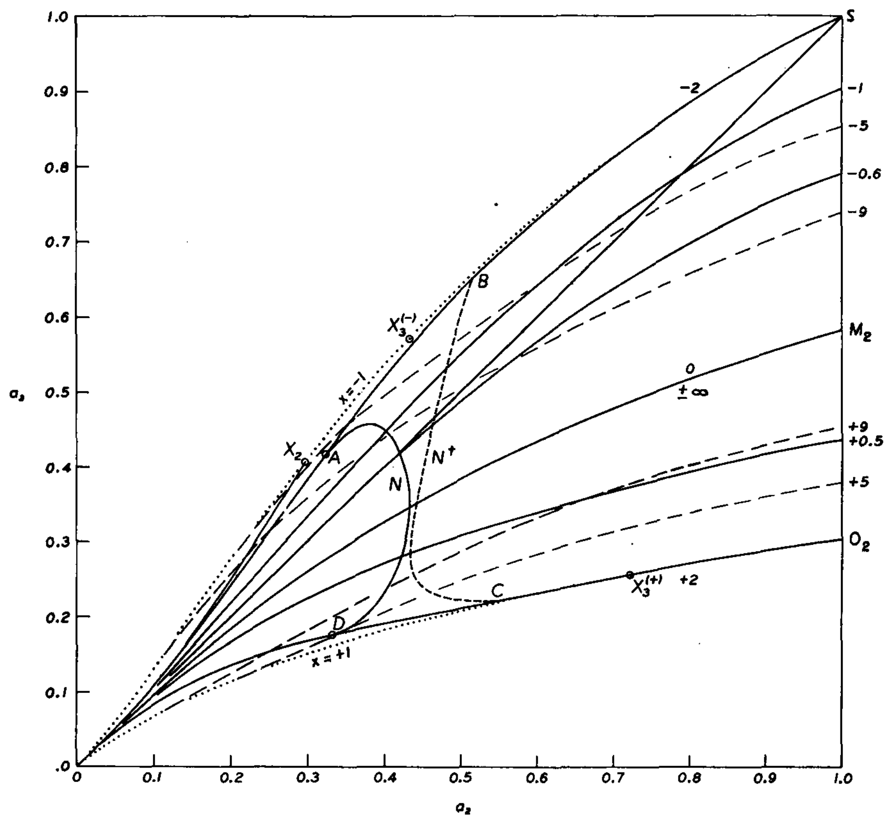

Figure 4. The Riemann ellipsoids of type S (for which the directions of rotation and vorticity coincide with the x_3-axis) can be arranged in sequences along which $f = \zeta/\Omega$ is a constant.

The stable part of the Maclaurin sequence is represented by the segment O_2S of the line $a_2 = 1$. At O_2 the Maclaurin spheroid becomes unstable by overstable oscillations and at M_2 the Jacobian and the Dedekind sequences bifurcate (labeled by "0, $\pm\infty$").

The different Riemann sequences are labeled by the values of f to which they belong; these sequences are bounded by the two selfadjoint sequences (the dotted curves labeled $x = -1$ and $x = +1$) along which $f = f^\dagger = \mp(a_1^2 + a_2^2)/a_1a_2$. The sequences belonging to f in the range $-2 \leqq f \leqq +2$ form a nonintersecting family of continuous curves which join points on the line O_2S to the origin. The sequences belonging to $f < -2$ and $f > +2$ are represented by curves which consist of two parts: a part which joins a point on the line SM_2 (or M_2O_2) to a point of the selfadjoint sequence for $x = -1$ (or $x = +1$) and a part which joins the point on the selfadjoint sequence to the origin. Along the selfadjoint sequence $x = -1$, instability by a mode of oscillation belonging to the second harmonics sets in at the point indicated by X_2 and the locus of points at which instability by this mode sets in is the curve which joins X_2 to the origin. The curve labeled AND is the locus of neutral points, belonging to the third harmonics, along the Riemann sequences for $-2 \leqq f \leqq +2$; and the curve labeled $BN^{\dagger}C$ is the corresponding locus for configurations adjoint to the Riemann ellipsoids represented in the domain included between the same sequences $f = -2$ and $f = +2$. The continuations of the loci AND and $BN^{\dagger}C$ into the domains included between the sequences $x = -1$ and $f = -2$ (and, similarly, between the sequences $x = +1$ and $f = +2$) are represented by curves (not shown) joining the points A and B to $X_3^{(-)}$ on the sequence $x = -1$ (and, similarly, by curves joining the points D and C to the point $X_3^{(+)}$ on the sequence $x = +1$); $X_3^{(-)}$ and $X_3^{(+)}$ are the neutral points, belonging to the third harmonics, along the selfadjoint sequences $x = -1$ and $x = +1$, respectively.

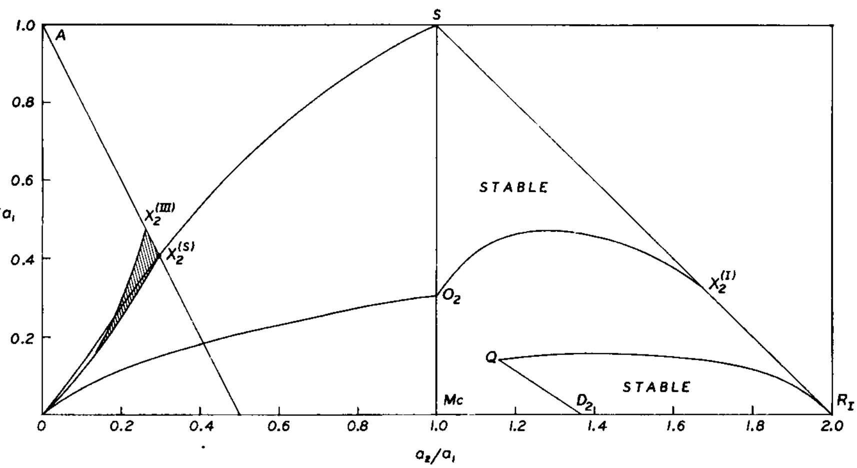

Figure 5. The loci of marginally stable configurations in the a_2/a_1, a_3/a_1-plane.

The type S ellipsoids are bounded by two selfadjoint sequences (SO and O_2O) and the stable part of the Maclaurin sequence represented by SO_2. Along the arc $X_2^{(S)}O$ the type S ellipsoids become unstable by a mode of oscillation belonging to the second harmonics; and along this same arc the stability passes to the type III ellipsoids whose domain of occupancy is $AX_2^{(S)}O$. The shaded region included between $X_2^{(III)}O$ and $X_2^{(S)}O$ represents the domain of stability for type III ellipsoids with respect to oscillations belonging to the second harmonics.

The type I ellipsoids occupy the triangle $SMcR_I$; and the region of the stable members is included in the two domains marked "stable". The domain $SO_2X_2^{(I)}$ of stable ellipsoids adjoining the stable Maclaurin spheroids is to be expected, but the domain D_2QR_I including disklike ellipsoids along D_2R_I is unexpected.

All type II ellipsoids are unstable.

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